PERMANENT CONFIGURATIONS IN THE
\textit{n}-BODY PROBLEM

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1. Introduction. The problem of two bodies for spheres, homogeneous in
concentric layers and finite in size, was first solved in a geometrical way by
Newton \cite{14}\footnote{Presented to the Society, September 13, 1943; received by the editors March 8, 1943 and,
in revised form, July 8, 1943.} about 1685. Euler \cite{4} gave the first detailed analytical solution of the problem in 1744. In 1772 Lagrange \cite{8} gave four particular solutions of the three-body problem in his prize memoir. All solutions of the two-body problem and Lagrange's particular solutions of the three-body problem belong to that special class called permanent configurations\footnote{(1) Numbers in brackets refer to the bibliography at the end of the paper.}, which we
we shall define presently for the case of \textit{n}-bodies.

Consider \textit{n} free particles in space which attract each other along lines
joining them according to any function of the distance. If all the masses are
projected with initial velocities, and thereafter are acted upon only by the
force of attraction, their orbits will be space curves. The problem of deter-
mining the positions and velocities of the bodies at any later time is the
problem of \textit{n} bodies. \textit{A configuration of \textit{n} bodies is said to be permanent if, as
the masses move in their respective orbits, the ratios of the mutual distances re-
main constant.} Such a configuration may change in size but not in shape.

The literature on permanent configuration problems contains two methods
of approach, (1) that of establishing continuity between the configurations for
\textit{n}—1 and \textit{n}-bodies, and (2) that of characterizing the configurations for any \textit{n}
by the necessary and sufficient conditions which the \textit{n}(\textit{n}—1)/2 mutual dis-
tances must satisfy. Collinear configurations were discovered for the case \textit{n}=3
by Euler \cite{5} and for any \textit{n} by Lehmann-Filhès \cite{9} and F. R. Moulton \cite{13},
the latter using the first approach. Noncollinear configurations for the case
\textit{n}=3 were discovered by Lagrange \cite{8}, who also treated the collinear case.

Dziobek \cite{3} discussed the general approach (2) and arrived at some results
for the case \textit{n}=4, which case was treated in detail by MacMillan and Bartky
\cite{12}. W. L. Williams \cite{16}, applying the same method, considered the non-
collinear case \textit{n}=5. Some noncollinear solutions of the \textit{n}-body problem have
been found by Hoppe \cite{7}, Andoyer \cite{1}, Longley \cite{10} and Emilia Breglia \cite{2},
but all have some element of symmetry.

Space permanent configurations are the scarcest and have the undesirable
property that the motion of all bodies is either toward their common center

\footnote{(2) MacMillan \cite{11, pp. 71–72, 75–78}.}
of gravity, in which case the configuration lasts for a finite time, or in the
the opposite direction, in which case the configuration is expanding without
limit(3). In fact it can be shown that all permanent configurations are either
space configurations of the dilating without rotation type or they are
planar(4). We shall treat only the plane configurations, and in particular only
those rotating without dilation, for if the law of attraction is the Newtonian
law and there exists a permanent configuration in which the \( n \) bodies revolve
in concentric circles about the common center of gravity, the bodies may also
move in similar conics with the common center of gravity as foci(6).

In most cases mentioned above it has not been necessary to restrict the
masses or mass ratios, but in this paper we shall, upon applying approach (1),
make certain restrictions which are sufficient, though they may not be neces-
sary, to insure continuity.

2. The equations for a permanent configuration. The problem of \( n \) bodies
belongs in the field of differential equations. If we require that the ratios of
the mutual distances remain constant, the problem becomes one of permanent
configurations and belongs in the theory of implicit functions. As pointed out
in §1, we need require only that the motion be circular, that is, the distances
themselves remain constant.

Let the origin be taken at the center of mass of the system, and let \( \omega \)
denote the constant angular velocity of the \( n \) masses about the origin. The
equations for a permanent configuration are(6)

\[
\begin{align*}
\sum_{j=1}^{n} \frac{x_i - x_j}{(r_{ij})^3} m_j - \lambda x_i &= 0, \\
\sum_{j=1}^{n} \frac{y_i - y_j}{(r_{ij})^3} m_j - \lambda y_i &= 0,
\end{align*}
\]

(2.1) \( i = 1, 2, \ldots, n, \)

where \( \lambda = \omega^2 \), \( (r_{ij})^2 = (x_i - x_j)^2 + (y_i - y_j)^2 \) and the prime on the summation
denotes the sum for all \( j \) except \( j \) equal \( i \). It is a problem now in implicit function
theory to solve equations (2.1) for the \( 2n \) coordinates \( x_i \) and \( y_i \) in terms of
the \( n+1 \) parameters \( m_i \) and \( \lambda \).

3. Existence of a point of libration. The first step in establishing continu-
ity between the solutions for \( n - 1 \) and \( n \) bodies consists of proving the exist-
ence of at least one point of libration for the case of \( n - 1 \) bodies, and this
section is devoted to that purpose.

Suppose that for a set of positive values \( m_1, m_2, \ldots, m_{n-1}, \lambda \), and for
\( n \geq 4 \), the first \( n-1 \) equations of each set (2.1) have a solution \( (x_0^p, y_0^p) \)

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(3) The space configuration for \( n = 4 \) was first noted by Lehmann-Filhes [9]. Cf. also Mac-
Millan [11, p. 74], and Wintner [17, p. 279].

(4) Wintner [17, pp. 287 ff.].

(5) Wintner [17, p. 300], or MacMillan [11, p. 74].

(6) Wintner [17, p. 302].
\[ (i = 1, 2, \cdots, n-1) \) and that these points \( P_i = (x_i^0, y_i^0) \) are not collinear. If an infinitesimal mass \( m_n \) be added to the system at the point \( (x_n, y_n) \) then \( x_n \) and \( y_n \) must satisfy the equations

\[
\phi_1(x_n, y_n) = \sum_{i=1}^{n-1} \frac{(x_n - x_i^0) m_i}{\sqrt{[(x_n - x_i^0)^2 + (y_n - y_i^0)^2]}} - \lambda x_n = 0, \\
\phi_2(x_n, y_n) = \sum_{i=1}^{n-1} \frac{(y_n - y_i^0) m_i}{\sqrt{[(x_n - x_i^0)^2 + (y_n - y_i^0)^2]}} - \lambda y_n = 0.
\]

(3.1)

The subscript \( n \) may be omitted, and the two equations

\[
\phi_1(x, y) = 0, \quad \phi_2(x, y) = 0,
\]

(3.2)

are the equations of two algebraic plane curves which we shall call \( C_1 \) and \( C_2 \) respectively. Since the \( n-1 \) points \( P_i^0 \) satisfy both equations (3.1), both curves pass through each of the \( P_i^0 \). It will be shown that \( C_1 \) and \( C_2 \) have at least one real intersection other than the \( n-1 \) points \( P_i^0 \).

In order to determine the behavior of the curves in the neighborhood of any one of the points \( P_i^0 \), we may write the equations for \( C_1 \) and \( C_2 \) in the form

\[
G_1(x, y) = (x - x_i^0) m_i + \frac{[(x - x_i^0)^2 + (y - y_i^0)^2]^{3/2}}{\sqrt{[(x - x_i^0)^2 + (y - y_i^0)^2]}} f_1(x, y) = 0, \\
G_2(x, y) = (y - y_i^0) m_i + \frac{[(x - x_i^0)^2 + (y - y_i^0)^2]^{3/2}}{\sqrt{[(x - x_i^0)^2 + (y - y_i^0)^2]}} f_2(x, y) = 0,
\]

where

\[
f_1(x, y) = \sum_{i=1}^{n-1} (x - x_i^0) m_i - \lambda x, \\
f_2(x, y) = \sum_{i=1}^{n-1} (y - y_i^0) m_i - \lambda y.
\]

(3.3)

Since \( f_1(x_i^0, y_i^0) = 0 \), and \( f_2(x_i^0, y_i^0) = 0 \) from the \( i \)th equation in each set (2.1), it follows that \( f_1(x, y) \) and \( f_2(x, y) \) may be expanded in powers of \( x - x_i^0 \) and \( y - y_i^0 \), vanishing for \( x = x_i^0, y = y_i^0 \). Thus

\[
\phi_1(x, y) = \frac{(x - x_i^0) m_i}{\sqrt{[(x - x_i^0)^2 + (y - y_i^0)^2]}} \left[ A_1(x - x_i^0) + B_1(y - y_i^0) + \cdots \right], \\
\phi_2(x, y) = \frac{(y - y_i^0) m_i}{\sqrt{[(x - x_i^0)^2 + (y - y_i^0)^2]}} \left[ A_2(x - x_i^0) + B_2(y - y_i^0) + \cdots \right].
\]

(3.4)
The slope of $C_1$ is
\[
\frac{\partial G_1}{\partial x} = m \alpha + 3 \left( (x-x_i)^2 + (y-y_i)^2 \right)^{3/2} f_1(x, y) + \left( (x-x_i)^2 + (y-y_i)^2 \right)^{3/2} [A_1 + \cdots]
\]
and the slope of $C_2$ is
\[
\frac{\partial G_2}{\partial x} = m \alpha + 3 \left( (x-x_i)^2 + (y-y_i)^2 \right)^{3/2} f_2(x, y) + \left( (x-x_i)^2 + (y-y_i)^2 \right)^{3/2} [B_2 + \cdots]
\]
On evaluating these slopes at $(x^0, y^0)$ we find that $C_1$ passes through $P^0_1$ tangent to $x = x_i^0$ and $C_2$ passes through $P^0_i$ tangent to $y = y_i^0$.

We shall determine next on which side of the tangent line the curve lies in the neighborhood of $P^0_i$. Let $x - x_i^0 = r \cos \theta$, $y - y_i^0 = r \sin \theta$ then equations (3.2) and (3.4) become, in polar coordinates,
\[
\psi_1(r, \theta) = \left( m_i/r^2 + A_1 r \right) \cos \theta + B_1 r \sin \theta + \cdots = 0,
\]
\[
\psi_2(r, \theta) = \left( m_i/r^2 + A_2 r \right) \cos \theta + B_2 r \sin \theta + \cdots = 0.
\]
From (3.3) we note that $\partial f_1/\partial y = \partial f_2/\partial x$, and hence $B_1 = A_2$. Since $m_i$ and $r$ are positive we may choose $r$ so small that both $(m_i/r^2 + A_1)$ and $(m_i/r^2 + B_2)$ are positive. Then as $\theta$ varies from 0 to $2\pi$, $\psi_1(r, \theta)$ changes sign at $\theta = \arctan \left( (m_i/r^2 + A_1)/( -B_1) \right)$, and $\psi_2$ changes sign at $\theta = \arctan \left( -B_1/(m_i/r^2 + A_2) \right)$. Hence the curvature of both $C_1$ and $C_2$ at $P^0_i$ depends only upon the sign of $B_1$. The behavior of the curves at any $P^0_i$ is either that of Fig. 1a or Fig. 1b. In case $B_1 = 0$ the behavior of the curves...
is governed by the sign of the first nonzero quantity in the sequence
\[ \frac{\partial^2f_1}{\partial y^2} = \frac{\partial^2f_2}{\partial x \partial y}, \quad \frac{\partial^3f_1}{\partial y^3} = \frac{\partial^2f_2}{\partial y^2 \partial x}, \ldots, \]
and the cases pictured in Figures 1a and 1b are all that can occur. In order to prove that there is at least one real solution of (3.2) other than the \( P_i \), let us denote by \( P^0 \) that \( P_i \) which has the largest ordinate. In case \( B_i \) is negative at \( P^0 \) the curves are as shown in Fig. 2, and we shall prove that there is at least one real intersection of \( C_1 \) and \( C_2 \) in that section of the plane for which \( y > y^0 \) and \( x > x^0 \).

By choice of notation there are no \( P_i \) in this region and suppose that there is no other intersection. Let \( P = (x, y) \) move from \( P^0 \) to \( +\infty \) along \( y = y^0 \). Along this ray \( P \) meets \( C_1 \) first, namely at \( P^0 \), and then meets \( C_2 \) later, since \( \phi_1 \) varies continuously from \( +\infty \) to \( -\infty \) as is easily verified in (3.1). Similarly, as \( P \) moves from \( P^0 \) to \( +\infty \) along \( x = x^0 \) it meets \( C_1 \) first and \( C_2 \) later. Let this region of the plane be covered by rays leading from \( P^0 \) to \( \infty \). As \( P \) moves from \( P^0 \) to \( \infty \) along rays \( P^0D_1, P^0D_2, \ldots \), it will meet \( C_2 \) first provided the slopes of the rays are sufficiently small, while along rays \( P^0B_1, P^0B_2, \ldots \), it will meet \( C_1 \) first provided the slopes of these rays are sufficiently large. If there is no intersection of \( C_1 \) and \( C_2 \) in the region there is a last ray \( P^0D \) of the rays \( P^0D_i \) along which \( P \) meets \( C_1 \) first and a last ray \( P^0B \) of rays \( P^0B_i \) along which \( P \) meets \( C_1 \) first. The areas covered by the rays \( P^0D_i \) and \( P^0B_i \) may (a) overlap, (b) be adjacent, in which case \( P^0D \) and \( P^0B \) coincide, (c) be separated by a sector \( BP^0D \).
Before taking up these three cases let us show that all rays from \( P^0 \) to \( \infty \) that lie in the region \( x > x^0 \) and \( y > y^0 \) cut both curves at least once. From (3.1) it is seen that \( \phi_1 \) and \( \phi_2 \) are continuous functions of \( x \) and \( y \) for \( P \) within this region, and that \( \phi_1 \) and \( \phi_2 \) each vary continuously from \( +\infty \) to \( -\infty \) as \( P \) moves from \( P^0 \) to \( \infty \) along any ray from \( P \) that lies in the region. Applying this result (a) to rays in the sector common to \( B_0P^0B \) and \( D_0P^0D \), we conclude that every ray drawn from \( P^0 \) and lying in this sector, since it meets both curves and meets each curve first, must pass through an intersection of \( C_1 \) and \( C_2 \); (b) to the single ray common to both sectors, we reach the same conclusion as in (a); (c) to rays in the sector \( BP^0D \) separating those already covered, we find that every ray, since it does not meet \( C_1 \) first and does not meet \( C_2 \) first but does meet them both, must pass through an intersection of \( C_1 \) and \( C_2 \). In all cases one is led to a contradiction of the hypothesis that there is no intersection.

It is evident that the same proof can be applied in case \( B_1 \) is positive at \( P^0 \). In case there are several such points \( P^0 \) with the same ordinate the proof can be applied at any one of them yielding an intersection with ordinate greater than \( y^0 \). Thus we have the following theorem.

**Theorem 3.1.** There exists at least one point of libration at which a zero mass \( m_n \) can be placed, so that it will, together with the given \( n-1 \) positive masses, form a permanent configuration.

4. **The solution as \( m_n \) becomes positive.** As shown in the last section there is a solution of (2.1) for positive \( \lambda, m_1, m_2, \ldots, m_{n-1} \), and \( m_n = 0 \). Let this solution be \( P_0^0 \). Since equations (2.1) are algebraic equations with coefficients functions of \( m_n \), the solution functions \( x_i^0 \) and \( y_i^0 \) are continuous functions of \( m_n \) as long as the roots are finite and the equations do not have indeterminate forms. Consequently, \( x_i^0 \) and \( y_i^0 \) are continuous functions of \( m_n \) if no \( x_i^0 \) or no \( y_i^0 \) become infinite, or if no pair \((x_i^0, y_i^0)\) becomes equal to another pair \((x_k^0, y_k^0)\).

Furthermore, the real roots \( x_i^0 \) and \( y_i^0 \) of algebraic equations (2.1) with real coefficients \( \lambda \) and \( m_i \) can disappear only by passing to infinity, or by an even number of real roots becoming complex conjugate quantities in pairs. Therefore, the problem is to determine as \( m_n \) varies whether

(1°) for all finite \( x_i^0 \) and \( y_i^0 \) any pair \((x_i^0, y_i^0)\) can become equal to another pair \((x_k^0, y_k^0)\),

(2°) any \( x_i^0 \) or \( y_i^0 \) can become infinite,

(3°) the solution \((x_i^0, y_i^0)\) can ever become imaginary.

5. **The \( P_0^0 \) remain distinct.** This section is devoted to problem (1°) of the last section. We shall treat the more general case in which any \( m_j \) may vary.

Let the notation be chosen so that

\[
x_1^0 \leq x_2^0 \leq x_3^0 \leq \cdots \leq x_{n-1}^0 \leq x_n^0
\]

where there is at least one inequality in the noncollinear case. Suppose that
as $m_i$ varies all $P_i^0$ remain finite, and $P_j^0$ approaches $P_k^0$ in a manner (a) such that $x_j^0 \neq x_k^0$, and that $j < k$. In the $j$th equation (2.1) there is a term involving $(x_j^0 - x_k^0)/(r_{jk}^0)^3 = \alpha_{jk}^0$ which becomes negatively infinite. If this $j$th equation is satisfied another term $\alpha_{jk}^0 m_l$ must become positively infinite, and according to our notation $l < j$. Now $\alpha_{jk}^0$ appears besides only in the $l$th equation and becomes negatively infinite in the term $\alpha_{jk}^0 m_l$. If the $l$th equation is satisfied a term $\alpha_{jk}^0 m_p$ must become positively infinite and according to the notation $p < l$. Continuing in this manner one arrives eventually at the situation in which a term involving $\alpha_{jk}^0$ with one of the subscripts $1$ becomes negatively infinite. But all terms in the first equation are negative except $-\lambda x_1^0$, which cannot become positively infinite under the hypothesis that the $P_i^0$ remain finite. Hence, the hypothesis that $P_j^0$ approaches $P_k^0$ in the manner (a) is false.

Suppose that $P_j^0$ approaches $P_k^0$ in the manner (b) such that $x_j^0 = x_k^0$. For this case let us choose the notation so that

\[(5.2) \quad y_1^0 \leq y_2^0 \leq y_3^0 \leq \ldots \leq y_{n-1}^0 \leq y_n^0,\]

where again there is at least one inequality. In this notation, if $P_j^0$ becomes $P_p^0$ and $P_k^0$ becomes $P_q^0$, our hypothesis is that $P_p^0$ approaches $P_q^0$ in the manner $x_p^0 = x_q^0$ and $p < q$. There is a term in the $(n+1)$th equation (2.1) involving $(y_p^0 - y_q^0)/(r_{pq}^0)^3 = \beta_{pq}^0$, which becomes negatively infinite. By an argument similar to that of the last paragraph one eventually arrives at the situation in which a term involving $\beta_{pq}^0$ with one of the subscripts $1$ becomes negatively infinite. But all terms in the $(n+1)$th equation are negative except $-\lambda y_1^0$, which cannot become positively infinite under the hypothesis that the $P_i^0$ remain finite. Hence, the hypothesis that $P_j^0$ approaches $P_k^0$ in the manner (b) is false. This completes the proof of the following theorem.

**Theorem 5.1.** As the $m_i$ vary and the solution functions $x_i^0$ and $y_i^0$ remain finite the $P_i^0$ remain distinct.

6. The $P_i^0$ remain finite. In order to show that as any $m_i$ varies no $x_i^0$ or $y_i^0$ can become infinite, let us again adopt the notation (5.1) and suppose that $x_j^0$ becomes positively infinite. From the center of gravity equation

\[(6.1) \quad \sum_{i=1}^n m_i x_i^0 = 0,\]

some $x_i^0$ must become negatively infinite, and from (5.1) it follows that $x_i^0$ approaches $-\infty$ and $x_n^0$ approaches $+\infty$. Consider the first equation (2.1). In order that it be satisfied $P_2^0$ must approach $P_2^0$ in a manner such that $x_1^0 \neq x_2^0$, that is, $x_2^0$ must approach $-\infty$. Now consider the second equation. If it is to be satisfied, $P_3^0$ must approach $P_3^0$ in a manner such that $x_2^0 \neq x_3^0$, that is, $x_3^0$ must approach $-\infty$. Continuing this process we are led to the conclusion that all $x_i^0$ approach $-\infty$, hence the hypothesis must be false.
In order to show that no $y_i^0$ can become infinite, we adopt the notation (5.2) and suppose that $y_i^0$ becomes positively infinite as $m_i$ varies. From the center of gravity equation

\begin{equation}
\sum_{i=1}^{n} m_i y_i^0 = 0,
\end{equation}

and the notation (5.2), it follows that $y_i^0$ approaches $-\infty$, and $y_i^0$ approaches $+\infty$. Consider the $(n+1)$th, $(n+2)$th, and so on of equations (2.1) and the proof is the same as that in the preceding paragraph with the corresponding changes in notation. Thus we have the following theorem.

**Theorem 6.1.** As the $m_i$ vary the $P_i^0$ remain finite.

7. The solution as $m_r$ vanishes. Before we take up the question of the solution becoming imaginary (3° of §4), it is to be shown that as $m_n$ approaches zero the equations (2.1) and the solution functions $x_i^0$ and $y_i^0$ remain regular, and that there is, accordingly, only one limiting position of the $P_i^0$ for the value $m_n = 0$. Let a solution for all positive $m_i$ and $\lambda$ be $P_i^0$, and let some $m_j$ approach zero.

First, if some $P_i^j$ approaches some $P_i^j$ in any manner then it must be that $P_i^j$ approaches $P_i^{j-1}$ or $P_i^{j+1}$, for all other possibilities lead to the fact that one of the equations cannot be satisfied by arguments used in §§5 and 6.

Secondly, if $P_i^j$ approaches $P_i^{j+1}$ in the manner $x_i^j = x_i^{j+1}$, and if the notation is that of (5.2) the $(n+j+1)$th equation (2.1) cannot be satisfied unless $P_i^{j-1}$ approaches $P_i^j$ in the manner $x_i^{j-1} = x_i^j$. Thus $P_i^{j-1}$ approaches $P_i^{j+1}$ in the manner $x_i^{j-1} = x_i^{j+1}$ as $m_i$ vanishes. But then the $(n+j+2)$th and $(n+j+1)$th equations cannot be satisfied unless $P_i^{j-2}$ approaches $P_i^{j-1}$ in the manner $x_i^{j-2} = x_i^{j-1}$ and $P_i^{j+2}$ approaches $P_i^{j+1}$ in the manner $x_i^{j+2} = x_i^{j+1}$. This shifts the difficulty to the $(n+j+2)$th and $(n+j+1)$th equations. On continuing this process one arrives at the $(n+1)$th and $(2n)$th equations which cannot be satisfied under the hypothesis it was necessary to make.

In case $P_i^j$ approaches $P_i^{j+1}$ in the manner $x_i^j \neq x_i^{j+1}$, then, if the notation is that of (5.1), the $j$th equation (2.1) cannot be satisfied unless $P_i^{j-1}$ approaches $P_i^j$ in the manner $x_i^{j-1} \neq x_i^j$ as $m_j$ vanishes. Thus $P_i^{j-1}$ approaches $P_i^{j+1}$ in the manner $x_i^{j-1} \neq x_i^{j+1}$. But the $(j+1)$th and $(j-1)$th equations cannot be satisfied unless $P_i^{j-2}$ approaches $P_i^{j-1}$ in the manner $x_i^{j-2} \neq x_i^{j-1}$, and $P_i^{j+2}$ approaches $P_i^{j+1}$ in the manner $x_i^{j+2} \neq x_i^{j+1}$. This shifts the difficulty to the $(j+2)$th and $(j-2)$th equations, and on continuing this process one eventually arrives at the 1st and $n$th equations which cannot be satisfied under the hypothesis it was necessary to make.

In the third place, if any $x_i^j$ or $y_i^j$ becomes positively infinite as $m_i$ vanishes, it must be that $x_i^j$ or $y_i^j$ becomes positively infinite, for all other possibilities lead to the fact that all $x_i^j$ or all $y_i^j$ become negatively infinite.
by arguments used in §6. If the notation is that of (5.1), the \( j \)th equation
cannot be satisfied as \( x_j' \) becomes positively infinite unless \( P_j' \) approaches
\( P_{j-1}' \) in the manner \( x_j' \neq x_{j-1}' \), that is, \( x_j' \) must approach \( +\infty \). Then the
\( (j-1) \)th equation cannot be satisfied unless \( P_{j-1}' \) approaches \( P_{j-2}' \) in the manner
\( x_{j-1}' \neq x_{j-2}' \), in which case \( x_{j-2}' \) approaches \( +\infty \). Continuing this process
one arrives at the conclusion that \( x_1', x_2', \ldots, x_{j-1}', x_j' \) become positively
infinite, and by choice of notation \( x_{j+1}', x_{j+2}', \ldots, x_n' \) become positively
infinite. Hence the center of gravity equation (6.1) cannot be satisfied, and
the hypothesis that \( x_j' \) become positively infinite is false.

The proof that \( y_j' \) cannot become positively infinite as \( m_j \) vanishes is pre-
cisely the same as that of the last paragraph with the notation that of (5.2)
and the argument beginning with the consideration of the \( (n+j) \)th equation
(2.1). The proof that no \( x_j' \) or \( y_j' \) can become negatively infinite as \( m_j \) vanishes
will read the same as the proof above with the words “positively infinite” re-
placed by “negatively infinite.” This concludes the proof of the following:

**Theorem 7.1.** Any solution \( P_j' \) of (2.1) for positive \( m_i \) and \( \lambda \) remains regu-
lar as any \( m_j \) vanishes.

In particular we shall use the following corollary.

**Corollary 7.1.** The solution \( P_j' \) for positive \( m_i \) and \( \lambda \) remains regular as
\( m_n \) vanishes.

8. **The condition that the \( P_i' \) remain real.** Since the solutions of (2.1) are
continuous functions of \( m_n \), it follows that no two solutions which are real
for \( m_n = 0 \) can ever become complex conjugate solutions for any positive value
of \( m_n \) without having first become equal. If a multiple solution of (2.1) is
impossible for a set of finite positive values of \( m_i \), then it is impossible for
any real solution to disappear by becoming complex or for any complex solu-
tion to become real.

The conditions that a set of simultaneous algebraic equations shall have
a multiple solution are that a set of values of the variable shall satisfy the
equations, and that the Jacobian of the functions with respect to the inde-
pendent variables shall vanish for the same set of values.

Since equations (2.1) are invariant under a rotation of axes, let the axes
be chosen so that \( y_2 = 0 \) in case \( n = 2 \), and so that \( y_1 = 0 \) for \( n \geq 3 \). It is to be
shown that for \( n = 2 \) the first three equations (2.1) are independent for all
positive values of \( \lambda, m_1 \) and \( m_2 \), while for \( n \geq 3 \) the \( 2n - 1 \) equations obtained
from (2.1) by omitting the \( (n+1) \)th form an independent set for any positive
\( \lambda, m_1, m_2 \), and sufficiently small positive \( m_3, m_4, \ldots, m_n \).

Let \( F_i \) and \( F_{n+i} \) denote the left members of (2.1) and let the Jacobian of
these functions be \( \Delta_n \). Furthermore, let the minor obtained by deleting the
\( (n+1) \)th row and \( (n+1) \)th column of \( \Delta_n \) be \( D_n \) and let \( n \geq 3 \). A real solution
\( P_i' \) of (2.1) for positive \( \lambda, m_1, m_2, \ldots, m_{n-1}, \) and \( m_n = 0 \) will remain real as \( m_n \)
increases from zero through positive values for which \( D_n \) does not vanish. This will be established when it is shown that \( D_n \neq 0 \) for sufficiently small positive \( m_{ji}, j = 3, 4, \ldots , n \).

The following notation will be used. \( \partial F_i/\partial x_j = a_{ij}, \partial F_{n+i}/\partial y_j = b_{ij}, \partial F_i/\partial y_j = \partial F_{n+i}/\partial x_j = c_{ij}. \) From (2.1) it follows that, for \( i \neq j \),

\[
\begin{align*}
a_{ij} &= \frac{3(x_i - x_j)^2 - (r_{ij})^2}{(r_{ij})^5} m_{ji}, \\
b_{ij} &= \frac{3(y_i - y_j)^2 - (r_{ij})^2}{(r_{ij})^5} m_{ji}, \\
c_{ij} &= \frac{3(x_i - y_j)(y_i - y_j)}{(r_{ij})^5} m_{ji},
\end{align*}
\]

(8.1)

and, for \( i = j \),

\[
\begin{align*}
a_{ii} &= -\sum_{k=1}^{n-1} a_{i+k} - \lambda, \\
b_{ii} &= -\sum_{k=1}^{n-1} b_{i+k} - \lambda, \\
c_{ii} &= -\sum_{k=1}^{n-1} c_{i+k},
\end{align*}
\]

(8.2)

where \( a_{i+k} = a_{i+k}, b_{i+k} = b_{i+k}, c_{i+k} = c_{i+k} \).

9. Case \( n = 2 \). Equations (2.1) in this case consist of four equations. Let us choose the axes so that \( y_2 = 0 \), then the solution of these equations is

\[
\begin{align*}
x_1 &= -\frac{m_2}{(\lambda M^2)^{1/3}}, \\
x_2 &= \frac{m_1}{(\lambda M^2)^{1/3}}, \\
y_1 &= 0, \\
y_2 &= 0,
\end{align*}
\]

(9.1)

where \( M = m_1 + m_2 \).

To compute the elements of \( A_2 \) the following quantities, \( x_1 - x_2 = -(M/\lambda)^{1/3}, y_1 - y_2 = 0, r_{12} = (M/\lambda)^{1/3}, \) are substituted in (8.1) and (8.2). We find

\[
\begin{pmatrix}
-2\lambda M_2/M - \lambda & 2\lambda M_2/M & 0 & 0 \\
2\lambda M_1/M & -2\lambda M_1/M - \lambda & 0 & 0 \\
0 & 0 & -\lambda M_1/M & -\lambda M_2/M \\
0 & 0 & -\lambda M_1/M & -\lambda M_2/M
\end{pmatrix}
\]

\( \Delta_2 \)

It is quite evident that \( \Delta_2 = 0 \) and it remains to be shown that the minor \( D_2 \) in the upper left-hand corner is different from zero. After adding the second column to the first, and then subtracting the first row from the second,

\[
\begin{pmatrix}
-\lambda & 2\lambda M_2/M & 0 \\
0 & -3\lambda & 0 \\
0 & 0 & -\lambda M_1/M
\end{pmatrix}
\]

\( D_2 \)

Since \( D_2 \neq 0 \) for all finite positive values of \( \lambda, m_1 \) and \( m_2 \) the solution of (2.1) for \( m_2 = 0 \), namely, \( x_1 = 0, y_1 = 0, x_2 = (m_1/\lambda)^{1/3}, y_2 = 0 \), remains real and varies continuously to the values given in (9.1) for all finite positive values of \( \lambda, m_1 \) and \( m_2 \).

10. Case \( n = 3 \). By choosing the axes so that \( y_1 = 0 \) the solution of the six equations (2.1) is
\[ x_1 = 2(m_2 + m_2m_3 + m_3^2)K, \quad y_1 = 0, \]
\[ x_2 = -(m_2m_3 + m_3 + 2m_1m_2 - m_3^2)K, \quad y_2 = -3^{1/2} M m_3 K, \]
\[ x_3 = -(2m_1m_3 + m_1m_2 + m_2m_3 - m_2^2)K, \quad y_3 = 3^{1/2} M m_2 K, \]

where \( M = m_1 + m_2 + m_3 \), and \( K = 1/(2^{1/n} M^{2/n}(m_1^2 + m_2 m_3 + m_3^2)^{1/n}) \).

If the elements of \( \Delta_3 \) are computed in terms of \( \lambda, m_1, m_2 \) and \( m_3 \), \( \Delta_3 \) will vanish, as can be shown by direct computation, since the system (2.1) is dependent and \( y_1 \) can be chosen arbitrarily. \( D_3 \) is the minor obtained by leaving out the 4th row and 4th column of \( \Delta_3 \) and has determinant different from zero for all finite positive \( \lambda, m_1, m_2 \), and \( m_3 \), as a rather long series of computations will show.

However, the proof that \( D_3 \) is different from zero for all \( \lambda, m_1, m_2 \) and sufficiently small \( m_3 \) is short and will be given here, since it is the same type of proof that will be used in the general case. In the notation (8.1) and (8.2),

\[
\Delta_3 = \begin{vmatrix}
a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13} \\
a_{21} & a_{22} & a_{23} & c_{21} & c_{22} & c_{23} \\
a_{31} & a_{32} & a_{33} & c_{31} & c_{32} & c_{33} \\
c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13} \\
c_{21} & c_{22} & c_{23} & b_{21} & b_{22} & b_{23} \\
c_{31} & c_{32} & c_{33} & b_{31} & b_{32} & b_{33} 
\end{vmatrix} \quad D_3 = \begin{vmatrix}
a_{11} & a_{12} & a_{13} & c_{12} & c_{13} \\
a_{21} & a_{22} & a_{23} & c_{22} & c_{23} \\
a_{31} & a_{32} & a_{33} & c_{32} & c_{33} \\
c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13} \\
c_{21} & c_{22} & c_{23} & b_{21} & b_{22} & b_{23} \\
c_{31} & c_{32} & c_{33} & b_{31} & b_{32} & b_{33} 
\end{vmatrix}
\]

After rearranging rows and columns in \( \Delta_3 \) we have

\[
\Delta_3 = \begin{vmatrix}
a_{11} & a_{12} & c_{11} & c_{12} & a_{13} & c_{13} \\
a_{21} & a_{22} & c_{21} & c_{22} & a_{23} & c_{23} \\
c_{11} & c_{12} & b_{11} & b_{12} & c_{13} & b_{13} \\
c_{21} & c_{22} & b_{21} & b_{22} & c_{23} & b_{23} \\
a_{31} & a_{32} & c_{31} & c_{32} & a_{33} & c_{33} \\
c_{31} & c_{32} & b_{31} & b_{32} & c_{33} & b_{33} 
\end{vmatrix}
\]

and if \( m_3 = 0, \Delta_3 \) and \( D_3 \) become

\[
\Delta_3 = \begin{vmatrix}
\Delta_2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & a_{32} & c_{32} \\
c_{31} & c_{32} & b_{31} & b_{32} & c_{33} & b_{33} 
\end{vmatrix}
\]

and
On expanding (10.3) we obtain

\[
D_3 = \begin{pmatrix}
\frac{m_1}{M} & \frac{m_2}{M} & 0 & 0 \\
\frac{m_2}{M} & \frac{m_3}{M} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{2m_2}{M} & \frac{2m_1}{M} & 0 & 0 \\
-\frac{2m_1}{M} & -\frac{2m_2}{M} & 0 & 0 \\
0 & 0 & \frac{3m_3}{M} & \frac{3m_2}{M} \\
0 & 0 & \frac{3m_2}{M} & \frac{3m_3}{M}
\end{pmatrix}\lambda.
\]

(10.4)

\[
D_3 = -\left(3\frac{m_3^2}{4M^2}\right)\lambda^6.
\]

Since \(D_3\) of (10.2) is a continuous function of \(m_3\) and, as (10.4) reveals, is different from zero for positive \(\lambda, m_1, m_2,\) and \(m_3 = 0,\) it is different from zero for sufficiently small positive \(m_3.\)

11. Case \(n=4.\) Using the notation (8.1) and (8.2) we have

\[
\Delta_4 = \begin{pmatrix}
a_{ij} & c_{ij} & b_{ij} \\
a_{ij} & c_{ij} & b_{ij} \\
a_{ij} & c_{ij} & b_{ij} \\
a_{ij} & c_{ij} & b_{ij}
\end{pmatrix}, \quad i, j = 1, 2, 3, 4,
\]

and

\[
D_4 = \begin{pmatrix}
a_{ij} & c_{ij} & c_{ij} & c_{ij} \\
0 & c_{ij} & c_{ij} & c_{ij} \\
0 & 0 & c_{ij} & c_{ij} \\
0 & 0 & 0 & c_{ij}
\end{pmatrix}, \quad i, j = 1, 2, 3, 4.
\]

After rearranging rows and columns, we have

\[
\Delta_4 = D_3
\]

\[
D_4
\]

For \(m_4=0\) these become
Since $x_i$ and $y_i$ are continuous functions of $m_i$, $D_i$ is a continuous function of $m_i$ and is different from zero for $m_i = 0$, provided that $\lambda$, $m_1$ and $m_2$ are positive, that $m_3$ is so chosen that $D_3 \neq 0$, and that $(x_4, y_4)$ is not a multiple solution of the 4th and 8th equations (2.1). Hence, $D_4 \neq 0$ for sufficiently small positive values of $m_4$, and the solution, real for positive $\lambda$, $m_1$, $m_2$, $m_3$, and $m_4 = 0$, varies continuously and remains real for sufficiently small positive values of $m_4$.

12. Existence of a solution for positive $m_n$. Continuing the process of §§9, 10, 11 it is possible for one to choose $m_3, m_6, \cdots, m_{n-1}$ successively so that $D_3, D_6, \cdots, D_{n-1}$ remain different from zero. In the general case $\Delta_n$ will be zero due to the dependence of equations (2.1). The minor $D_n$ obtained from $\Delta_n$ by deleting the $(n+1)$th row and $(n+1)$th column will be, in the notation of (8.1) and (8.2) and after rows and columns have been rearranged,

$$
\Delta_4 = \begin{vmatrix}
\Delta_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix}
$$

$$
D_4 = \begin{vmatrix}
D_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix}
$$

$$
\Delta_4 = \begin{vmatrix}
a_{41} & a_{42} & a_{43} & a_{44} \\
a_{42} & a_{43} & a_{44} & a_{45} \\
a_{43} & a_{44} & a_{45} & a_{46} \\
a_{44} & a_{45} & a_{46} & a_{47}
\end{vmatrix}
$$

$$
D_4 = \begin{vmatrix}
d_{41} & d_{42} & d_{43} & d_{44} \\
d_{42} & d_{43} & d_{44} & d_{45} \\
d_{43} & d_{44} & d_{45} & d_{46} \\
d_{44} & d_{45} & d_{46} & d_{47}
\end{vmatrix}
$$

Since $x_i$ and $y_i$ are continuous functions of $m_i$, $D_i$ is a continuous function
of \( m_n \); and, if \((x_n, y_n)\) is not a multiple solution of the \(n\)th and \((2n)\)th equations (2.1), \(D_n \neq 0\) for \(m_n = 0\). Therefore, \(D_n \neq 0\) for sufficiently small values of \(m_n\), and the solution, real for positive \(\lambda, m_1, m_2, \ldots, m_{n-1}\), and \(m_n = 0\), varies continuously and remains real for sufficiently small positive values of \(m_n\). This concludes the proof of the following theorem.

**Theorem 12.1.** For arbitrary finite positive \(\lambda, m_1, m_2, \) and sufficiently small positive \(m_3, m_4, \ldots, m_n\) there exists at least one real solution of equations (2.1).

It should be noted that, from the Theorem of Lagrange and our remark in §10, \(m_3\) need not be restricted and hence we have the slightly more general theorem.

**Theorem 12.2.** For arbitrary finite positive masses \(m_1, m_2, m_3\), arbitrary finite angular velocity, and sufficiently small positive masses \(m_4, m_5, \ldots, m_n\), there exists at least one noncollinear plane permanent configuration.

13. A special form for the equations. In §12 we showed that under certain conditions \(D_n \neq 0\). It follows from this result that for any \(\lambda, m_1, m_2, \) and properly chosen \(m_3, m_4, \ldots, m_n\), equations (2.1) with the \((n+1)\)th deleted form an independent set. It is easy to show, by processes similar to those of §§8 to 12, that the first and \((n+2)\)th may be replaced by the center of gravity equations (6.1) and (6.2) yielding an independent set of \(2n-1\) equations equivalent to the one just mentioned.

It is our purpose in this section to put this latter set of \(2n-1\) equations into a form convenient for calculating a solution. We shall then solve them for the coordinates in terms of \(\lambda\) and the masses \(m_i\). Let the mass ratios be \(1, m_2/m_1 = \mu, m_4/m_2 = \sigma_i (i = 1, 2, \ldots, n-2)\) and for the particular solution we wish to construct we shall take \(\lambda = m_1\). The equations then become

\[
\begin{align*}
  x_1 + \mu x_2 + \sum x_{2+1} \sigma_j &= 0, \\
  y_1 + \mu y_2 + \sum y_{2+1} \sigma_j &= 0, \\
  -x_2 + \frac{x_2 - x_1}{(r_{21})^3} + \sum \frac{x_2 - x_{2+j}}{(r_{2+1})^3} \sigma_j &= 0, \\
  -x_i + \frac{x_i - x_1}{(r_{i1})^3} + \frac{x_i - x_2}{(r_{i2})^3} \mu + \sum^* \frac{x_i - x_{2+j}}{(r_{i+1})^3} \sigma_j &= 0, \quad i = 3, 4, \ldots, n \\
  -y_i + \frac{y_i - y_1}{(r_{i1})^3} + \frac{y_i - y_2}{(r_{i2})^3} \mu + \sum^* \frac{y_i - y_{2+j}}{(r_{i+1})^3} \sigma_j &= 0,
\end{align*}
\]

where \(\sum^*\) denotes \(\sum_{j=2}^{n-2}\) and \(\sum^*\) denotes the sum over the same \(j\) except for \(j = i - 2\).

Let us multiply the next to last equation by \(y_i\) and the last by \(-x_i\) and
form the sum. The resulting equation,

\[ \frac{x_i y_1 - x_1 y_i}{(r_{i1})^3} + \frac{x_2 y_2 - x_2 y_i}{(r_{i2})^3} + \sum_{j=3}^{n} \frac{x_j y_{2+j} - x_{2+j} y_j}{(r_{j2})^3} \mu + \sum_{j=3}^{n} \frac{x_{2+j} y_j - x_j y_{2+j}}{(r_{j1})^3} \sigma_j = 0, \]

is used to replace the next to last above. By means of the complex numbers

\[ z_{ij} = (y_i - y_j + (-1)^{1/2}(x_i y_j - x_j y_i))/(r_{ij})^3 \]

we are now able to write the \(2n-1\) equations (13.1) in the form

\[ x_1 + \mu x_2 + \sum x_{2+j} \sigma_j = 0, \]
\[ y_1 + \mu y_2 + \sum y_{2+j} \sigma_j = 0, \]
\[ -x_2 + \frac{x_2 - x_1}{(r_{12})^3} + \sum \frac{x_2 - x_{2+j}}{(r_{2j})^3} \sigma_j = 0, \]
\[ -y_1 + z_{11} + z_{12} \mu + \sum z_{1,2+j} \sigma_j = 0, \quad i = 3, 4, \ldots, n, \]

which is convenient for computing the solution.

In the case \(n=4\) the coordinates may be expressed as power series in the mass ratios \(\sigma_1\) and \(\sigma_2\) with coefficients which are Laurent series in \(\mu\). For \(n>4\) the coordinates may be expressed as power series in \((\sigma_i/3)^{1/3}\) with coefficients Laurent series in \(\mu\).

14. Solution for \(n=4\). In this case the seven equations (13.2) are

\[ x_1 + \mu x_2 + \sigma_1 x_3 + \sigma_2 x_4 = 0, \]
\[ y_1 + \mu y_2 + \sigma_1 y_3 + \sigma_2 y_4 = 0, \]
\[ -x_2 + \frac{x_2 - x_1}{(r_{12})^3} + \frac{x_2 - x_3}{(r_{23})^3} + \frac{x_3 - x_4}{(r_{24})^3} \sigma_1 + \frac{x_3 - x_4}{(r_{24})^3} \sigma_2 = 0, \]
\[ \frac{x_3 y_1 - x_1 y_3}{(r_{31})^3} + \frac{x_3 y_2 - x_2 y_3}{(r_{32})^3} + \frac{x_4 y_1 - x_1 y_4}{(r_{41})^3} + \frac{x_4 y_2 - x_2 y_4}{(r_{42})^3} \mu + \frac{x_3 y_4 - x_4 y_3}{(r_{44})^3} \sigma_2 = 0, \]
\[ -y_3 + \frac{y_3 - y_1}{(r_{13})^3} + \frac{y_3 - y_2}{(r_{23})^3} + \frac{y_4 - y_3}{(r_{34})^3} \mu + \frac{y_4 - y_3}{(r_{34})^3} \sigma_2 = 0, \]
\[ \frac{x_4 y_1 - x_1 y_4}{(r_{41})^3} + \frac{x_4 y_2 - x_2 y_4}{(r_{42})^3} + \frac{x_4 y_3 - x_3 y_4}{(r_{44})^3} \mu + \frac{x_4 y_4 - x_3 y_4}{(r_{44})^3} \sigma_1 = 0, \]
\[ -y_4 + \frac{y_4 - y_1}{(r_{14})^3} + \frac{y_4 - y_2}{(r_{24})^3} + \frac{y_4 - y_3}{(r_{34})^3} \mu + \frac{y_4 - y_3}{(r_{34})^3} \sigma_1 = 0. \]

Since one of the coordinates is arbitrary we may choose \(y_1=0\). The solution of these equations may be obtained as follows. With \(\sigma_2=0\) the first five equations (14.1) form an independent set of the six equations (2.1), and the solution functions may be obtained from (10.1) by proper change of parame-
The solution functions thus obtained and arranged in powers of $\sigma_1$ are

$$x_1 = \mu(1 + \mu)^{-2/3} + \frac{3 - \mu}{6} (1 + \mu)^{-5/3}\sigma_1 + \cdots,$$

$$x_2 = -(1 + \mu)^{-2/3} + \frac{1 - 3\mu}{6} (1 + \mu)^{-5/3}\sigma_1 + \cdots,$$

(14.2)$$y_2 = \frac{\sigma_1^{3/2}}{2} \left[ 0 - \frac{(1 + \mu)^{1/3}}{\mu} \sigma_1 + \cdots \right],$$

$$x_3 = -\frac{1 - \mu}{2} (1 + \mu)^{-2/3} - \frac{9 + 14\mu + 13\mu^2}{12\mu} (1 + \mu)^{-5/3}\sigma_1 + \cdots,$$

$$y_3 = \frac{\sigma_1^{3/2}}{2} \left[ (1 + \mu)^{1/3} - \frac{3 + \mu}{6\mu} (1 + \mu)^{-2/3}\sigma_1 + \cdots \right].$$

The last two equations (14.1) may be obtained from the preceding two by the transformation, in two rowed notation,

(14.3)$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & y_2 & y_3 & y_4 & \mu & \sigma_1 & \sigma_2 \\ x_1 & x_2 & x_3 & x_4 & -y_2 & -y_3 & -y_4 & \mu & \sigma_2 & \sigma_1 \end{pmatrix},$$

and then setting $\sigma_1 = 0$. Hence the solution functions $x_4, y_4$ of the last two equations may be obtained from the solution functions $x_3, y_3$ except for the coefficients of the powers of $\sigma_1$. On substituting

$$x_4 = -\left( (1 - \mu)/2 \right) (1 + \mu)^{-2/3} + c_1(\mu)\sigma_1 + \cdots,$$

$$y_4 = -\left( \sigma_1^{3/2}/2 \right) \left[ (1 + \mu)^{1/3} + c_2(\mu)\sigma_1 + \cdots \right]$$

in the last two equations (14.1) and equating coefficients we find

$$c_1(\mu) = \left( \frac{3^{1/2}}{27} + \frac{5}{12} \right) \frac{1}{\mu} - \left( \frac{2(3)^{1/2}}{81} - \frac{29}{36} \right),$$

$$- \left( \frac{3^{1/2}}{81} + \frac{7}{9} \right) \mu + \cdots,$$

(14.4)$$c_2(\mu) = \left( \frac{2(3)^{1/2}}{81} + \frac{5}{18} \right) \frac{1}{\mu} + \left( \frac{8(3)^{1/2}}{243} + \frac{11}{54} \right)$$

$$+ \left( \frac{2(3)^{1/2}}{81} - \frac{8}{27} \right) \mu + \cdots.$$

This completes the solution of (14.1) as functions of $\mu$ and $\sigma_1$.

In order to obtain the coefficients of $\sigma_2$ in the solution functions of (14.1) we may use the fact that the system (14.1) remains unchanged by the transformation (14.3) and that the solution functions must have this same prop-
Coefficients of higher powers of the parameters may be computed, and herein lies the convenience of the form (14.1). The coefficients of any term $\sigma_1^2\sigma_2^2$ in the expansion of $x_1, y_2, x_3, x_4, y_4$ may be determined successively, and in this order, by substituting series with undetermined coefficients into the equations in the order given in (14.1).

The solution for $n = 4$, computed as described above and arranged in powers of $\sigma_3$, will follow immediately. The coefficients in this series are themselves arranged in powers of $\sigma_1$ with coefficients which are Laurent series in $\mu$.

\[
x_1 = \mu(1 + \mu)^{-2/3} + \frac{3 - \mu}{6} (1 + \mu)^{-5/3}\sigma_1 + \cdots + \left\{ \frac{3 - \mu}{6} (1 + \mu)^{-5/3} \right\} \sigma_2 + \cdots,
\]

\[
x_2 = -(1 + \mu)^{-2/3} + \frac{1 - 3\mu}{6} (1 + \mu)^{-5/3}\sigma_1 + \cdots + \left\{ \frac{1 - 3\mu}{6} (1 + \mu)^{-5/3} \right\} \sigma_2 + \cdots,
\]

\[
y_2 = \frac{3^{1/2}}{2} \left( \frac{0 - (1 + \mu)^{1/3}}{\mu} \sigma_1 + \cdots + \left\{ \frac{(1 + \mu)^{1/3}}{\mu} \right\} \right) + \left[ \sigma_1 + \cdots \right) \sigma_2 + \cdots,
\]

(14.5)

\[
x_3 = -\frac{1 - \mu}{2} (1 + \mu)^{-2/3} - \frac{9 + 14\mu + 13\mu^2}{12\mu} (1 + \mu)^{-5/3}\sigma_1 + \cdots
\]

\[
y_3 = \frac{3^{1/2}}{2} \left( (1 + \mu)^{1/3} - \frac{3 + \mu}{6\mu} (1 + \mu)^{-2/3}\sigma_1 + \cdots
\]

\[
+ \left\{ \frac{2(3)^{1/2}}{81} + \frac{5}{18} \right\} \frac{1}{\mu} + \left( \frac{8(3)^{1/2}}{243} + \frac{11}{54} \right) \right\} \sigma_1 + \cdots \sigma_2 + \cdots,
\]

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\begin{align*}
x_4 &= - \frac{1 - \mu}{2} (1 + \mu)^{-2/3} + \left( \frac{3^{1/2}}{27} + \frac{5}{12} \right) \frac{1}{\mu} - \left( \frac{2(3)^{1/2}}{81} - \frac{29}{36} \right) \\
&\quad - \left( \frac{3^{1/2}}{81} + \frac{7}{9} \right) \mu + \cdots \right) \sigma_1 + \cdots \nonumber \\
&+ \left\{ - \frac{9 + 14\mu + 13\mu^2}{12\mu} (1 + \mu)^{-2/3} \right. \nonumber \\
&\left. + \left[ \left( \frac{5(3)^{1/2}}{2 \cdot 3^6} - \frac{1687}{2 \cdot 3^6} \right) \frac{1}{\mu^2} + \cdots \right] \sigma_1 + \cdots \right\} \sigma_2 + \cdots, \quad (14.5) \
y_4 &= - \frac{3^{1/2}}{2} (1 + \mu)^{-1/3} + \left( \frac{2(3)^{1/2}}{81} + \frac{5}{18} \right) \frac{1}{\mu} + \left( \frac{8(3)^{1/2}}{243} + \frac{11}{54} \right) \\
&\quad + \left( \frac{2(3)^{1/2}}{81} - \frac{8}{27} \right) \mu + \cdots \right) \sigma_1 + \cdots \right. \nonumber \\
&\left. + \left[ \left( \frac{41(3)^{1/2}}{3^6} + \frac{3173}{2 \cdot 3^7} \right) \frac{1}{\mu^2} + \cdots \right] \sigma_1 + \cdots \right\} \sigma_2 + \cdots. \nonumber 
\end{align*}

15. Points of libration for \( n = 4, 5 \). In order to find a point where \( m_5 = 0 \) may be introduced, we must solve the pair of equations given by the last of equations (13.2) for the case \( n = 5 \), namely,

\begin{align*}
\frac{A_{61}}{(r_{61})^3} + \frac{A_{62}}{(r_{62})^3} \mu + \frac{A_{63}}{(r_{63})^3} \sigma_1 + \frac{A_{64}}{(r_{64})^3} \sigma_2 &= 0, \\
- y_6 + \frac{y_6 - y_2}{(r_{62})^3} \mu + \frac{y_6 - y_2}{(r_{63})^3} \sigma_1 + \frac{y_6 - y_4}{(r_{64})^3} \sigma_2 &= 0, 
\end{align*}

(15.1)

where \( A_{ij} = x_{ij}y_i - x_iy_j \).

If \( \sigma_1 = 0 \) these equations are precisely those which determine \( P_3 \), hence \( P_5 \) coincides with \( P_3 \) for \( \sigma_1 = 0 \). Furthermore, \( P_5 = P_3 \) is a solution of equations (15.1) for \( \sigma_1 \neq 0 \), the identity being in \( \mu, \sigma_1 \) and \( \sigma_2 \). In order to obtain a solution of (15.1) having the property (in §5) that \( P_5 \neq P_3 \), we let \( \sigma_1 = 3\nu^4 \). Upon substituting

\begin{align*}
x_5 &= - \frac{1 - \mu}{2} (1 + \mu)^{-2/3} + p_1(\mu)\nu + p_2(\mu)\nu^2 \\
&\quad + p_3(\mu)\nu^3 + \cdots + c_1(\mu)\sigma_2 + \cdots, \\
y_5 &= \frac{3^{1/2}}{2} \left[ (1 + \mu)^{1/3} + q_1(\mu)\nu + q_2(\mu)\nu^2 \\
&\quad + q_3(\mu)\nu^3 + \cdots + c_2(\mu)\sigma_2 + \cdots \right]
\end{align*}

(15.2)

in (15.1) the \( p_i(\mu) \) and \( q_i(\mu) \) may be determined by equating coefficients of \( \nu \).
The functions \( c_1(\mu) \) and \( c_2(\mu) \) are the same as in (14.4). The result is

\[
\rho_1(\mu) = \pm \left( \frac{1}{2} - \frac{1}{4} \mu + \cdots \right), \quad \rho_2(\mu) = \pm \left( \frac{1}{6} + \frac{125}{144} \mu + \cdots \right),
\]

\[
\rho_3(\mu) = -\frac{9 + 14\mu + 13\mu^2}{4\mu} (1 + \mu)^{-5/3},
\]

(15.3)

\[
q_1(\mu) = \pm \left( -1 - \frac{1}{2} \mu + \cdots \right), \quad q_2(\mu) = \pm \left( \frac{1}{3} + \frac{25}{72} \mu + \cdots \right),
\]

\[
q_3(\mu) = -\frac{3 + \mu}{2\mu} (1 + \mu)^{-2/3}.
\]

Thus, as \( \sigma_1 \) increases from zero to a positive value, a point of libration, denoted by \( P^*_3 \), branches off from \( P_3 \) and moves to a finite distance from \( P_3 \). We shall use only the upper sign in (15.3).

The solution of

\[
\frac{A_{61}}{(r_{61})^3} + \frac{A_{62}}{(r_{62})^3} \mu + \frac{A_{63}}{(r_{63})^3} \sigma_1 + \frac{A_{64}}{(r_{64})^3} \sigma_2 = 0,
\]

(15.4)

\[-y_6 + \frac{y_6 - y_2}{(r_{61})^3} \mu + \frac{y_6 - y_2}{(r_{62})^3} \sigma_1 + \frac{y_6 - y_4}{(r_{64})^3} \sigma_2 = 0\]

for the libration point \( P^*_6 \) may now be obtained from (15.2) by the transformation

\[
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y_2 & y_3 & y_4 & y_5 & y_6 & \mu & \sigma_1 & \sigma_2 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -y_2 & -y_4 - y_5 - y_6 & \mu & \sigma_2 & \sigma_1
\end{pmatrix},
\]

which reduces (15.4) to (15.1). Therefore, as \( \sigma_2 \) increases from zero to a positive value, a point of libration \( P^*_6 \), whose coordinates are

\[
x_8 = -\frac{1 - \mu}{2} (1 + \mu)^{-5/3} + c_1(\mu) \sigma_1 + \cdots + \rho_1(\mu) \left( \frac{\sigma_2}{3} \right)^{1/3}
\]

\[
+ \rho_2(\mu) \left( \frac{\sigma_2}{3} \right)^{3/2} + \rho_3(\mu) \left( \frac{\sigma_2}{3} \right) + \cdots,
\]

(15.5)

\[
y_8 = -\frac{3^{1/2}}{2} \left[ (1 + \mu)^{1/3} + c_2(\mu) \sigma_1 + \cdots + q_1(\mu) \left( \frac{\sigma_2}{3} \right)^{1/3}
\]

\[
+ q_2(\mu) \left( \frac{\sigma_2}{3} \right)^{3/2} + q_3(\mu) \left( \frac{\sigma_2}{3} \right) + \cdots \right],
\]

branches off from \( P_4 \) and moves to a finite distance from \( P_4 \).

As the masses \( m_6 \) and \( m_8 \) increase from zero to positive values, the solution
functions (14.5), (15.2) and (15.5) must be extended to include powers of $(\sigma_{2}/3)^{1/3}$ and $(\sigma_{4}/3)^{1/3}$. This can readily be accomplished by the substitution

$$
\begin{pmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 & y_2 & y_3 & y_4 & y_5 & y_6 & \mu & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
  x_1 & x_2 & x_3 & x_4 & x_5 & y_2 & y_3 & y_4 & y_5 & y_6 & \mu & \sigma_3 & \sigma_4 & \sigma_1 & \sigma_2
\end{pmatrix},
$$

which leaves the system (13.1), for $n=6$, unchanged.

16. The solution functions for any $n$. It is evident that the process carried out in the last section may be repeated until the system contains any finite number of masses. Concerning the points of libration at any stage, we have the following theorem.

**Theorem 16.1.** As masses $m_{2k-1}$ and $m_{2k}$ are added to the system, by allowing them to increase from zero to some sufficiently small positive values at points $P_{2k-1}$ and $P_{2k}$ respectively, two points of libration $P_{2k+1}^{*}$ and $P_{2k+2}^{*}$ branch off from $P_{2k-1}$ and $P_{2k}$ respectively.

In regard to the solution functions at any stage the following theorem is evident.

**Theorem 16.2.** As masses $m_{2k+1}$ and $m_{2k+2}$ increase from zero to some sufficiently small positive values, the solution functions $x_{i}$ and $y_{i}$ ($i=1, 2, \ldots, 2k+2$) vary continuously and may be expressed as power series in $(\sigma_{2k-1}/3)^{1/3}$ and $(\sigma_{2k}/3)^{1/3}$, where $\sigma_{2k-1}$ and $\sigma_{2k}$ are the mass ratios $m_{2k+1}/m_{1}$ and $m_{2k+2}/m_{1}$ respectively.

As pointed out in §14, all the solution functions for $n=4$ are power series in $\sigma_{1}$ and $\sigma_{2}$. This is due to the fact that equations (13.1) are invariant under the transformations

$$
\begin{pmatrix}
x_i \\
\pm y_i
\end{pmatrix}, \quad i, j = 3, 4, \ldots, n,
$$

where the sign is to be taken "+" if $i, j$ are both even or both odd, otherwise "−" and also to the fact that the solution functions if $n=4$ are power series in $\sigma_{2}$. For $n>4$ we note that only $x_{1}, x_{2}$ and $y_{2}$ are power series in $\sigma_{1}$, and hence only these three functions are power series in the mass ratios $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n-2}$. Since $x_{5}$ and $y_{6}$ are power series in $(\sigma_{1}/3)^{1/3}$, it follows that $x_{i}$ and $y_{i}$ ($i=3, 4, \ldots, n$) are power series in $(\sigma_{1}/3)^{1/3}, (\sigma_{2}/3)^{1/3}, \ldots, (\sigma_{n-2}/3)^{1/3}$. It should be noted, however, that $x_{i}$ and $y_{i}$ ($i=3, 4, \ldots, n$) may be power series in $(\sigma_{1}/3)^{1/3}, (\sigma_{2}/3)^{1/3}, \ldots, (\sigma_{n-3}/3)^{1/3}, (\sigma_{n-2}/3)$, but this solution does not have the property stated in Theorem 5.1, for as $m_{n-2}$ vanishes $P_{n-1}^{*}$ and $P_{n}^{*}$ coincide.

Let $(\sigma_{i}/3)^{1/3} = \tau_{i}^{1/3}$. In case the solution functions are power series in $\tau_{i}^{1/3}$, $i=1, 2, \ldots, n-2$, the following functions have all the properties stated in theorems of §§5, 6, 7, 8 and constitute a solution of (13.1).
\begin{align*}
x_1 &= H_{1,0} + H_{1,1} \sum \frac{1}{\sqrt{3}} t_{2j-1} + \overline{H}_{1,1} \sum \frac{1}{\sqrt{3}} t_{2j} + H_{1,2} \sum \frac{2}{\sqrt{3}} t_{2j-1} + \overline{H}_{1,2} \sum \frac{2}{\sqrt{3}} t_{2j} \\
&\quad + H_{1,3} \sum \frac{3}{\sqrt{3}} t_{2j-1} + \overline{H}_{1,3} \sum \frac{3}{\sqrt{3}} t_{2j} + \cdots,
\quad \\
y_1 &= K_{1,0} + K_{1,1} \sum \frac{1}{\sqrt{3}} t_{2j-1} + \overline{K}_{1,1} \sum \frac{1}{\sqrt{3}} t_{2j} + K_{1,2} \sum \frac{2}{\sqrt{3}} t_{2j-1} + \overline{K}_{1,2} \sum \frac{2}{\sqrt{3}} t_{2j} \\
&\quad + K_{1,3} \sum \frac{3}{\sqrt{3}} t_{2j-1} + \overline{K}_{1,3} \sum \frac{3}{\sqrt{3}} t_{2j} + \cdots,
\quad \\
x_2 &= H_{2,0} + H_{2,1} \sum \frac{1}{\sqrt{3}} t_{2j-1} + \overline{H}_{2,1} \sum \frac{1}{\sqrt{3}} t_{2j} + H_{2,2} \sum \frac{2}{\sqrt{3}} t_{2j-1} + \overline{H}_{2,2} \sum \frac{2}{\sqrt{3}} t_{2j} \\
&\quad + H_{2,3} \sum \frac{3}{\sqrt{3}} t_{2j-1} + \overline{H}_{2,3} \sum \frac{3}{\sqrt{3}} t_{2j} + \cdots,
\quad \\
y_2 &= K_{2,0} + K_{2,1} \sum \frac{1}{\sqrt{3}} t_{2j-1} + \overline{K}_{2,1} \sum \frac{1}{\sqrt{3}} t_{2j} + K_{2,2} \sum \frac{2}{\sqrt{3}} t_{2j-1} + \overline{K}_{2,2} \sum \frac{2}{\sqrt{3}} t_{2j} \\
&\quad + K_{2,3} \sum \frac{3}{\sqrt{3}} t_{2j-1} + \overline{K}_{2,3} \sum \frac{3}{\sqrt{3}} t_{2j} + \cdots,
\quad \\
\quad & (16.1)
\end{align*}

where \( \sum \) denotes the sum over \( j \) from 1 to the largest integer in \( (n-2)/2 \), and \( \sum' \) the same sum over \( j \) except for \( j = i \). The \( H \)'s and \( K \)'s are functions of \( \mu \) and the nonzero ones are

\begin{align*}
H_{1,0} &= \mu (1 + \mu)^{-\sqrt{3}}, \\
H_{1,1} &= \overline{H}_{1,1} = \frac{3}{2} \mu (1 + \mu)^{-\sqrt{3}}, \\
H_{1,2} &= \overline{H}_{1,2} = \frac{3}{2} \mu (1 + \mu)^{-\sqrt{3}}, \\
H_{1,3} &= \overline{H}_{1,3} = \frac{3}{2} \mu (1 + \mu)^{-\sqrt{3}},
\end{align*}

\begin{align*}
H_{2,0} &= -(1 + \mu)^{-\sqrt{3}}, \\
H_{2,1} &= \overline{H}_{2,1} = \frac{1 - 3\mu}{2} (1 + \mu)^{-\sqrt{3}}, \\
H_{2,2} &= \overline{H}_{2,2} = \frac{1}{2} (1 + \mu)^{-\sqrt{3}}, \\
H_{2,3} &= \overline{H}_{2,3} = \frac{3}{2} \mu (1 + \mu)^{-\sqrt{3}},
\end{align*}

\begin{align*}
K_{2i+1,0} &= H_{2i+2,0} = \frac{3}{2} (1 + \mu)^{-\sqrt{3}}, \\
K_{2i+1,1} &= \overline{H}_{2i+2,1} = \frac{1 - 3\mu}{2} (1 + \mu)^{-\sqrt{3}}, \\
K_{2i+1,2} &= \overline{H}_{2i+2,2} = \frac{1}{2} (1 + \mu)^{-\sqrt{3}}, \\
K_{2i+1,3} &= \overline{H}_{2i+2,3} = \frac{3}{2} \mu (1 + \mu)^{-\sqrt{3}},
\end{align*}
\[ \begin{align*}
H_{2i+1,1} &= H_{2i+2,1} = \frac{1}{2} - \frac{1}{4} \mu + \cdots, \\
K_{2i+1,1} &= -K_{2i+2,1} = \frac{3^{1/2}}{2} \left(-1 - \frac{1}{2} \mu + \cdots\right), \\
H_{2i+1,2} &= H_{2i+2,2} = -\frac{1}{6} + \frac{125}{144} \mu + \cdots, \\
K_{2i+1,2} &= -K_{2i+2,2} = \frac{3^{1/2}}{2} \left(\frac{1}{3} + \frac{25}{72} \mu + \cdots\right), \\
H_{2i+1,3} &= H_{2i+2,3} = -\frac{9 + 14\mu + 13\mu^2}{4\mu} (1 + \mu)^{-5/3}, \\
K_{2i+2,3} &= -K_{2i+3,3} = -\frac{3^{1/2}}{4\mu} (3 + \mu) (1 + \mu)^{-2/3}, \\
H_{2i+2,3} &= H_{2i+3,3} = \frac{3^{1/2}}{9\mu} + \frac{5}{4\mu} - \frac{2(3)^{1/2}}{27} + \frac{29}{12} - \frac{3^{1/2}}{27} \mu - \frac{7}{3} \mu + \cdots, \\
K_{2i+2,3} &= -K_{2i+3,3} = -\frac{3^{1/2}}{2} \left(\frac{2(3)^{1/2}}{27\mu} + \frac{5}{6\mu} + \frac{8(3)^{1/2}}{81} + \frac{11}{18} + \frac{2(3)^{1/2}}{27} \mu - \frac{8}{9} \mu + \cdots\right).
\end{align*} \]

Since (16.1) is a solution of (13.1), and hence of (2.1), for values of \(\sigma_1, \sigma_2, \cdots, \sigma_{n-2}\) for which the series converge, and these values may be attained by choosing \(m_1\) sufficiently large, we may state Theorem 12.2 as follows:

**Theorem 16.3.** Given \(n-1\) arbitrary finite masses it is possible to choose an additional mass sufficiently large and an angular velocity sufficiently great, so that the \(n\) masses form a noncollinear plane permanent configuration.

**17. The polygon formed by the \(P_i\).** A diagram illustrates clearly the properties of a solution. In case \(n=1\) the polygon, Fig. 3a, consists of one point \(P_1\) at the origin. There is one point of libration \(P_2^*\) at a distance \((\lambda/m_1)^{1/3}=1\) from \(P_1\). As a zero mass \(m_2\) is placed at \(P_2^*\) and allowed to increase to a positive value, so that \((m_2/m_1)=\mu>0\), the configuration for \(n=2\) is obtained. The polygon now consists of the line segment \(P_1P_2\), Fig. 3b, with two points of libration \(P_2^*\) and \(P_1^*\) not collinear\(^{(7)}\) with \(P_1\) and \(P_2\). These points are the vertices of equilateral triangles with side \(P_1P_2=(1+\mu)^{1/3}\).

\(\text{(7)}\) The points of libration collinear with \(P_1P_2\) are obtained by solving the first three equations (2.1) for \(x_3, y_3\), where \(n=3\) and \(x_1=y_1=y_2=m_2=0\). For fixed order \(x_1, x_2, x_3\) this is equivalent to Lagrange's quintic, cf. Lagrange \([8, p. 277]\), or Tisserand \([15, p. 155]\).
tive value, so that \((m_3/m_1) = \sigma_1 > 0\), the configuration for \(n = 3\) is obtained. The polygon in this case is the triangle \(P_1P_2P_3\), Fig. 3c, which by the Theorem of Lagrange is an equilateral triangle of side \((1+\mu+\sigma_1)^{1/3}\). There are at least two(\(^8\)) points of libration \(P_4^*\) and \(P_8^*\). In general there is an \((n+2)\)-sided polygon consisting of \(n\) points \(P_i\) and two libration points \(P_{n+1}^*\) and \(P_{n+2}^*\).

From the solution functions (16.1) we observe an interesting property which we state in the following theorem.

(\(^8\)) According to Henrichsen [6], if the three masses are equal there are ten points of libration.
Theorem 17.1. As the masses \( m_i \) \((i = 3, 4, \ldots, n)\), located respectively at points \( P_i \), approach zero the points \( P_i \) cluster about the two libration points for the two body problem.

The geometric properties of these \( n \)-sided polygons, whether they are convex or concave, have not been determined. Furthermore, the possibility of removing the restrictions on the mass ratios seems to depend upon whether the determinant \( D_n \) in §12 is different from zero for all values of the masses.

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