SEGMENTS OF ORDERED SETS(1)

BY

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SECTION I. INTRODUCTION

A set P of elements $a, b, c, \cdots$ in which is defined a binary relation "$\leqq$" having the properties (1) $a \leqq a$ for all $a \in P$, (2) $a \leqq b$ and $b \leqq a$ imply $a = b$, (3) $a \leqq b$, $b \leqq c$ imply $a \leqq c$, will be called an *ordered set*. If for any pair of elements $a, b \in P$ one of the three relations $a \leqq b$, $b \leqq a$, $a = b$ necessarily holds, P is said to be completely ordered(2). In such a set, the subset of all elements x which satisfy the condition $a \leqq x \leqq b$ is the segment joining a and b; however, this definition cannot be used for any pair of elements of an arbitrary ordered set, since it implies that $a \leqq b$. In Section II a definition of a segment is given which is applicable to any pair of elements of an ordered set. Lattices and their fundamental properties can then be defined entirely in terms of the notion of segment. In Section III, the lattice of segments of a lattice is defined, generalizing the lattice of quotient lattices of Ore [3]. A significant difference between our theory and Ore's is that our ordering relation for segments preserves the relation of set-theoretic inclusion between the segments as sets.

Section IV contains a discussion of the convex subsets of a lattice which may be defined in the natural way in terms of segments. An imbedding of any lattice in a complete lattice is obtained by use of the lattice of its convex subsets. In Section V we define in terms of segments two new types of lattice with interesting geometric interpretations.

The following definitions and notation are used: A *lattice* L is an ordered set which contains with any pair of elements a and b their *least upper bound* $a+b$ (an element such that $a+b \geqq a$, b and $c \geqq a$, b implies $c \geqq a+b$) and their *greatest lower bound* ab (an element such that $ab \leqq a$, b and $d \leqq a$, b implies $d \leqq ab$). An element $0 \in L$ such that $0 \leqq a$ for all $a \in L$ is the *zero* of L; dually an element $1 \in L$ such that $1 \geqq a$ for all $a \in L$ is the *unit* of L. Two elements a and b such that $ab = 0$ are called μ-independent; if $a+b=1$, they are α-independent(3); and if they are both α-independent, and μ-independent, they are

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(1) This paper is the major portion of the author's doctoral thesis, Duthie [9]. (Numbers in brackets refer to the bibliography at the end of the paper.) The omitted portion consists of a fuller treatment of Section II and a discussion of segments and their applications in Boolean algebras.

(2) This terminology differs from the usual designation of *partially ordered* and *linearly ordered*, respectively, for the set P; since this paper is concerned only incidentally with the latter type of order, it is more convenient to make specific mention of it when it occurs.

(3) The α and μ abbreviations for *additive* and *multiplicative* follow a convention of Stone [5], where they are used to denote additive and multiplicative ideals.

complements. A lattice in which every element has a complement is called *complemented*. L is *modular* if $a \leqq c$ implies $(a+b)c=a+bc$ for all $b, c \in L$; *distributive* if $(a+b)c=ac+bc$ for all $a, b, c \in L$; *complete* if any subset of elements of L has a least upper and greatest lower bound in L. Two lattices L and L^* are *isomorphic* if there is a one-one mapping of their elements such that $a \leftrightarrow a^*$, $b \leftrightarrow b^*$ implies $a+b \leftrightarrow a^*+b^*$ and $ab \leftrightarrow a^*b^*$; they are *dually isomorphic* if $a+b \leftrightarrow a^*b^*$ and $ab \leftrightarrow a^*+b^*$. The set-theoretical intersection of two subsets A and B of elements of a lattice will be denoted by $A \cap B$. $a \prec b$ means $a<b$ and $a<x<b$ has no solution.

Section II. Lattice properties in terms of segments

1. **Definition of a segment and a lattice.** Let a, b be any two elements of an ordered set P. If $a \leqq b$, "the segment $[a, b]$ joining a and b" is defined to be the set of all elements x of P such that $a \leqq x \leqq b$. We order such segments so as to preserve set-theoretic inclusion; that is, $[a, b] \leqq [c, d]$ if and only if $a \geqq c$ and $b \leqq d$ and hence if and only if the set $[c, d]$ contains the set $[a, b]$.

If neither $a \leqq b$ nor $b \leqq a$, we define the symbol $[a, b]$ as the set-theoretic intersection of all segments $[u, v]$ containing both a and b. It is easily seen that $[a, b]$ will be a segment if and only if a and b have a least upper bound $a+b$ and a greatest lower bound ab in P, and in this case, it equals $[ab, a+b]$. We thus arrive at the following definition:

Definition 2.1. *The segment joining any pair of elements x and y of an ordered set P is the set of all elements $z \in P$ which satisfy the condition*

$$xy \leqq z \leqq x+y.$$

This set will be denoted by the symbol $[x, y]$. The elements xy and $x+y$ will be called the lower and upper extremities, respectively, of the segment $[x, y]$.

Corollary 2.2. *An ordered set P is a lattice if and only if there is a segment $[x, y]$ joining every pair of elements $x, y \in P$.*

Because of the uniqueness of least upper and greatest lower bounds in a lattice L, the following corollaries are also immediate consequences of Definition 2.1.

Corollary 2.3. *There is one and only one segment joining each pair of elements of a lattice L.*

Corollary 2.4. *Two segments coincide if and only if their extremities coincide.*

Since the condition $xy \leqq z \leqq x+y$ is transformed into itself by dualization, being a segment is a self-dual property of subsets of an ordered set.

2. **Principal ideals as segments.** As an example of a segment in a lattice L, consider the principal μ-ideal or α-ideal generated by an element $a \in L$, which

is denoted by $(a)_\mu$ or by $(a)_\alpha$ respectively. Then $(a)_\mu$ ($(a)_\alpha$) is the set of all elements of the form xa ($x+a$), $x \in L$; but that is the same as the set of all $y \in L$ such that $y \leqq a$ ($y \geqq a$). Hence if L has a zero element (unit element), then $(a)_\mu = [0, a]$ ($(a)_\alpha = [0, 1]$) proving

THEOREM 2.5. *If L is a lattice with a zero (unit), then a segment of L is a principal μ-ideal (α-ideal) if and only if it contains the zero (unit) element.*

This theorem is used in what follows as the definition of principal ideals, and other concepts of lattice theory will also be characterized by means of segments; hence it is desirable to show that all the fundamental definitions of the theory can be stated in terms of segments, and the remainder of this section is devoted to that purpose.

3. **Modularity.** Given a segment and a point in a completely ordered set such as the real line, it is trivial that if the segments joining the point to the extremities of the given segment coincide, then the segment is a point. On the other hand, in the case of segments and elements of a lattice, this fact is not only not trivial but even untrue, as is shown by

THEOREM 2.6. *A lattice L is modular if and only if the identity of the segments joining an element to the extremities of the segment joining two other elements implies the identity of these two elements.*

Proof. It is known (Ore [3, p. 413]) that L is modular if and only if for any a, b, c, $a \geqq b$ and $a+c=b+c$, $ac=bc$ imply $a=b$. Thus by Corollary 2.4, L is modular if and only if $a \geqq b$ and $[a, c]=[b, c]$ imply $a=b$. Since $a+b=ab$ if and only if $a=b$, and since $a+b \geqq ab$ always, L is modular if and only if for any a, b, c $[a+b, c]=[ab, c]$ implies $a=b$. This last statement is equivalent to the theorem.

4. **Distributivity.** Since the definition of distributivity to be given below involves set-theoretic intersection of segments, the following simple lemma will be needed.

LEMMA 2.7. $[a, b] \cap [c, d] = [ab+cd, (a+b)(c+d)]$ *if the intersection is non-empty.*

Proof. Any element $x \in L$ which belongs to both $[a, b]$ and $[c, d]$ is subjected to the simultaneous conditions $ab \leqq x \leqq a+b$ and $cd \leqq x \leqq c+d$. Hence the smallest x which satisfies these conditions is $x=ab+cd$ and the largest is $x=(a+b)(c+d)$. The intersection will be non-empty if and only if

$$ab + cd \leqq (a + b)(c + d).$$

In a completely ordered set such as the real line, a point lying on the segment joining two points is uniquely represented as the intersection of the segments joining it to each of these points. That this is likewise a non-trivial property of elements and segments of a lattice is shown by

THEOREM 2.8. *A lattice is distributive if and only if any element belonging to the segment joining two elements a, b is the intersection of the segments joining it to each element a, b.*

Proof(4). If $c \in [a, b]$, then by Lemma 2.7, $[a, c] \cap [c, b] = [ac+bc, (a+c)(b+c)]$. Hence it suffices to show that L is distributive if and only if

(i) $a+b \geqq c \geqq ab$ implies $ac+bc=c$ and $(a+c)(b+c)=c$.

Now (i) is easily seen to be equivalent to

(ii) L contains no non-distributive sublattice of order five.

But (ii) is a necessary and sufficient condition for distributivity, (G. Birkhoff [2, Theorem 3.7, Theorem 5.2 with second corollary]).

5. **Complementation.** The fact that an element a of a lattice has a complement is easily translated into segment terms by the following definition.

DEFINITION 2.9. *An element a of a lattice L with zero and unit elements is a complement of another element b if and only if the segment joining a and b is the whole lattice L.*

The following proof of the well known fact that complements in a distributive lattice are unique illustrates the use of segments in deriving other lattice properties: Assume that an element a has two complements, a' and a''. Then $[a'', a'] = [a'', a'] \cap [0, 1] = [a'', a'] \cap [a', a] = [a', a']$, where the last equality follows from Theorem 2.8. Hence by Corollary 2.4, $a''+a' = a' = a''a'$, and so $a''=a'$.

SECTION III. FORMAL PROPERTIES OF SEGMENTS

1. **Segments as lattices.** As an immediate corollary of Definition 2.1, we have the following properties of the symbol $[a, b]$:

$$[a, b] = [b, a] = [ab, a+b] = [a+b, ab].$$

Thus there is no loss in generality if the symbol $[a, b]$ is considered to be an *ordered pair* with the additional condition $a \leqq b$; then $[a, b]$ is a sublattice of L whose zero element is a and whose unit element is b. Such a sublattice has been called by Ore [3] a "quotient structure."

Properties of L such as modularity, distributivity, completeness, and complementation carry over from L to $[a, b]$(5).

THEOREM 3.1. *A sublattice of the direct sum of two lattices is a segment if and only if it is the direct sum of a segment in one by a segment in the other.*

Proof. Let (a, b) and (c, d) be the lower and upper extremities of a segment in the direct sum. Then an arbitrary element of the segment satisfies the con-

(4) A longer, but purely combinatorial, proof of this theorem due in part to J. von Neumann is given in Duthie [9]. This proof was supplied by the referee, to whom the author is indebted for considerable revision of this section.

(5) The proofs are given in von Neumann [6, vol. 1].

dition $(a, b) \leqq (x, y) \leqq (c, d)$. But from the definition of direct sum, this is equivalent to the two conditions $a \leqq x \leqq c$ and $b \leqq y \leqq d$ together.

This theorem is used in the following example.

EXAMPLE 3.2. Let E_1 be the completely ordered real line, and consider the real plane as the (lattice) direct sum $E_1 \oplus E_1$. With respect to this ordering, segments in the plane are of three types: (1) single points, (2) closed segments parallel to either axis, (3) closed rectangles with sides parallel to the axes.

The usual theorems on the modularity, distributivity, and complementation of the direct sum of two lattices are corollaries of Theorem 3.1, in view of the formulation of these properties given in Section II.

2. **Ordered sets of segments.** The set of segments of a lattice, which will be denoted by L_s, may be ordered as follows:

DEFINITION 3.3. *There is an element $\phi \in L_s$ such that for all $[a, b] \in L_s$, $\phi < [a, b]$.*

The element ϕ should not be confused with the element $[0, 0]$, in case L has a zero element.

DEFINITION 3.4. *$[a, b] \leqq [c, d]$ if and only if $a \geqq c$ and $b \leqq d$. The segment $[c, d]$ is called an extension of $[a, b]$.*

It is immediately apparent from Definition 3.4 that the set $L_s - \phi$ is closely related to the direct sum of L_d (the dual of L obtained by interchanging $\leqq$ and $\geqq$ in L) and L. In fact, the only thing which prevents $L_s - \phi$ and $L_d \oplus L$ from being isomorphic is the restriction $a \leqq b$ on the elements $[a, b]$ of L_s. We are thus led to the consideration of a subset S of elements (a, b) of $L_d \oplus L$ subjected to a similar restriction. Since $(a, b) + (c, d) = (ac, b+d)$ and $ac \leqq b+d$ if $a \leqq b$ and $c \leqq d$, the set S is closed under addition. However, it is not closed under multiplication, for $(a, b) \cdot (c, d) = (a+c, bd)$ and the condition $a+c \leqq bd$ will not in general hold. Hence S is not a *sublattice* of $L_d \oplus L$, but it is possible to make it into a *lattice* by adjoining an element θ with the properties

$$\begin{aligned} \theta \cdot (a, b) &= (a, b) \cdot \theta = \theta, \\ \theta + (a, b) &= (a, b) + \theta = (a, b), \end{aligned} \qquad \text{for all } (a, b) \in S,$$

$$(a, b) \cdot (c, d) = \theta \qquad \text{if } a + c \not\leqq bd, \text{ and } a + c \neq bd.$$

This discussion may then be summarized as follows:

THEOREM 3.5. *The set of segments of a lattice L, ordered as in Definitions 3.3 and 3.4, is a lattice isomorphic to the subset S of $L_d \oplus L$ with the element θ adjoined, and the elements ϕ and θ correspond under the isomorphism.*

Furthermore, the operation of multiplication in L_s is equivalent to set-theoretical intersection of segments, by virtue of Lemma 2.7.

If the sum of two segments is defined as the intersection of all segments containing them, then the same L_s is obtained as above. However, the derivation by use of the direct sum of L and L_d makes some later results more easily proved.

3. **Relations between L and L_s.** In looking for special properties of L_s, it is natural to seek first those properties of L which carry over to L_s. By definition, L_s always has a zero element, even if L does not, and it is easily seen that L_s will have a unit if and only if L has both a zero and a unit, in which case the unit is $[0, 1]$. It is also obvious from the definition of the lattice operations in L_s that if L is complete, then L_s is also complete. Another property of L_s which carries over from L is given in

THEOREM 3.6. *If L is complemented, then L_s is complemented.*

Proof. Two elements $[a, b]$ and $[c, d]$ of L_s are complements if and only if

$$[a, b]\cdot[c, d] = \phi = [a + c, bd],$$
$$[a, b] + [c, d] = [0, 1] = [ac, b + d].$$

Hence $[a, b]$ and $[c, d]$ will be complements if their lower extremities are μ-independent in L and their upper extremities are α-independent in L, and $a+c \not< bd$, and $a+c \neq bd$, conditions which can obviously be satisfied if L is complemented.

This proof also shows that an element of L_s may have more than one complement, even though complementation in L is unique; hence there is no hope that distributivity will carry over from L to L_s in general. That it does not even carry over into modularity is shown by

EXAMPLE 3.7. Let L be the completely ordered set $0 \prec a \prec 1$, and consider the elements $[0, 0]$, $[0, a]$, and $[1, 1]$ of L_s. Since $[0, 0] < [0, a]$, the modular law may be applied to these three elements, giving

$$\{[0, 0] + [1, 1]\}[0, a] = [0, 1]\cdot[0, a] = [0, a];$$

but

$$[0, 0] + [1, 1]\cdot[0, a] = [0, 0] + \phi = [0, 0].$$

Hence L_s is not modular. Incidentally, L is not complemented, and neither is L_s.

Two more general types of lattices have recently been defined and discussed by Wilcox [8, pp. 495–496]. Making use of a binary relation $(b, c)M$, which means $(a+b)c = a+bc$ for all $a \leqq c$, a lattice is said to be *symmetric* if it satisfies

AXIOM A. *If $bc=0$ and $(b, c)M$, then $(c, b)M$.*

L is called *semi-modular* if, in addition, it satisfies

AXIOM B. *If* $bc \neq 0$, *then* $(b, c)M$.

THEOREM 3.8. *If L is modular and contains more than two elements, then L_s satisfies Axiom* B *but not Axiom* A.

Proof. In Example 3.7, it was shown that $([1, 1], [0, a])M$ does not hold; but $[0, a] \cdot [1, 1] = \phi$ and $([0, a], [1, 1])M$ since $[1, 1]$ is atomic(6). If L has only two elements, then L_s has the four elements $[0, 1]$, $[1, 1]$, $[0, 0]$, ϕ and is a Boolean algebra—hence *a fortiori* satisfies Axiom A.

Now consider the so-called one-sided modular law, valid in any lattice whatsoever, written in terms of elements of L_s: $[a_1, a_2] \leqq [c_1, c_2]$ implies

$$\{[a_1, a_2] + [b_1, b_2]\}[c_1, c_2] \geqq [a_1, a_2] + [b_1, b_2][c_1, c_2].$$

Assuming that $[b_1, b_2] \cdot [c_1, c_2] \neq \phi$ and $[a_1, a_2] \neq \phi$ (the equality of the two sides of the above expression is trivial if $[a_1, a_2] = \phi$) all elements and combinations of elements in the inequality are different from ϕ, and hence the rules for combining elements in L_s will be valid. Therefore

$$\{[a_1, a_2] + [b_1, b_2]\}[c_1, c_2] = [a_1b_1, a_2 + b_2][c_1, c_2] = [a_1b_1 + c_1, (a_2 + b_2)c_2].$$

Since L is modular, and $c_1 \leqq a_1 \leqq a_2 \leqq c_2$,

$$\begin{aligned}[a_1b_1 + c_1, (a_2 + b_2)c_2] &= [a_1(b_1 + c_1), a_2 + b_2c_2] = [a_1, a_2] + [(b_1 + c_1), b_2c_2] \\ &= [a_1, a_2] + [b_1, b_2][c_1, c_2];\end{aligned}$$

proving that Axiom B holds in L_s.

L_s thus belongs to a class of lattices satisfying Axiom B but not Axiom A. Since the term *semi-modular* has already been applied to those lattices satisfying both axioms, the larger class satisfying Axiom B will be called *pseudo-modular*. This property and the analogous one of *pseudo-distributivity* will be discussed in Section V.

4. **Special properties of L_s.** It has already been noted that L_s has a zero even if L does not. L_s has other properties not necessarily shared by L, some of which will be exhibited in the theorems of this paragraph.

THEOREM 3.9. *There is a one-to-one correspondence between elements of L and atomic elements of L_s, and every non-atomic element of L_s can be expressed uniquely as the sum of exactly two atomic elements.*

Proof. The correspondence is $a \leftrightarrow [a, a]$; furthermore, $[a, a] + [b, b] = [ab, a+b] = [a, b]$ if $a \leqq b$, and $[a, a] \cdot [b, b] = \phi$ if $a \neq b$.

THEOREM 3.10. *The principal α-ideal generated by an element $[a, b]$ of L_s is isomorphic to the direct sum of the principal μ-ideal generated by its lower extremity and the principal α-ideal generated by its upper extremity.*

(6) An element p of a lattice L with a zero is atomic if $p \succ 0$.

Proof. $([a, b])_\alpha$ is composed of all $[x, y]$ such that $x \leqq a$ and $y \geqq b$; hence from the definition of direct sum, it is isomorphic to $(a)_\mu \oplus (b)_\alpha$.

COROLLARY 3.11. *The set of all principal μ-ideals (α-ideals) of a lattice L with a zero (unit) element is a divisorless*(7) *principal α-ideal of L_s, which is isomorphic (dually isomorphic) to L.*

Proof. Theorems 3.10 and 2.5. The ideal is divisorless because $[0, 0]$ (or $[1, 1]$) is an atomic element.

COROLLARY 3.12. *Every principal α-ideal of L_s can be represented uniquely as the intersection of two divisorless principal α-ideals.*

Proof. Theorem 3.9.

5. **Representation of segments by principal ideals.** Since $(a)_\alpha$ is the set of all $x \in L$ such that $x \geqq a$ and $(b)_\mu$ is the set of all $x \in L$ such that $x \leqq b$, it is possible to reverse the procedure of §2 of Section II and express segments in terms of principal ideals by

THEOREM 3.13. *Every segment $[a, b]$ is uniquely represented by the set-theoretical intersection of the principal α-ideal generated by its lower extremity and the principal μ-ideal generated by its upper extremity.*

Lemma 2.7 then becomes a corollary of this theorem, since

$$\begin{aligned}[a, b] \cap [c, d] &= \{(a)_\alpha \cap (b)_\mu\} \cap \{(c)_\alpha \cap (d)_\mu\} \\ &= \{(a)_\alpha \cap (c)_\alpha\} \cap \{(b)_\mu \cap (d)_\mu\},\end{aligned}$$

which by Corollary 3.11

$$\begin{aligned}&= (a + c)_\alpha \cap (bd)_\mu \\ &= [a + c, bd].\end{aligned}$$

SECTION IV. CONVEX SUBSETS OF A LATTICE

1. **Convexity.** Segments are used to define the convex subsets of a lattice in the same manner as in metric or linear spaces.

DEFINITION 4.1. *A subset S of elements of a lattice L is called convex if it contains the segment joining any pair of its elements.*

Since $a, b \in S$ implies $[a, b] = [ab, a+b] \subset S$, we have the obvious

COROLLARY 4.2. *A convex subset of L is a sublattice of L, and if it is not a segment it will lack either a zero or unit or both*(8).

(7) A μ-ideal (α-ideal) is divisorless if it is contained in no other μ-ideal (α-ideal) except L itself.

(8) This corollary shows that "convex subset" as here defined is equivalent to what Ore [3] calls "dense substructure." It is not however equivalent to the definition of convexity given by Birkhoff [2], since there the convex subsets need not be lattices.

COROLLARY 4.3. *If a lattice L has finite dimension, then all convex sets are segments.*

2. **Ideals and convex sets.** In the two preceding sections, the relationship between segments and principal ideals was discussed. Analogous relationships will now be derived for convex sets and ideals.

THEOREM 4.4. *The intersection of an α-ideal and a μ-ideal is a convex set and every convex set is the intersection of the least α- and μ-ideals containing it,*

Proof. Let C be the intersection of an α-ideal A and a μ-ideal M. Since A and M are sublattices of L, C is also a sublattice, and if $a, b \in C$, then $ab, a+b \in c$. Now M contains $a+b$ and therefore all $x \in L$ such that $x \leqq a+b$; dually, A contains ab and with it all $y \in L$ such that $ab \leqq y$. Hence $A \cap M$ contains all $x \in L$ such that $ab \leqq x \leqq a+b$.

Now assume that A and M are the least α- and μ-ideals containing C. Then M (A) is composed of elements $x \in C$ and all elements $v \in L$ such that $v \leqq x$ $(v \geqq x)$ for any $x \in C$. Hence $C = A \cap M$.

COROLLARY 4.5. *All ideals are convex sets.*

COROLLARY 4.6. *Every convex set is the intersection of all the ideals containing it.*

3. **The lattice of convex subsets of** L. The set-theoretical intersection of a set of convex sets is again convex and so the set of convex subsets of a lattice L can be made into a complete lattice, which will be denoted by L_{cs}, by the usual process of defining the lattice sum of any number of convex sets as the intersection of all convex sets containing each of the sets. The symbols $\wedge$ and $\vee$ will be used to denote the lattice operations in L_{cs}. The zero and unit elements of L_{cs} are the empty set and L itself.

Most of the significance of this lattice lies in

THEOREM 4.7. *L_s is a sublattice, but not in general a complete sublattice, of L_{cs}.*

Proof. It has already been pointed out that L_s is not necessarily complete. If $A = [a, b]$ and $B = [c, d]$, then $A \wedge B = [a, b] \cdot [c, d]$ by Lemma 2.7. Likewise, $A \vee B = [a, b] + [c, d]$, since any convex set containing both A and B must contain the segments joining the elements ab, cd and $a+b$, $c+d$, hence $abcd$ and $a+b+c+d$.

In particular, L_{cs} has as atomic elements the atomic elements of L_s (although they no longer form a basis for L_{cs}) and hence a class of divisorless principal α-ideals each of which contains as a sublattice the corresponding divisorless principal α-ideal of L_s. Hence if L has a zero or unit element, then by Corollary 3.11 we have an imbedding of a lattice isomorphic or dually iso-

morphic to L in a complete lattice. Moreover, this lattice is isomorphic to the lattice of μ-ideals or α-ideals of L, by virtue of

THEOREM 4.8. *If L is a lattice with a zero (unit) element, then a convex subset A of L is a μ-ideal (α-ideal) of L if and only if it contains the zero (unit) element of L.*

Proof. A being convex, it must contain with each of its elements a the segment $[0, a]$ ($[a, 1]$). Hence with each a it contains all elements of the form xa $(x+a)$, $x \in L$, and with each a, $b \in A$, $a+b$ (ab). The converse is obvious.

In the case of distributive lattices, the lattice of μ- or α-ideals is known to be distributive; hence $([0, 0])_\alpha$ and $([1, 1])_\alpha$ (in L_{cs}) are also distributive, and the above imbedding process preserves this lattice property.

If L is a Boolean algebra, then by introducing an operation called orthocomplementation, it is possible to identify ideals in such a way as to make this operation actually a complementation (at the same time preserving the completeness and distributivity of the lattice of ideals). Hence one obtains an imbedding of a Boolean algebra in a complete Boolean algebra(9).

SECTION V. PSEUDO-MODULAR AND PSEUDO-DISTRIBUTIVE LATTICES

1. **Definitions and examples.** In Section III, it was shown that the L_s of a modular lattice satisfied the condition of Axiom B, and such a lattice was called *pseudo-modular*. The following corollary is an immediate consequence of Axiom B and gives an equivalent formulation of the property of pseudo-modularity which is more conveniently specialized to the corresponding notion of pseudo-distributivity.

COROLLARY 5.1. *If every sublattice of a lattice L with a zero element which does not contain the zero element is modular, then L is pseudo-modular, and conversely.*

The method of defining the analogous property of pseudo-distributivity is then obvious:

DEFINITION 5.2. *A lattice with zero element is said to be pseudo-distributive if and only if every sublattice not containing the zero element is distributive.*

It is now possible to make a formal distinction between the lattices of segments of modular and distributive lattices by

THEOREM 5.3. *The lattice of segments of a modular (distributive) lattice is pseudo-modular (pseudo-distributive).*

(The theorem has already been proved for modularity, but the following proof is applicable to both cases.)

(9) A detailed discussion of the lattices of ideals of Boolean algebras and distributive lattices will be found in Stone [4] and [5].

Proof. Every sublattice of an L_s which does not contain the zero of L_s is also a sublattice of one of the divisorless principal α-ideals generated by an atomic element of L_s. Hence by Theorem 3.10 it is isomorphic to a sublattice of the direct sum $(a)_\mu \oplus (a)_\alpha$, which is modular (distributive) if L is modular (distributive).

This theorem completes the discussion of lattices of segments, and the remainder of this section will be devoted to the development of properties of pseudo-modularity and pseudo-distributivity, which seem to have some interest in themselves, especially the latter, since it is applicable to modular lattices.

Examples of pseudo-modular lattices which are not lattices of segments of some modular lattices are the semi-modular lattices, one type of which can be constructed from modular lattices by a process due to Wilcox [8, p. 497 ff.]. Pseudo-distributive lattices may be constructed from distributive lattices by the same process, the restrictions on the distributive lattices being less stringent since there is no need to satisfy Axiom A of Section III. The method is as follows:

Let D be an arbitrary distributive lattice with zero element. If S is a μ-ideal of D with 0 deleted, then the set $L = D - S$ is a lattice with its operations $\cup, \cap$ defined by

$$a \cup b = a + b,$$

$$a \cap b = \begin{cases} ab & \text{if} \quad ab \in L, \\ 0 & \text{if} \quad ab \in S. \end{cases}$$

The proof that $a \cup b$ and $a \cap b$ are actually effective as least upper and greatest lower bounds in L depends only on the fact that S is a μ-ideal with 0 deleted, and not on the distributivity of D, so the argument given by Wilcox applies directly. The distributivity of any sublattice of L not containing 0 is immediate from the definitions of $a \cup b$ and $a \cap b$ and the distributivity of sublattices of D.

2. **Duality considerations.** The above examples show immediately that pseudo-modularity and pseudo-distributivity, unlike modularity and distributivity, are neither self-dual properties of a lattice nor do they imply their duals. A repetition of the above construction using an α-ideal in place of the μ-ideal will yield examples of *dual pseudo-modular* or *dual pseudo-distributive* lattices from modular or distributive lattices with unit elements. However, under certain conditions pseudo-distributivity is equivalent to dual pseudo-distributivity. We first give examples to show that modularity and pseudo-distributivity are independent properties of a lattice.

EXAMPLE 5.4. L_5 is a 5-element lattice with order relations as follows: $0 \prec a \prec 1$, $0 \prec b \prec c \prec 1$, $a \not\prec c$.

EXAMPLE 5.5. Define order in a 6-element lattice L_6 by $0 \prec a_i \prec b \prec 1$, $i=1, 2, 3$.

L_5 is a non-modular pseudo-distributive lattice, while L_6 is modular and pseudo-distributive.

THEOREM 5.6. *In a complete complemented modular lattice, pseudo-distributivity is equivalent to dual pseudo-distributivity.*

To prove this theorem, the following lemma is needed:

LEMMA 5.7. *In a modular lattice L, the segment $[ab, a]$ is isomorphic to the segment $[b, a+b]$ for all $a, b \in L$.*

The proof of this lemma is given in Birkhoff [1] and in Ore [3, p. 418, Theorem 2].

Proof of the theorem. By duality, we need only prove that pseudo-distributivity implies dual pseudo-distributivity. Assume that L is not dual pseudo-distributive; then there is some sublattice S of L which does not contain the unit of L and which is not distributive. Since L is complete, S has a unit element $x \neq 1$, and S is a sublattice of $[0, x]$; also L is complemented, so there exists an element $y \in L$ $(\neq 0)$ such that $x+y=1$, $xy=0$. Then by Lemma 5.7, $[0, x]$ is isomorphic to $[y, 1]$, which is distributive if L is pseudo-distributive, contradicting the assumption that S is not distributive.

Remark. The hypotheses of this theorem can be weakened somewhat; for instance, modularity may be replaced by the condition $[ab, a]$ isomorphic to $[b, a+b]$ for all $a, b \in L$. Ward [7, p. 448] has shown that such a lattice need not be modular, but is modular if one of the chain conditions holds. Example 5.5 shows that it is not possible to omit the requirement of complementation, since L_6 is pseudo-distributive but not dual pseudo-distributive.

3. **m-distributivity in modular lattices.** In lattices on which there is defined a positive monotone dimension or measure function[10] $d(a)$ it is possible to make a slight generalization of the notions of pseudo-distributivity and its dual as follows:

DEFINITION 5.8. *A lattice L with a dimension function $d(x)$ is said to be m-distributive (dual m-distributive) if all sublattices whose zero elements (unit elements) have dimension $d(a) \geqq m$ $(d(a) \geqq (n-m)$ in the finite case and $d(a) \geqq (1-m)$ in the continuous case) are distributive.*

COROLLARY 5.9. *In a lattice in which $d(x)$ has the range $0, 1, 2, \cdots, n$, 1-distributivity (dual 1-distributivity) is equivalent to pseudo-distributivity (dual pseudo-distributivity).*

[10] The function $d(a)$ is a numerical function whose range is either discrete $(0, 1, 2, \cdots, n)$ or continuous $(0 \leqq d(a) \leqq 1)$, and has the properties (1) $a \leqq b$ implies $d(a) \leqq d(b)$, (2) $d(a)+d(b) = d(a+b)+d(ab)$ for all $a, b \in L$. For more details, see von Neumann [6, vol. 1], or Birkhoff [2, chap. 4].

When $d(x)$ is continuous, it never has a value m such that m-distributivity or its dual is equivalent to pseudo-distributivity or its dual.

COROLLARY 5.10. *Every lattice of dimension n is $(n-1)$-distributive and dual $(n-1)$-distributive. In particular, all lattices of dimension 2 are both pseudo-distributive and dual pseudo-distributive.*

It is obvious that in either the finite or continuous case, 0-distributivity is equivalent to distributivity as is also dual 0-distributivity.

EXAMPLE 5.11. An example of an m-distributive lattice which is not pseudo-distributive is the free modular lattice generated by three elements. It is of finite dimension $(n=8)$ and is both 4-distributive and dual 4-distributive.

Since lattices of finite or continuous dimension are complete, a slight modification of the proof of Theorem 5.6 yields

THEOREM 5.12. *In a complemented modular lattice of finite or continuous dimension, m-distributivity is equivalent to dual m-distributivity.*

The 4-distributive lattice of Example 5.11, while complete and modular, is not complemented. The class of m-distributive lattices which are complete, modular, and complemented is greatly restricted by the following,

LEMMA 5.13. *If a complete complemented modular lattice is irreducible, then for any $a, b \in L$, $[a, b]$ is irreducible*(11).

An immediate consequence of this lemma and Corollary 5.10 is

THEOREM 5.14. *An irreducible complemented modular lattice of finite dimension n is m-distributive only if $m=n-1$, and there are no irreducible m-distributive or pseudo-distributive complemented modular lattices of continuous dimension.*

In particular, the only pseudo-distributive projective geometries are one-dimensional, and their pseudo-distributivity is trivial.

Now consider the reducible(12) complemented modular lattices of finite or continuous dimension. The *index* of such a lattice is defined as follows:

DEFINITION 5.15. *If Z is the center of a reducible complemented modular lattice L of finite or continuous dimension, then $m=\max_{x \in Z} \{\min [d(x), d(x')]\}$ (where x' denotes the (unique) complement of x) is the index of L.*

THEOREM 5.16. *A complemented modular lattice of index m which is m-distributive is a Boolean algebra.*

(11) For proof of this lemma, see von Neumann [6, vol. 1].

(12) These lattices are discussed in von Neumann [9, vol. 2, Part III], and Birkhoff [2 chap. 4].

Proof. Let a be an element of Z such that $d(a)=m$. Since $a \in Z$, $L=[0, a]+[0, a']$; also $d(a)=m$ and $d(a') \geqq m$, making both $[0, a]$ and $[0, a']$ distributive by Theorem 5.12. Hence L is distributive and is a Boolean algebra.

COROLLARY 5.17. *The only pseudo-distributive reducible complemented modular lattices of finite or continuous dimension are Boolean algebras.*

The preceding sequence of theorems and corollaries indicates that if significant examples of pseudo- and m-distributive modular lattices and their duals are to be found, the requirement of complementation must be dropped. The lattices of invariant subgroups of a group and ideals of a ring are lattices of this type, and the determination of what, if any, algebraic properties of groups and rings are implied by the various degrees of distributivity of their lattices of invariant subgroups and ideals is a problem which might be of some significance. For instance, consider the following example of the lattice of (invariant) subgroups of the quaternion group ± 1, $\pm i$, $\pm j$, $\pm k$. Its proper invariant subgroups are $A_1=(+1, +i, -1, -i)$, $A_2=(+1, +j, -1, -j)$, $A_3=(+1, +k, -1, -k)$, and $B=(+1, -1)$, and these together with the whole group and the unit element constitute a lattice dually isomorphic to the lattice L_6 of Example 5.5. Hence it is dual pseudo-distributive.

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