ON THE PARTIAL SUMS OF HARMONIC DEVELOPMENTS
AND OF POWER SERIES

BY

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1. Introduction. Consider the class $E$ of power series $f(z) = \sum c_{n}z^{n}$, convergent for $|z| < 1$ and such that $|f(z)| \leq 1$. The following result is due to I. Schur and G. Szegö [5](1).

For any series of the class $E$,

$$|s_{n}(z)| = \left| \sum_{0}^{n} c_{n}z^{n} \right| \leq 1$$

in $|z| \leq r_{n}$, but not always in $|z| < r_{n} + \epsilon$, $\epsilon > 0$, where $r_{n}$ is the largest $r$ for which

$$T_{n}(r, \theta) = \frac{1}{2} + \sum_{1}^{n} r^{n} \cos n\theta \geq 0$$

for all $\theta$.

The $r_{n}$ are non-decreasing,

$$r_{n} > 1 - \frac{\log 2n}{n}, \quad n = 1, 2, 3, \ldots$$

$$r_{n} = 1 - \frac{\log 2n - \log \log 2n + \epsilon_{n}}{n}, \quad \lim_{n \to \infty} \epsilon_{n} = 0.$$

We obtain the same constant $r_{n}$ if we assume $Rf(z) \geq 0$ and require $R_{1}s_{n}(z) \geq 0$.

Here $R_{u}$ means the real part of $u$; $Iu$ will denote the imaginary part.

In what follows, we consider harmonic sine developments

$$H(r, \theta) = \sum_{1}^{n} b_{n}r^{n} \sin n\theta,$$

convergent for $0 < r < 1$, and non-negative for $0 < \theta < \pi$. Evidently there exists an $R_{n}$ with the following properties:

(a) Whenever

$$H(r, \theta) \geq 0, \quad 0 < r < 1; 0 < \theta < \pi,$$

then,

$$s_{n}(r, \theta) = \sum_{1}^{n} b_{n}r^{n} \sin n\theta \geq 0, \quad 0 < r \leq R_{n}; 0 < \theta < \pi.$$
(b) For any \( \varepsilon > 0 \) we can find an \( H \) satisfying (1.1) and such that \( s_n(r, \theta) \) becomes negative for some \( \theta \) and some \( r < R_n + \varepsilon \).

We denote the class of harmonic functions satisfying (1.1) by \( T \). On writing \( f(z) = \sum b_n z^n \), the power series \( f(z) \) is regular in \( |z| < 1 \), has all its coefficients real, and \( |f(z)| \geq 0 \) in \( |z| < 1, Iz > 0 \). The class \( T \) has been discussed by Rogosinski [4]; the function \( f(z) \) is called typically real. (Cf. also S. Mandelbrojt [2].)

One of the results of the present paper is

\[
R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n}{n} + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty.
\]

M. S. Robertson [3] gave the erroneous estimate

\[
R_n \geq 1 - 2 \log \frac{n}{n}
\]

for \( n > 12 \). His calculation yields however, as is seen easily, \( R_n \geq 1 - 4 \log \frac{n}{n} \), for \( n > n_0(2) \). We then apply the properties of \( R_n \) to Fourier series of convex functions and to a certain class of power series.

Note that if \( \phi(\theta) = \sum b_n \sin \nu \theta, \phi(\theta) \geq 0, 0 < \theta < \pi \), then

\[
H(r, \theta) = \frac{2}{\pi} \int_0^\pi \phi(x) \left( \sum_1^n r^\nu \sin \nu \sin \nu x \right) dx
\]

\[
= \frac{1}{\pi} \int_0^\pi \phi(x) \sum_1^n r^\nu \left[ \cos \nu (\theta - x) - \cos \nu (\theta + x) \right] dx
\]

\[
= \frac{1 - r^2}{2\pi} \int_0^\pi \phi(x) \left( \frac{1}{1 - 2r \cos (\theta - x) + r^2} - \frac{1}{1 - 2r \cos (\theta + x) + r^2} \right) dx
\]

\[
= \frac{2r(1 - r^2)}{\pi} \sin \theta \int_0^\pi \phi(x) \sin x dx
\]

Hence \( H(r, \theta) \) belongs to the class \( T \). (Cf. Zygmund [8, p. 57].)

2. Characterization of \( R_n \). We quote the following lemma, due to Fejér (Turán [7]).

**Lemma 1.** In order that

\[
\sum \lambda_x \sin \nu x \sin \nu y \geq 0 \quad \text{for } 0 < x < \pi, 0 < y < \pi,
\]

it is necessary and sufficient that

\[
\sum \nu \lambda_x \sin \nu \theta \geq 0 \quad \text{for } 0 < \theta < \pi.
\]

We now prove

**Theorem 1.** The quantity $R_n$ as defined in §1 is the largest $r$ for which

\[(2.1) \quad S_n(r, \theta) = \sum_{\nu=1}^{n} \nu r^\nu \sin \nu \theta \geq 0 \quad \text{for} \quad 0 < \theta < \pi.\]

We have for $0 < \rho < 1$,

$$\rho^\nu b_\nu = \frac{2}{\pi} \int_{0}^{\pi} H(\rho, x) \sin \nu x \, dx, \quad \nu = 1, 2, 3, \ldots;$$

hence

$$s_n(r, \theta) = \frac{2}{\pi} \int_{0}^{\pi} H(r, x) \left( \sum_{\nu=1}^{n} \left( \frac{r}{\rho} \right)^\nu \sin \nu \theta \sin \nu x \right) \, dx.$$  

For any $r < R_n$ we can choose $\rho < 1$ so that $r/\rho < R_n$; we then obtain by Lemma 1 (for $\lambda = r$) $s_n(r, \theta) \geq 0$ for any $r < R_n$ and for $0 < \theta < \pi$; hence (a) holds for $r \leq R_n$. Conversely, for the function

$$H(r, \theta) = \sum_{\nu=1}^{n} \nu r^\nu \sin \nu \theta = r \sin \theta \frac{1 - r^2}{(1 - 2r \cos \theta + r^2)^2} > 0$$

the function

$$s_n(r, \theta) = \sum_{\nu=1}^{n} \nu r^\nu \sin \nu \theta$$

becomes negative for any $r > R_n$ and for some $\theta$ in $(0, \pi)$. This proves Theorem 1. To estimate $R_n$ we first give another characterization for it. An easy calculation yields

$$\frac{(1 - 2r \cos \theta + r^2)^2}{r \sin \theta} \sum_{\nu=1}^{n} \nu r^\nu \sin \nu \theta$$

$$= 1 - r^2 - (n + 1) r^{n+2} \frac{\sin (n - 1) \theta}{\sin \theta} + n r^{n+1} (2n + 2 + nr^2) \frac{\sin n \theta}{\sin \theta}$$

$$- r^n (n + 1 + 2nr^2) \frac{\sin (n + 1) \theta}{\sin \theta} + n r^{n+1} (n + 2) \frac{\sin (n + 2) \theta}{\sin \theta}$$

$$= C_n(r, \theta).$$

This furnishes

**Theorem 2.** $R_n$ is the largest $r$ for which

$$C_n(r, \theta) \geq 0 \quad \text{for all} \quad \theta.$$
\[ C_n(r, \theta) = 1 - r^2 + (n^2 - 1) r^{n+2} - (n + 1) r^n (2n + 2 + nr^2) + (n + 1) r^{n+1} (n + 1 + 2nr^2) + n(n+2) r^{n+1}. \]

Thus

\[ C_n(r, \theta) \geq 1 - r^2 + (n^2 - 1) r^{n+2} + nr^{n+1} (2n + 2 + nr^2) + (n + 1) r^{n+1} (n + 1 + 2nr^2) + n(n+2) r^{n+1}, \quad n \geq 1, \]

and equality holds if \( n = 2k, \) and \( \theta = \pi. \) This yields

**Theorem 3.** Denote the unique positive root of the equation

\[ p_n(r) = 1 - r^2 - (n + 1) r^n - (3n^2 + 2n - 1) r^{n+1} - n^2 r^{n+2} = 0 \]

by \( p_n. \) Then \( R_n \geq p_n, \) and equality holds for \( n = 2k, k \geq 1. \)

Note that \( p_n(0) = 1, p_n(1) < 0, p_n'(r) < 0. \) Hence \( p_n \) is unique and

\[ 0 < p_n < 1. \]

Evidently \( p_n(-1) = 0, \) hence \( 1 + r \) can be factored out, and we get

\[ p_n(r) = 1 - r - (n + 1) r^n - (2n^2 + 2n - 1) r^{n+1} - n^2 r^{n+2} = q_n(r), \]

so that \( q_n(p_n) = 0. \)

3. **Estimation of \( p_n \) and \( R_n.** Direct calculation gives

\[ R_1 = 1; \rho_1 = 0.182 \cdots. \]

Also \( R_2 = R_3, \) and

\[ S_2(r, \theta) = r \sin \theta + 2r^2 \sin 2\theta = r \sin \theta (1 + 4r \cos \theta), \]

which yields by Theorem 1: \( R_2 = 1/4 = \rho_2. \) A similar calculation yields

\[ R_3 = 2^{1/3} / 3. \]

We shall prove

\[ p_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \log 3/4 + \epsilon_n}{n}, \quad \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Let \( c \) be a constant, and

\[ r_n(c) = 1 - \frac{3 \log n}{n} + \frac{\log n + c}{n}; \]

then from

\[ \log (1 - x) = -x + O(x^2) \quad \text{as } x \rightarrow 0, \]
we conclude
\[ r_n(c)^n = \exp\left\{-3 \log n + \log \log n + c + O(n^{-1} \log^2 n)\right\} \]
\[ = n^{-3} \log n \cdot e^c \left\{1 + O(n^{-1} \log^2 n)\right\} \quad \text{as} \quad n \to \infty. \]

Furthermore, from (2.3), (3.2), and (3.3)
\[ q_n\{r_n(c)\} = \frac{3 \log n}{n} - \frac{\log \log n + c}{n} - \frac{4 \log n}{n} \cdot e^c \left\{1 + o(1)\right\}, \]

hence
\[ \frac{nq_n\{r_n(c)\}}{\log n} \to 3 - 4e^c \quad \text{as} \quad n \to \infty. \]

Thus for
\[ c = \log 3/4 + \epsilon, \]
\(\epsilon\) a given small number, and for sufficiently large values of \(n\)
\[ \text{sgn } q_n\{r_n(c)\} = \text{sgn } \epsilon, \]
from which follows (3.1).

We have thus proved

**Theorem 4.** If \(\rho > 0\) and \(\rho_n(\rho_n) = 0\), then
\[ \rho_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \log 3/4 + \epsilon_n}{n}, \quad \text{where} \quad \epsilon_n \to 0 \quad \text{as} \quad n \to \infty. \]

4. **Derivation of an asymptotic estimate for \(R_n\).** On writing
\[ R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \delta_n}{n}, \quad n > 1, \]
it follows from Theorem 3 that
\[ \delta_n \geq \log 3/4 + \epsilon_n, \]
and equality holds for \(n = 2k, k \geq 1\); hence from Theorem 4
\[ \liminf_{n \to \infty} \delta_n = \log 3/4, \quad \lim_{k \to \infty} \delta_{2k} = \log 3/4. \]

It remains to give an estimate for \(R_{2k-1}\) from above.

If for a particular value of \(\theta\) and \(r\), \(C_{2k-1}(r, \theta) < 0\), then by Theorem 2, evidently \(R_{2k-1} < r\). We now choose \(\theta = \pi - (3\pi/4k)\); then
\[ C_{2k-1}(r, \theta) = 1 - r^2 + \frac{1}{\sin \left(\frac{3\pi}{4k}\right)} \left( 2kr^{2k+1} \sin \frac{3(k-1)\pi}{2k} ight) \]
\[ + r^{2k} \left[ 4k + (2k - 1)r^2 \right] \sin \frac{3(2k - 1)\pi}{4k} \]
\[ + r^{2k-1} \left[ 2k + 2(2k - 1)r^2 \right] \sin \frac{3\pi}{2} \]
\[ + (2k - 1)r^{2k} \sin \frac{3(2k + 1)\pi}{4k} \]
\[ = 1 - r^2 - \frac{1}{\sin \left(\frac{3\pi}{4k}\right)} \left( 2kr^{2k+1} \cos \frac{3\pi}{2k} \right) \]
\[ + r^{2k} \left[ 4k + (2k - 1)r^2 \right] \left[ \cos \frac{3\pi}{4k} \right] \]
\[ + 2kr^{2k-1} + (4k - 2)r^{2k+1} + (2k - 1)r^{2k} \cos \frac{3\pi}{4k} \]
\[ < 1 - r^2 - \frac{4k}{3\pi} \left( 2kr^{2k+1} \left( 1 - \frac{9\pi^2}{8k^2} \right) \right) \]
\[ + r^{2k} \left[ 4k + (2k - 1)r^2 \right] \left( 1 - \frac{9\pi^2}{32k^2} \right) \]
\[ + 2kr^{2k-1} + (4k - 2)r^{2k+1} + (2k - 1)r^{2k} \left( 1 - \frac{9\pi^2}{32k^2} \right) \], \quad k \geq 3 \]

(since \( \cos x > 1 - x^2/2 \) for all \( x \)). Hence

\[ C_{2k-1}(r, \theta) < 1 - r^2 - 2k(5/2) \left( k^2r^{2k+1} + 2kr^{2k} + (k - 1/2)r^{2k+2} + 2kr^{2k-1} \right) + (4k - 2)r^{2k+1} + (k - 1/2)r^{2k} \], \quad k \geq 5, \]

thus

\[ C_{2k-1}(r, \theta) < 1 - r^2 - (2k/5)(11k - 3)r^{2k+2} < 1 - r^2 - 4k^2r^{2k+2}. \]

Choosing \( r \) so that

\[ (4.1) \quad 1 - r^2 - 4k^2r^{2k+2} \leq 0, \]

we get

\[ C_{2k-1}(r, \theta) < 0, \quad R_{2k-1} < r. \]

To find an upper bound for \( r \), we put

\[ (4.2) \quad r = 1 - \frac{3\log(2k - 1)}{2k - 1} + \frac{\log \log(2k - 1) + c}{2k - 1}; \]
we obtain as in (3.3)
\[ r^{2k-1} = \exp \left\{ -3 \log (2k - 1) + \log \log (2k - 1) + c + O(k^{-1} \log^2 k) \right\} \]
\[ = (2k - 1)^{-3} \log (2k - 1) \cdot e^c \left\{ 1 + O(k^{-1} \log^2 k) \right\} . \]

Thus, using (4.2),
\[ \frac{4k^2r^{2k+2}}{1 - r^2} = \left( \frac{2k}{2k - 1} \right)^2 \cdot \frac{r^2}{1 + r} \cdot \frac{r^{2k-1}(2k - 1)^2}{1 - r} \]
\[ = \frac{1 + o(1)}{2 + o(1)} \cdot \frac{1}{3} e^c \left\{ 1 + o(1) \right\} \to \frac{1}{6} e^c \text{ as } k \to \infty. \]

Hence (4.1) is satisfied for all sufficiently large \( k \) provided \( e^c/6 > 1 \), that is, \( c > \log 6 \). It now follows that \( \limsup_{k \to \infty} \delta_{2k-1} \leq 6 \). Summarizing we have

**Theorem 5.** Let
\[ R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \delta_n}{n}, \quad n > 1; \]
then \( \lim_{k \to \infty} \delta_{2k} = \log 3/4 \), and
\[ \log 3/4 \leq \liminf_{k \to \infty} \delta_{2k-1} \leq \limsup_{k \to \infty} \delta_{2k-1} \leq 6. \]

5. **Application to Fourier series.** Consider the roof-function
\[ \frac{2b}{a(\pi - a)} \sum_{1}^{\infty} \frac{\sin v \theta \sin v \phi}{v^2} = \begin{cases} \frac{b \theta}{a} & \text{for } 0 \leq \theta \leq a, \\ \frac{b}{\pi - a} & \text{for } a \leq \theta \leq \pi, \end{cases} \]
where \( 0 < a < \pi, 0 < b \), and the corresponding harmonic function
\[ \frac{2b}{a(\pi - a)} \sum_{1}^{\infty} r^v \frac{\sin v \phi \sin v \theta}{v^2} = \theta ; \]
Denote its partial sums by
\[ H_n(r, \theta) = \frac{2b}{a(\pi - a)} \sum_{1}^{n} r^v \frac{\sin v \phi \sin v \theta}{v^2}; \]
then
\[ \frac{\partial^2 H_n(r, \theta)}{\partial \theta^2} = - \frac{2b}{a(\pi - a)} \sum_{1}^{n} r^v \sin v \phi \sin v \theta \leq 0 \]
for \( 0 < r \leq R_n, 0 < \theta < \pi \), by Lemma 1 and Theorem 1. Hence \( H_n(r, \theta) \) is con-
vex upwards for $0 < \theta < \pi$, $r \leq R_n$; but not convex for $r > R_n$. The same is true for the limiting cases $a \to 0$ and $a \to \pi$. In which cases

$$H(r; 0, b) = \frac{2b}{\pi} \sum_{1}^{\infty} \frac{r^{\nu} \sin \nu \theta}{\nu},$$

$$H(r; \pi, b) = \frac{2b}{\pi} \sum_{1}^{\infty} \frac{r^{\nu} \sin \nu(\pi - \theta)}{\nu}.$$

Moreover every polygon convex upwards and lying above the axis of abscissae is expressible as a finite sum with positive coefficients of roof-functions. Hence the partial sums of the corresponding harmonic development are convex upwards for $r \leq R_n$. Finally any function positive in $0 < \theta < \pi$, and convex upwards can be approximated uniformly by such polygons; hence we have

**Theorem 6.** If $f(\theta) > 0$ in $0 < \theta < \pi$, and is convex upwards, and if $f(\theta) \sim \sum \nu b \sin \nu \theta$, then $\sum \nu r^{\nu} \sin \nu \theta$ is convex upwards in $0 < \theta < \pi$, $r \leq R_n$; but not always for $r < R_n + \epsilon$, $\epsilon > 0$.

6. **Cosine series.** We now consider the cosine series of the step function

$$\frac{2b}{\pi} \left\{ \frac{\pi - a}{2} - \sum_{1}^{\infty} \frac{\sin \nu \theta}{\nu} \right\} = \begin{cases} 0 & \text{for } 0 \leq \theta < a, \\ b & \text{for } a < \theta \leq \pi, \end{cases}$$

where $0 < a < \pi$, $b > 0$; and the corresponding harmonic development

$$K(r, \theta) = \frac{b}{\pi} (\pi - a) - \frac{2b}{\pi} \sum_{1}^{\infty} \frac{r^{\nu} \sin \nu \theta}{\nu}.$$

For the partial sums $K_n(r, \theta)$ of this series we have

$$\frac{\partial K_n(r, \theta)}{\partial \theta} = \frac{2b}{\pi} \sum_{1}^{\infty} r^{\nu} \sin \nu \sin \nu \theta \geq 0 \quad \text{for } 0 < r \leq R_n, \ 0 < \theta < \pi,$$

hence $K_n(r, \theta)$ is monotonic increasing in the same domain; $R_n$ cannot be replaced by $R_n + \epsilon$, $\epsilon > 0$. The same statement for any monotonic increasing function follows now in an obvious way. Hence we have

**Theorem 7.** If $f(\theta)$ is monotonic in $0 < \theta < \pi$, and

$$f(\theta) \sim a_0/2 + \sum_{1}^{\infty} a_\nu \cos \nu \theta,$$

then the $n$th partial sum of $a_0/2 + \sum_{1}^{\infty} a_\nu r^{\nu} \cos \nu \theta$ is monotonic in the same sense for $0 < r \leq R_n$, and here $R_n$ cannot be replaced by $R_n + \epsilon$, $\epsilon > 0$.

7. **Curves convex in direction of the $v$-axis.** We say that a curve in the $(u, v)$-plane is convex in the direction of the $v$-axis if any parallel to the $v$-axis...
has at most two points in common with the curve. This class of mappings was considered by L. Fejér [1] and the author [6]. We now prove

**Theorem 8.** Suppose the power series \( \sum a_n z^n = f(z) = w = u + iv \) is regular in \( |z| < 1 \), and all \( a_n \) are real. Suppose further that the images \( K_r \) of the circles \( |z| = r, 0 < r < 1 \), are convex in the direction of the v-axis (thus \( f(z) \) is univalent). Then the partial sum \( \sum a_n z^n \) has the same property in \( |z| \leq R_n \), but—in general—not in a larger circle.

For the proof we may assume without loss of generality that the upper half of the circle \( |z| < 1 \) is mapped onto the upper half of the image in the \( w \)-plane. On writing \( w(e^{i\theta}) = u(\theta) + iv(\theta) \sim \sum a_n \cos \theta + i \sum a_n \sin \theta \), we find that \( v(\theta) \) is positive for \( 0 < \theta < \pi \), and (from the assumption) \( u(\theta) \) is decreasing in the same interval. Our theorem follows now from Theorems 5 and 7.

8. Conclusion. Suppose \( f(z) = \sum \alpha_n z^n \) is a typically real function, that is,

\[
\sum \alpha_n z^n \sin n\theta \geq 0 \quad \text{for} \quad 0 < r < 1, \ 0 < \theta < \pi.
\]

Then the Riesz means of second order

\[
P_n(z) = (n + 1)^{-2} \sum_{r=1}^{n} (n - r + 1) \alpha_r z^r,
\]

are typically real in \( |z| \leq 1 \) (Szász [6]; cf. Theorem 1). Evidently \( \lim_{n \to \infty} P_n(z) = f(z) \) in \( |z| < 1 \), uniformly in \( |z| \leq r, r < 1 \). Another such sequence of polynomials is

\[
s_n(R_n z) = \sum_{r=1}^{n} \alpha_r R_n^r z^r,
\]

These polynomials are typical real in \( |z| \leq 1 \) by property (a) of §1. Furthermore for \( |z| \leq r < 1 \)

\[
|f(z) - s_n(R_n z)| \leq \sum_{r=1}^{n} |\alpha_r| r^r(1 - R_n) + \sum_{n+1}^{\infty} |\alpha_r| r^r
\]

\[
< (1 - R_n) \sum_{r=1}^{n} |\alpha_r| r^r + \sum_{n+1}^{\infty} |\alpha_r| r^r \to 0, \text{ as } n \to \infty.
\]

Hence

\[
\lim_{n \to \infty} s_n(R_n z) = f(z)
\]

uniformly in \( |z| \leq r < 1 \).

**References**


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