ON SOME TRIGONOMETRIC SUMMABILITY METHODS
AND GIBBS' PHENOMENON

BY

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1. Introduction. We consider Fourier sine series

\[ f(\theta) \sim \sum_{1}^{\infty} b_{n} \sin n\theta, \quad \sum_{1}^{n} b_{n} \sin n\theta = s_{n}(\theta). \]

Suppose that \( f(\pm 0) \) exists and is greater than 0, that is \( f(\theta) \) has a simple jump \( 2f(\pm 0) \) at \( \theta = 0 \). We say that the series (1.1) presents Gibbs' phenomenon, if

\[ \limsup s_{n}(\theta_{n}) > f(\pm 0), \quad \text{as} \quad \theta_{n} \downarrow 0. \]

More generally, if only for some \( k \geq 0 \)

\[ \lim_{n \to 0} \frac{2(k + 1)}{\theta^{k+1}} \int_{0}^{\theta} (\theta - t)^{k} f(t) dt = j > 0 \]

exists, and \( \limsup s_{n}(\theta_{n}) > j/2 \), we say that the series (1.1) presents a generalized Gibbs' phenomenon at \( \theta = 0 \); \( j \) is the generalized jump of \( f(\theta) \) at \( \theta = 0 \). Our aim is to find general conditions for \( f(\theta) \) or its Fourier coefficients, which imply a Gibbs' phenomenon. It is known that the jump \( j \) is closely connected with the asymptotic behavior of the sequence \( \{nb_{n}\} \); on the other hand

\[ s_{n}(\theta_{n}) = \sum_{1}^{n} nb_{n} \sin n\theta_{n}/n \]

is a linear transform of \( \{nb_{n}\} \), with the triangular matrix \( a_{n}\nu = \nu^{-1} \sin n\theta_{n}, \) \( \nu = 1, 2, \cdots, n \). In a previous paper [3] we have discussed the relationship of the transform

\[ T_{n}(\theta_{n}) = \sum_{1}^{n} \tau_{\nu} \sin \nu\theta_{n}/\nu \]

to the Cesàro means of the sequence \( \{\tau_{\nu}\} \). As an application we have proved the presence of a Gibbs' phenomenon under general assumptions and we gave new formulae for the determination of the jump \( 2f(\pm 0) \). More general results are given in the present paper; the knowledge of (3) is not assumed.

We consider trigonometric linear transforms, related to \( T_{n} \). We discuss in particular the transforms

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(*) Numbers in brackets refer to the literature cited at the end of this paper.

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\[ B_n = \frac{1}{2} \{ T_n(\theta_n) + T_n(\phi_n) \}, \quad H_n = \frac{1}{n} \sum_{\nu=1}^{n} T_\nu(\theta_\nu), \]
\[ S_n = \frac{1}{\theta_n} \int_{0}^{\theta_n} T_n(t) dt. \]

In §5 we prove a theorem on \((C, 2)\) summability of the sequence \(\{nb_n\}\) for Fourier series.

We shall make repeated use of the well known theorem:

**Theorem A.** The convergence of a sequence \(\{\tau_n\}\) implies the convergence of the transform \(T_n = \sum a_n \tau_n\), if and only if

\[ \lim_{n \to \infty} a_n = a, \quad \text{exists for } \nu = 1, 2, 3, \ldots, \]
\[ \sum_{\nu=1}^{n} |a_\nu| = O(1), \quad \text{as } n \to \infty, \]
\[ \lim_{n \to \infty} \sum_{\nu=1}^{n} a_\nu = \sigma, \quad \text{exists.} \]

We then have \(\lim T_n = \sigma \lim \tau_n + \sum a_\nu (\tau_\nu - \lim \tau_\nu)\).

2. The average \((1/2) \{ T_n(\theta_n) + T_n(\phi_n) \} = B_n\). On putting \(\rho_n = 1\) in Theorem 2 of our paper [3] we get easily the following theorem.

**Theorem 1.** Suppose that

\[ (2.1) \quad \frac{1}{n} \sum_{\nu=1}^{n} \tau_\nu \to \tau, \quad \phi_n > 0, \quad \theta_n \geq 0, \quad n\theta_n = O(1), \quad n\phi_n = O(1), \]

and that \(\sigma(\alpha) = \int_{0}^{\alpha} (t^{-1} \sin t) dt\) has the same value for all limit points \(\alpha\) of the sequence \(\{n\theta_n\}\), and the same value for all limit points \(\beta\) of the sequence \(\{n\phi_n\}\); then

\[ (2.2) \quad \lim_{n \to \infty} B_n = (\tau/2)(\sigma(\alpha) + \sigma(\beta)). \]

For the special case \(\phi_n = \pi/n\) see Rogosinski [1, 2].

We next assume \((C, 2)\) \(\lim \tau_n = \tau\). Let \(\tau'_n = \sum \tau_\nu, \tau''_n = \sum \tau'_\nu\), then our assumption is: \(2\tau'_n/n^2 \to \tau\). We write \(\Delta^2 \tau_n = \tau_n, \Delta \tau_n = \tau_n - \tau_{n+1}, \Delta^2 \tau_n = \Delta(\Delta \tau_n)\); then

\[ B_n = \sum_{\nu=1}^{n-2} \tau_n A'_n = \frac{1}{2} \sin \theta_n + \sin \nu \phi_n + \frac{2}{n-1} \left( \frac{\sin (n - 1) \theta_n + \sin (n - 1) \phi_n}{n} \right) - \frac{2}{n} \sin \frac{n \theta_n + \sin \nu \phi_n}{n} + \tau_n \frac{\sin n \theta_n + \sin n \phi_n}{n}. \]

(*) The statement at the beginning of §2 in [3] should be corrected accordingly; it does not affect the rest of the paper.
It follows from Theorem A that, if in addition to the assumptions of Theorem 1,
\[
\sum_{\nu=1}^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu \theta_n + \sin \nu \phi_n}{\nu} \right| + n^2 \left| \Delta \frac{\sin (n-1) \theta_n + \sin (n-1) \phi_n}{n-1} \right|
+ n \left| \sin n \theta_n + \sin n \phi_n \right| = O(1),
\]
then again (2.2) holds.

We now assume \( n(\phi_n - \theta_n) = (2\mu - 1)\pi + \delta_n, \delta_n = O(1/n), \mu \) an integer \( \geq 1 \). Then
\[
\sin n \theta_n + \sin n \phi_n = 2 \sin \frac{n}{2} (\theta_n + \phi_n) \cos \frac{n}{2} (\theta_n - \phi_n) = O \left( \frac{1}{n} \right);
\]
also
\[
\cos \frac{n-1}{2} (\theta_n - \phi_n) = \cos \frac{n}{2} (\theta_n - \phi_n) \cos \frac{\theta_n - \phi_n}{2}
+ \sin \frac{n}{2} (\theta_n - \phi_n) \sin \frac{\theta_n - \phi_n}{2} = O \left( \frac{1}{n} \right).
\]
Furthermore, for \( \theta > 0, \)
\[
\Delta^2 \left( \frac{\sin \nu \theta}{\nu} \right) = \Delta^2 \int_0^\theta \cos \nu t dt = R \int_0^\theta \Delta^2 z^* dt, \quad z = c^{*t}.
\]
Hence
\[
\Delta^2 \frac{\sin \nu \theta}{\nu} = R \int_0^\theta z^*(1-z)^2 dt,
\]
and
\[
\left| \Delta^2 \frac{\sin \nu \theta}{\nu} \right| < \int_0^\theta |1-z|^2 dt = \int_0^\theta |e^{-it/2} - c^{it/2}|^2 dt
= 4 \int_0^\theta \sin^2 \frac{t}{2} dt < \int_0^\theta t^2 dt = \frac{\theta^3}{3}.
\]
Thus in view of (2.1)
\[
\sum_{\nu=1}^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu \theta_n + \sin \nu \phi_n}{\nu} \right| < (\theta_n^3 + \phi_n^3) \sum_{\nu=1}^{n-2} \nu^2 = O(1).
\]
We have thus proved the following theorem.

**Theorem 2.** Suppose that \( (C, 2) \lim \tau_n = \tau, \phi_n > 0, \theta_n \geq 0, n \theta_n = O(1), n \phi_n = n \theta_n + (2\mu - 1)\pi + O(1/n), \mu \) an integer greater than or equal to 1, and that \( \sigma(\alpha) = \int_0^\alpha t^{-1} \sin t dt \) has the same value for all limit points \( \alpha \) of the sequence \( \{n \theta_n\} \). Then \( \lim B_n = (\tau/2) \{\sigma(\alpha) + \sigma(\alpha + (2\mu - 1)\pi)\} \).

3. The average \( (1/n) \sum_{\nu=1}^n T_{\nu}(\theta) = H_n \). We have
where

\[ C_{n\nu} = \nu^{-1} \sum_{\nu=1}^{n} \sin \nu \theta_k \]

Thus \( nH_n = \sum_{\nu=1}^{n-1} (C_{n\nu} - C_{n\nu+1}) \tau_{\nu} + C_{nn} \tau_n \). We first prove the following theorem.

**Theorem 3.** If \((C, 1)\) \( \lim \tau_n = \tau \), and

\[ \frac{1}{n} \sum_{\nu=1}^{n} \mid \nu\theta_n - \alpha \mid \to 0, \]

then

\[ H_n \to \tau \int_{0}^{\alpha} \frac{\sin t}{t} \, dt. \]

(3.3) is strong summability \((C, 1)\) of the sequence \( \{n\theta_n\} \) to \( \alpha \); it implies that \( \theta_n \to 0 \), hence \( \lim_{n \to \infty} n^{-1} C_{n\nu} = 0 \) for each \( \nu \). Furthermore

\[ \sum_{1}^{n-1} \nu \mid C_{n\nu} - C_{n\nu+1} \mid + n \mid C_{nn} \mid \]

\[ = \sum_{1}^{n-1} \left[ \sum_{\nu=1}^{\nu} \sin \nu \theta_k - \frac{\nu}{\nu + 1} \sum_{\nu=1}^{\nu+1} \sin (\nu + 1) \theta_k \right] + \left| \sin n \theta_n \right| \]

\[ < \sum_{1}^{n-1} \left| \sin \nu \theta_k \right| + \sum_{1}^{n-1} \left[ \sum_{\nu=1}^{\nu+1} \left( \sin \nu \theta_k - \sin (\nu + 1) \theta_k \right) \right] \]

\[ + \sum_{1}^{n-1} \frac{1}{\nu + 1} \left[ \sum_{\nu=1}^{\nu+1} \sin (\nu + 1) \theta_k \right] + 1 \]

\[ < \sum_{1}^{n} \nu \theta_n + 2 \sum_{1}^{n} \left( \sin \frac{\theta_k}{2} \cos \frac{2\nu + 1}{2} \theta_k \right) + \sum_{1}^{n} \sum_{\nu=1}^{\nu+1} \theta_k + 1 \]

\[ < 3 \sum_{1}^{n} \nu \theta_n + 1 = O(n), \text{ as } n \to \infty, \]

hence \( H_n = O(1) \). Thus, in order that \( H_n \) has a limit whenever \( n^{-1} \tau_n \) tends to a limit \( \tau \), the additional condition is that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} C_{n\nu} = \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} \nu^{-1} \sum_{\nu=1}^{\nu} \sin \nu \theta_k = \sigma \]

exists. We then have \( H_n \to \sigma \tau \) (cf. Theorem A). Now
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\[ \frac{1}{n} \sum_{k=1}^{n} C_{nk} = \frac{1}{n} \sum_{k=1}^{n} S_{k}(\theta_{k}), \]

and

\[ S_{k}(\theta_{k}) = \sum_{\lambda=1}^{\infty} \lambda^{-1} \sin \lambda \theta_{k} = \int_{0}^{\theta_{k}} \left( \sum_{\lambda=1}^{\infty} \cos \lambda t \right) dt \]

\[ = -\frac{\theta_{k}}{2} + \int_{0}^{\theta_{k}} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt. \]

Furthermore

\[ \int_{0}^{\theta_{k}} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt = \int_{0}^{\theta_{k}} \frac{\sin (\kappa + 1/2)t}{t} dt \]

\[ + \int_{0}^{\theta_{k}} \sin (\kappa + 1/2)t \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} dt, \]

and, from the mean value theorem,

\[ 0 < \frac{1}{2 \sin (t/2)} - \frac{1}{t} = \frac{t - 2 \sin (t/2)}{2t \sin (t/2)} < \frac{t}{12}, \quad \theta < t < \pi. \]

Hence

\[ \left| \int_{0}^{\theta_{k}} \sin (\kappa + 1/2)t \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} dt \right| < \frac{1}{12} \int_{0}^{\theta_{k}} t dt = \frac{\theta_{k}^{3}}{24}, \]

and

\[ S_{k}(\theta_{k}) = -\frac{\theta_{k}}{2} + \int_{0}^{(\kappa+1/2)\theta_{k}} \frac{\sin t}{t} dt + \delta \theta_{k}^{2} \]

\[ = \int_{0}^{\alpha} \frac{\sin t}{t} dt + \int_{\alpha}^{(\kappa+1/2)\theta_{k}} \frac{\sin t}{t} dt - \frac{\theta_{k}}{2} + \delta \theta_{k}^{2}, \]

where \( |\delta_{k}| < 1/24 \). Thus, for \( 0 < \theta_{k} < 1 \),

\[ \left| S_{k}(\theta_{k}) - \int_{0}^{\alpha} \frac{\sin t}{t} dt \right| < |(\kappa + 1/2)\theta_{k} - \alpha| + |\theta_{k} - |k\theta_{k} - \alpha|| + 2\theta_{k}, \]

and from (3.3)

\[ \left| \frac{1}{n} \sum_{k=1}^{n} S_{k}(\theta_{k}) - \int_{0}^{\alpha} \frac{\sin t}{t} dt \right| < \frac{1}{n} \sum_{k=1}^{n} |k\theta_{k} - \alpha| + \frac{2}{n} \sum_{k=1}^{n} \theta_{k} \to 0; \]

which proves Theorem 3. For the case of \((C, 2)\) summability we prove the following theorem.

**Theorem 4.** If \((C, 2)\) \( \lim \tau_{n} = \tau \),
and if (3.3) holds, then (3.4) holds.

Evidently (3.3) implies \( \sum \nu \theta = O(n) \); furthermore

\[
q_n = \sum_{\nu=1}^{n} \nu^2 (\theta_{\nu} - \theta_{\nu+1}) = \sum_{\nu=1}^{n} (2\nu - 1) \theta_{\nu} - n^2 \theta_{n+1} = O(n) - n^2 \theta_{n+1},
\]

hence from (3.6)

\[
q_n^2 = O(1) \quad \text{as} \quad n \to \infty.
\]

We have from (3.1)

\[
nH_n = \sum_{\nu=1}^{n-2} \tau_n \Delta^2 \nu \theta_{\nu} + \tau_n \nu (C_n, n-1 - 2C_{nn}) + \tau_n \nu \theta_{nn},
\]

where \( nC_{nn} = \sin n \theta_n = O(1) \), and

\[
n(C_n, n-1 - 2C_{nn}) = O(1) + \left( n/(n-1) \right) \sin (n-1) \theta_{n-1} + \sin (n-1) \theta_n = O(1).
\]

Furthermore, using (2.4) and (3.8),

\[
\sum_{\nu=1}^{n-2} \nu^2 | \Delta^2 \nu \theta_{\nu} | \\
= \sum_{\nu=1}^{n-2} \nu^2 \left| \frac{1}{\nu} \sum_{\nu}^{n} \sin \nu \theta_{\nu} - \frac{2}{\nu+1} \sum_{\nu=1}^{n} \sin (\nu+1) \theta_{\nu} + \frac{1}{\nu+2} \sum_{\nu=2}^{n} \sin (\nu+2) \theta_{\nu} \right| \\
\leq \sum_{\nu=1}^{n-2} \nu^2 \left| \frac{1}{\nu} \sin \nu \theta_{\nu} + \sin \nu \theta_{\nu+1} \right| - \frac{2}{\nu+1} \sin (\nu+1) \theta_{\nu+1} + \frac{1}{\nu+2} \sin (\nu+2) \theta_{\nu+2} \\
+ \sum_{\nu=1}^{n-2} \nu^2 \left| \sum_{\nu+2}^{n} \left( \frac{\sin \nu \theta_{\nu}}{\nu} - \frac{2 \sin (\nu+1) \theta_{\nu+1}}{\nu+1} + \frac{\sin (\nu+2) \theta_{\nu+2}}{\nu+2} \right) \right| \\
< O(n) + \sum_{\nu=1}^{n-2} \nu | \sin \nu \theta_{\nu} + \sin \nu \theta_{\nu+1} - 2 \sin (\nu+1) \theta_{\nu+1} | \\
+ \sum_{\nu=1}^{n-2} \nu^2 \sum_{\nu+2}^{n} | \Delta^2 \nu^{-1} \sin \nu \theta_{\nu} | \\
< O(n) + \sum_{\nu=1}^{n-2} \nu (\theta_{\nu+1} + \theta_{\nu+1} + \nu | \theta_{\nu} - \theta_{\nu+1} | ) + \sum_{\nu=1}^{n} \nu^2 \left( \sum_{\nu=\nu}^{n} \theta_{\nu} \right) \\
= O(n) + O\left( \sum_{\nu=1}^{n} \nu^2 \theta_{\nu} \right) = O(n).
Now (3.4) follows as in Theorem 3. This proves Theorem 4. In this connection the following lemma is of interest:

**Lemma.** If for some \( c > 0 \)

\[
0 \leq \theta_{n+1} \leq (1 + c/n)\theta_n, \quad n = 1, 2, 3, \ldots,
\]

and

\[
\sum_1^n \nu \theta_\nu = O(n), \quad \text{as } n \to \infty,
\]

then (3.6) and (3.8) hold.

We write \( \sum_1^n (\theta_\nu - \theta_{n+1}) = \sum' + \sum'' \), where \( \sum' \) contains all terms with \( \theta_\nu \leq \theta_{n+1} \), and \( \sum'' \) the rest. Now, using (3.9) and (3.10),

\[
\sum' \leq c \sum_1^n \nu \theta_\nu = O(n),
\]

and from (3.7)

\[
\sum'' - \sum' < \sum_1^n (2\nu - 1)\theta_\nu = O(n),
\]

hence \( \sum'' = O(n) \) and (3.6) is proved. This and (3.10) yield (3.8). In particular in Theorem 4 assumption (3.6) can be replaced by \( \theta_n \downarrow \).

**4. The integral mean** \( (1/\theta_n) \int_0^\theta \sum_1^n \nu \sin \nu t \, dt \). To a given sequence \( \{ \tau_n \} \) we consider the transform

\[
S_n = \frac{1}{\theta_n} \int_0^{\theta_n} \left( \sum_1^n \tau_\nu \sin \nu t \right) \, dt = \sum_1^n \frac{1 - \cos \nu \theta_n}{\nu^2 \theta_n} \tau_\nu = \sum_1^n a_n \tau_\nu, \quad \theta_\nu \to 0;
\]

so that now

\[
a_n \tau_\nu = (1 - \cos \nu \theta_n)/\nu^2 \theta_n, \quad \nu = 1, 2, \ldots, n.
\]

Evidently \( 0 < a_n \tau_\nu \to 0 \) as \( n \to \infty \); hence a necessary and sufficient condition that for every convergent sequence \( \tau_n \to \tau \) the sequence \( S_n \) has a limit \( \sigma \tau \) is that \( \lim_n A_n = \sigma \) exists (from Theorem A). Now

\[
\sum_1^n \frac{1 - \cos \nu \theta_n}{\nu^2} = \int_0^{\theta_n} \left( \sum_1^n \frac{\sin \nu t}{\nu} \right) \, dt
\]

\[
= \int_0^{\theta_n} \left\{ -\frac{t}{2} + \int_0^t \frac{\sin (n + 1/2)k}{2 \sin (t/2)} \, dk \right\} \, dt
\]

\[
= -\frac{\theta_n^2}{4} + \int_0^{\theta_n} (\theta_n - t) \frac{\sin (n + 1/2)t}{2 \sin (t/2)} \, dt,
\]
thus
\[ \sum_{1}^{n} a_{\theta_{n}} = -\frac{\theta_{n}^{2}}{4} + \theta_{n}^{-1} \int_{0}^{\theta_{n}} (\theta_{n} - t) \frac{\sin (n + 1/2)t}{t} dt + \theta_{n}^{-1} \int_{0}^{\theta_{n}} (\theta_{n} - t) \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} \sin (n + 1/2)t dt, \]
and using (3.5)
\[ \theta_{n}^{-1} \left| \int_{0}^{\theta_{n}} (\theta_{n} - t) \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} \sin (2n + 1) \frac{t}{2} dt \right| < \theta_{n}^{-1} \int_{0}^{\theta_{n}} (\theta_{n} - t) \frac{dt}{12} < \frac{1}{24} \theta_{n}^{2} \to 0. \]
Hence
\[ \sum_{1}^{n} a_{\theta_{n}} = o(1) + \theta_{n}^{-1} \int_{0}^{(n+1/2)\theta_{n}} \left( \theta_{n} - \frac{t}{n + 1/2} \right) \frac{\sin t}{t} dt = o(1) + \int_{0}^{(n+1/2)\theta_{n}} \frac{\sin t}{t} dt - \frac{2}{(2n + 1)\theta_{n}} \left[ 1 - \cos (n + 1/2)\theta_{n} \right]. \]
If \( \beta \leq \infty \) is a limit point of the sequence \( \{n\theta_{n}\} \), then for a subsequence of integers \( n \)
\[ \sum_{1}^{n} a_{\theta_{n}} \to \int_{0}^{\beta} \frac{\sin t}{t} dt - \frac{1 - \cos \beta}{\beta} = g(\beta). \]
Now \( g'(\beta) = (1 - \cos \beta) / \beta^{2} \geq 0 \), hence \( g(\beta) \uparrow \), as \( \beta \) increases, to \( g(\infty) = \pi / 2 \). We have thus proved:

**Theorem 5.** In order that for every convergent sequence \( \tau_{n} \to \tau \) the transform \( S_{n} = \sum_{1}^{n} \tau_{n} (1 - \cos \nu \theta_{n}) / \nu^{2} \theta_{n} \) has a limit, it is necessary and sufficient that \( \lim_{n \to \infty} \theta_{n} = \beta, \beta \leq \infty, \) exists. We then have \( S_{n} \to g(\beta) \), where \( g(\beta) = \int_{0}^{\beta} \sin t dt / t - (1 - \cos \beta) / \beta \leq \pi / 2 \).

We now assume only \( n^{-1} \sum_{1}^{n} \tau_{n} \to \tau', \) or \( n^{-1} \tau_{n} \to \tau. \) We have \( S_{n} = \sum_{1}^{n-1} (a_{\theta_{n}} - a_{\theta_{n+1}}) \tau'_{n} + a_{\theta_{n}} \tau'_{n} ; \) and the additional condition for the existence of \( \lim S_{n} \) is by Theorem A:
\[ \sum_{1}^{n-1} \nu \left| \frac{1 - \cos \nu \theta_{n}}{\nu^{2} \theta_{n}} - \frac{1 - \cos (\nu + 1)\theta_{n}}{(\nu + 1)^{2} \theta_{n}} \right| + \frac{1 - \cos n\theta_{n}}{n\theta_{n}} = O(1), \]
or
\[ \theta_{n}^{-1} \sum_{1}^{n-1} \nu^{2} \left| (\nu + 1)^{2} (1 - \cos \nu \theta_{n}) - \nu^{2} (1 - \cos (\nu + 1)\theta_{n}) \right| = O(1). \]
But
\[\theta_n^{-1} \sum_{1}^{n} \nu^{-1} (2\nu + 1)(1 - \cos \nu \theta_n) < 3 \sum_{1}^{n} \frac{1 - \cos \nu \theta_n}{\nu^2 \theta_n} = 3 \sum_{1}^{n} a_n = O(1),\]
hence (4.3) reduces to \(\theta_n^{-1} \sum_{1}^{n} \nu^{-1} |\cos \nu \theta_n - \cos (\nu + 1)\theta_n| = O(1), \) as \(n \to \infty,\) or
\[\sum_{1}^{n} \nu^{-1} |\sin (\nu + 1/2)\theta_n| = O(1).\]
But this condition is equivalent to \(n \theta_n = O(1)\) (cf. [3, §2]). Thus we have the following:

**Theorem 6.** The transform \(S_n = \sum_{1}^{n} \tau_n (1 - \cos \nu \theta_n) / \nu^2 \theta_n\) has a limit for every sequence \(\{\tau_n\}\) which is summable (C, 1) to \(\tau,\) if and only if \(\lim n \theta_n = \beta < \infty\) exists. We then have \(S_n \to \tau \gamma(\beta).\)

We finally assume (C, 2) \(\lim \tau_n = \tau;\) using the formula
\[\sum_{1}^{n} a_n \tau_n = \sum_{1}^{n-2} \nu^2 \tau_n \Delta^2 a_n + \Delta \tau_{n-1}(a_{n-1} - 2a_n) + \tau_n a_n,\]
we now get for the existence of \(\lim S_n\) the additional conditions
\[\theta_n^{-1} \sum_{1}^{n-2} \nu^2 \left| \frac{1 - \cos \nu \theta_n}{\nu^2} - \frac{1 - \cos (\nu + 1)\theta_n}{(\nu + 1)^2} + \frac{1 - \cos (\nu + 2)\theta_n}{(\nu + 2)^2} \right| = O(1),\]
\[\theta_n^{-1} \frac{1 - \cos (n - 1)\theta_n}{(n - 1)^2} - \frac{1 - \cos n \theta_n}{n^2} + \frac{1 - \cos n \theta_n}{\theta_n} = O(1).\]

In particular \(\sin^2 \frac{n \theta_n}{2} = O(\theta_n) = o(1),\) hence \(n \theta_n \to \beta = 2\lambda \pi,\) \(\lambda\) an integer. We assume \(\lambda > 0;\) on putting \(n \theta_n = 2\lambda \pi + 2\epsilon_n,\) \(\epsilon_n \to 0,\) we must have
\[\sin^2 (\lambda \pi + \epsilon_n) = \sin^2 \epsilon_n = O((\lambda \pi + \epsilon_n)/n) = O(1/n),\]
or \(\epsilon_n = O(1/n^{1/2});\) to satisfy (4.5) the additional condition is \(\sin^2 (n - 1)\theta_n = O(\theta_n),\) or \((\sin n \theta_n, \cos \theta_n, - \cos n \theta_n, \sin \theta_n)^2 = O(1/n),\) which reduces to our previous condition. We finally show that now (4.4) is also satisfied. We have
\[\frac{1 - \cos \nu \theta}{\nu^2} = \int_{0}^{\theta} dt \int_{0}^{t} \cos \nu u \cos \nu t dt = \int_{0}^{\theta} (\theta - t) \cos \nu t dt,\]
hence
\[\Delta^2 \frac{1 - \cos \nu \theta}{\nu^2} = \int_{0}^{\theta} (\theta - t) \Delta^2 \cos \nu t dt = R \int_{0}^{\theta} (\theta - t) z'(1 - z)^2 dt, \quad z = e^{\nu t},\]
and
\[\Delta^2 \frac{1 - \cos \nu \theta}{\nu^2} < \int_{0}^{\theta} (\theta - t) |1 - e^{\nu t}|^2 dt < \int_{0}^{\theta} (\theta - t) t^2 dt = \frac{\theta^4}{12}.\]
Using this inequality, the left side of (4.4) is less than \( \sum_{n=1}^{n-2} n^2 \theta_n^3 < n^3 \theta_n^3 = O(1) \). This proves the following theorem.

**Theorem 7.** The transform \( S_n \) has a limit \( \sigma \) (with \( \sigma \neq 0 \)) for every sequence \( \{\tau_n\} \) which is summable \((C, 2)\) to \( \tau \) if and only if \( \theta_n = 2(\lambda \pi + \epsilon_n) \), \( \epsilon_n = O(n^{-1/2}) \), \( \lambda \) an integer greater than 0. We then have \( S_n \to \tau \int_0^{2\pi} t^{-1} \sin t \, dt = \pi \sigma (2\lambda \pi) \).

5. \((C, 2)\) summability of \( \{nb_n\} \) for sine series. We shall prove the following theorem.

**Theorem 8.** If \( f(\theta) \sim \sum \nu b_n \) sin \( \nu \theta \), and for some \( j \)

\[
\int_0^\theta \left| f(t) - j/2 \right| \, dt = O(\theta), \quad \int_0^\theta \left\{ f(t) - j/2 \right\} \, dt = o(\theta), \quad \text{as } \theta \downarrow 0,
\]

\[
(5.1) \quad (C, 2) \lim n b_n = j/\pi.
\]

We have, for \( \tau_n = nb_n \),

\[
\tau_n = \sum_{\nu=1}^n (n - \nu + 1) \nu b_n = \frac{2}{\pi} \int_0^\pi f(t) \left( \sum_{\nu=1}^n (n - \nu + 1) \nu \sin \nu t \right) \, dt,
\]

and \( \sum (n - \nu + 1) \nu \sin \nu t = -(d/dt) \sum (n - \nu + 1) \cos \nu t \). On putting

\[
\sum_{\nu=1}^n \cos \nu t = -\frac{1}{2} + \frac{\sin (n + 1/2)t}{2 \sin (t/2)} = \sigma_n(t),
\]

we have

\[
\sum_{\nu=1}^n (n - \nu + 1) \cos \nu t = \sum_{\nu=1}^n \sigma_n(t) = -\frac{n}{2} + \frac{1}{2 \sin (t/2)} \sum_{\nu=1}^n \sin (\nu + 1/2)t
\]

\[
= -\frac{n}{2} + \frac{1}{2 \sin (t/2)} \left\{ \sin^3 (n + 1)(t/2) - \sin^3 (t/2) \right\}
\]

\[
= -\frac{n + 1}{2} + \frac{1}{2} \frac{\sin^2 (n + 1)(t/2)}{\sin^2 (t/2)},
\]

hence

\[
\sum_{\nu=1}^n (n - \nu + 1) \nu \sin \nu t
\]

\[
= \frac{1}{2} \frac{\sin (n+1)(t/2)}{\sin^2 (t/2)} \left\{ \cos \frac{t}{2} \sin (n+1) \frac{t}{2} - (n+1) \cos (n+1) \frac{t}{2} \sin \frac{t}{2} \right\}.
\]

Thus
\[ (5.4) \quad \tau_n^2 = \frac{1}{\pi} \int_0^\infty f(t) \frac{\sin (n+1)(t/2)}{\sin^2 (t/2)} \left\{ \sin \frac{n}{2} - n \cos (n+1) \frac{t}{2} \sin \frac{t}{2} \right\} dt; \]

furthermore

\[ \int_0^\infty \left( \sum_{1}^{n} (n - \nu + 1) \nu \sin \nu t \right) dt = \sum_{1}^{n} (n - \nu + 1) \left\{ 1 - (-1)^\nu \right\} = 2 \sum_{\lambda=0}^{(n-1)/2}(n - 2\lambda) \]

\[ = 4 \sum_{\lambda=0}^{(n-1)/2} \left( \frac{n}{2} - \lambda \right) = h_n, \]

say, where \( 2hn/n^2 \to 1 \) as \( n \to \infty \). Our aim is to prove

\[ (5.6) \quad \tau_n^2/h_n \to j/\pi, \quad \text{or} \quad (\tau_n^2 - jh_n/\pi)/h_n \to 0. \]

From (5.3), (5.4) and (5.5)

\[ \tau_n^2 - \frac{1}{\pi} jh_n = \frac{1}{\pi} \int_0^\infty \left\{ f(t) - j/2 \right\} \frac{\sin (n+1)(t/2)}{\sin^2 (t/2)} K_n(t) dt, \]

where

\[ K_n(t) = \sin (nt/2) - n \cos ((n+1)t/2) \sin (t/2) = n \sin (t/2)(1 - \cos ((n+1)t/2)) - (n \sin (t/2) - \sin (nt/2)). \]

Now from (3.5)

\[ 0 < \frac{1}{\sin^2 (t/2)} - \frac{1}{(t/2)^2} = \left( \frac{1}{\sin (t/2)} - \frac{2}{t} \right) \left( \frac{1}{\sin^2 (t/2)} + \frac{2}{t \sin (t/2)} + \frac{4}{t^2} \right) < \frac{t}{6} \cdot \frac{3}{\sin^2 (t/2)} < \frac{\pi}{2 \sin (t/2)}, \]

and

\[ |K_n(t)| < n \sin (t/2) + |\sin n(t/2)|. \]

Hence

\[ \left| \int_0^\infty \left\{ f(t) - j/2 \right\} \left( \frac{1}{\sin^2 (t/2)} - \frac{1}{(t/2)^2} \right) \sin (n+1) \frac{t}{2} K_n(t) dt \right| < \frac{\pi}{2} \int_0^\infty \left| f(t) - j/2 \right| \left( n + \frac{|\sin (nt/2)|}{\sin (t/2)} \right) dt = O(n) = o(h_n); \]

thus (5.6) will follow if we prove
In view of (5.7) this will follow from

\[ \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2)}{t^2} K_n(t) dt = o(n^2). \]

(5.8) \[ \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2)}{t^2} \sin \frac{t}{2} (1 - \cos (n + 1) \frac{t}{2}) dt = o(n), \]

and

(5.9) \[ \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2)}{t^2} (n \sin \frac{t}{2} - \sin n \frac{t}{2}) dt = o(n^2). \]

Again, using (3.5),

\[ \left| \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2)}{t^2} (\frac{t}{2} - \sin \frac{t}{2}) (1 - \cos (n + 1) \frac{t}{2}) dt \right| < \int_0^\infty | f(t) - j/2 | dt = O(1), \]

thus (5.8) is equivalent to

\[ \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2)}{t^2} (1 - \cos (n + 1) \frac{t}{2}) dt = o(n), \]

or

\[ I_n = \int_0^\infty \{ f(t) - j/2 \} \frac{\sin ((n + 1)t/2) - 2^{-1} \sin (n + 1) t}{t^2} dt = o(n). \]

We write \( I_n = \int_0^{c/n} f(t) dt + \int_{c/n}^\infty = L_1 + L_2 \), say, where \( c \) is a constant, arbitrarily large. On putting \( \int_0^\infty \{ f(t) - j/2 \} dt = F(\theta) \), and using integration by parts, we get from (5.1)

\[ L_1 = \frac{F(\theta)}{\theta} \sin (n + 1)(\theta/2) - 2^{-1} \sin (n + 1) \int_0^{\theta/n} \]

\[ - (n + 1) \int_0^{\theta/n} F(t) \frac{\cos ((n + 1)t/2) - \cos (n + 1) t}{2t^2} dt + 2 \int_0^{\theta/n} F(t) \frac{\sin ((n + 1)t/2) - 2^{-1} \sin (n + 1) t}{t^2} dt \]

\[ = o(n) + o(n) \cdot \int_0^{\theta/n} t^{-1} | \sin (3(n + 1)t/4) \sin ((n + 1)t/4) | dt + o(1) \int_0^{\theta/n} t^{-2} \sin^2 ((n + 1)t/4) dt = o(n), \]
as \( n \to \infty \). Furthermore \(|L_2| < \int_{c/n}^{\infty} |f(t) - j/2| 2 \sin^2 \left(\frac{(n+1)t}{4}\right) t^2 dt\); we use here Fejér's inequality, \( \sin^2 x < \left(\frac{2x}{1+x}\right)^2 \) for \( x>0 \). Then

\[
|L_2| < \frac{(n+1)^2}{2} \int_{c/n}^{\infty} \left| f(t) - j/2 \right| \frac{dt}{(1 + ((n+1)t/4))^2}
\]

\[
= \frac{(n+1)^2}{2} \int_{c/n}^{\infty} \phi(t) \frac{dt}{(1 + ((n+1)t/4))^2}
\]

\[
+ \frac{(n+1)^2}{2} \int_{c/n}^{\infty} \frac{(n+1)\phi(t)dt}{4(1 + ((n+1)t/4))^2},
\]

where \( \phi(\theta) = \int_0^\theta |f(t) - j/2| dt \). But from (5.1), \( \phi(\theta) < \gamma \theta, \gamma \) an absolute constant, hence

\[
|L_2| < O(1) + \frac{(n+1)^3}{8} \gamma \int_{c/n}^{\infty} \frac{tdt}{(1 + ((n+1)t/4))^2}
\]

\[
< O(1) + \frac{(n+1)^3}{2} \gamma \int_{c/n}^{\infty} \frac{dt}{(1 + ((n+1)t/4))^2} < \frac{2\gamma(n+1)^3}{1 + c/4} + O(1),
\]

and \( \lim \sup_{n \to \infty} n^{-1} L_2 \leq 4\gamma/(4+c) \). Thus \( \lim \sup_{n \to \infty} n^{-1} |I_n| \leq 4\gamma/(4+c) \); but \( c \) being arbitrary, now (5.8) follows.

Similarly we decompose the left side of (5.9) into \( \int_0^{c/n} + \int_{c/n}^{\infty} = M_1 + M_2 \) say, where

\[
M_1 = \int_0^{c/n} \{f(t) - j/2\} t^2 \left[ \frac{n}{2} \left( \cos \frac{n}{2} t - \cos \frac{n+2}{2} \frac{t}{2} \right) \right.
\]

\[
+ \frac{1}{2} \left( \cos (2n+1) \frac{t}{2} - \cos \frac{t}{2} \right) \] \( dt \)

\[
= \int_0^{c/n} \{f(t) - j/2\} t^2 Q_n(t) dt
\]

\[
= F(t) t^{-2} Q_n(t) \bigg|_{c/n}^{c/n} - \int_0^{c/n} F(t) t^{-2} Q_n(t) dt
\]

\[
+ 3 \int_0^{c/n} t^{-2} Q_n(t) F(t) dt.
\]

The mean value theorem yields

\[
\frac{Q_n(t)}{n^4 t^4} \sim \frac{Q_n'(t)}{4n^4 t^3} \sim \frac{Q_n''(t)}{12n^4 t^2} \sim \frac{Q_n'''(t)}{24n^4 t} \sim \frac{1}{96}
\]

for \( t \to 0 \), hence, using (5.1), we find that \( M_1 = o(n^2) \), as \( n \to \infty \).
To estimate $M_2$ we use the formula

$$n \sin x - \sin nx = \sin x \sum_{n=1}^{\infty} \left\{1 - \cos (n - 2\nu + 1)x\right\} > 0, \quad 0 < x < \pi,$$

which follows from

$$\frac{\sin nx}{\sin x} = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = \sum_{n=1}^{\infty} e^{i(n-2\nu+1)x} = \sum_{n=1}^{\infty} \cos (n - 2\nu + 1)x.$$

Now

$$\left| M_2 \right| < \int_{c/n}^{\tau} \left| f(t) - j/2 \right| t^{-\frac{3}{2}} \sum_{n=1}^{\infty} \left\{1 - \cos (n - 2\nu + 1) \frac{t}{2}\right\} dt$$

$$= \int_{c/n}^{\tau} \left| f(t) - j/2 \right| t^{-\frac{3}{2}} \sum_{2n \leq n} \left\{1 - \cos (n - 2\nu + 1) \frac{t}{2}\right\} dt = \sum_{2n \leq n} D_n,$$

say, and, as in the estimate of $L_2$

$$D_n < \phi(\pi) \frac{(n - 2\nu + 1)^2}{2(1 + (n - 2\nu + 1)\pi/4)^2} + \frac{2\gamma(n - 2\nu + 1)}{1 + c/4}$$

$$< \phi(\pi) + \frac{8\gamma(n - 2\nu + 1)}{4 + c}.$$

Hence $\left| M_2 \right| < \phi(\pi) + 4\gamma n^2/(4+c)$, and $\lim \sup_{n \to \infty} |M_2|/n^2 \leq 4\gamma/(4+c) ; c$ being arbitrarily large, we finally get (5.9), and Theorem 8 is proved.

6. The jump of $f(\theta)$ and Gibbs’ phenomenon. The foregoing results enable us to give new formulae for the jump of $f(\theta)$ and to prove a Gibbs phenomenon.

**Theorem 9.** Under the assumptions (5.1), $j$ is determined by any one of the formulae

\begin{align*}
(6.1) & \quad (C, 2) \lim n b_n = \frac{1}{\pi} j, \\
(6.2) & \quad \sum_{n=1}^{\infty} b_n \sin \nu \frac{\pi}{n} \to \frac{1}{\pi} j \int_{0}^{\pi} \frac{\sin t}{t} dt, \\
(6.3) & \quad \lim_{n \to \infty} \frac{1}{n} \sum_{n=1}^{\infty} s_n \left(\frac{\pi}{\nu}\right) = \frac{1}{\pi} j \int_{0}^{\pi} \frac{\sin t}{t} dt, \\
(6.4) & \quad \lim_{n \to \infty} \frac{\pi}{2} \int_{0}^{2\pi/n} s_n(t) dt = \frac{1}{\pi} j \int_{0}^{2\pi} \frac{\sin t}{t} dt,
\end{align*}

where $s_n(t) = \sum_{n=1}^{\infty} b_n \sin nt$. 
(6.1) is the statement of Theorem 8; (6.2) follows from Theorem 2 for \( \theta_n = 0, \phi_n = \pi/n, \tau_n = nb_n \), using (6.1). Similarly (6.3) follows from Theorem 4 for \( \tau_n = nb_n, \theta_n = \pi/n, \) and (6.4) follows from Theorem 7 for \( \theta_n = 2\pi/n \).

To prove a Gibbs’ phenomenon under the same assumptions, put, in Theorem 2, \( \tau_n = nb_n, \theta_n = 0, n\phi_n = \pi + O(1/n) \), then

\[
s_n(\phi_n) \to \frac{1}{\pi} \int_0^\pi t^{-1} \sin t \, dt,
\]

hence

\[
\lim \sup s_n(\theta_n) \geq \frac{j/2 \times 1.08949 \cdots}{\theta_n \downarrow 0}.
\]

Similarly from Theorem 4 for \( \alpha = \pi \)

\[
\frac{1}{n} \sum_{i=1}^n s_n(\theta_i) \to \frac{1}{\pi} \int_0^\pi t^{-1} \sin t \, dt,
\]

hence

\[
\lim \sup s_n(\theta_n) \geq \frac{1}{\pi} \int_0^\pi t^{-1} \sin t \, dt.
\]

Theorem 7 also proves the presence of a Gibbs' phenomenon, however with a smaller constant.

We may also combine Theorem 8 with Theorem 3 of [3], putting there \( \rho_n = 1, \kappa = 2, n\theta_n = \pi + O(n^{-1}) \). Summarizing, we have the following theorem.

**Theorem 10. Under the assumptions (5.1)**

\[
\lim \sup_{\theta_n \to 0} \sum_{i=1}^n b_i \sin \nu \theta_n \geq \frac{jg}{2},
\]

where

\[
g = \frac{2}{\pi} \int_0^\pi t^{-1} \sin t \, dt = 1.08949 \cdots.
\]

**Literature**


**University of Cincinnati, Cincinnati, Ohio.**