ON THE PARTIAL SUMS OF FOURIER SERIES AT
POINTS OF DISCONTINUITY

BY

OTTO SZÁSZ

1. Introduction. Consider a Fourier sine series

\[ f(\theta) \sim \sum_{\nu=1}^{\infty} b_{\nu} \sin \nu \theta, \quad b_{\nu} = (2/\pi) \int_{0}^{\pi} f(\theta) \sin \nu \theta \, d\theta, \quad 0 < \theta < \pi, \]

and write

\[ s_{n}(\theta) = \sum_{\nu=1}^{n} b_{\nu} \sin \nu \theta, \quad n = 1, 2, 3, \ldots. \]

Fejér proved (cf. Zygmund [5, p. 181])\(^{(1)}\) that if \(f(\theta)\) is of bounded variation, and if \(n\theta_{n} \to \alpha\) as \(\theta_{n} \to 0\), then

\[ s_{n}(\theta_{n}) \to (2/\pi)f(+0) \int_{0}^{\pi} \frac{\sin t}{t} \, dt = (2/\pi)f(+0)I(\alpha). \]

In particular, choosing \(\alpha = \pi/2 = \int_{0}^{\pi} t^{-1} \sin t \, dt\) (thus \(\int_{0}^{\pi} t^{-1} \sin t \, dt = 0\)), we get \(s_{n}(\theta_{n}) \to f(+0)\), which is half of the jump of \(f(\theta)\) at \(\theta = 0\).

On the other hand for \(\alpha = \pi\), which gives \(I(\alpha)\) its maximal value

\[ s_{n}(\theta_{n}) \to (2/\pi)f(+0) \int_{0}^{\pi} \frac{\sin t}{t} \, dt = f(+0) \times 1.08949 \cdots. \]

Thus the limit points of the partial sums as \(\theta_{n} \to 0\) cover an interval which extends beyond \(f(+0)\), if \(f(+0) \neq 0\). This is called Gibbs' phenomenon.

It was also proved by Fejér and Csillag (for references and further results see Szász [4]) that for functions of bounded variation

\[ n^{-1} \sum_{\nu=1}^{n} \nu b_{\nu} \to (2/\pi)f(+0), \quad \text{as} \quad n \to \infty. \]

These facts suggest the consideration of

\[ s_{n}(\theta_{n}) = \sum_{\nu=1}^{n} \nu b_{\nu} \frac{\sin \nu \theta_{n}}{\nu}, \quad \theta_{n} \to 0, \]

as a transform of the sequence \(\{n\theta_{n}\}\), that is, as a special case of the triangular type transform

---

\(^{(1)}\) Numbers in brackets refer to the literature at the end of this paper.

Presented to the Society, September 8, 1942; received by the editors September 8, 1942.
(1.5) \[ T_n = \sum_{1}^{n} a_n \tau_n \]

where now \( \tau_n = \nu b_n, a_{n\nu} = \nu^{-1} \sin \nu \theta_n \). We shall not restrict ourselves to regularity conditions, and we shall not assume convergence of the sequence \( \{ \tau_n \} \), but merely Cesàro summability of some order. We then seek simple necessary and sufficient conditions for the convergence of the transform \( T_n \) (in general to a different limit). The application to Fourier sine series yields a generalized Gibbs' phenomenon, and also a new device to determine the generalized jump of a function. Our results are in close relationship with some results of Rogosinski [1, 2].

We consider more generally the transform

(1.6) \[ T(\rho_n, \theta_n) = \sum_{1}^{n} \tau_n \rho_n^{1/2} \sin \nu \theta_n, \quad \rho_n \to 1, \theta_n \to 0, \]

which in the case \( \tau_n = \nu b_n \) becomes \( \sum_{1}^{n} \nu b_n \sin \nu \theta_n = s_n(\rho_n, \theta_n) \), where \( s_n(\rho, \theta) \) is the \( n \)th partial sum of the harmonic series \( \sum_{1}^{n} \rho b_n \sin \nu \theta \).

2. Permanency with respect to convergent sequences. It is well known that the convergence of the sequence \( \{ \tau_n \} \) implies the convergence of the transform \( T_n \), if and only if

\[ \lim_{n \to \infty} a_{n\nu} = 0, \quad \text{for} \quad \nu = 1, 2, 3, \ldots ; \]

\[ \sum_{1}^{n} | a_{n\nu} | = O(1), \quad \text{as} \quad n \to \infty ; \]

\[ \lim_{n \to \infty} \sum_{1}^{n} a_{n\nu} = \sigma \text{ exists.} \]

We then have \( \lim T_n = \sigma \lim \tau_n \). If we restrict ourselves to sequences \( \tau_n \to 0 \), then the last condition can be omitted. Applied to (1.6) this yields the necessary and sufficient conditions:

(2.1) \[ \sum_{1}^{n} \rho_n^{1/2} | \sin \nu \theta_n | = O(1), \quad \text{as} \quad n \to \infty ; \]

(2.2) \[ \lim \sum_{1}^{n} \rho_n^{1/2} \sin \nu \theta_n = \sigma. \]

In particular the last condition is \( s_n(\rho_n, \theta_n) \to \sigma \) for the harmonic series \( \sum_{1}^{n} \rho b_n \sin \nu \theta = \arctan \left\{ (\rho \sin \theta) / (1 - \rho \cos \theta) \right\} \).

We first assume

(2.3) \[ 0 < \lim \inf \rho_n^n \leq \lim \sup \rho_n^n < \infty; \]
in this case for some $c_1 > 0$, $c_2 > 0$

$$c_1 \sum_{1}^{n} \nu^{-1} \sin \nu \theta < \sum_{1}^{n} \rho_{n} \nu^{-1} \sin \nu \theta < c_2 \sum_{1}^{n} \nu^{-1} \sin \nu \theta,$$

thus (2.1) reduces to

$$\sum_{1}^{n} \nu^{-1} \sin \nu \theta_n = O(1), \quad \text{as } n \to \infty.$$ 

Now for any $\theta > 0$

$$\sum_{1}^{n} \nu^{-1} \sin \nu \theta < n \theta,$$

hence $n \theta_n = O(1)$ implies (2.4). To prove the converse let $\theta_n < 1 < \theta_n (n-1)$, and put $[\theta_n^{-1}] = \kappa_n = \kappa$, so that $\kappa \leq \theta_n^{-1} < \kappa + 1 \leq n$. Now $\sum_{1}^{n} \nu^{-1} |\sin \nu \theta_n| > (1/2) \sum_{1}^{n} \nu^{-1} (1 - \cos 2 \nu \theta_n)$, and

$$\left| \sum_{1}^{n} \nu^{-1} \cos 2 \nu \theta_n \right| < \sum_{1}^{n} \nu^{-1} + \left| \sum_{\kappa + 1}^{n} \nu^{-1} \cos 2 \nu \theta_n \right|$$

$$< 1 + \log \kappa + (1/(\kappa + 1)) \max_{\kappa \leq \lambda \leq n} \left| \sum_{\kappa + 1}^{\lambda} \cos 2 \nu \theta_n \right|$$

$$< 1 + \log \theta_n^{-1} + \theta_n / \sin \theta_n < 3 - \log \theta_n.$$

Thus

$$2 \sum_{1}^{n} \nu^{-1} |\sin \nu \theta_n| > \log n + \log n \theta_n - 3 = -3 + \log (n \theta_n);$$

hence (2.4) implies $n \theta_n = O(1)$. For null sequences only this is required.

To satisfy (2.2) consider the case that 0 is a limit point of the sequence $\{ n \theta_n \}$; for a subsequence of indices $n: n \theta_n \to 0$, and for that subsequence, using (2.5)

$$\sum_{1}^{n} \nu^{-1} \sin \nu \theta_n = O \left( \sum_{1}^{n} \nu^{-1} |\sin \nu \theta_n| \right) = O(n \theta_n) = o(1).$$

Hence $\sigma$, if it exists, is 0 and then every convergent sequence is transformed into a null sequence. Next assume lim inf $n \theta_n > 0$. We choose a subsequence of integers $n = n'$ for which $\rho_{n'}$ and $n \theta_n$ have limits $n' \theta_{n'} \to \beta > 0$, $\rho_{n'} \to e^{\gamma}$ say; by (2.3) $\gamma$ is finite. Furthermore from $\log \rho / (\rho - 1) \to 1$ as $\rho \to 1$, $n' (\rho_{n'} - 1) \to \gamma$.

Suppose first $\gamma = 0$, that is $\rho_{n'} \to 1$, and $n' (\rho_{n'} - 1) \to 0$. Now, as $n$ runs through the sequence $\{ n' \}$

$$\left| \sum_{1}^{n} (\rho_{n'} - 1) \nu^{-1} \sin \nu \theta_n \right| < | \rho_{n'} - 1 | O(n) \sum_{1}^{n} \nu^{-1} |\sin \nu \theta_n| = o(1) O(n \theta_n) = o(1),$$
hence
\[ \lim \sum_{1}^{n} r^{n-1} \sin \theta_n = \lim \sum_{1}^{n} r^{-1} \sin \theta_n \]
for \( n = n' \to \infty \), if either side exists. But
\[
\sum_{1}^{n} r^{-1} \sin \theta = \int_{0}^{\beta} \left( \sum_{1}^{n} \cos nt \right) dt = - (1/2)\theta + \int_{0}^{\beta} \frac{\sin (n + 1/2)t}{2 \sin (t/2)} dt \\
= - (1/2)\theta + \int_{0}^{(n+1/2)\beta} \frac{\sin u du}{(2n + 1) \sin (u/(2n + 1))},
\]
hence
\[
\sum_{1}^{n} r^{-1} \sin \theta_n = - (1/2)\theta_n + \int_{0}^{\beta} \frac{\sin u du}{u (2n + 1) \sin (u/(2n + 1))} + o(1) \\
\to \int_{0}^{\beta} \frac{\sin u du}{u} du
\]
as \( n \to \infty \) through the sequence \( \{n'\} \). The consideration of the case \( \gamma \neq 0 \) remains; we write
\[
\sum_{1}^{n} \rho^{n-1} \sin \theta
\]
\[ = \int_{0}^{\beta} \left( \sum_{1}^{n} \rho^{n} \cos nt \right) dt \\
= \int_{0}^{\beta} \frac{\cos t - \rho^{n+1} \cos nt - \rho^{n+1} \cos (n+1)t}{1 - 2\rho \cos t + \rho^{2}} dt \\
= \int_{0}^{\beta} \frac{1 - \rho^{2} (1 - \cos t) + \rho^{n+1} \cos nt - \cos (n+1)t}{(1 - \rho^{2} + 2\rho (1 - \cos t))} dt,
\]
thus
\[
\sum_{1}^{n} \rho^{n-1} \sin \theta = - (1 - \rho^{2}) \int_{0}^{\beta} \frac{dt}{(1 - \rho^{2} + 4\rho \sin^{2} (t/2)} \\
- 2 \int_{0}^{\beta} \frac{\sin^{2} (t/2) dt}{(1 - \rho^{2} + 4\rho \sin^{2} (t/2))} \\
- (1 - \rho) \rho^{n+1} \int_{0}^{\beta} \frac{\cos nt dt}{(1 - \rho^{2} + 4\rho \sin^{2} (t/2))} \\
+ 2\rho^{n+1} \int_{0}^{\beta} \frac{\sin (t/2) \sin (n + 1/2) dt}{(1 - \rho^{2} + 4\rho \sin^{2} (t/2))}. \]
Now
\[
\int_0^{\theta_n} \frac{\sin^2 (t/2) dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2 (t/2)} < \frac{\theta_n}{4\rho_n} \to 0 \quad \text{as} \quad n \to \infty.
\]

Next
\[
(r_n^2 - 1) \int_0^{\theta_n} \frac{dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2 (t/2)} = n(r_n - 1)(r_n + 1) \int_0^{\theta_n} \frac{du}{n^2[(r_n - 1)^2 + 4\rho_n \sin^2 (u/2n)]} \to 2\gamma \int_0^\beta \frac{du}{\gamma^2 + u^2} = 2 \arctan (\beta/\gamma).
\]

Similarly
\[
(r_n - 1)^{n+1} \int_0^{\theta_n} \frac{\cos nt \, dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2 (t/2)} = n(r_n - 1)^{n+1} \int_0^{\theta_n} \frac{\cos u \, du}{n^2[(r_n - 1)^2 + 4\rho_n \sin^2 (u/2n)]} \to \gamma e^\gamma \int_0^\beta \frac{\cos u \, du}{\gamma^2 + u^2},
\]
and
\[
\frac{n+1}{\rho_n} \int_0^{\theta_n} \frac{\sin (t/2) \sin (n + 1/2)t \, dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2 (t/2)} = \frac{n+1}{\rho_n} \int_0^{(n+1/2)\theta_n} \frac{(2n + 1) \sin \{u/(2n + 1)\} \cdot \sin u \, du}{n(2n + 1)[(r_n - 1)^2 + 4\rho_n \sin^2 (u/(2n+1))]}
\]
\[
\to (1/2)e^\gamma \int_0^\beta \frac{u \sin u \, du}{\gamma^2 + u^2}.
\]

Summarizing
\[
\sum_1^n \frac{\rho_n^{n+1}}{\rho_n} \sin n\theta \to \int_0^\beta \frac{\gamma e^\gamma \cos u + e^\gamma \sin u - 2\gamma}{\gamma^2 + u^2} \, du
\]
\[
= \int_0^\beta \frac{e^\gamma(t \sin \eta + \cos \gamma t) - 2}{1 + t^2} \, dt.
\]

The case (2.3) is now completely discussed. We next assume
\[
\limsup_{n \to \infty} \rho_n^n = \infty,
\]
so that for a subsequence \(n = n':\rho_n' \to \infty\). We first prove that (2.1) implies \(n':\theta_n' \to 0\). Otherwise for a subsequence \(n''\) of \(n':n':\theta_n' \to 0\). For these indices
\[ \sum_{n=1}^{\infty} \rho_n^{\nu-1} | \sin \nu \theta_n | = \sum_{\nu \geq \alpha / \beta} \rho_n^{\nu-1} | \sin \nu \theta_n |, \]

where \( \alpha \) is so chosen that \( 0 < \alpha < \beta \) and \( \alpha \leq \pi / 2 \). Now

\[ \sum_{\nu \geq \alpha / \beta} \rho_n^{\nu-1} | \sin \nu \theta_n | > \left( \frac{2}{\pi} \right) \theta_n \sum_{\nu} \rho_n^{\nu-1} = \frac{2 \rho_n \theta_n}{\pi} \frac{\rho_n^{[\nu^{\nu-1}]} - 1}{\rho_n - 1}, \]

hence (2.1) implies

\[ \theta_n \rho_n^{\alpha / \beta} = O(\rho_n - 1), \]

which by virtue of \( \log \rho_n / (\rho_n - 1) \to 1 \) yields \( n \theta_n = o(1) \). Furthermore

\[ \left( \frac{2}{\pi} \right) \theta_n \sum_{\nu} \rho_n^{\nu} < \sum_{\nu} \rho_n^{\nu-1} | \sin \nu \theta_n | < \theta_n \sum_{\nu} \rho_n^{\nu}, \]

thus, if for a subsequence of indices \( \rho_n^\alpha \to \infty \), then for these indices (2.1) is equivalent to

\[ \theta_n \rho_n = O(\rho_n - 1). \]

If this condition is satisfied, then in view of \( n \theta_n \to 0 \)

\[ 0 < \theta_n \sum_{\nu=1}^{n} \rho_n^{\nu} - \sum_{\nu=1}^{n} \rho_n^{\nu-1} | \sin \nu \theta_n | = \theta_n \sum_{\nu=1}^{n} \rho_n^{\nu} \left( 1 - \frac{\sin \nu \theta_n}{\nu \theta_n} \right), \]

hence (2.2) holds if and only if \( \lim \theta_n \rho_n^\alpha / (\rho_n - 1) \) exists, which is then the value of \( \sigma \).

Finally assume that \( \lim \inf \rho_n^\alpha = 0 \); thus for a subsequence of indices \( \rho_n^\alpha \to 0 \) \( (\gamma = -\infty) \). If \( n \theta_n = O(1) \), then

\[ \sum_{\nu} \rho_n^{\nu-1} | \sin \nu \theta_n | = O(\sum_{\nu} ^{n} | \sin \nu \theta_n |) = O(1), \]

which is (2.1). If on the other hand for a subsequence \( n \theta_n \to \infty \), then

\[ \sum_{\nu=1}^{n} \rho_n^{\nu-1} | \sin \nu \theta_n | > \sum_{\nu=1}^{n} \rho_n^{\nu-1} \frac{1 - \cos 2 \nu \theta_n}{2}; \]

but \( \rho_n^{\nu-1} \downarrow \) as \( \nu \uparrow \), hence for \( \theta_n < 1 \)

\[ \left| \sum_{\nu=1}^{n} \rho_n^{\nu-1} \cos 2 \nu \theta_n \right| < \frac{\theta_n}{\sin \theta_n} < \frac{1}{1 + [\theta_n^{-1}]} \max_{\nu=1}^{[\nu]} \cos 2 \nu \theta_n \]

Furthermore
\[ \sum_{n<1} \rho_n^{-1} = \sum_{u<1} u^{-1} \exp (\nu \log \rho_n) > \int_{1}^{n+1} u^{-1} \exp (-u \log \rho_n) du = \int_{1}^{n+1} \rho_n^{-1} e^{-u \log \rho_n} du = \int_{1}^{n+1} \rho_n^{-1} e^{-u \log \rho_n} du; \]

thus in this case \((2.1)\) implies \(\theta_n = O(\log 1/\rho_n),\) or \(\theta_n = O(1-\rho_n).\) If this condition is satisfied, then

\[ \sum_{n=1}^{\infty} \rho_n^{-1} \sin \nu \theta_n = O\left( \frac{n}{n+1} \right) = O(\theta_n/(1-\rho_n)) = O(1), \]

hence \((2.1)\) holds. To satisfy \((2.2)\) now, we note that

\[ \sum_{n=1}^{\infty} \rho_n^{-1} \sin \nu \theta_n = O\left( \frac{\rho_n}{1-\rho_n} \right) = O(\rho_n) = o(1), \]

hence \((2.2)\) holds if and only if

\[ \sum_{n=1}^{\infty} \rho_n^{-1} \sin \nu \theta_n = \arctan \frac{\rho_n \sin \theta_n}{1 - \rho_n \cos \theta_n} \]

has a limit, and \(\sigma\) is then this limit. But

\[ \frac{\rho_n \sin \theta_n}{1 - \rho_n \cos \theta_n} = \frac{\rho_n \sin \theta_n}{1 - \rho_n + \rho_n(1 - \cos \theta_n)} \sim \frac{\theta_n}{1 - \rho_n(1 + O(1 - \rho_n))} \sim \frac{\theta_n}{1 - \rho_n}, \]

hence \(\sigma\) exists, if and only if \(\lim \theta_n/(1-\rho_n) = \delta < +\infty.\) We then have \(\sigma = \lim \arctan \left\{ \theta_n/(1-\rho_n) \right\} = \arctan \delta.\) To summarize our results put

\[ \sigma(\beta , 0) = \int_{0}^{\beta} \sin \frac{u}{u} du, \]

(a)

\[ \sigma(\beta , \gamma) = \int_{0}^{\beta} \frac{\gamma e^{\gamma} \cos u + e^{\gamma} \sin u - 2\gamma}{\gamma^2 + u^2} du, \text{for finite } \gamma \neq 0, \]

(b)

\[ \sigma(0, \infty) = \lim \frac{\theta_n}{\rho_n - 1}, \]

(c)

\[ \sigma(\delta, -\infty) = \lim \arctan \frac{\theta_n}{1 - \rho_n} = \arctan \delta < \pi/2. \]

We then have

**Theorem 1.** Necessary and sufficient conditions that for every convergent sequence \(n \theta_n\) the transform \(\sum_{n=1}^{\infty} \rho_n \theta_n \sin \nu \theta_n\) has a limit, are that one of the following three cases holds:
(a') \( n(\rho_n - 1) \to \gamma \) finite, \( n\theta_n \to \beta < \infty \),
(b') \( n(\rho_n - 1) \to +\infty \), \( \lim \theta_n \rho_n(\rho_n - 1)^{-1} \) exists,
(c') \( n(\rho_n - 1) \to -\infty \), \( \lim \theta_n(1 - \rho_n)^{-1} - \delta \) exists, \( 0 \leq \delta < \infty \).

The limit of the transform is then \( \tau \sigma \), where \( \sigma \) is defined above for the respective cases. Different subsequences may belong to different cases \((\beta, \gamma)\) if only the corresponding \( \sigma \) attain the same value, and with the restriction \( n\theta_n = O(1) \) in case (a').

3. Permanency with respect to \((C, \kappa)\) summability. Given the sequence \( \{\tau_n\} \), write

\[
\tau_n = \tau_n, \quad \tau_n^b = \sum_{r=1}^{n} \tau_n = n, \quad n, \quad k = 1, 2, 3, \ldots ;
\]

also

\[
(3.1) \quad A_n^c = C_{n+k} = \frac{(k + 1) \cdots (k + n)}{n!} \sim \frac{n^\kappa}{k!}, \quad \text{as } n \to \infty.
\]

The sequence \( \{\tau_n\} \) is summable \((C, \kappa)\) to the value \( \tau \), if \( \tau_n/A_n^c \to \tau \) as \( n \to \infty \);
\((C, 0)\) is evidently convergence.

We write

\[
\Delta^0 \tau_n = \tau_n, \quad \Delta^1 \tau_n = \Delta \tau_n = \tau_n - \tau_{n+1}, \quad \Delta^k \tau_n = \Delta(\Delta^{k-1} \tau_n);
\]

then by induction

\[
(3.2) \quad \Delta^k \tau_n = \sum_{r=0}^{k} (-1)^r C_{n,n+k} \tau_{n+r}, \quad k = 0, 1, 2, \ldots .
\]

Abel's transformation yields for finite sums

\[
\sum_{r=1}^{n} \alpha_r \tau_r = \sum_{r=1}^{n} \tau_r \Delta \alpha_r = \sum_{r=1}^{n} \tau_r \Delta^2 \alpha_r = \cdots ,
\]

where \( \alpha_{n+1} = 0, \alpha_{n+2} = 0, \ldots \). Applying this to (1.5) we get

\[
T_n = \sum_{r=1}^{n} \tau_r \Delta^c \alpha_n,
\]

where \( \alpha_n = 0 \) for \( \nu > n \). Thus the transform converges for every \((C, \kappa)\) summable sequence if in addition to the conditions of §2

\[
\sum_{r=1}^{n} A_n^c | \Delta^c \alpha_n | = O(1) \quad \text{as } n \to \infty.
\]

In particular for the transform (1.6) we have the conditions (2.1), (2.2) and

\[
(3.3) \quad \sum_{r=1}^{n-\varepsilon} A_n^c | \Delta^c \rho_n^{p-1} \sin \rho_{n} | + \sum_{n-\varepsilon+1}^{n} A_n^c | \delta_n | = O(1) \quad \text{as } n \to \infty,
\]

where \( n\theta_n = O(1) \) in case (a').
where, from (3.2)

\[ \delta_\lambda = \sum_{\nu=0}^{n-\lambda} (-1)^\nu C_{\lambda,\nu} \frac{\lambda^\nu \sin (\lambda + \nu) \theta_n}{\lambda + \nu}, \quad n - \nu < \lambda \leq n. \]

We first consider \((C, 1)\) summability \((\kappa = 1)\). Now (3.3) becomes

\[ \sum_{\nu=1}^{n-1} (\nu + 1) | Ap_n^{(n-\nu)} \sin \nu \theta_n + (n + 1) \rho_n^n \sin n \theta_n | = O(1), \]

or

\[ (3.4) \sum_{\nu=1}^{n-1} \nu | Ap_n^{(n-\nu)} \sin \nu \theta_n - \rho_n^{n-1} (\nu + 1) \sin (\nu + 1) \theta_n | + \rho_n^n \sin n \theta_n | = O(1). \]

We consider in succession the different cases of Theorem 1.

\((a')\) For a sequence of indices \(n \theta_n \rightarrow \beta < \infty, n(\rho_n - 1) \rightarrow \gamma \) finite, that is, \(\rho_n^n \sin n \theta_n = O(1)\), and

\[ \sum_{\nu=1}^{n-1} \nu \rho_n^n \sin \nu \theta_n - \rho_n^{n-1} (\nu + 1) \sin (\nu + 1) \theta_n | \]

\[ \leq \sum_{\nu=1}^{n-1} \nu \rho_n^n \sin \nu \theta_n - (\nu + 1) \sin (\nu + 1) \theta_n | \]

\[ + | 1 - \rho_n | \sum_{\nu=1}^{n-1} \rho_n^n \sin (\nu + 1) \theta_n |. \]

Now

\[ | 1 - \rho_n | \sum_{\nu=1}^{n-1} \rho_n^n \sin (\nu + 1) \theta_n | < | 1 - \rho_n | \sum_{\nu=1}^{n-1} \rho_n^n \sin (\nu + 1) \theta_n | = O(1), \]

and

\[ \nu \sin \nu \theta_n - (\nu + 1) \sin (\nu + 1) \theta_n | \]

\[ = | (\nu + 1) \sin (\nu + 1) \theta_n - 2 \sin (1/2) \theta_n \cos ((2\nu + 1)/2) \theta_n | < 2 \theta_n ; \]

hence

\[ \sum_{\nu=1}^{n-1} \nu \rho_n^n \sin \nu \theta_n - (\nu + 1) \sin (\nu + 1) \theta_n | < 2 \theta_n \sum_{\nu=1}^{n-1} \rho_n^n = O(n \theta_n) = O(1). \]

Hence in this case no additional condition results.

\((b')\) \(n(\rho_n - 1) \rightarrow + \infty, \theta_n \rho_n^n (\rho_n - 1) -1 \rightarrow \sigma\). Hence \(n \theta_n \rightarrow 0\), and now \(\rho_n^n \sin n \theta_n = O(1)\) is equivalent to \(n \theta_n \rho_n^n = O(1)\). Thus \(\theta_n \rho_n^n (\rho_n - 1) -1 = n \theta_n \rho_n^n n -1 (\rho_n - 1) -1 \rightarrow 0\), that is, \(\sigma = 0\). Now

\[ | 1 - \rho_n | \sum_{\nu=1}^{n-1} \rho_n^n \sin (\nu + 1) \theta_n | = O \left[ (\rho_n - 1) \theta_n \sum_{\nu=1}^{n-1} \nu \rho_n^n \right] \]

\[ = O[(\theta_n \rho_n^n (\rho_n - 1) -1)] = o(1); \]
furthermore
\[ \theta_n \sum_{r=1}^{n} \varphi_{n} = O[\theta_n \rho_n (\rho_n - 1)^{-1}] = o(1); \]
hence (3.4) holds. Finally:

(c') If \( \lim \theta_n/(1 - \rho_n) < \infty \) exists, and \( n(\rho_n - 1) \rightarrow -\infty \), that is, \( \rho_n \rightarrow 0 \), then
\[ \rho_n \sin n\theta_n \rightarrow 0, \quad \left| 1 - \rho_n \right| \sum_{r=1}^{n-1} \rho_n^r \left| \sin (r+1)\theta_n \right| < (1 - \rho_n) \sum_{r=0}^{\infty} \rho_n^r = 1, \]
and
\[ \theta_n \sum_{r=1}^{n} \varphi_{n} < \frac{\theta_n}{1 - \rho_n} = O(1). \]

No additional condition appears in this case. Summarizing, we have

**Theorem 2.** Necessary and sufficient conditions that when \( \lim n^{-1} \sum_{n} \varphi_{n} = \tau \) exists the transform \( \sum_{n} \varphi_{n} \sin n\theta_n \) has a limit: \( \tau \sigma \), are either of the alternatives:

(a'') \( n(\rho_n - 1) \rightarrow \gamma \), finite, \( n\theta_n \rightarrow \beta < \infty \),

(b'') \( n(\rho_n - 1) \rightarrow +\infty \), \( n\theta_n \rho_n = O(1) \),

(c'') \( n(\rho_n - 1) \rightarrow -\infty \), \( \lim \theta_n/(1 - \rho_n) \rightarrow 0 \).

The value of \( \sigma \) is in the cases (a'') and (c'') given by (a) and (c). In case (b'') \( \sigma = 0 \). Different subsequences may belong to different cases if only \( \sigma \) has the same value, with the restriction \( n\theta_n = O(1) \) in case (a'').

We now consider \( (C, \kappa) \) summability for \( \kappa > 1 \). First of all, to satisfy (3.3) we must have

\( \sum_{r=0}^{\infty} (-1)^{r} C_{k, \kappa} \rho_{n}^{-m+v} \sin (n - m + v)\theta_n/n - m + v = O(1), m = 0, 1, \ldots, k - 1. \)

Or
\[ \sum_{r=0}^{\infty} (-1)^{r} C_{k, \kappa} \rho_{n}^{-m+v} \sin (n - m + v)\theta_n/n - m + v = O(1), m = 0, 1, \ldots, k - 1. \]

This is equivalent to
\[ \sum_{r=0}^{\infty} (-1)^{r} C_{k, \kappa} \rho_{n}^{-m+v} \sin (n - m + v)\theta_n/n - m + v = O(1), m = 0, 1, \ldots, k - 1. \]
In case (a'') the first condition becomes \( n\theta_n = O(n^{1-\varepsilon}) \), as \( n \to \infty \); in particular \( n\theta_n \to 0 \), thus in view of (a') \( n\theta_n \to \lambda \pi \), \( \lambda \) a positive integer or zero. On putting \( n\theta_n = \lambda \pi + \varepsilon_n \), we get \( \varepsilon_n = O(n^{1-\varepsilon}) \), or \( n\theta_n - \lambda \pi = O(n^{1-\varepsilon}) \). From the second condition now \( \cos n\theta_n \sin \theta_n = O(n^{1-\varepsilon}) \), as \( n \to \infty \), or \( \lambda \pi + \varepsilon_n = O(n^{1-\varepsilon}) \); hence for \( \kappa = 2 \), (3.6) reduces to

\[
(3.7) \quad n\theta_n = \lambda \pi + O(n^{-1}).
\]

For \( \kappa > 2 \) we must have

\[
n\theta_n - \lambda \pi = \varepsilon_n = O(n^{1-\varepsilon}) \quad \text{and} \quad \lambda \pi + \varepsilon_n = O(n^{1-\varepsilon}),
\]

hence \( \lambda = 0 \), and

\[
(3.7') \quad n\theta_n = O(n^{1-\varepsilon}).
\]

It then follows that

\[
n^{1-\varepsilon} \sin (n - \nu)\theta_n = O(1) \quad \text{for} \quad \nu = 0, 1, \ldots, \kappa - 1.
\]

Furthermore, for the rest of (3.3)

\[
\sum_{\nu=1}^{n-\varepsilon} \Delta_{\rho_n^\nu}^{-1} | \sin \nu \theta_n | = O \left( \sum_{\nu=1}^{n-\varepsilon} | \Delta_{\rho_n^\nu}^{-1} \sin \nu \theta_n | \right).
\]

Now

\[
\Delta_{\rho_n^\nu}^{-1} \sin \theta = \Delta_{\rho_n^\nu} \int_0^\theta \cos vtdt = R \int_0^\theta \Delta_{\rho_n^\nu} z^\nu dt, 
\]

and, using (3.2)

\[
\Delta_{\rho_n^\nu}^{-1} \sin \theta = R \int_0^\theta \sum_{\lambda=0}^\nu (-1)^\lambda C_{\lambda,\lambda} z^{\nu+\lambda} dt = R \int_0^\theta z^\nu (1 - z)^\nu dt,
\]

hence

\[
| \Delta_{\rho_n^\nu}^{-1} \sin \theta | < \rho^\nu \int_0^\theta | 1 - \rho e^{it} | \nu dt < \rho^\nu \int_0^\theta \{(1 - \rho)^2 + \rho^2 \}^{\nu/2} dt
\]

\[
< \theta \rho^\nu \{(1 - \rho)^2 + \rho^2 \}^{\nu/2}.
\]

Thus

\[
\sum_{\nu=1}^{n-\varepsilon} | \Delta_{\rho_n^\nu}^{-1} \sin \nu \theta_n | < \left( \sum_{\nu=1}^{n-\varepsilon} \rho_n^\nu \right) \{(1 - \rho_n)^2 + \rho_n^2 \}^{\nu/2} \theta_n
\]

\[
< \left\{ n^2 (1 - \rho_n)^2 + \rho_n n^2 \theta_n \right\}^{\nu/2} \theta_n \sum_{\nu=1}^{n-\varepsilon} \rho_n^\nu
\]

\[
= O \left( \theta_n \sum_{\nu=1}^{n-\varepsilon} \rho_n^\nu \right),
\]

and, from \( \rho_n^\nu = O(1) \),
Hence in case (a'') the additional condition is (3.7) for κ = 2, and (3.7') for
κ > 2.
In case (b'): \( n(\rho_n - 1) \to +\infty \), \( n\theta_n\rho_n^\kappa = O(1) \), as \( n \to \infty \); hence \( \rho_n^\kappa \to +\infty \),
and \( n\theta_n \to 0 \). Now (3.6) becomes

\[(3.9) \quad n \rho_n \theta_n = O(1).\]

For large \( n \) evidently \( \rho_n > 1 \), and

\[\theta_n \sum_{j=1}^{n} \rho_j = O(n \theta_n) = O(1)\]
(from (3.9)). In view of (3.8) now (3.3) holds. Thus in this case the additional
condition is (3.9) (for κ ≥ 2).

Finally, in case (c'') \( n(\rho_n - 1) \to -\infty \) (that is \( \rho_n \to 0 \)), and \( \lim \theta_n/(1 - \rho_n) = \delta < \infty \) exists. Now \( \rho_n < 1/(n(1 - \rho_n)) \), hence \( n\theta_n\rho_n^\kappa = O(1) \); thus for κ = 2 con-
dition (3.6) reduces to \( n\rho_n^\kappa \sin n\theta_n = O(1) \). While for κ > 2 (3.6) reduces to
\( n^{\kappa-1}\rho_n^\kappa \sin n\theta_n = O(1) \) and \( n^{\kappa-2}\theta_n\rho_n^\kappa = O(1) \). Furthermore, as \( \rho_n < 1 \), \( \theta_n \sum_{j=1}^{n} \rho_j \theta_j < n\theta_n \rho_n = O(1) \), hence, in view of (3.8) now (3.3) is satisfied.

We summarize our results in

**Theorem 3.** In order that \( \lim n^{\kappa} \theta_n \sum_{j=1}^{n} \rho_j \theta_j \sin n\theta_j = r \) exists, whenever (C, κ)
lim \( n\theta_n \to r \) for some \( κ \geq 2 \), necessary and sufficient conditions are the alternatives:
(a'') \( n(\rho_n - 1) \to 0 \) (that is \( \rho_n \to 0 \)), and for κ = 2: \( \theta_n = \lambda \pi + O(n^{-1}) \), \( \lambda \) an integer, for
κ > 2: \( \theta_n = O(n^{-r}) \);
(b'') \( n(\rho_n - 1) \to +\infty \), \( n^\kappa \theta_n = O(1) \);
(c'') \( n(\rho_n - 1) \to -\infty \), \( \lim \theta_n/(1 - \rho_n) = \delta < \infty \) exists, and for κ = 2:
\( n\rho_n^\kappa \sin n\theta_n = O(1) \), for κ > 2: \( n^{\kappa-1}\rho_n^\kappa(n^\kappa + |\sin n\theta_n|) = O(1) \).

The value of \( \sigma \) is given in case (a'') by (a), where for κ = 2: \( \beta = \lambda \pi \), for
κ > 2: \( \beta = 0 \), \( \sigma = 0 \); in case (b'') \( \sigma = 0 \); in case (c'') \( \sigma = \arctan \delta \).

**4. Application to Fourier series.** First consider a function of bounded
variation and its Fourier sine series (1.1). It follows from the introduction
that \( \lim nb_n \), if it exists, is \( (2/\pi)f(0+) \). Under the assumptions of Theorem 1
on \( \rho_n \) and \( \theta_n \), \( s_n(\rho_n, \theta_n) \to r \sigma = (2\sigma/\pi)f(0+ \).
In particular whenever \( \sigma > \pi/2 \), then we have an analogue of Gibbs' phenomenon. It is known that for
functions of bounded variation

\[ (1/n) \sum_{j=1}^{n} nb_j \to (2/\pi)f(0+) \]

more generally if (cf. Szász [3, Lemma 6])
(4.1) \[ 2f_1(\theta) = \frac{2}{\theta} \int_0^\theta f(t)dt \to j, \quad \text{as } \theta \downarrow 0, \]

and

(4.2) \[ \lim_{\theta \downarrow 0} \lim_{n \to \infty} \min_{0 < \xi < \xi_n} \sum_{n} b_n \geq 0, \]

then

\[ (1/n) \sum_{1}^{n} v b_n \to j/\pi. \]

Hence, applying Theorem 2 we have

\[ s_n(\rho_n, \theta_n) \to (j/\pi) \sigma(\beta, \gamma), \quad \text{as } n(\rho_n - 1) \to \gamma \text{ and } n\theta_n \to \beta; \]

\( j \) is the generalized jump of \( f(\theta) \) at \( \theta = 0 \). For \( \gamma = 0 \) this yields a generalization of formula (1.4). Note that

\[ f_1(\theta) = \theta^{-1} \int_0^\theta f(t)dt = \sum_{1}^{\infty} b_n \frac{1 - \cos \nu \theta}{\nu \theta} = \left( \frac{\sin (\nu \theta /2)}{\nu \theta /2} \right)^2; \]

\( (2\theta/\pi) \left\{ s_0/2 + \sum_{1}^{\nu} \left( (\sin \nu \theta /\nu \theta)^2 \right) s_n \right\} \) is called the Riemannian mean of the second kind corresponding to the sequence \( \{s_n\} \). It is a regular transform, as is seen from the identity

\[ \frac{2\theta}{\pi} \left\{ 1/2 + \sum_{1}^{\nu} \left( (\sin \nu \theta /\nu \theta)^2 \right) \right\} = 1. \]

If we assume only that \( (C, 2) \lim n b_n = j/\pi \) exists, then Theorem 3 yields again a Gibbs' phenomenon in the case \( \lambda > 0 \).

In this connection we introduce two lemmas.

**Lemma 1.** If

(4.3) \[ (1 - r) \sum_{1}^{\nu} \tau_n r^n \to \tau \quad \text{as } r \uparrow 1, \]

and

(4.4) \[ \tau'_n = \sum_{1}^{\nu} \tau_n > -p n, \]

for some \( p > 0 \), and all \( n > 0 \), then

(4.5) \[ (C, 2) \lim \tau_n = \tau. \]

We have from (4.3)
1943] THE PARTIAL SUMS OF FOURIER SERIES 453

\[ (1 - r)^2 \sum_{n=1}^{\infty} r^n r^n \to r \quad \text{as } r \uparrow 1, \]

hence

\[ (1 - r)^2 \sum_{n=1}^{\infty} (r^n + pn)r^n \to r + p \quad \text{as } r \uparrow 1; \]

in view of (4.4) a theorem of Hardy and Littlewood yields

\[ \sum_{n=1}^{n} (r^n + pn) \sim (1/2)(r + p)n^2, \]

or

\[ \sum_{n=1}^{n} r^n \sim (1/2)n^2, \quad \text{as } n \to \infty, \]

which is (4.5).

**Lemma 2.** If (4.1) holds, then \((1 - r)\sum_{n=1}^{\infty} n b_n r^n \to j/\pi\). [3, Lemma 5].

Combining these two lemmas it is seen that (4.1) and the assumption

\[ \sum_{n=1}^{n} nb_n > -pn \quad \text{for some } p > 0 \text{ and all } n > 0, \]

imply \((C, 2) \lim nb_n = j/\pi\). With reference to Theorem 3 the assumptions (4.1) and (4.6) again yield a Gibbs' phenomenon.

In closing we remark that the existence of \(f(+0)\) implies itself \((C, 2) \lim nb_n = (1/2)f(+0)\). A more general result will be given elsewhere.

**References**


University of Cincinnati, Cincinnati, Ohio