TENSOR THEORY OF INVARIANTS FOR THE PROJECTIVE DIFFERENTIAL GEOMETRY OF A CURVED SURFACE

BY

THOMAS C. DOYLE

Introduction. In 1907 E. J. Wilczynski [1](1) announced an analytic basis for the projective differential geometry of a curved surface. The present paper has resulted from a study of Wilczynski's work in the light of tensor analysis. His first memoir was restricted to the invariant theory of a system of partial differential equations defining a curved surface under the subgroup of transformations preserving their canonical form, but his fourth memoir undertook the development of a general invariant theory of these same equations under their full geometric group of transformations. There are some problems in the field of differential invariants whose complete solution would be discouragingly burdensome without the aid of the tensor calculus. The general projective invariant theory of surfaces begun by Wilczynski is probably one of these which was destined to remain uncompleted under the longhand methods at his disposal. It happens, however, to be particularly amenable to tensor methods as this paper will show. To aid readability we have recast some of Wilczynski's methods and results into tensor notation from whence we introduce covariant differentiation sufficing for the formation of tensors from which invariants are then formed by contraction. The Lie theory of groups serves to point out the number of existing invariants of any specified order and all such invariants are then explicitly exhibited. In this way all projective differential invariants and covariants of a curved surface are obtained.

In addition to the aid afforded by Wilczynski's papers, the author wishes to acknowledge the assistance derived from the excellent text of E. P. Lane [2]. Here again some of Lane's work has been put in tensor form to make it available for our use. Subsequent references to Lane will be from this text with author and page number cited.

It should be remarked that G. Fubini and E. Čech were the first to apply the conventional tensor analysis of Ricci and Levi-Civita to projective differential geometry. The differences and similarities between the Italian and American schools in their point of view and method of attack have been elucidated by Lane in his report *Present tendencies in projective geometry*, Amer. Math. Monthly vol. 37 (1930) pp. 212–216. As one of these tendencies Lane cited the gradual knitting together of the American and Italian schools. Our paper belongs in this category in that while pursuing the Wilczynski

Presented to the Society, April 24, 1943; received by the editors November 7, 1942.

(1) Numbers in brackets refer to the references listed at the end of the paper.

method one arrives at the tensor analysis of Fubini and Čech. It is inevitable that some of our identities and tensor equations should duplicate those of earlier writers but it is hoped that there will also appear much that is direct, unifying and new.

1. **The first fundamental surface tensor.** Let $y^A = y^A(u^1, u^2)$, $A = 0, 1, 2, \infty$, be four analytic functions of two independent variables u^i, $i = 1, 2$, to be interpreted as the four homogeneous coordinates of the points of a surface S in a three-dimensional linear space S_3. The points y^A of S_3 will be subjected to the group G_3 of projective transformations, $y'^A = p^A{}_Q y^Q$, $|p| \neq 0$, with constant coefficients. Under this group of point transformations the surface $y^A(u)$ will be projected into the surface $y'^A(u) = p^A{}_Q y^Q(u)$. The projective geometry of S consists of those properties of S which persist when it is projected into the totality of surfaces S'. The various elements of the set S' will be regarded as equivalent and hence indistinguishable from one another. The analytic parameterization of points of S as functions of u^i may be subjected to the group G_2, $\bar{u}^i = \bar{u}^i(u)$, $|\partial\bar{u}/\partial u| \neq 0$, without disturbing the geometric point y^A, thus $\bar{y}^A(\bar{u}) = y^A(u(\bar{u}))$. In addition the functions y^{*A} defined by the group G_1, $y^A = e^{\phi(u)} y^{*A}$, $\phi(u)$ an arbitrary analytic function of u^i, represent the same geometric point as y^A. *The projective differential geometry of the surface S will find its analytic structure in the differential invariant theory of the functions $y^A(u)$ under the transformations of the groups G_3, G_2 and G_1.*

So far as G_3 is concerned, our only immediate concern is the relative covariant tensor ϵ_{ABCD} with components everywhere $+1$ or -1 according as the indices represent an even or odd permutation respectively of the natural order 0, 1, 2, ∞ and zero otherwise. Under $y'^A = p^A{}_Q y^Q$ with the inverse, $y^A = q^A{}_Q y'^Q$, $p^A{}_Q q^Q{}_B = \delta^A{}_B$, the components ϵ_{ABCD} transform by $\epsilon'_{ABCD} = \epsilon_{ABCD} = |q|^{-1} \epsilon_{QRST} q^Q{}_A q^R{}_B q^S{}_C q^T{}_D$. Analogous to ϵ_{ABCD} in S_3, there is the relative contravariant tensor ϵ^{ik} of S, defined by $\epsilon^{ik} = -\epsilon^{ki}$, $\epsilon^{12} = 1$, in all coordinate systems u^i and transforming under $\bar{u}^i = \bar{u}^i(u)$ by $\bar{\epsilon}^{ik} = \epsilon^{ik} = |\partial u/\partial \bar{u}|\, \epsilon^{rs} \partial_r \bar{u}^i \partial_s \bar{u}^k$.

Now let the two analytic functions $u^i = u^i(t)$ define an analytic curve C, $y^A = y^A(u(t))$, on S. The tangent plane to S at the point t of C has a four-rowed determinant equation $|y, y_1, y_2, Y| = 0$, the subscripts denoting partial derivatives evaluated at the point t. The osculating plane to C at t is represented by $|y, \dot{y}, \ddot{y}, Y| = 0$, where a dot denotes a total derivative with respect to t, thus $\dot{y} = \sum_{r=1}^{2} y_r \dot{u}^r = y_r \dot{u}^r$. Since y and $\dot{y}$ are common to both the tangent plane to S and the osculating plane to C, a necessary condition that these two planes coincide is that $\ddot{y} = y_r \ddot{u}^r + y_{rs} \dot{u}^r \dot{u}^s$, a point of the osculating plane, be likewise on the tangent plane, the condition being $|y, y_1, y_2, y_{rs}| \dot{u}^r \dot{u}^s = 0$. A curve C along which this condition holds is called an *asymptotic curve* of S. We define

$$G_{jk} = |\, y, y_1, y_2, y_{jk} \,| = (1/2) \epsilon_{QRST} \epsilon^{rs} y^Q y^R{}_r y^S{}_s y^T{}_{jk},$$

which transforms under G_3 by $G'_{jk} = |q|^{-1} G_{jk}$, under G_2, for which $\bar{y}_j = y_r \bar{\partial}_j u^r$, $\bar{y}_{jk} = y_r \bar{\partial}_{jk} u^r + y_{rs} \bar{\partial}_j u^r \bar{\partial}_k u^s$, by $\bar{G}_{jk} = |\partial u/\partial \bar{u}|\, G_{rs} \bar{\partial}_j u^r \bar{\partial}_k u^s$ and under G_1, for which

$$(1.1)\qquad \begin{aligned} y &= e^{\phi}y^*, \qquad y_j = e^{\phi}(y^*{}_j + \phi_j y^*),\\ y_{jk} &= e^{\phi}[y^*{}_{jk} + \phi_j y^*{}_k + \phi_k y^*{}_j + (\phi_{jk} + \phi_j\phi_k)y^*], \end{aligned}$$

by $G^*{}_{jk} = e^{-4\phi}G_{jk}$. The determinant of the G's may be seen to transform by $|G'| = |q|^{-2}|G|$, $|\bar{G}| = |\partial u/\partial\bar{u}|^4|G|$, $|G^*| = e^{-8\phi}|G|$. At any point P of S the differential form $G_{rs}du^r du^s = 0$ determines a pair of distinct asymptotic directions providing $|G| \neq 0$, a condition invariant under G_3, G_2, and G_1.

THEOREM. *The condition $|G| \neq 0$ is necessary and sufficient that S should not be a developable surface.*

Proof. Lane (p. 38). Henceforth we shall consider only nondevelopable surfaces for which $|G| \neq 0$.

Let $g_{jk} = (-|G|)^{-1/2}G_{jk}$ define g_{jk}. A coordinate system in which the asymptotic net is chosen as the parametric lines will be called *canonical*. In such a system the differential equation of the net is $2G_{12}du^1du^2 = 0$. The canonical form of g_{jk}, denoted by $\bar{g}_{jk}$, is thus $\bar{g}_{11} = \bar{g}_{22} = 0$, $\bar{g}_{12} = 1$. Under G_3, G_2 and G_1 the quantities g_{jk} transform by $g'_{jk} = g_{jk}$, $\bar{g}_{jk} = |\partial u/\partial\bar{u}|^{-1}g_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s$, $g^*{}_{jk} = g_{jk}$, respectively, and we note that $|\bar{g}| = |g| = -1$. Because of the analogy between the rôle of this tensor in the projective differential geometry of a surface and that of the metric tensor in metric differential geometry, g_{jk} will be called the *first fundamental surface tensor*.

2. **The basic system of differential equations determined by a given surface.** If y designates a point free to move over S, we seek a reference frame moving with y and having y, y_1, y_2 as three of its four vertices. For the fourth we select the point $w = (1/2)g^{rs}y_{rs} = (1/2)(-|G|)^{1/2}G^{rs}y_{rs}$, where g^{jk} and G^{jk} are defined by $g^{ir}g_{rj} = G^{ir}G_{rj} = \delta^i{}_j$. It remains to verify that $|y, y_1, y_2, w| \neq 0$;

$$\begin{aligned} |y, y_1, y_2, w| &= (1/2)(-|G|)^{1/2}G^{rs}|y, y_1, y_2, y_{rs}| = (1/2)(-|G|)^{1/2}G^{rs}G_{rs}\\ &= (-|G|)^{1/2} \neq 0. \end{aligned}$$

It is now possible to express any point as a suitable linear combination of the vertices of the moving tetrahedron. In particular, the point y_{jk} may be expressed in the form $y_{jk} = c_{jk}y + b^r{}_{jk}y_r + d_{jk}w$, where the 12 coefficients, symmetric in their lower indices, are to be determined in sets of 4 each as the unique solutions of three ($jk = 11$, 12, 22) systems of linear equations with the common determinant $|y, y_1, y_2, w| \neq 0$. Thus

$$\begin{aligned} c_{jk} &= (-|G|)^{-1/2}|y_{jk}, y_1, y_2, w|, & b^1{}_{jk} &= (-|G|)^{-1/2}|y, y_{jk}, y_2, w|,\\ b^2{}_{jk} &= (-|G|)^{-1/2}|y, y_1, y_{jk}, w|, & d_{jk} &= (-|G|)^{-1/2}|y, y_1, y_2, y_{jk}|\\ & & &= (-|G|)^{-1/2}G_{jk} = g_{jk}. \end{aligned}$$

Under G_3 these transform by $c'_{jk} = c_{jk}$, $b'^i{}_{jk} = b^i{}_{jk}$ and $d'_{jk} = d_{jk}$. Their behavior under G_2 and G_1 will be examined later. The four identities $g^{rs}c_{rs} = 0$, $g^{rs}b^i{}_{rs} = 0$,

$|g| = -1$ may be seen by inspection to hold among the 12 coefficients, leaving 8 of these as essential.

We proceed now to the elimination of the rather artificial term w from the system $y_{jk} = c_{jk}y + b^r{}_{jk}y_r + g_{jk}w$ to arrive at a system of two linear homogeneous equations in one dependent variable y of the form $A^{rs}{}_\alpha y_{rs} + A^r{}_\alpha y_r + A_\alpha y = 0$, $\alpha = 1, 2$, which Wilczynski took as basic for his theory. We emphasize the scalar character of the lower indices of the A's under G_1, G_2, G_3 by employing Greek letters. To this end we first seek a matrix $\|a^i{}_\alpha\|$ of functions which will satisfy the quadratic relations $g_{rs}a^r{}_\alpha a^s{}_\beta = |a| d_{\alpha\beta}$, where $d_{11} = d_{22} = 0$, $d_{12} = 1$. Consider the equation $g_{11}x^1x^1 + 2g_{12}x^1x^2 + g_{22}x^2x^2 = 0$ which will be quadratic if $g_{11} \neq 0$. If both g_{11} and g_{22} vanish identically, then g_{jk} is in canonical form already so that the matrix $\|a^i{}_\alpha\|$ may be taken as the unit matrix. Assume that $g_{22} \neq 0$. On applying the coordinate transformation $\bar{u}^1 = u^2$, $\bar{u}^2 = -u^1$, the tensor transformation $\bar{g}_{jk} = |\partial u/\partial \bar{u}|^{-1} g_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s$ yields $\bar{g}_{11} = g_{22} \neq 0$ so that one may always find a coordinate system for which $g_{11} \neq 0$. In such a system solve the quadratic equation for the roots $x^1{}_1/x^2{}_1 = -(g_{12}+1)/g_{11}$, $x^1{}_2/x^2{}_2 = (-g_{12}+1)/g_{11}$. Choose

$$a^1{}_1 = -t(1 + g_{12}), \qquad a^2{}_1 = tg_{11},$$
$$a^1{}_2 = s(1 - g_{12}), \qquad a^2{}_2 = sg_{11},$$

so that $|a| = -2stg_{11}$. These values of $a^i{}_\alpha$ determined to within two factors of proportionality then satisfy the desired relations $g_{rs}a^r{}_\alpha a^s{}_\beta = |a| d_{\alpha\beta}$, $|a| g^{jk} = d^{\rho\sigma}a^j{}_\rho a^k{}_\sigma = a^j{}_1 a^k{}_2 + a^j{}_2 a^k{}_1$, the summations on the scalar indices ρ and σ extending through their range 1, 2. Contraction with these a's gives the three equations $a^j{}_\alpha a^k{}_\beta(y_{jk} - c_{jk}y - b^r{}_{jk}y_r - g_{jk}w) = 0$, one of which, for $\alpha = 1$ and $\beta = 2$, is merely the identity

$$(1/2)d^{\rho\sigma}a^j{}_\rho a^k{}_\sigma(y_{jk} - c_{jk}y - b^r{}_{jk}y_r - g_{jk}w) = |a| ((1/2)g^{jk}y_{jk} - w) = 0,$$

while the remaining two, $a^j{}_\alpha a^k{}_\alpha(y_{jk} - c_{jk}y - b^r{}_{jk}y_r) = 0$, are free of w. On expanding

$$(a^1{}_1)^2 y_{11} + 2a^1{}_1 a^2{}_1 y_{12} + (a^2{}_1)^2 y_{22} + \cdots = 0,$$
$$(a^1{}_2)^2 y_{11} + 2a^1{}_2 a^2{}_2 y_{12} + (a^2{}_2)^2 y_{22} + \cdots = 0,$$

we note that the determinants I and I' of the coefficients of the y_{jk} vanish. Furthermore, if J is defined as the apolarity invariant of a pair of binary quadratic forms,

$$\phi = ax^2 + 2bxy + cy^2, \qquad \phi' = a'x^2 + 2b'xy + c'y^2, \qquad J = ac' - 2bb' + a'c,$$

then for the system under consideration, $K = J^2 - 4II' = |a|^4 \neq 0$. *Thus it has been shown that given the equations* $y'^A = p^A{}_Q y^Q(u)$ *of a class of projectively equivalent nondevelopable surfaces, there is determined a system of two linear homogeneous differential equations with nonvanishing* K *whose solutions are the*

class of functions y'^A. These equations contain 12 coefficients determined to within two factors of proportionality and satisfying two quadratic dependencies so that their essential number is 8 which agrees with our previous knowledge of the system in its "w" form.

The next few sections will be devoted to a demonstration of the converse statement that *a given pair of differential equations of the form* $A^{rs}{}_\alpha y_{rs}+A^r{}_\alpha y_r+A_\alpha y=0$ *with nonvanishing* K *and satisfying conditions for complete integrability serve to define an integrating nondevelopable curved surface to within a projective transformation.* This will be accomplished by an investigation of the transformation character of such a system which will lead in turn to a canonical form from which the desired conclusion may be drawn.

3. **Normalization of the basic system.** Let there be given a pair of completely integrable linear homogeneous partial differential equations of the second order, $Y_\alpha=A^{rs}{}_\alpha y_{rs}+A^r{}_\alpha y_r+A_\alpha y=0$. Any solution $y(u)$ of the system $Y_\alpha=0$ will likewise be a solution of the system $Y'_\alpha=\lambda^\rho{}_\alpha(u)Y_\rho=0$, $|\lambda|\neq 0$, and conversely. We follow Wilczynski [1, p. 176] with tensor methods in selecting the multipliers λ so as to simplify the system. The covariant relative tensor ϵ_{jk} defined by $\epsilon^{jr}\epsilon_{kr}=\delta^j{}_k$ is now introduced. If we define $D_{\alpha\beta}$ by $D_{\alpha\beta}=(1/2)\epsilon_{rs}\epsilon_{tu}A^{rt}{}_\alpha A^{su}{}_\beta$, then

$$D_{11}=A^{11}{}_1A^{22}{}_1-(A^{12}{}_1)^2,\qquad D_{12}=A^{11}{}_1A^{22}{}_2-2A^{12}{}_1A^{12}{}_2+A^{22}{}_1A^{11}{}_2,$$
$$D_{22}=A^{11}{}_2A^{22}{}_2-(A^{12}{}_2)^2.$$

Since $A'^{ij}{}_\alpha=\lambda^\rho{}_\alpha A^{ij}{}_\rho$, the quantities $D_{\alpha\beta}$ transform by $D'_{\alpha\beta}=D_{\rho\sigma}\lambda^\rho{}_\alpha\lambda^\sigma{}_\beta$, $|D'|=|\lambda|^2|D|$. We confine our discussion henceforth to a system for which $|D|=-K\neq 0$. This condition will be interpreted geometrically in §9. Let $d_{\alpha\beta}=(-|D|)^{-1/2}D_{\alpha\beta}$, then $d'_{\alpha\beta}=|\lambda|^{-1}d_{\rho\sigma}\lambda^\rho{}_\alpha\lambda^\sigma{}_\beta$, $|d'|=|d|$. By the same normalization process as was used in the preceding section it is possible to determine the matrix $\|\lambda^\alpha{}_\beta\|$ to within two factors of proportionality so that the components $d_{\alpha\beta}$ will be of the form $d_{\alpha\alpha}=0$, $d_{12}=1$. Assume that this has been done. We shall say that the system is now in normal form, and for all subsequent discussions we retain this form, characterized by $D_{\alpha\alpha}=|A^{ik}{}_\alpha|=0$, $\alpha=1, 2$. But this normalization implies that $A^{ik}{}_\alpha$ is of the form $A^{ik}{}_\alpha=a^i{}_\alpha a^k{}_\alpha$ from whence $D_{12}=(1/2)\epsilon_{rs}\epsilon_{tu}a^r{}_1a^t{}_1a^s{}_2a^u{}_2=|a|^2$, $D=-|a|^4\neq 0$, $d_{11}=d_{22}=0$, $d_{12}=1$. Define g^{jk} by $g^{jk}=|a|^{-1}d^{\rho\sigma}a^j{}_\rho a^k{}_\sigma$, then $g_{jk}=|a|d_{\rho\sigma}a^\rho{}_j a^\sigma{}_k$ where $a^\alpha{}_r a^r{}_\beta=\delta^\alpha{}_\beta$ and $|g|=|d|=-1$. We introduce double subscripts according to the scheme $Y_{\alpha\alpha}=Y_\alpha$, $A^i{}_{\alpha\alpha}=A^i{}_\alpha$, $A_{\alpha\alpha}=A_\alpha$ and a quantity $w=(1/2)g^{rs}y_{rs}$. Annex now to the normalized system of equations

$$Y_{\alpha\alpha}=a^r{}_\alpha a^s{}_\alpha y_{rs}-A^r{}_{\alpha\alpha}y_r-A_{\alpha\alpha}y-|a|d_{\alpha\alpha}w=0$$

a third equation

$$Y_{12}=a^r{}_1a^s{}_2y_{rs}-A^r{}_{12}y_r-A_{12}y-|a|d_{12}w=0$$

which is merely a definition of w when we define $A^i{}_{12}=A_{12}=0$ and recall

that $a^r{}_1 a^s{}_2 y_{rs} = (1/2) d^{\rho\sigma} a^r{}_\rho a^s{}_\sigma y_{rs} = (1/2) |a| g^{rs} y_{rs}$. The augmented system is now

$$Y_{\alpha\beta} = a^r{}_\alpha a^s{}_\beta y_{rs} - A^r{}_{\alpha\beta} y_r - A_{\alpha\beta} y - |a| d_{\alpha\beta} w = 0 \tag{3.1}$$

and contraction with $a^\alpha{}_j a^\beta{}_k$ yields a system of the form $y_{jk} = c_{jk} y + b^r{}_{jk} y_r + g_{jk} w$ with the same four identities among the coefficients which were found in the second section,

$$\begin{aligned} g^{rs} b^i{}_{rs} &= g^{rs} a^\rho{}_r a^\sigma{}_s A^i{}_{\rho\sigma} = |a|^{-1} d^{\rho\sigma} A^i{}_{\rho\sigma} = 2|a|^{-1} A^i{}_{12} = 0, \\ g^{rs} c_{rs} &= g^{rs} a^\rho{}_r a^\sigma{}_s A_{\rho\sigma} = |a|^{-1} d^{\rho\sigma} A_{\rho\sigma} = 2|a|^{-1} A_{12} = 0, \\ |g_{jk}| &= \big| |a| d_{\rho\sigma} a^\rho{}_j a^\sigma{}_k \big| = |d| = -1. \end{aligned}$$

4. **Transformation of the dependent variable.** To prepare the basic system in its "w" form for the formal operations of tensor calculus, write $\|y, y_1, y_2, w\|$ as $\|y_0, y_1, y_2, y_\infty\|$. Then $\partial_j y_0 = y_j$ and $\partial_j y_k = c_{jk} y_0 + b^r{}_{jk} y_r + g_{jk} y_\infty$. To complete the formulas for the derivatives of the vertices of the relative reference tetrahedron it is necessary to compute $\partial_i w$, which may be done by forming

$$\begin{aligned} \partial_k \partial_i y_j = {} & b^r{}_{ij}(c_{rk} y_0 + b^s{}_{rk} y_s + g_{rk} w) + \partial_k b^r{}_{ij} y_r \\ & + c_{ij} \delta^r{}_k y_r + \partial_k c_{ij} y_0 + g_{ij} \partial_k w + w \partial_k g_{ij}, \end{aligned}$$

permuting i and k and subtracting to eliminate the second derivatives,

$$\begin{aligned} & (\partial_k c_{ij} - \partial_i c_{kj} + c_{kr} b^r{}_{ij} - c_{ir} b^r{}_{kj}) y_0 \\ & \quad + (\partial_k b^r{}_{ij} - \partial_i b^r{}_{kj} + b^s{}_{ij} b^r{}_{sk} - b^s{}_{kj} b^r{}_{si} + c_{ij} \delta^r{}_k - c_{kj} \delta^r{}_i) y_r \\ & \quad + (\partial_k g_{ij} - \partial_i g_{kj} + g_{kr} b^r{}_{ij} - g_{ir} b^r{}_{kj}) w + g_{ji} \partial_k w - g_{jk} \partial_i w = 0. \end{aligned}$$

Let

$$\begin{aligned} c_{ijk} &= \partial_k c_{ij} - \partial_j c_{ki} + c_{kr} b^r{}_{ji} - c_{jr} b^r{}_{ki}, \qquad g_k = g^{rs} \partial_r g_{sk}, \\ b^l{}_{ijk} &= \partial_k b^l{}_{ji} - \partial_j b^l{}_{ki} + b^l{}_{kr} b^r{}_{ji} - b^l{}_{jr} b^r{}_{ki}, \end{aligned} \tag{4.1}$$

then contraction with g^{jk} yields a solution for $\partial_i w$,

$$g^{rs} c_{ris} y_0 + (b^r{}_{sit} g^{st} + c^r{}_i) y_r + (b^r{}_{ir} + g_i) w - \partial_i w = 0.$$

Let two matrices $\|\Gamma^A{}_{iB}\|$, $i = 1, 2$, be defined by

$$\begin{Vmatrix} \Gamma^0{}_{i0} & \Gamma^k{}_{i0} & \Gamma^\infty{}_{i0} \\ \Gamma^0{}_{ij} & \Gamma^k{}_{ij} & \Gamma^\infty{}_{ij} \\ \Gamma^0{}_{i\infty} & \Gamma^k{}_{i\infty} & \Gamma^\infty{}_{i\infty} \end{Vmatrix} = \begin{Vmatrix} 0 & \delta^k{}_i & 0 \\ c_{ij} & b^k{}_{ij} & g_{ij} \\ g^{rs} c_{ris} & c^k{}_i - g^{st} b^k{}_{sti} & b^r{}_{ir} + g_i \end{Vmatrix} \tag{4.2}$$

so that the differential equations may be written

$$\partial_i y_A = \Gamma^Q{}_{iA} y_Q, \qquad A = 0, 1, 2, \infty.$$

A transformation of the dependent variable is easily applied to the system

when written in this form, for from $y=e^{\phi}y^*$ follows

$$
\begin{aligned}
y_j &= e^{\phi}(\phi_j y^* + y^*_j),\\
\partial_i y_j &= e^{\phi}[(\phi_{ij} + \phi_i\phi_j)y^* + (\delta^r{}_i\phi_j + \delta^r{}_j\phi_i)y^*_r + \partial_i y^*_j],\\
w &= (1/2)g^{ij}\partial_i y_j = e^{\phi}[(1/2)g^{rs}(\phi_{rs} + \phi_r\phi_s)y^* + \phi^r y^*_r + w^*]
\end{aligned}
\tag{4.3}
$$

when we recall that $g_{ij}=g^*_{ij}$. The relations $g^{rs}\partial_j g_{rs}=0$ are a consequence of $|g|=-1$, for $\partial_j|g|=(\partial|g|/\partial g_{rs})\partial_j g_{rs}=|g|g^{rs}\partial_j g_{rs}=0$. On defining a matrix $\|\lambda^A{}_B\|$ by

$$
\begin{Vmatrix} \lambda^0{}_0 & \lambda^i{}_0 & \lambda^\infty{}_0 \\ \lambda^0{}_j & \lambda^i{}_j & \lambda^\infty{}_j \\ \lambda^0{}_\infty & \lambda^i{}_\infty & \lambda^\infty{}_\infty \end{Vmatrix}
=
\begin{Vmatrix} 1 & 0 & 0 \\ \phi_j & \delta^i{}_j & 0 \\ (1/2)g^{rs}(\phi_r\phi_s + \phi_{rs}) & \phi^i & 1 \end{Vmatrix}
\tag{4.4}
$$

with the inverse, $\|\Lambda^A{}_B\|$, for which $\Lambda^A{}_Q\lambda^Q{}_B=\delta^A{}_B$,

$$
\begin{Vmatrix} \Lambda^0{}_0 & \Lambda^i{}_0 & \Lambda^\infty{}_0 \\ \Lambda^0{}_j & \Lambda^i{}_j & \Lambda^\infty{}_j \\ \Lambda^0{}_\infty & \Lambda^i{}_\infty & \Lambda^\infty{}_\infty \end{Vmatrix}
=
\begin{Vmatrix} 1 & 0 & 0 \\ -\phi_j & \delta^i{}_j & 0 \\ (1/2)g^{rs}(\phi_r\phi_s - \phi_{rs}) & -\phi^i & 1 \end{Vmatrix},
\tag{4.5}
$$

the transformation of the dependent variable becomes $y_A=e^{\phi}y^*{}_Q\lambda^Q{}_A$. Applying this to $\partial_j y_A=\Gamma^Q{}_{jA}y_Q$ gives

$$
e^{\phi}(y^*{}_Q\partial_j\lambda^Q{}_A + \lambda^Q{}_A\partial_j y^*{}_Q + y^*{}_Q\lambda^Q{}_A\phi_j) = e^{\phi}\Gamma^R{}_{jA}\lambda^Q{}_R y^*{}_Q
$$

from whence

$$
\partial_j y^*{}_A = [\Lambda^R{}_A(\Gamma^Q{}_{jR}\lambda^B{}_Q - \partial_j\lambda^B{}_R) - \delta^B{}_A\phi_j]y^*{}_B = \Gamma^{*B}{}_{jA}y^*{}_B.
$$

From these, the transformation law of the coefficients is seen to be

$$
\Gamma^{*B}{}_{jA} = \Lambda^R{}_A(\Gamma^Q{}_{jR}\lambda^B{}_Q - \partial_j\lambda^B{}_R) - \delta^B{}_A\phi_j.
\tag{4.6}
$$

Expansion gives

$$
\begin{aligned}
c^*{}_{jk} &= c_{jk} + b^r{}_{jk}\phi_r + (1/2)g_{jk}g^{rs}(\phi_r\phi_s + \phi_{rs}) - (\phi_j\phi_k + \phi_{jk})\\
b^{*i}{}_{jk} &= b^i{}_{jk} - (\delta^i{}_j\phi_k + \delta^i{}_k\phi_j) + g_{jk}\phi^i, \qquad g^*{}_{jk} = g_{jk}.
\end{aligned}
\tag{4.7}
$$

Since $g^{*rs}c^*{}_{rs}=g^{rs}c_{rs}$ and $g^{*rs}b^{*i}{}_{rs}=g^{rs}b^i{}_{rs}$, the four identities $g^{rs}c_{rs}=0$, $g^{rs}b^i{}_{rs}=0$, $|g|=-1$ persist.

5. **The semi-canonical form of the basic system.** The integrability conditions of the system $\partial_j y_A=\Gamma^Q{}_{jA}y_Q$ are found by forming $\partial_j\partial_k y_A=\Gamma^Q{}_{kA}\Gamma^R{}_{jQ}y_R+\partial_j\Gamma^R{}_{kA}y_R$ and insisting on commutability of the order of differentiation of y_A for all y_A; $\Gamma^B{}_{Ajk}=\partial_k\Gamma^B{}_{jA}-\partial_j\Gamma^B{}_{kA}+\Gamma^B{}_{kQ}\Gamma^Q{}_{jA}-\Gamma^B{}_{jQ}\Gamma^Q{}_{kA}=0$. Their tensor character under $y\to y^*$ may be seen by writing (4.6) in the form $\partial_j\lambda^B{}_A=\Gamma^Q{}_{jA}\lambda^B{}_Q-\Gamma^{*B}{}_{jQ}\lambda^Q{}_A-\lambda^B{}_A\phi_j$, differentiating and eliminating $\partial_{jk}\lambda^B{}_A$ to obtain $\Gamma^Q{}_{Ajk}\lambda^B{}_Q=\Gamma^{*B}{}_{Qjk}\lambda^Q{}_A$. Henceforth we consider only completely in-

tegrable systems characterized by $\Gamma^B{}_{Ajk}=0$. These conditions will be examined in detail in §16. For the present it suffices to observe that contraction gives $\Gamma^Q{}_{Qjk}=\partial_k\Gamma^Q{}_{jQ}-\partial_j\Gamma^Q{}_{Qk}=0$ so that $\Gamma^Q{}_{jQ}=2b^r{}_{jr}+g_j$ are components of a gradient. Contraction of (4.6) gives $\Gamma^{*Q}{}_{jQ}=\Gamma^Q{}_{jQ}-4\phi_j$ when it is noted that $|\lambda|=1$ and hence $\Lambda^R{}_Q\partial_j\lambda^Q{}_R=0$. Since $\Gamma^Q{}_{jQ}$ are components of a gradient, there exists a function, call it $\log\Gamma$, for which $\partial_j\log\Gamma=\Gamma^Q{}_{jQ}$. To simplify the coefficients of the equations choose

$$\phi = \log\Gamma^{1/4}, \qquad \phi_j = (1/4)\Gamma^Q{}_{jQ}, \tag{5.1}$$

for which choice $\Gamma^{*Q}{}_{jQ}=0$. Denote the new dependent variables by Y_A and the coefficients resulting from (4.7) for this choice of ϕ by C_{jk}, $B^i{}_{jk}$ and g_{jk}, writing their total matrices analogous to (4.2) as $\|\Pi^A{}_{jB}\|$, $j=1, 2$. The system resulting from this particular choice of ϕ is then of the form

$$\partial_jY_A = \Pi^Q{}_{jA}Y_Q, \qquad \Pi^Q{}_{jQ} = 2B^r{}_{jr} + g_j = 0. \tag{5.2}$$

This will be called the *semi-canonical form* of the equations. The variables Y_A result from $y^*{}_A=e^{-\phi}\Lambda^Q{}_Ay_Q$ on setting $\phi_j=(1/4)\Gamma^Q{}_{jQ}$ in the matrix (4.5) of $\Lambda^A{}_B$; writing $\Gamma_j=\Gamma^Q{}_{jQ}$,

$$\begin{gathered} Y_0 = \Gamma^{-1/4}y_0, \qquad Y_j = \Gamma^{-1/4}(y_j - \Gamma_jy_0), \\ Y_\infty = \Gamma^{-1/4}[y_\infty - (1/4)\Gamma^ry_r + (1/32)(\Gamma^r\Gamma_r - 4\partial_r\Gamma^r)y_0]. \end{gathered} \tag{5.3}$$

To see how the semi-canonical form is affected by a change of dependent variable, $y=e^\phi y^*$, substitute (4.3) into the expressions (5.3) for Y_A, remembering that $(1/4)\Gamma^Q{}_{jQ}=(1/4)\Gamma^{*Q}{}_{jQ}+\phi_j$, $\log\Gamma^{1/4}=\log\Gamma^{*1/4}+\phi-\log k$, where k is a constant of integration, to obtain

$$Y_0 = k\Gamma^{*-1/4}y^*{}_0 = kY^*{}_0,\ Y_j = k\Gamma^{*-1/4}(y^*{}_j - (1/4)\Gamma^{*Q}{}_{jQ}y^*{}_0) = kY^*{}_j,\ Y_\infty = kY^*{}_\infty.$$

The system $\partial_jY_B=\Pi^Q{}_{jB}Y_Q$ then transforms into $\partial_jY^*{}_B=\Pi^Q{}_{jB}Y^*{}_Q$ so that $\Pi^A{}_{jB}=\Pi^{*A}{}_{jB}$. But for $\Pi^A{}_{jB}=\Pi^A{}_{jB}(c_{jk}, b^i{}_{jk}, g_{jk}$, and derivatives) we have by definition $\Pi^{*A}{}_{jB}=\Pi^A{}_{jB}(c^*{}_{jk}, b^{*i}{}_{jk}, g^*{}_{jk}$ and derivatives) and now it has been proved that these two sets of functions, the same in form, are moreover equal in value. This leads to the idea of a *seminvariant* which will be discussed in the next section.

6. **A complete system of seminvariants and semicovariants.** Any function S of the coefficients Γ and their derivatives which transforms under $y=e^\phi y^*$ by the law $S(\Gamma)=e^{w\phi}S(\Gamma^*)$ will be called a *relative seminvariant of weight w* of the system. For example, the coefficients of the semi-canonical form

$$\begin{aligned} C_{jk} &= c_{jk} + (1/4)b^r{}_{jk}\Gamma_r + (1/2)g_{jk}((1/16)\Gamma_r\Gamma^r + (1/4)\partial_r\Gamma^r) \\ &\quad - ((1/16)\Gamma_j\Gamma_k + (1/4)\partial_j\Gamma_k), \\ B^i{}_{jk} &= b^i{}_{jk} - (1/4)(\delta^i{}_j\Gamma_k + \delta^i{}_k\Gamma_j) + (1/4)g_{jk}\Gamma^i, \qquad g_{jk}, \end{aligned} \tag{6.1}$$

have been found to be absolute seminvariants. Similarly, any function C of

both the coefficients Γ and the dependent variable y which transforms by $C(\Gamma, y)=e^{w\phi}C(\Gamma^*, y^*)$ will be called a *semicovariant* of weight w. For example, y itself is a semicovariant of weight 1 and the ratios of the semi-canonical variables

$$(6.2)\qquad \begin{aligned} Y_j/Y_0 &= (y_j - (1/4)\,\Gamma_j y_0)/y_0, \\ Y_\infty/Y_0 &= [y_\infty - (1/4)\Gamma^s y_s + (1/32)(\Gamma_r\Gamma^r - 4\partial_r\Gamma^r)y_0]/y_0 \end{aligned}$$

have been shown to be absolute semicovariants.

FIRST REPLACEMENT THEOREM. *Any absolute seminvariant $S(\Gamma)$ is expressible in terms of the absolute seminvariants C_{jk}, $B^i{}_{jk}$, g_{jk} and their derivatives. Moreover, $S(\Gamma)=S(\Pi)$, where the Π's are formed from* (4.2) *for the values* (6.1).

Proof. Let $S(\Gamma)$ be any absolute seminvariant of the system $\partial_i y_A=\Gamma^Q{}_{iA}y_Q$. Then when the system is transformed into $\partial_i y^*{}_A=\Gamma^{*Q}{}_{iA}y^*{}_Q$ under $y=e^\phi y^*$, by the definition of an absolute seminvariant we must have $S(\Gamma)=S(\Gamma^*)$ for arbitrary choice of ϕ. In particular the law $S(\Gamma)=S(\Gamma^*)$ must hold for the special choice of ϕ leading to the semi-canonical form, for which the coefficients $\Gamma^{*B}{}_{iA}$ become $\Pi^B{}_{iA}$, so that $S(\Gamma)=S(\Pi)$ which establishes the theorem.

As an application of this replacement theorem, substitute the semi-canonical coefficients Π into the expressions (6.1) for the seminvariants C_{jk} and $B^i{}_{jk}$ to obtain $C_{jk}=C_{jk}$ and $B^i{}_{jk}=B^i{}_{jk}$ in virtue of the relations $\Pi^Q{}_{jQ}=0$ which characterize the semi-canonical system. Because of the identities $g^{rs}C_{rs}=0$, $g^{rs}B^i{}_{rs}=0$, $|g|=-1$, there are but 8 independent quantities among the set C_{jk}, $B^i{}_{jk}$ and g_{jk}.

SECOND REPLACEMENT THEOREM. *Any absolute semicovariant $C(\Gamma, y)$ is expressible in terms of the seminvariants C_{jk}, $B^i{}_{jk}$, g_{jk}, their derivatives, and the semicovariants Y_j/Y_0 and Y_∞/Y_0. In fact, C must be homogeneous of degree* 0 *in Y_A and $C(\Gamma, y_j/y_0, y_\infty/y_0)=C(\Pi, Y_j/Y_0, Y_\infty/Y_0)$.*

Proof. Since $\partial_j y_A$ is expressible linearly in terms of y_A, derivatives of y_A need not occur in any semicovariant. To show that any absolute semicovariant $C(\Gamma, y_A)$, transforming by $C(\Gamma, y_A)=C(\Gamma^*, y^*{}_A)$, must be homogeneous of degree 0 in y_A, apply the transformation $y=e^k y^*$, k a constant. Then $y_A=e^k y^*{}_A$, $\Gamma^{*A}{}_{jB}=\Gamma^A{}_{jB}$ and the transformation law of C becomes $C(\Gamma, y_A)=C(\Gamma, e^{-k}y_A)$ which is the criterion for homogeneity of degree 0. Now let $C(\Gamma, y_i/y_0, y_\infty/y_0)$ be any absolute semicovariant transforming under $y=e^\phi y^*$ for all ϕ by the law $C(\Gamma, y_i/y_0, y_\infty/y_0)=C(\Gamma^*, y^*{}_i/y^*{}_0, y^*{}_\infty/y^*{}_0)$. Then for the special choice of ϕ leading to the semi-canonical form, $C(\Gamma, y_i/y_0, y_\infty/y_0)=C(\Pi, Y_i/Y_0, Y_\infty/Y_0)$.

As an application of this replacement theorem, substitute the semi-canonical coefficients and variables into the expressions (6.2) for the semicovariants Y_j/Y_0 and Y_∞/Y_0 to obtain $Y_j/Y_0=Y_j/Y_0$, $Y_\infty/Y_0=Y_\infty/Y_0$ as a consequence of $\Pi^Q{}_{jQ}=0$.

Any function of the coefficients whose form and value are unaltered by transformations of both the dependent and independent variables will be called an *absolute invariant. Since any absolute invariant must necessarily be a seminvariant, the set of all absolute invariants will constitute a subset of the totality of seminvariants. They will be found by selecting those seminvariants which are unaltered by a transformation of the independent variables.* This we proceed to do.

7. **Transformation of the independent variables.** The transformation $\bar{u}^i = \bar{u}^i(u)$, $|\partial\bar{u}/\partial u| \neq 0$, of the independent variables induces

$$y(u) = y(u(\bar{u})) = \bar{y}(\bar{u}), \quad \partial_j y = \bar{\partial}_r \bar{y} \partial_j \bar{u}^r, \quad \partial_{jk} y = \bar{\partial}_r \bar{y} \partial_{jk} \bar{u}^r + \bar{\partial}_{rs} \bar{y} \partial_j \bar{u}^r \partial_k \bar{u}^s.$$

On applying this to the system in the form (3.1),

$$a^r{}_\alpha a^s{}_\beta y_{rs} - A^r{}_{\alpha\beta} y_r - A_{\alpha\beta} y - |a| d_{\alpha\beta} w = 0,$$

we find $a^r{}_\alpha a^s{}_\beta \partial_r \bar{u}^j \partial_s \bar{u}^k \bar{\partial}_{jk} \bar{y} + \cdots = \bar{a}^j{}_\alpha \bar{a}^k{}_\beta \bar{\partial}_{jk} \bar{y} + \cdots = 0$, so that $\bar{a}^i{}_\alpha = a^r{}_\alpha \partial_r \bar{u}^i$, $|\bar{a}| = |a| \, |\partial u/\partial \bar{u}|^{-1}$. But from §3, $g^{jk} = |a|^{-1} d^{\rho\sigma} a^j{}_\rho a^k{}_\sigma$, so that $\bar{g}^{jk} = |a|^{-1} \cdot |\partial u/\partial \bar{u}| d^{\rho\sigma} (a^r{}_\rho \partial_r \bar{u}^j)(a^s{}_\sigma \partial_s \bar{u}^k) = |\partial u/\partial \bar{u}| g^{rs} \partial_r \bar{u}^j \partial_s \bar{u}^k$. These laws and their covariant equivalents will be used so repeatedly that we display them here:

$$(7.1) \qquad \bar{g}_{jk} = |\partial u/\partial \bar{u}|^{-1} \bar{g}_{rs} \bar{\partial}_j u^r \bar{\partial}_k u^s, \qquad \bar{g}^{jk} = |\partial u/\partial \bar{u}| \, g^{rs} \partial_r \bar{u}^j \partial_s \bar{u}^k.$$

The quantity w transforms by

$$\begin{aligned} w &= (1/2) g^{rs} y_{rs} = (1/2) g^{rs} \partial_{rs} \bar{u}^t \bar{y}_t + (1/2) |\partial \bar{u}/\partial u| \, \bar{g}^{rs} \bar{y}_{rs} \\ &= (1/2) g^{rs} \partial_{rs} \bar{u}^t \bar{y}_t + |\partial \bar{u}/\partial u| \, \bar{w}. \end{aligned}$$

We now introduce a matrix $\|\sigma^A{}_B\|$ defined by

$$(7.2) \qquad \left\| \begin{matrix} \sigma^0{}_0 & \sigma^i{}_0 & \sigma^\infty{}_0 \\ \sigma^0{}_j & \sigma^i{}_j & \sigma^\infty{}_j \\ \sigma^0{}_\infty & \sigma^i{}_\infty & \sigma^\infty{}_\infty \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & \partial_j \bar{u}^i & 0 \\ 0 & (1/2) g^{rs} \partial_{rs} \bar{u}^i & |\partial \bar{u}/\partial u| \end{matrix} \right\|,$$

and its inverse, $\|\Sigma^A{}_B\|$, satisfying $\Sigma^A{}_Q \sigma^Q{}_B = \delta^A{}_B$,

$$(7.3) \qquad \left\| \begin{matrix} \Sigma^0{}_0 & \Sigma^i{}_0 & \Sigma^\infty{}_0 \\ \Sigma^0{}_j & \Sigma^i{}_j & \Sigma^\infty{}_j \\ \Sigma^0{}_\infty & \Sigma^i{}_\infty & \Sigma^\infty{}_\infty \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & \bar{\partial}_j u^i & 0 \\ 0 & (1/2) \bar{g}^{rs} \bar{\partial}_{rs} u^i & |\partial u/\partial \bar{u}| \end{matrix} \right\|.$$

The transformations $y_A \to \bar{y}_A$ may now be written $y_A = \sigma^Q{}_A \bar{y}_Q$, from whence $\partial_j y_A = \sigma^Q{}_A \partial_j \bar{u}^r \bar{\partial}_r \bar{y}_Q + \bar{y}_Q \partial_j \sigma^Q{}_A$. Substitution into $\partial_j y_A = \Gamma^Q{}_{jA} y_Q$ gives

$$\bar{\partial}_j \bar{y}_A = \Sigma^R{}_A (\Gamma^Q{}_{tR} \sigma^B{}_Q - \partial_s \sigma^B{}_R) \bar{\partial}_j u^s \bar{y}_B = \bar{\Gamma}^B{}_{jA} \bar{y}_B,$$

and therefore

$$(7.4) \qquad \bar{\Gamma}^B{}_{jA} = \Sigma^R{}_A (\Gamma^Q{}_{tR} \sigma^B{}_Q - \partial_t \sigma^B{}_R) \bar{\partial}_j u^t.$$

On expanding,

$$
(7.5)\qquad
\begin{aligned}
\bar{c}_{jk} &= c_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s,\\
\bar{b}^i{}_{jk} &= b^r{}_{st}\partial_r \bar{u}^i \bar{\partial}_j u^s \bar{\partial}_k u^t - \partial_{st}\bar{u}^i \bar{\partial}_j u^s \bar{\partial}_k u^t + (1/2)\,|\,\partial u/\partial \bar{u}\,|\,\bar{g}_{jk} g^{rs}\partial_{rs}\bar{u}^i,
\end{aligned}
$$

we find $\bar{g}^{rs}\bar{c}_{rs} = |\partial u/\partial \bar{u}|\, g^{rs}c_{rs}$, $\bar{g}^{rs}\bar{b}^i{}_{rs} = |\partial u/\partial \bar{u}|\, g^{rs}b^t{}_{rs}\partial_t \bar{u}^i$, $|\bar{g}| = |g|$, so that the four identities $g^{rs}c_{rs}=0$, $g^{rs}b^i{}_{rs}=0$, $|g| = -1$ hold in all coordinate systems.

8. **Wilczynski's canonical form of the basic system.** Thus far we have arrived at a semi-canonical form of the partial differential equations characterized by $\Pi^Q{}_{iQ}=0$ and obtained by applying a specially adapted transformation of the dependent variable. Now we investigate the possibility of making a suitable choice of the independent variables which will still further simplify the system. Namely, is it possible to find a transformation $\bar{u}^1 = U(u^1, u^2)$, $\bar{u}^2 = V(u^1, u^2)$, $|\partial \bar{u}/\partial u| \neq 0$, to coordinates $\bar{u}^i$ for which $\bar{g}^{jk}$ will have the components $\bar{g}^{11} = \bar{g}^{22} = 0$, $\bar{g}^{12} = 1$? Following Wilczynski [1, p. 243], the transformation law $|\partial \bar{u}/\partial u|\,\bar{g}^{jk} = g^{rs}\partial_r \bar{u}^j \partial_s \bar{u}^k$ shows that U and V must satisfy

$$
\begin{aligned}
g^{11}U_1U_1 + 2g^{12}U_1U_2 + g^{22}U_2U_2 &= 0,\\
g^{11}V_1V_1 + 2g^{12}V_1V_2 + g^{22}V_2V_2 &= 0,
\end{aligned}
$$

and since

$$
\left\| \begin{matrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{matrix} \right\| = \left\| \begin{matrix} -g_{22} & g_{12} \\ g_{12} & -g_{11} \end{matrix} \right\|,
$$

the ratios U_1/U_2 and V_1/V_2 must be roots of the quadratic $g_{22}x^2 - 2g_{12}x + g_{11} = 0$. If both g_{11} and g_{22} vanish, the system is already in the desired form. Otherwise, as previously remarked in §2, there is readily found a coordinate system in which $g_{22} \neq 0$. Assume that u^i is such a coordinate system, then U and V must be chosen to satisfy

$$
g_{22}U_1 - (g_{12}+1)U_2 = 0, \qquad g_{22}V_1 - (g_{12}-1)V_2 = 0,
$$

and since the determinant $\Delta = 2g_{22}$ of this system does not vanish, the solutions U and V will have a nonvanishing Jacobian and hence will constitute a proper transformation to coordinates $\bar{u}^i$ for which $\bar{g}^{11} = \bar{g}^{22} = 0$. It remains to verify that $\bar{g}^{12} = 1$;

$$
\begin{aligned}
|\,\partial \bar{u}/\partial u\,|\,\bar{g}^{12} &= -g_{22}U_1V_1 + g_{12}(U_1V_2 + U_2V_1) - g_{11}U_2V_2\\
&= (U_2V_2/g_{22})\left[-(g^2{}_{12}-1) + g_{12}(2g_{12}) - g_{11}g_{22}\right]\\
&= 2U_2V_2/g_{22} = |\,\partial \bar{u}/\partial u\,|,
\end{aligned}
$$

and therefore $\bar{g}^{12}=1$. Henceforth the canonical components of a quantity will be indicated by placing a tilda over the symbol, thus $\tilde{g}_{11} = \tilde{g}_{22} = 0$, $\tilde{g}_{12} = 1$.

Having chosen the independent variables to yield this simple form for g^{jk}, transform the dependent variable to produce the semi-canonical form. Since

$g^{jk}=g^{*jk}$, our present normalization of g^{jk} will not be disturbed. But the identities $g^{rs}C_{rs}=0, g^{rs}B^i{}_{rs}=0, \Pi^Q{}_{jQ}=2B^r{}_{jr}+g_j=0$ simplify to $\tilde{C}_{12}=0$, $\tilde{B}^i{}_{12}=0$, $\tilde{B}^r{}_{jr}=0$, recalling that $\tilde{g}_j=\tilde{g}^{rs}\partial_r\tilde{g}_{sj}=0$. Finally then $\tilde{C}_{12}=\tilde{B}^i{}_{12}=\tilde{B}^1{}_{11}=\tilde{B}^2{}_{22}=0$ so that the system reduces to

$$\bar{y}_{11} = \tilde{C}_{11}\bar{y} + \tilde{B}^2{}_{11}\bar{y}_2, \qquad \bar{y}_{22} = \tilde{C}_{22}\bar{y} + \tilde{B}^1{}_{22}\bar{y}_1. \tag{8.1}$$

This is Wilczynski's canonical form [1, p. 246] of the given system.

9. **Geometric interpretation.** Wilczynski has shown [1, pp. 233–236] that a system $A^{rs}{}_\alpha y_{rs}+A^r{}_\alpha y_r+A_\alpha y=0$ which is completely integrable and for which $K\neq 0$ admits precisely four linearly independent particular solutions $y^A(u)$ and that the most general solution system y'^A is of the form $y'^A=p^A{}_Q y^Q$ with constant p's. Regarding the particular solutions y^A as homogeneous coordinates of a point on an integrating surface S, the generality in the choice of the solution system y'^A determines S to within a projective transformation. The differential equations of the asymptotic net, $|y, y_1, y_2, y_{rs}|du^r du^s=0$, become in Wilczynski's canonical form $|\bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_{12}|d\bar{u}^1 d\bar{u}^2=0$. If we define G_{jk} by $G_{jk}=|y, y_1, y_2, y_{jk}|$ and h_{jk} by $h_{jk}=(-|G|)^{-1/2}G_{jk}$, then in canonical coordinates $\tilde{h}_{jk}=\tilde{g}_{jk}$ and hence $h_{jk}=g_{jk}$, from whence $|h|=-1$ and therefore $|\bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_{12}|\neq 0$. It follows then that *the canonical parametric net is the asymptotic net.* Furthermore, contraction of $|\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{y}_{jk}| = |\partial u/\partial\bar{u}|\,|y_0, y_1, y_2, y_{rs}|\bar{\partial}_j u^r \bar{\partial}_k u^s$ with $\bar{g}^{jk}$ gives $|\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{y}_\infty| = |\partial u/\partial\bar{u}|^2 \cdot |y_0, y_1, y_2, y_\infty|$ and under $y=e^\phi y^*$, $|y_0, y_1, y_2, y_\infty| = e^{4\phi}|y_0^*, y_1^*, y_2^*, y_\infty^*|$. Because $|\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{y}_\infty| = |\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{y}_{12}|\neq 0$, it follows that in all coordinates $|y_0, y_1, y_2, y_\infty|\neq 0$.

10. **The most general transformation preserving the semi-canonical form.** In §7 it was shown that a transformation $u^i\to\bar{u}^i$ of the independent variables induces a transformation $y_B\to\bar{y}_B$ for which $y_B=\sigma^Q{}_B\bar{y}_Q$. Let this be done and follow with a transformation $\bar{y}\to\bar{y}^*$ of the dependent variable as explained in §4 for which $\bar{y}_Q=e^\phi\lambda^A{}_Q\bar{y}^*{}_A$. *The combined transformation* $y_B=e^\phi\sigma^Q{}_B\lambda^A{}_Q\bar{y}^*{}_A$ *with the coefficients* $e^\phi\mu^A{}_B=e^\phi\sigma^Q{}_B\lambda^A{}_Q$ *is the most general transformation of the representation preserving the form of the basic system.* Since the matrices $\|\lambda\|$ and $\|\sigma\|$ are commutable, we may also write $\mu^A{}_B=\sigma^A{}_Q\lambda^Q{}_B$ and inspection of (4.4) and (7.2) shows that

$$\begin{aligned}\left\|\begin{matrix}\mu^0{}_0 & \mu^i{}_0 & \mu^\infty{}_0\\ \mu^0{}_j & \mu^i{}_j & \mu^\infty{}_j\\ \mu^0{}_\infty & \mu^i{}_\infty & \mu^\infty{}_\infty\end{matrix}\right\| &= \left\|\begin{matrix}1 & 0 & 0\\ \phi_j & \partial_j\bar{u}^i & 0\\ (1/2)g^{rs}(\phi_r\phi_s+\phi_{rs}) & g^{rs}((1/2)\partial_{rs}\bar{u}^i+\phi_r\partial_s\bar{u}^i) & |\partial\bar{u}/\partial u|\end{matrix}\right\|.\end{aligned} \tag{10.1}$$

Let $\|M^A{}_B\|$ be the inverse of $\|\mu^A{}_B\|$, $M^A{}_Q\mu^Q{}_B=\delta^A{}_B$. Then

$$
(10.2)\quad \left\| \begin{matrix} M^0{}_0 & M^i{}_0 & M^\infty{}_0 \\ M^0{}_j & M^i{}_j & M^\infty{}_j \\ M^0{}_\infty & M^i{}_\infty & M^\infty{}_\infty \end{matrix} \right\|
= \left\| \begin{matrix} 1 & 0 & 0 \\ -\bar\partial_j\phi & \bar\partial_j u^i & 0 \\ (1/2)\bar g^{rs}(\bar\partial_r\phi\bar\partial_s\phi - \bar\partial_{rs}\phi) & \bar g^{rs}((1/2)\bar\partial_{rs}u^i - \bar\partial_r\phi\bar\partial_s u^i) & |\partial u/\partial\bar u| \end{matrix} \right\|.
$$

Analogously to (4.6) and (7.4) the transformation law of the coefficients is

$$
(10.3)\qquad \Gamma^{*B}{}_{jA} = M^R{}_A(\Gamma^Q{}_{tR}\mu^B{}_Q - \partial_t\mu^B{}_R)\bar\partial\, u^t - \delta^B{}_A\bar\partial_j\phi.
$$

Contraction gives

$$
(10.4)\qquad \Gamma^{*Q}{}_{jQ} = \Gamma^Q{}_{tQ}\bar\partial_j u^t - \partial_t \log|\mu|\,\bar\partial_j u^t - 4\bar\partial_j\phi
$$

when we recall that $\partial_j|\mu| = (\partial|\mu|/\partial\mu^Q{}_R)\partial_j\mu^Q{}_R = |\mu|\,M^R{}_Q\partial_j\mu^Q{}_R$. Since ultimately we shall seek invariants of the system which in turn must necessarily be functions of the seminvariants $\Pi^A{}_{jB}$, we shall be concerned with the transformation law of these quantities. It will be found by asking for the most general transformation preserving the semi-canonical form, as characterized by the conditions $\Gamma^Q{}_{jQ}=0$ before and after the transformation. Equations (10.4) together with the relation $|\mu| = |\partial\bar u/\partial u|^2$ show that ϕ must be taken as a solution of the equation

$$
(10.5)\qquad \bar\partial_j\psi = (1/2)\bar\partial_j \log|\partial u/\partial\bar u| \quad\text{or}\quad 2|\partial u/\partial\bar u|\,\bar\partial_j\psi = \bar\partial_j|\partial u/\partial\bar u|.
$$

This simply means that starting with the system in semi-canonical form, a transformation of the independent variables destroys this form and that to regain it a transformation $y = e^\psi y^*$, ψ satisfying (10.5), must be applied to the dependent variable. Hence, finally, the seminvariants $\Pi^A{}_{jB}$ transform by

$$
(10.6)\qquad \begin{aligned} \overline{\Pi}^B{}_{jA} &= M^R{}_A(\Pi^Q{}_{tR}\mu^B{}_Q - \partial_t\mu^B{}_R)\bar\partial_j u^t - (1/2)\delta^B{}_A\bar\partial_j \log|\partial u/\partial\bar u|, \\ \overline{\Pi}^Q{}_{jQ} &= \Pi^Q{}_{rQ}\bar\partial_j u^r, \end{aligned}
$$

and the relative semicovariants Y_B by $Y_B = e^\psi\mu^Q{}_B\overline{Y}_Q$, where $\|\mu^A{}_B\|$ and $\|M^A{}_B\|$ are given by (10.1) and (10.2) with $\bar\partial_j\phi$ replaced by $\bar\partial_j\psi = (1/2)\bar\partial_j \log|\partial u/\partial\bar u|$ and $\psi = \log|\partial u/\partial\bar u|^{1/2}$ on choosing the constant of integration to be zero.

Expansion of (10.6) gives

$$
(10.7)\qquad \begin{aligned} \bar C_{jk} &= [C_{rs} + B^t{}_{rs}\psi_t + (1/2)g_{rs}g^{uv}(\psi_u\psi_v + \psi_{uv}) - (\psi_r\psi_s + \psi_{rs})]\bar\partial_j u^r\bar\partial_k u^s, \\ \bar B^i{}_{jk} &= [B^s{}_{rt}\partial_s\bar u^i - \partial_{rt}\bar u^i + g_{rt}g^{uv}(\psi_u\partial_v\bar u^i + (1/2)\partial_{uv}\bar u^i)]\bar\partial_j u^r\bar\partial_k u^t \\ &\quad - (\delta^i{}_j\bar\partial_k\psi + \delta^i{}_k\bar\partial_j\psi), \\ \bar g_{jk} &= |\partial\bar u/\partial u|\, g_{rs}\bar\partial_j u^r\bar\partial_k u^s, \end{aligned}
$$

from whence $\bar{g}^{rs}\overline{B}^i{}_{rs} = |\partial u/\partial \bar{u}|\, g^{rs}B^t{}_{rs}\partial_t \bar{u}^i$, $\bar{g}^{rs}\overline{C}_{rs} = |\partial u/\partial \bar{u}|\,(g^{rs}C_{rs} + g^{rs}B^t{}_{rs}\psi_t)$, $|\bar{g}| = |g|$, so that the identities $g^{rs}C_{rs} = 0$, $g^{rs}B^i{}_{rs} = 0$, $|g| = -1$ persist.

11. The second fundamental surface tensor and its relative invariant. The Christoffel symbols of the second kind formed from the relative tensor g_{jk}, defined by

$$\left\{ {i \atop jk} \right\} = (1/2)g^{ir}(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}),$$

transform by

$$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {r \atop st} \right\} \partial_r \bar{u}^i \bar{\partial}_j u^s \bar{\partial}_k u^t + \bar{\partial}_{jk} u^r \partial_r \bar{u}^i - (\delta^i{}_j \bar{\partial}_k \psi + \delta^i{}_k \bar{\partial}_j \psi - \bar{g}_{jk}\bar{g}^{ir}\bar{\partial}_r \psi),$$

and contraction with $\bar{g}^{jk}$ yields

$$\bar{g}^{rs}\overline{\left\{ {i \atop rs} \right\}} = |\partial u/\partial \bar{u}| \left(g^{rs} \left\{ {t \atop rs} \right\} \partial_t \bar{u}^i - g^{rs}\partial_{rs}\bar{u}^i \right) \tag{11.1}$$

from whence

$$\begin{aligned} \overline{\left\{ {i \atop jk} \right\}} - (1/2)\bar{g}_{jk}\bar{g}^{rs}\overline{\left\{ {i \atop rs} \right\}} &= \left[\left(\left\{ {t \atop rs} \right\} - (1/2)g_{rs}g^{uv}\left\{ {t \atop uv} \right\} \right) \partial_t \bar{u}^i - \partial_{rs}\bar{u}^i \right] \bar{\partial}_j u^r \bar{\partial}_k u^s \\ &\quad - (\delta^i{}_j \bar{\partial}_k \psi + \delta^i{}_k \bar{\partial}_j \psi - \bar{g}_{jk}\bar{\partial}^i \psi) + (1/2)\,|\partial u/\partial \bar{u}|\, \bar{g}_{jk} g^{rs}\partial_{rs}\bar{u}^i. \end{aligned}$$

With the help of the relations $\bar{\partial}^i \psi = \bar{g}^{ir}\bar{\partial}_r \psi = |\partial u/\partial \bar{u}|\, \partial^s \psi \partial_s \bar{u}^i$ it may be seen from (10.7) that $B^i{}_{jk}$ transform by the same law as

$$\left\{ {i \atop jk} \right\} - (1/2)g_{jk}g^{rs}\left\{ {i \atop rs} \right\}.$$

Subtraction reveals the tensor character of

$$T^i{}_{jk} = B^i{}_{jk} - \left(\left\{ {i \atop jk} \right\} - (1/2)g_{jk}g^{rs}\left\{ {i \atop rs} \right\} \right), \tag{11.2}$$

which will be called the *second fundamental surface tensor*. Now

$$g^{rs}\left\{ {i \atop rs} \right\} = g^{it}g^{rs}\partial_r g_{st} = g^{\cdot}$$

and

$$\left\{ {r \atop jr} \right\} = (1/2)\partial_j \log |g| = 0$$

so that

$$T^r{}_{jr} = B^r{}_{jr} + (1/2)g_j = (1/2)\Pi^Q{}_{jQ} = 0, \qquad g^{rs}T^i{}_{rs} = 0.$$

This leaves but two nonvanishing components in canonical coordinates, namely $\tilde{T}^1{}_{22}$ and $\tilde{T}^2{}_{11}$. The following theorem on the vanishing of $T^i{}_{jk}$ is due to Wilczynski [1, p. 260].

THEOREM. *The vanishing of the tensor $T^i{}_{jk}$ implies that the integrating surface is a ruled quadric.*

Proof. The conditions $T^i{}_{jk}=0$ reduce in canonical coordinates to $\tilde{B}^2{}_{11}=\tilde{B}^1{}_{22}=0$. The canonical system (8.1) reduces to the pair of differential equations $\tilde{y}_{11}=\tilde{C}_{11}\tilde{y}$, $\tilde{y}_{22}=\tilde{C}_{22}\tilde{y}$. If $\phi(u^1)$ and $\psi(u^1)$ are two particular solutions of the first of these equations, any set of four particular solutions will be of the form $\tilde{y}^A(u^1)=a^A\phi+b^A\psi$, where a^A and b^A are arbitrary sets of constants. $y^A(u^1)$ thus defines the line joining the two points a^A and b^A and the parametric curves $u^2=$constant consist of straight lines. Similarly the curves $u^1=$constant are straight lines, but the only surface with two sets of straight line rulings is a quadric.

The tensor $T^i{}_{jk}$ and its relative invariant of weight one,

$$\theta = g^{tu}T^r{}_{st}T^s{}_{ru}, \qquad \bar{\theta} = \left|\partial u/\partial \bar{u}\right| \theta, \tag{11.3}$$

with the canonical form

$$\tilde{\theta} = 2\tilde{B}^2{}_{11}\tilde{B}^1{}_{22} \tag{11.4}$$

will be of outstanding importance in the invariant theory of the system. The relations

$$T^{jrs}T_{rsk} = (1/2)\delta^j{}_k\theta \tag{11.5}$$

should be noted. The vanishing of θ implies that either $\tilde{B}^2{}_{11}=0$, $\tilde{B}^1{}_{22}=0$ or both. In any case, the integrating surface is then ruled. Since Wilczynski has developed the ruled surface theory by an entirely different method, we shall henceforth restrict our discussion to non-ruled surfaces for which $\theta\neq 0$.

The forms $\phi=g_{rs}du^rdu^s$, $\psi=T_{rst}du^rdu^sdu^t$, reducing in canonical coordinates to $\tilde{\phi}=2d\tilde{u}^1d\tilde{u}^2$, $\tilde{\psi}=\tilde{T}_{111}(d\tilde{u}^1)^3+\tilde{T}_{222}(d\tilde{u}^2)^3$, may be recognized at once as the two ground forms ϕ and ψ of Blaschke [3, pp. 121–125]. The three null directions of $\psi=0$ are the three tangents of Darboux. Our relative invariant θ is the invariant J of Blaschke which he has called the "Pick invariant" and he has shown that $J=0$ characterizes a ruled surface.

12. **Covariant differentiation.** The Christoffel symbols, $\lambda^i{}_{jk}$, formed from the absolute covariant tensor θg_{jk} are

$$\begin{aligned}\lambda^i{}_{jk} &= (1/2)\theta^{-1}g^{ir}[\partial_j(\theta g_{kr}) + \partial_k(\theta g_{jr}) - \partial_r(\theta g_{jk})] \\ &= \left\{ {i \atop jk} \right\} + (1/2)(\delta^i{}_j\partial_k \log\theta + \delta^i{}_k\partial_j \log\theta - g_{jk}\partial^i \log\theta)\end{aligned} \tag{12.1}$$

and they transform under $u^i \to \bar{u}^i$ by the familiar law,

$$\bar{\lambda}^r{}_{jk}\bar{\partial}_r u^i = \lambda^i{}_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s + \bar{\partial}_{jk} u^i, \tag{12.2}$$

of metric differential geometry. Moreover, the components $\lambda^i{}_{jk}$ being built from seminvariants are unaffected by a transformation of the dependent variable and hence (12.1) is also their law of transformation under the most general transformation of the representation preserving the semi-canonical form as explained in §10.

Consider now any relative simple mixed tensor, $\bar{T}^r{}_j\bar{\partial}_r u^i = |\partial u/\partial \bar{u}|^w T^i{}_r \bar{\partial}_j u^r$ (of weight w). On differentiating with respect to $\bar{u}^k$ and applying

$$\bar{\theta} = |\partial u/\partial \bar{u}|\,\theta \ (w = 1); \quad \bar{\partial}_j \log \bar{\theta} - \partial_r \log \theta \bar{\partial}_j u^r = \bar{\partial}_j \log |\partial u/\partial \bar{u}| \tag{12.3}$$

we obtain $\bar{T}^r{}_{j,k}\bar{\partial}_r u^i = |\partial u/\partial \bar{u}|^w T^i{}_{r,s}\bar{\partial}_j u^r \bar{\partial}_k u^s$ (of weight w), where

$$T^i{}_{j,k} = \partial_k T^i{}_j + T^r{}_j \lambda^i{}_{rk} - T^i{}_r \lambda^r{}_{jk} - wT^i{}_j \partial_k \log \theta \quad (T^i{}_j \text{ of weight } w). \tag{12.4}$$

The generalization of this formula to tensors of any rank is obvious. When applied to the relative invariant θ of weight 1, this gives $\theta_{,k} = \partial_k \theta - \theta \partial_k \log \theta = 0$. Similarly, remembering that the covariant derivative of the absolute tensor θg_{jk} with respect to its own Christoffel symbols must necessarily vanish, $(\theta g_{jk})_{,l} = \theta g_{jk,l} = 0$, we see that likewise $g_{jk,l} = 0$. It may be verified that the relative tensor ϵ^{jk} of weight 1 also has this property, $\epsilon^{jk}{}_{,l} = 0$. Summarizing, it may be said that θ, g_{jk} and ϵ^{jk} behave as constants under covariant differentiation.

Perhaps an indication here of the reason behind the choice of θg_{jk} as the basis for covariant differentiation may be appropriate. It should be recalled from the transformation law (7.1) of g_{jk} that this tensor is of weight -1. Hence the transformation law of its Christoffel symbols as given at the beginning of §11 is complicated and unsuited for covariant differentiation. An affine connection with the desired transformation law (12.2) essential for covariant differentiation can be constructed from Christoffel symbols only when the basic tensor is of weight 0. But such a tensor may be formed from g_{jk} of weight -1 by multiplication with a relative scalar of weight 1. Since θ is such a scalar, and the *unique* one of lowest differential order as will be shown in §17, it becomes the simplest choice for the required multiplier.

It may be noticed that the covariant differentiation introduced in this section is formally precisely that used in the Riemannian geometry of a two-dimensional surface where the affine connection consists of Christoffel symbols formed from a covariant tensor of weight 0. In particular, no use has been made of the type of covariant differentiation of projective vectors introduced by O. Veblen [4, p. 19], for Veblen's method assigns to an S_n but one more formal dimension to allow for gauge transformations analogous to our factor of homogeneity $y = e^{\phi} y^*$. We, on the other hand, are concerned

with the geometry of a *subspace S_2 immersed in S_3*, so that with indices for S_3 covering the homogeneous range 0, 1, 2, ∞ our formalism assigns not one but rather two extra indices to the geometric objects of the two-dimensional space to be studied. It will appear in §25 that under $u^i \to \bar{u}^i$ two of these indices, 0 and ∞, are invariant in character while the remaining pair, 1 and 2, are tensor in character. Another essential difference beyond that of formal dimensionality between our method and Veblen's is that in our development the gauge transformation $y = e^{\phi} y^*$ has been completely removed from the picture by constructing the invariant framework from quantities unaltered by these transformations whereas Veblen's theory is designed to retain the gauge transformations.

13. **The intrinsic relative reference frame and the projective normal.** It was seen in §10 that the relative semicovariants Y_A transform under the most general change of representation preserving the semi-canonical form by $Y_A = e^{\psi} \mu^Q{}_B \overline{Y}_Q$ with $\|\mu^A{}_B\|$ given by (10.1) wherein ϕ is replaced by $\psi = \log |\partial u/\partial \bar{u}|^{1/2}$. Defining a *relative covariant* as a function $C(\Pi, Y)$ of seminvariants and semicovariants and their derivatives which transforms under $Y_A \to \overline{Y}_A$ by the law $C(\Pi, Y) = |\partial u/\partial \bar{u}|^w C(\overline{\Pi}, \overline{Y})$, and similarly a *relative vector covariant* as an entity with components $V_j(\Pi, Y)$ transforming by $V_j(\Pi, Y) = |\partial u/\partial \bar{u}|^w V_r(\overline{\Pi}, \overline{Y}) \partial_j \bar{u}^r$, we seek in this section a pair of covariants and a vector covariant which will be analogous to the semicovariants Y_0, Y_∞ and Y_j respectively.

First, $Y_0 = |\partial u/\partial \bar{u}|^{1/2} \overline{Y}_0$ shows that Y_0 is a relative covariant of weight 1/2. Second, the elimination of $\partial_j \psi$ from $Y_j = |\partial u/\partial \bar{u}|^{1/2} (\partial_j \psi \overline{Y}_0 + \overline{Y}_r \partial_j \bar{u}^r)$ by means of $\partial_j \psi = (1/2) \bar{\partial}_r \log \bar{\theta} \partial_j \bar{u}^r - (1/2) \partial_j \log \theta$ resulting from (12.3) yields

$$Y_j + (1/2) \partial_j \log \theta Y_0 = |\partial u/\partial \bar{u}|^{1/2} (\overline{Y}_r + (1/2) \bar{\partial}_r \log \bar{\theta}) \partial_j \bar{u}^r.$$

The derivation of the last covariant from $Y_\infty = |\partial u/\partial \bar{u}|^{1/2} [(1/2) g^{rs} (\psi_r \psi_s + \psi_{rs}) \overline{Y}_0 + g^{rs} ((1/2) \partial_{rs} \bar{u}^t + \psi_r \partial_s \bar{u}^t) \overline{Y}_t + |\partial u/\partial \bar{u}|^{-1} \overline{Y}_\infty]$ is more detailed. We need the transformation law of

$$g^i = g^{rs} \begin{Bmatrix} i \\ rs \end{Bmatrix}$$

as given by (11.1), $g^{rs} \partial_{rs} \bar{u}^i = g^r \partial_r \bar{u}^i - |\partial \bar{u}/\partial u| \bar{g}^i$. Then

$$\begin{aligned} g^{rs}((1/2) \partial_{rs} \bar{u}^t &+ \psi_r \partial_s \bar{u}^t) \overline{Y}_t \\ &= (1/2)(g^r - \partial^r \log \theta) \partial_r \bar{u}^t \overline{Y}_t - (1/2) |\partial \bar{u}/\partial u| (\bar{g}^t - \bar{\partial}^t \log \bar{\theta}) \overline{Y}_t \\ &= (1/2)(g^r - \partial^r \log \theta) \partial_r \bar{u}^t (\overline{Y}_t + (1/2) \bar{\partial}_t \log \bar{\theta} \overline{Y}_0) \\ &\quad - (1/2) |\partial \bar{u}/\partial u| (\bar{g}^t - \bar{\partial}^t \log \bar{\theta})(\overline{Y}_t + (1/2) \bar{\partial}_t \log \bar{\theta} \overline{Y}_0) \\ &\quad - (1/4)(g^r - \partial^r \log \theta) \partial_r \bar{u}^t \bar{\partial}_t \log \bar{\theta} \overline{Y}_0 \\ &\quad + (1/4) |\partial \bar{u}/\partial u| (\bar{g}^t - \bar{\partial}^t \log \bar{\theta}) \bar{\partial}_t \log \bar{\theta} \overline{Y}_0, \end{aligned}$$

$$(1/2)g^{rs}\psi_r\psi_s = (1/8)(\partial_r \log \theta \partial^r \log \theta - |\partial\bar{u}/\partial u|\, \bar{\partial}_r \log \bar{\theta}\bar{\partial}^r \log \bar{\theta})$$
$$+ (1/4)(|\partial\bar{u}/\partial u|\, \bar{\partial}_r \log \bar{\theta}\bar{\partial}^r \log \bar{\theta} - \partial^r \log \theta \partial_r \log \bar{\theta}),$$
$$(1/2)g^{rs}\psi_{rs} = (1/4)[\,|\partial\bar{u}/\partial u|\, \bar{g}^{rs}\bar{\partial}_{rs} \log \bar{\theta} - g^{rs}\partial_{rs} \log \theta$$
$$+ \bar{\partial}_r \log \bar{\theta}(g^s\partial_s\bar{u}^r - |\partial\bar{u}/\partial u|\, \bar{g}^r)].$$

On substituting these into the expression for Y_∞ and rearranging we find

$$(1/4)g^{rs}(\partial_{rs} \log \theta - (1/2)\partial_r \log \theta \partial_s \log \theta)Y_0$$
$$- (1/2)(g^r - \partial^r \log \theta)(Y_r + (1/2)\partial_r \log \theta Y_0) + Y_\infty$$
$$= |\partial\bar{u}/\partial u|^{1/2}[(1/4)\bar{g}^{rs}(\bar{\partial}_{rs} \log \bar{\theta} - (1/2)\bar{\partial}_r \log \bar{\theta}\bar{\partial}_s \log \bar{\theta})\bar{Y}_0$$
$$- (1/2)(\bar{g}^r - \bar{\partial}^r \log \bar{\theta})(\bar{Y}_r + (1/2)\bar{\partial}_r \log \bar{\theta}\bar{Y}_0) + \bar{Y}_\infty].$$

Defining Z_A by

$$Z_0 = \theta^{1/2}Y_0, \qquad Z_j = \theta^{1/2}((1/2)\partial_j \log \theta Y_0 + \delta^r{}_j Y_r),$$
$$(13.1) \qquad Z_\infty = \theta^{1/2}[(1/4)g^{rs}(\partial_{rs} \log \theta + (1/2)\partial_r \log \theta \partial_s \log \theta - g_r\partial_s \log \theta)Y_0$$
$$+ (1/2)(\partial^r \log \theta - g^r)Y_r + Y_\infty],$$

we have $Z_B = e^{(1/2)\log\theta}N^A{}_B Y_A$, $Y_B = e^{-(1/2)\log\theta}\nu^A{}_B Z_A$, $N^A{}_Q\nu^Q{}_B = \delta^A{}_B$, where

$$(13.2) \qquad \begin{Vmatrix} N^0{}_0 & N^i{}_0 & N^\infty{}_0 \\ N^0{}_j & N^i{}_j & N^\infty{}_j \\ N^0{}_\infty & N^i{}_\infty & N^\infty{}_\infty \end{Vmatrix}$$
$$= \begin{Vmatrix} 1 & 0 & 0 \\ (1/2)\partial_j \log \theta & \delta^i{}_j & 0 \\ (1/4)g^{rs}(\partial_{rs} \log \theta + (1/2)\partial_r \log \theta \partial_s \log \theta - g_r\partial_s \log \theta) & (1/2)(\partial^i \log \theta - g^i) & 1 \end{Vmatrix},$$

$$(13.3) \qquad \begin{Vmatrix} \nu^0{}_0 & \nu^i{}_0 & \nu^\infty{}_0 \\ \nu^0{}_j & \nu^i{}_j & \nu^\infty{}_j \\ \nu^0{}_\infty & \nu^i{}_\infty & \nu^\infty{}_\infty \end{Vmatrix}$$
$$= \begin{Vmatrix} 1 & 0 & 0 \\ -(1/2)\partial_j \log \theta & \delta^i{}_j & 0 \\ (1/4)g^{rs}((1/2)\partial_r \log \theta \partial_s \log \theta - \partial_{rs} \log \theta) & -(1/2)(\partial^i \log \theta - g^i) & 1 \end{Vmatrix}.$$

The quantities Z_A transform by

$$(13.4) \qquad \bar{Z}_0 = Z_0, \qquad \bar{Z}_j = Z_r\bar{\partial}_j u^r, \qquad \bar{Z}_\infty = |\partial u/\partial\bar{u}|\, Z_\infty \quad (w = 1).$$

The combined effect of the successive transformations on y_A is the single composite operation $y_A \to Z_A$ as given by

$$y_B = \Gamma^{1/4}\lambda^Q{}_B(\phi_i = (1/4)\Gamma^R{}_{iR})Y_Q = \theta^{-1/2}\Gamma^{1/4}\lambda^Q{}_B\nu^R{}_Q Z_R$$

from whence $|y_A| = \theta^{-2}\Gamma|Z_A|$, so that $|Z_A| \neq 0$ is a consequence of $|y_A| \neq 0$. Under (13.4), $\overline{Z}_A = |\partial u/\partial \bar{u}|^2|Z_A|$, and consequently $|Z_A| \neq 0$ is an intrinsic property of the surface. Henceforth the vertices of the relative reference frame will be chosen as the points Z_A and since $Z_0 = \theta^{1/2}\Gamma^{-1/4}y$ this vertex is the point y itself while $Z_j = \theta^{1/2}\Gamma^{-1/4}[((1/2)\partial_j \log \theta - (1/4)\Gamma^Q{}_{jQ})y + y_j]$, being a linear combination of y and y_j, is, for given j, necessarily a point on the tangent to the parametric u^j curve. The three points Z_0 and Z_j determine the tangent plane at y. Since $|Z_A| \neq 0$, the fourth vertex Z_∞ is not incident with the tangent plane. Furthermore, the geometric invariance of the points Z_0 and Z_∞ assures the invariance of the line $L_{0\infty}$ joining them. Because this edge, $L_{0\infty}$, of the relative reference tetrahedron is intrinsically determined by the surface, it will be called the *projective normal* to the surface at the point y. For a geometric characterization of the projective normal see Lane (p. 93).

Now there are many lines through the point Z_0 of S and not in the tangent plane which have geometric significance and any one of these would be acceptable geometrically as an edge of the reference tetrahedron. The reader may then wonder why the formal tensor methods of this section have yielded that particular intrinsic line which has been commonly called the *projective normal*. Indeed, its identity as the projective normal does not become obvious until the canonical form of the underlying differential equations is displayed in §15. And yet not entirely accidental has been its derivation by the analytic approach of choosing linear combinations of the Y_A which would yield a reference frame with vertices transforming as geometric objects under $u^i \to \bar{u}^t$, for such a procedure will lead to some kind of a normal form for the differential equations and it is well known that the projective normal is an edge of the reference tetrahedron when the system has been put into Fubini's canonical form. The identification in §15 of our canonical form resulting from the processes of tensor analysis with Fubini's canonical form at the same time identifies our intrinsic line $L_{0\infty}$ with the projective normal.

14. **Covariant form of the differential equations; the third fundamental surface tensor.** Under the transformation $Y_B = e^{(-1/2)\log\theta}\nu^Q{}_B Z_Q$ of the previous section, the differential equations $\partial_j Y_A = \Pi^Q{}_{jA} Y_Q$ for the seminvariants Y_A are transformed into

$$\partial_j Z_A = E^Q{}_{jA} Z_Q, \tag{14.1}$$

where the new coefficients E are expressed in terms of the old coefficients Π by the formulas

$$E^B{}_{jA} = N^Q{}_A(\Pi^R{}_{jQ}\nu^B{}_R - \partial_j \nu^B{}_Q) + (1/2)\delta^B{}_A \partial_j \log \theta \tag{14.2}$$

analogous to (4.6) on setting $\phi_j = -(1/2)\partial_j \log \theta$. Expansion gives the two matrices for $j = 1, 2$:

$$(14.3)\qquad \left\| \begin{matrix} E^0{}_{j0} & E^i{}_{j0} & E^\infty{}_{j0} \\ E^0{}_{jk} & E^i{}_{jk} & E^\infty{}_{jk} \\ E^0{}_{j\infty} & E^i{}_{j\infty} & E^\infty{}_{j\infty} \end{matrix} \right\| = \left\| \begin{matrix} 0 & \delta^i{}_j & 0 \\ E^0{}_{jk} & T^i{}_{jk} + \lambda^i{}_{jk} & g_{jk} \\ E^0{}_{j\infty} & E^i{}_{j\infty} & \partial_j \log \theta \end{matrix} \right\|,$$

wherein

$$(14.4)\qquad \begin{aligned} E^0{}_{jk} = {} & C_{jk} - (1/2)B^r{}_{jk}\partial_r \log \theta \\ & + (1/4)g_{jk}g^{rs}((1/2)\partial_r \log \theta \partial_s \log \theta - \partial_{rs} \log \theta) \\ & - (1/2)((1/2)\partial_j \log \theta \partial_k \log \theta - \partial_{jk} \log \theta). \end{aligned}$$

The coefficients in the last line of the matrix may be read from (14.2) if desired, but it is preferable to express them in terms of the elements of the second line. This may be done by differentiating

$$\partial_i Z_j = E^0{}_{ij}Z_0 + (T^r{}_{ij} + \lambda^r{}_{ij})Z_r + g_{ij}Z_\infty$$

with respect to u^k and eliminating $\partial_{ik}Z_j$ by interchanging i and k and subtracting. Solving for $\partial_i Z_\infty$ will yield a solution of the form

$$\partial_i Z_\infty = E^0{}_{i\infty}Z_0 + E^r{}_{i\infty}Z_r + E^\infty{}_{i\infty}Z_\infty.$$

Now at the beginning of §4, a computation of this very same nature was performed for the system $\partial_j y_k = c_{jk}y_0 + b^r{}_{jk}y_r + g_{jk}y_\infty$ with the result $\partial_i y_\infty = \Gamma^0{}_{i\infty}y_0 + \Gamma^r{}_{i\infty}y_r + \Gamma^\infty{}_{i\infty}y_\infty$, the Γ's given by the last line of (4.2). To emphasize the analogy between the system of §4 with the system just following equation (14.4) we define coefficients γ_{ij} and $\beta^k{}_{ij}$ by

$$(14.5)\qquad \|\gamma_{ij}, \beta^k{}_{ij}, g_{ij}\| = \|E^0{}_{ij}, T^k{}_{ij} + \lambda^k{}_{ij}, g_{ij}\|,$$

so that the system of this section is now $\partial_i Z_j = \gamma_{ij}Z_0 + \beta^r{}_{ij}Z_r + g_{ij}Z_\infty$. Then the present analogy with (4.2) gives

$$(14.6)\qquad \|E^0{}_{i\infty}, E^k{}_{i\infty}, E^\infty{}_{i\infty}\| = \|g^{rs}\gamma_{ris}, \gamma^k{}_i - g^{st}\beta^k{}_{sti}, \beta^r{}_{ir}\|,$$

with an irregularity in the analogy between $E^\infty{}_{i\infty}$ and $\Gamma^\infty{}_{i\infty}$ which will be explained presently and where γ_{jik} and $\beta^k{}_{ij}$ are defined analogously to (4.1):

$$(14.7)\qquad \begin{aligned} \gamma_{ijk} &= (\partial_k\gamma_{ji} - \gamma_{jr}\beta^r{}_{ik} - \gamma_{ri}\beta^r{}_{jk}) - (\partial_j\gamma_{ki} - \gamma_{kr}\beta^r{}_{ij} - \gamma_{ri}\beta^r{}_{kj}), \\ \beta^l{}_{ijk} &= \partial_k\beta^l{}_{ji} - \partial_j\beta^l{}_{ki} + \beta^l{}_{kr}\beta^r{}_{ji} - \beta^l{}_{jr}\beta^r{}_{ki}. \end{aligned}$$

The derivation of $\Gamma^\infty{}_{i\infty}$ just preceding equations (4.2) was given by

$$\Gamma^\infty{}_{i\infty} = g^{rs}(\partial_s g_{ir} - \partial_i g_{sr} + g_{st}b^t{}_{ir} - g_{it}b^t{}_{sr}) = g_i + b^r{}_{ir}$$

in virtue of the identity $g^{rs}b^i{}_{rs} = 0$, but now

$$E^\infty{}_{i\infty} = g^{rs}(\partial_s g_{ir} - \partial_i g_{sr} + g_{st}\beta^t{}_{ir} - g_{it}\beta^t{}_{sr}) = \beta^r{}_{ir} = \partial_i \log \theta$$

since $g^{rs}\beta^i{}_{rs} = g^i$.

The vertices Z_A of the relative reference frame transform by $Z_B = \eta^A{}_B \bar{Z}_A$, $\bar{Z}_B = H^A{}_B Z_A$, $\eta^A{}_Q H^Q{}_B = \delta^A{}_B$, where

$$\left\| \begin{matrix} \eta^0{}_0 & \eta^i{}_0 & \eta^\infty{}_0 \\ \eta^0{}_j & \eta^i{}_j & \eta^\infty{}_j \\ \eta^0{}_\infty & \eta^i{}_\infty & \eta^\infty{}_\infty \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & \partial_j \bar{u}^i & 0 \\ 0 & 0 & |\partial \bar{u}/\partial u| \end{matrix} \right\|, \tag{14.8}$$

$$\left\| \begin{matrix} H^0{}_0 & H^i{}_0 & H^\infty{}_0 \\ H^0{}_j & H^i{}_j & H^\infty{}_j \\ H^0{}_\infty & H^i{}_\infty & H^\infty{}_\infty \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & \bar{\partial}_j u^i & 0 \\ 0 & 0 & |\partial u/\partial \bar{u}| \end{matrix} \right\|, \tag{14.9}$$

so that, by analogy with (7.4),

$$\bar{E}^B{}_{jA} = H^R{}_A (E^Q{}_{tR} \eta^B{}_Q - \partial_t \eta^B{}_R) \bar{\partial}_j u^t, \tag{14.10}$$

from whence $\bar{\gamma}_{jk} = \gamma_{rs} \bar{\partial}_j u^r \bar{\partial}_k u^s$. This tensor γ_{jk} will be called the *third fundamental surface tensor*. The identity $g^{rs}\gamma_{rs} = 0$ holds in all coordinate systems, but $g^{rs}\beta^k{}_{rs} = g^k$ vanishes only in canonical coordinates. Since the components $\beta^i{}_{jk}$ transform by the same law as $\lambda^i{}_{jk}$, the quantities γ_{ijk} and $\beta^l{}_{ijk}$ are tensors as are $E^0{}_{i\infty}$ and $E^k{}_{i\infty}$ likewise. This also follows from expansion of (14.10). The definitions (14.7) may now be written

$$\begin{aligned} \gamma_{ijk} &= \gamma_{ij,k} - \gamma_{ik,j} + \gamma_{kr} T^r{}_{ji} - \gamma_{jr} T^r{}_{ki}, \\ \beta^l{}_{ijk} &= \lambda^l{}_{ijk} + T^l{}_{ij,k} - T^l{}_{ik,j} + T^l{}_{kr} T^r{}_{ji} - T^l{}_{jr} T^r{}_{ki}, \end{aligned} \tag{14.11}$$

and the elements of the matrix (14.6) are

$$\begin{aligned} E^0{}_{i\infty} &= g^{rs}(\gamma_{ri,s} + \gamma_{rt} T^t{}_{is}), \\ E^k{}_{i\infty} &= g^{st}(-\lambda^k{}_{sti} + T^k{}_{si,t} + T^k{}_{sr} T^r{}_{it}) + \gamma^k{}_i, \\ E^\infty{}_{i\infty} &= \partial_i \log \theta. \end{aligned} \tag{14.12}$$

The system $\partial_j Z_A = E^Q{}_{jA} Z_Q$ will be said to be in *covariant form*.

15. **Fubini's canonical form.** Let the basic system be expressed in the covariant form of the preceding section and apply now a transformation of the independent variables to normalize the tensor g_{jk}. Since there will no longer be any occasion for referring to Wilczynski's canonical form as developed in §8, we shall henceforth employ the symbol $\sim$ to designate the canonical system of this section. Thus the basic system is now

$$\begin{aligned} \partial_{11}\tilde{Z}_0 &= \tilde{\gamma}_{11}\tilde{Z}_0 + \tilde{\lambda}^1{}_{11}\partial_1\tilde{Z}_0 + \tilde{T}_{111}\partial_2\tilde{Z}_0, \\ \partial_{12}\tilde{Z}_0 &= \tilde{Z}_\infty, \\ \partial_{22}\tilde{Z}_0 &= \tilde{\gamma}_{22}\tilde{Z}_0 + \tilde{T}_{222}\partial_1\tilde{Z}_0 + \tilde{\lambda}^2{}_{22}\partial_2\tilde{Z}_0, \end{aligned} \tag{15.1}$$

when we recall that the identities $T^r{}_{jr} = 0$ and $g^{rs}T^i{}_{rs} = 0$ result in

$$\tilde{T}^j{}_{kk} = \tilde{T}_{kkk} \neq 0 \qquad (j \neq k),$$

all other components vanishing, and that the definitions (12.1) of $\lambda^i{}_{jk}$ become $\tilde{\lambda}^i{}_{jk}=(1/2)(\delta^i{}_j\partial_k \log \tilde{\theta}+\delta^i{}_k\partial_j \log \tilde{\theta}-\tilde{g}_{jk}\partial^i \log \tilde{\theta})$ from whence

$$\tilde{\lambda}^j{}_{jj} = \partial_j \log \tilde{\theta}, \qquad \tilde{\theta} = 2\tilde{T}_{111}\tilde{T}_{222},$$

all other components vanishing. Here as elsewhere when evaluating in canonical coordinates, it becomes necessary to suspend the convention of summing on repeated indices. *Also $j\neq k$ will be understood in all canonical formulas of this section involving these two indices and the range of j and k will be the set* 1, 2.

The form (15.1) is now recognized to be Fubini's canonical form which Lane (p. 69) has written in essentially the notation

$$\begin{aligned} Z_{uu} &= pZ + (\log \theta)_u Z_u + \beta Z_v, \qquad \tilde{\gamma}_{11} = p, \qquad \tilde{T}_{111} = \beta, \\ Z_{vv} &= qZ + \gamma Z_u + (\log \theta)_v Z_v, \qquad \tilde{\gamma}_{22} = q, \qquad \tilde{T}_{222} = \gamma, \end{aligned} \tag{15.2}$$
$$\theta = 2\beta\gamma.$$

We shall understand henceforth by the expression *canonical form* the system (15.1).

With a minimum of introduction we record in this section the canonical form of tensors and related quantities that will be needed in sections to follow. From (14.4) defining $E^0{}_{jk}=\gamma_{jk}$ and from the identity $g^{rs}\gamma_{rs}=0$,

$$\begin{aligned} \tilde{\gamma}_{jj} &= \tilde{C}_{jj} - (1/2)\tilde{T}_{jjj}\partial_k \log \tilde{\theta} - (1/4)(\partial_j \log \tilde{\theta})^2 + (1/2)\partial_{jj} \log \tilde{\theta}, \\ \tilde{\gamma}_{12} &= 0. \end{aligned} \tag{15.3}$$

For the covariant derivative defined by

$$\begin{aligned} \gamma_{ij,k} &= \partial_k\gamma_{ij} - \gamma_{rj}\lambda^r{}_{ik} - \gamma_{ir}\lambda^r{}_{jk}, \\ \tilde{\gamma}_{jj,k} &= \partial_k\tilde{\gamma}_{jj}, \qquad \tilde{\gamma}_{jj,j} = \partial_j\tilde{\gamma}_{jj} - 2\tilde{\gamma}_{jj}\partial_j \log \tilde{\theta}, \end{aligned}$$

all other components vanish. For the "curvature tensor" of θg_{jk},

$$\begin{aligned} \lambda^l{}_{ijk} &= \partial_k\lambda^l{}_{ji} - \partial_j\lambda^l{}_{ki} + \lambda^l{}_{kr}\lambda^r{}_{ji} - \lambda^l{}_{jr}\lambda^r{}_{ki}, \\ \tilde{\lambda}^j{}_{jjk} &= \partial_{jk} \log \tilde{\theta}, \end{aligned} \tag{15.4}$$

all other components vanish. Defining $\lambda=g^{st}\lambda^r{}_{str}$, a reduction to canonical form verifies

$$g^{rs}\lambda_{rjks} = g^{rs}\lambda_{jrsk} = (1/2)g_{jk}\lambda, \qquad \tilde{\lambda} = -2\partial_{12} \log \tilde{\theta} \qquad (w = 1). \tag{15.5}$$

For the covariant derivatives

$$T^i{}_{jk,l} = \partial_l T^i{}_{jk} + T^r{}_{jk}\lambda^i{}_{rl} - T^i{}_{rk}\lambda^r{}_{jl} - T^i{}_{jr}\lambda^r{}_{kl},$$

the only nonvanishing components are

$$\begin{aligned} \tilde{T}^j{}_{kk,j} &= \tilde{T}_{kkk,j} = \partial_j\tilde{T}_{kkk} + \tilde{T}_{kkk}\partial_j \log \tilde{\theta}, \\ \tilde{T}^j{}_{kk,k} &= \tilde{T}_{kkk,k} = \partial_k\tilde{T}_{kkk} - 2\tilde{T}_{kkk}\partial_k \log \tilde{\theta}. \end{aligned} \tag{15.6}$$

From (14.12):

$$
\begin{aligned}
\tilde{E}^0{}_{j\infty} &= \tilde{\gamma}_{jj,k} + \tilde{\gamma}_{kk}\tilde{T}_{jjj}, \\
\tilde{E}^j{}_{j\infty} &= \partial_{12} \log \tilde{\theta} + (1/2)\tilde{\theta}, \\
\tilde{E}^k{}_{j\infty} &= \tilde{T}_{jjj,k} + \tilde{\gamma}_{jj}.
\end{aligned} \tag{15.7}
$$

For the covariant derivatives

$$
\begin{aligned}
E^0{}_{j\infty,k} &= \partial_k E^0{}_{j\infty} - E^0{}_{r\infty}\lambda^r{}_{jk} - E^0{}_{j\infty}\partial_k \log \theta, \\
E^i{}_{j\infty,k} &= \partial_k E^i{}_{j\infty} + E^r{}_{j\infty}\lambda^i{}_{rk} - E^i{}_{r\infty}\lambda^r{}_{jk} - E^i{}_{j\infty}\partial_k \log \theta,
\end{aligned}
$$

we shall be interested only in

$$
\begin{aligned}
\tilde{E}^0{}_{j\infty,k} &= \partial_k \tilde{E}^0{}_{j\infty} - \tilde{E}^0{}_{j\infty}\partial_k \log \tilde{\theta}, \\
\tilde{E}^j{}_{j\infty,k} &= \partial_k \tilde{E}^j{}_{j\infty} - \tilde{E}^j{}_{j\infty}\partial_k \log \tilde{\theta}, \qquad \tilde{E}^k{}_{j\infty,k} = \partial_k \tilde{E}^k{}_{j\infty}.
\end{aligned} \tag{15.8}
$$

16. **Tensor form of the differential equations and their integrability conditions.** The differential equations satisfied by the vertices of the relative reference tetrahedron have the canonical form

$$
\begin{aligned}
&(1) \qquad \partial_1 \tilde{Z}_0 = \tilde{Z}_1, \\
&(2) \qquad \partial_2 \tilde{Z}_0 = \tilde{Z}_2, \\
&(3) \qquad \partial_1 \tilde{Z}_1 = \tilde{\gamma}_{11}\tilde{Z}_0 + \partial_1 \log \tilde{\theta}\tilde{Z}_1 + \tilde{T}_{111}\tilde{Z}_2, \\
&(4) \qquad \partial_2 \tilde{Z}_1 = \tilde{Z}_\infty, \\
&(5) \qquad \partial_1 \tilde{Z}_2 = \tilde{Z}_\infty, \\
&(6) \qquad \partial_2 \tilde{Z}_2 = \tilde{\gamma}_{22}\tilde{Z}_0 + \tilde{T}_{222}\tilde{Z}_1 + \partial_2 \log \tilde{\theta}\tilde{Z}_2, \\
&(7) \qquad \partial_1 \tilde{Z}_\infty = \tilde{E}^0{}_{1\infty}\tilde{Z}_0 + \tilde{E}^1{}_{1\infty}\tilde{Z}_1 + \tilde{E}^2{}_{1\infty}\tilde{Z}_2 + \partial_1 \log \tilde{\theta}\tilde{Z}_\infty, \\
&(8) \qquad \partial_2 \tilde{Z}_\infty = \tilde{E}^0{}_{2\infty}\tilde{Z}_0 + \tilde{E}^1{}_{2\infty}\tilde{Z}_1 + \tilde{E}^2{}_{2\infty}\tilde{Z}_2 + \partial_2 \log \tilde{\theta}\tilde{Z}_\infty,
\end{aligned}
$$

where the values of $\tilde{E}^A{}_{j\infty}$ are as listed in the preceding section. The integrability conditions of (1) and (2) are satisfied identically by (4) and (5), those of (3) and (4) by (7) and those of (5) and (6) by (8). Finally from

$$
\begin{aligned}
\partial_2(\partial_1 \tilde{Z}_\infty) = {} & (\tilde{\gamma}_{22}\tilde{E}^2{}_{1\infty} + \tilde{E}^0{}_{2\infty}\partial_1 \log \tilde{\theta} + \partial_2 \tilde{E}^0{}_{1\infty})\tilde{Z}_0 \\
& + (\tilde{T}_{222}\tilde{E}^2{}_{1\infty} + \tilde{E}^1{}_{2\infty}\partial_1 \log \tilde{\theta} + \partial_2 \tilde{E}^1{}_{1\infty})\tilde{Z}_1 \\
& + (\tilde{E}^0{}_{1\infty} + \tilde{E}^2{}_{1\infty}\partial_2 \log \tilde{\theta} + \tilde{E}^2{}_{2\infty}\partial_1 \log \tilde{\theta} + \partial_2 \tilde{E}^2{}_{1\infty})\tilde{Z}_2 \\
& + (\tilde{E}^1{}_{1\infty} + \partial_1 \log \tilde{\theta}\partial_2 \log \tilde{\theta} + \partial_{12} \log \tilde{\theta})\tilde{Z}_\infty, \\
\partial_1(\partial_2 \tilde{Z}_\infty) = {} & (\tilde{\gamma}_{11}\tilde{E}^1{}_{2\infty} + \tilde{E}^0{}_{1\infty}\partial_2 \log \tilde{\theta} + \partial_1 \tilde{E}^0{}_{2\infty})\tilde{Z}_0 \\
& + (\tilde{E}^0{}_{2\infty} + \tilde{E}^1{}_{2\infty}\partial_1 \log \tilde{\theta} + \tilde{E}^1{}_{1\infty}\partial_2 \log \tilde{\theta} + \partial_1 \tilde{E}^1{}_{2\infty})\tilde{Z}_1 \\
& + (\tilde{T}_{111}\tilde{E}^1{}_{2\infty} + \tilde{E}^2{}_{1\infty}\partial_2 \log \tilde{\theta} + \partial_1 \tilde{E}^2{}_{2\infty})\tilde{Z}_2 \\
& + (\tilde{E}^2{}_{2\infty} + \partial_1 \log \tilde{\theta}\partial_2 \log \tilde{\theta} + \partial_{12} \log \tilde{\theta})\tilde{Z}_\infty
\end{aligned}
$$

the integrability conditions of (7) and (8) are

$$(16.1)\quad \begin{aligned}(\partial_2\tilde{E}^0{}_1 - \tilde{E}^0{}_1\partial_2 \log\tilde{\theta} + \tilde{\gamma}_{22}\tilde{E}^2{}_{1\infty}) - (\partial_1\tilde{E}^0{}_{2\infty} - \tilde{E}^0{}_{2\infty}\partial_1 \log\tilde{\theta} + \tilde{\gamma}_{11}\tilde{E}^1{}_{2\infty}) &= 0,\\ (\partial_j\tilde{E}^j{}_{k\infty} + \tilde{E}^0{}_{k\infty}) - (\partial_k\tilde{E}^j{}_{j\infty} - \tilde{E}^j{}_{j\infty}\partial_k \log\tilde{\theta} + \tilde{T}_{kkk}\tilde{E}^k{}_{j\infty}) &= 0.\end{aligned}$$

These three equations comprise the nonvanishing integrability conditions of the system in canonical form. We proceed next to their tensor formulation.

On forming the covariant derivatives, as defined in the case of a mixed tensor by (12.4), of the vertices Z_A, the equations $\partial_j Z_A = E^Q{}_{jA}Z_Q$ become

$$(16.2)\qquad Z_{0,j} = Z_j;\quad Z_{j,k} = \gamma_{jk}Z_0 + T^r{}_{jk}Z_r + g_{jk}Z_\infty;\quad Z_{\infty,k} = E^0{}_{k\infty}Z_0 + E^r{}_{k\infty}Z_r.$$

When so written, the equations will be said to be in *tensor form.* To arrive at the integrability conditions of the system we first develop a familiar set of identities existing between the second covariant derivatives of a relative vector T_i of weight w transforming by $T_i = |\partial u/\partial\bar{u}|^w T_r\bar{\partial}_i u^r$. From (12.4),

$$\begin{aligned}T_{i,j} &= \partial_j T_i - T_r\lambda^r{}_{ij} - wT_i\lambda^r{}_{rj},\\ T_{i,j,k} &= (\partial_k\partial_j T_i - T_r\partial_k\lambda^r{}_{ij} - wT_i\partial_k\lambda^r{}_{rj})\\ &\quad - \lambda^r{}_{ij}(T_{r,k} + T_s\lambda^s{}_{rk} + wT_r\lambda^s{}_{sk})\\ &\quad - w\lambda^r{}_{rj}(T_{i,k} + T_s\lambda^s{}_{ik} + wT_i\lambda^s{}_{sk})\\ &\quad - T_{r,j}\lambda^r{}_{ik} - T_{i,r}\lambda^r{}_{jk} - wT_{i,j}\lambda^r{}_{rk}.\end{aligned}$$

Then the integrability conditions $\partial_k\partial_j T_i - \partial_j\partial_k T_i = 0$ are equivalent to

$$T_{i,j,k} - T_{i,k,j} = -T_t\lambda^t{}_{ijk} - wT_i\lambda^r{}_{rjk}.$$

But from (15.4), $\tilde{\lambda}^r{}_{rjk} = \tilde{\lambda}^j{}_{jjk} + \tilde{\lambda}^k{}_{kjk} = \partial_{jk}\log\tilde{\theta} - \partial_{jk}\log\tilde{\theta} = 0$, and so in all coordinates $\lambda^r{}_{rjk} = 0$, giving

$$(16.3)\qquad T_{i,j,k} - T_{i,k,j} = -T_r\lambda^r{}_{ijk},\qquad I_{,j,k} - I_{,k,j} = 0,$$

for a covariant vector of weight w and an invariant of weight w respectively. The integrability conditions of (16.2) are thus obtained from

$$\begin{aligned}Z_{0,j,k} &= \gamma_{jk}Z_0 + T^r{}_{jk}Z_r + g_{jk}Z_\infty,\\ Z_{i,j,k} &= (\gamma_{ij,k} + T^r{}_{ij}\gamma_{rk} + g_{ij}E^0{}_{k\infty})Z_0\\ &\quad + (\gamma_{ij}\delta^r{}_k + T^s{}_{ij}T^r{}_{sk} + T^r{}_{ij,k} + g_{ij}E^r{}_{k\infty})Z_r + g_{rk}T^r{}_{ij}Z_\infty,\\ Z_{\infty,j,k} &= (E^0{}_{j\infty,k} + E^r{}_{j\infty}\gamma_{rk})Z_0\\ &\quad + (E^0{}_{j\infty}\delta^r{}_k + E^s{}_{j\infty}T^r{}_{sk} + E^r{}_{j\infty,k})Z_r + g_{rk}E^r{}_{j\infty}Z_\infty,\end{aligned}$$

by insisting that equations of the type (16.3) be satisfied identically in Z_A. They are

$$(16.4)\quad \begin{aligned}&I_{jkl} - I_{jlk} = 0, && I^i{}_{jkl} - I^i{}_{jlk} = 0, && T_{jkl} - T_{jlk} = 0,\\ &E_{lk\infty} - E_{kl\infty} = 0, && I_{kl} - I_{lk} = 0, && I^i{}_{kl} - I^i{}_{lk} = 0,\end{aligned}$$

where

$$
\begin{aligned}
I_{jkl} &= \gamma_{jk,l} + T^r{}_{jk}\gamma_{rl} + g_{jk}E^0{}_{l\infty}, \\
I^i{}_{jkl} &= T^i{}_{jk,l} + T^r{}_{jk}T^i{}_{rl} + \gamma_{jk}\delta^i{}_l + (1/3)(\lambda^i{}_{jkl} + \lambda^i{}_{kjl}) + g_{jk}E^i{}_{l\infty}, \\
T_{jkl} &= T^r{}_{jk}g_{rl}, \qquad E_{lk\infty} = g_{lr}E^r{}_{k\infty}, \\
I_{kl} &= E^0{}_{k\infty,l} + E^r{}_{k\infty}\gamma_{rl}, \qquad I^i{}_{kl} = E^i{}_{k\infty,l} + E^r{}_{k\infty}T^i{}_{rl} + E^0{}_{k\infty}\delta^i{}_l.
\end{aligned} \tag{16.5}
$$

Thus the integrability conditions express the complete symmetry of all these tensors in their lower indices. The relations

$$
\begin{aligned}
(1/3)(\lambda^i{}_{jkl} + \lambda^i{}_{kjl}) - (1/3)(\lambda^i{}_{jlk} + \lambda^i{}_{ljk}) \\
= (1/3)(\lambda^i{}_{kjl} + \lambda^i{}_{jlk} + \lambda^i{}_{lkj}) + \lambda^i{}_{jkl} = \lambda^i{}_{jkl}
\end{aligned}
$$

result from the fundamental identities $\lambda^i{}_{jkl}+\lambda^i{}_{jlk}=0$ and $\lambda^i{}_{kjl}+\lambda^i{}_{jlk}+\lambda^i{}_{lkj}=0$ of a curvature tensor. Of these 12 integrability conditions we shall now show that the first 9 vanish identically, leaving only the last 3 which express the symmetry of I_{kl} and $I^i{}_{kl}$. Namely, the two conditions $I_{jkl}-I_{jlk}=0$ on contraction with g^{jl} yield the two equivalent conditions $g^{jl}(\gamma_{jk,l}+\gamma_{lr}T^r{}_{jk})-E^0{}_{k\infty}=0$ which are satisfied identically by the definitions (14.12) of $E^0{}_{k\infty}$. Similarly the four conditions $I^i{}_{jkl}-I^i{}_{jlk}=0$ on contraction with g^{jl} yield the four equivalent conditions $g^{jl}(T^i{}_{jk,l}+T^r{}_{jk}T^i{}_{rl}+\lambda^i{}_{jkl})+\gamma^i{}_k-E^i{}_{k\infty}=0$ which are likewise satisfied identically by the definitions (14.12) of $E^i{}_{k\infty}$. The symmetry conditions $T_{jkl}-T_{jlk}=0$ may be verified from the observation just following equations (11.3) and $E_{lk\infty}-E_{kl\infty}=0$ may be seen from the second of (15.7) to hold in canonical coordinates.

It remains to compare the canonical forms of the last two sets, namely

$$
\begin{aligned}
(\tilde{E}^0{}_{1\infty,2} + \tilde{E}^2{}_{1\infty}\tilde{\gamma}_{22}) - (\tilde{E}^0{}_{2\infty,1} + \tilde{E}^1{}_{2\infty}\tilde{\gamma}_{11}) &= 0, \\
(\tilde{E}^j{}_{k\infty,j} + \tilde{E}^0{}_{k\infty}) - (\tilde{E}^j{}_{j\infty,k} + \tilde{E}^k{}_{j\infty}\tilde{T}_{kkk}) &= 0,
\end{aligned}
$$

and identify them with (16.1) by referring to (15.8).

17. **Preliminary enumeration of invariants by the Lie theory.** We now consider the differential equations

$$
\begin{aligned}
\partial_k Z_0 = Z_k, \qquad \partial_j Z_k &= \gamma_{jk}Z_0 + (T^r{}_{jk} + \lambda^r{}_{jk})Z_r + g_{jk}Z_\infty, \\
\partial_j Z_\infty &= E^0{}_{j\infty}Z_0 + E^r{}_{j\infty}Z_r,
\end{aligned} \tag{17.1}
$$

and ask first for the number of absolute invariants which are functions of the coefficients g_{jk} and T_{ijk}. This number will be found by applying Lie's theory of continuous groups. Namely, under the infinitesimal transformation $\bar{u}^i=u^i+\epsilon\xi^i(u)$ and its inverse $u^i=\bar{u}^i-\epsilon\xi^i(\bar{u})$, the transformation laws $\bar{g}_{jk}=|\partial\bar{u}/\partial u|\, g_{rs}\bar{\partial}_j u^r\bar{\partial}_k u^s$ and $\bar{T}_{ijk}=|\partial\bar{u}/\partial u|\, T_{rst}\bar{\partial}_i u^r\bar{\partial}_j u^s\bar{\partial}_k u^t$ reduce to

$$
\begin{aligned}
\delta g_{jk} &= \epsilon(g_{jk}\delta^r{}_s - g_{sk}\delta^r{}_j - g_{js}\delta^r{}_k)\partial_r\xi^s, \\
\delta T_{ijk} &= \epsilon(T_{ijk}\delta^r{}_s - T_{sjk}\delta^r{}_i - T_{isk}\delta^r{}_j - T_{ijs}\delta^r{}_k)\partial_r\xi^s.
\end{aligned}
$$

Any absolute invariant $I(g, T)$ must satisfy $\delta I=0$ for all $\partial_j\xi^k$, the conditions being

$$X^\alpha{}_\beta I = \lambda^\alpha{}_{\beta jk}(\partial I/\partial g_{jk}) + \lambda^\alpha{}_{\beta ijk}(\partial I/\partial T_{ijk}) = 0,$$

where

$$\begin{aligned} \lambda^\alpha{}_{\beta jk} &= g_{jk}\delta^\alpha{}_\beta - g_{\beta k}\delta^\alpha{}_j - g_{j\beta}\delta^\alpha{}_k, \\ \lambda^\alpha{}_{\beta ijk} &= T_{ijk}\delta^\alpha{}_\beta - T_{\beta jk}\delta^\alpha{}_i - T_{i\beta k}\delta^\alpha{}_j - T_{ij\beta}\delta^\alpha{}_k. \end{aligned} \tag{17.2}$$

From the general theory of such a system, these equations will be completely integrable if the commutator $X^\alpha{}_\beta(X^\gamma{}_\delta I) - X^\gamma{}_\delta(X^\alpha{}_\beta I)$ is a linear combination of $X^\alpha{}_\beta I$. Computation yields $X^\alpha{}_\beta(X^\gamma{}_\delta I) - X^\gamma{}_\delta(X^\alpha{}_\beta I) = \delta^\gamma{}_\beta X^\alpha{}_\delta I - \delta^\alpha{}_\delta X^\gamma{}_\beta I$ and hence the system is completely integrable. Its arguments are the 3 quantities g_{ij} and the 4 quantities T_{ijk} satisfying the 3 identities $|g_{ij}| = -1$, $g^{rs}T_{irs} = 0$. Thus the system consists of 4 equations in 4 independent variables, g_{11}, g_{22}, T_{111} and T_{222}. Moreover, the 4 equations are independent as may be seen by expanding the matrix $\| -2g_{\beta j}\delta^\alpha{}_j,\ T_{jjj}\delta^\alpha{}_\beta - 3T_{\beta jj}\delta^\alpha{}_j \|$ of coefficients of the derivatives of I with respect to the independent variables in canonical coordinates. For its determinant we find

$$\begin{vmatrix} 0 & 0 & -2\tilde{T}_{111} & \tilde{T}_{222} \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \tilde{T}_{111} & -2\tilde{T}_{222} \end{vmatrix} = 6\tilde{\theta} \neq 0.$$

The general theory of linear homogeneous partial differential equations then states that a completely integrable system of q independent equations in N independent variables admits $N-q$ independent solutions. In this case, therefore, there is no absolute invariant which is a function of only g_{jk} and T_{ijk}. *We have already found one relative invariant, $\theta = T_{rst}T^{rst}$, and this we now know is the only such relative invariant depending on these arguments.*

We must next inquire into the number of absolute invariants which are functions of g_{jk}, T_{ijk} and $T_{ijk,l}$, then into the number of those which are functions of g_{jk}, T_{ijk}, $T_{ijk,l}$ and $T_{ijk,l,m}$ and so on. At each adjunction of a higher order covariant derivative it would become necessary to demonstrate anew the complete integrability of the system defining the invariants. Direct computation analogous to that at the beginning of this section would show that if $T_{ij\ldots l}$ be any covariant tensor, the system $X^\alpha{}_\beta I = 0$ defining its invariants obeys the commutator identities $X^\alpha{}_\beta(X^\gamma{}_\delta I) - X^\gamma{}_\delta(X^\alpha{}_\beta I) = \delta^\gamma{}_\beta X^\alpha{}_\delta I - \delta^\alpha{}_\delta X^\gamma{}_\beta I$. We omit this computation since Lie [*Über Differentialinvarianten*, Math. Ann. vol. 24 (1884) p. 566] has shown that the system of partial differential equations defining the invariants of an infinite group of the type considered here is always completely integrable.

18. **Enumeration of invariants of g_{jk}, T_{ijk} and its covariant derivatives.** Having found in the preceding section that there is but one relative invariant, namely θ, which is a function of g_{jk} and T_{ijk} only, we proceed to the deter-

mination of all absolute invariants which are functions of the tensors g_{jk}, T_{ijk}, and $T_{ijk,l}$, all relative of weight -1. The resulting system of partial differential equations whose solutions yield all the desired invariants is

$$X_1{}^\alpha{}_\beta I = \lambda^\alpha{}_{\beta k_1 k_2}(\partial I/\partial g_{k_1 k_2}) + \lambda^\alpha{}_{\beta k_1 k_2 k_3}(\partial I/\partial T_{k_1 k_2 k_3}) + \lambda^\alpha{}_{\beta k_1 k_2 k_3 k_4}(\partial I/\partial T_{k_1 k_2 k_3,k_4}) = 0$$

where $\lambda^\alpha{}_{\beta k_1 k_2}$ and $\lambda^\alpha{}_{\beta k_1 k_2 k_3}$ have been defined by (17.2) and $\lambda^\alpha{}_{\beta k_1 k_2 k_3 k_4} = \delta^\alpha{}_\beta T_{k_1 k_2 k_3,k_4} - \sum_{m=1}^4 \delta^\alpha{}_{k_m} T_{k_1 \cdots k_{m-1}\beta \cdots,k_4}$. The commutator identities $X_1^\alpha{}_\beta X_1^\gamma{}_\delta I - X_1^\gamma{}_\delta X_1^\alpha{}_\beta I = \delta^\gamma{}_\beta X_1^\alpha{}_\delta I - \delta^\alpha{}_\delta X_1^\gamma{}_\beta I$ still hold, from whence the system of 4 equations is completely integrable. Moreover, the 4 equations are independent since the matrix $\|\lambda^\alpha{}_{\beta k_1 k_2}, \lambda^\alpha{}_{\beta k_1 k_2 k_3}, \lambda^\alpha{}_{\beta k_1 k_2 k_3 k_4}\|$ has already been found to be of rank 4 by inspection of its elements $\lambda^\alpha{}_{\beta k_1 k_2}, \lambda^\alpha{}_{\beta k_1 k_2 k_3}$. The arguments g_{jk}, T_{ijk}, $T_{ijk,l}$ are $3+4+8=15$ in number with the identities $|g| = -1$, $g^{rs}T_{rsi}=0$, $\theta_{,j}=2T^{rst}T_{rst,j}=0$, which are $1+2+4+2=9$ in number. This leaves 6 independent arguments, 4 of which we take as in the preceding section to be g_{jj} and T_{jjj}. The two additional independent arguments are chosen as the components of the vector T_j defined by the equations

$$T_r T^r{}_{jk} = T_{jk}, \quad \text{where} \quad T_{jk} = T^r{}_{jk,r}, \tag{18.1}$$

which may be solved for T^i by contracting with T^{ijk} and using (11.5):

$$T^i = 2\theta^{-1}T^{irs}T_{rs} \ (w = 1), \qquad \tilde{T}^j = \tilde{T}_k = \tilde{T}_{jj}/\tilde{T}_{jjj}. \tag{18.2}$$

From the definition $T_{jk}=T^r{}_{jk,r}$ and the identities $(1/2)\tilde{\theta}_{,k}=\tilde{T}^{rst}\tilde{T}_{rst,k} = \tilde{T}_{jjj}\tilde{T}_{kkk,k}+\tilde{T}_{kkk}\tilde{T}_{jjj,k}=0$ there follows

$$\tilde{T}_{jjj,k} = \tilde{T}_{jj} = \tilde{T}_{jjj}\tilde{T}_k, \qquad \tilde{T}_{kkk,k} = -\tilde{T}_{kkk}\tilde{T}_k. \tag{18.3}$$

The 8 components $T_{jkl,m}$ related by the 6 identities $g^{rs}T_{rsj,k}=0$, $T^{rst}T_{rst,j}=0$ are expressible in terms of the vector T_i and the tensor T_{jkl} by the equations

$$\begin{aligned} T_{jkl,m} &= (2/3)(g_{mj}T_{kl} + g_{mk}T_{lj} + g_{ml}T_{jk}) \\ &\quad - (1/3)(g_{jk}T_{lm} + g_{kl}T_{jm} + g_{lj}T_{km}) - T_{jkl}T_m, \end{aligned} \tag{18.4}$$

as may be verified in canonical coordinates. Similarly the relations

$$T_{jkr}T^r{}_{lm} = (1/4)(g_{jl}g_{mk} + g_{jm}g_{lk} - g_{jk}g_{lm})\theta \tag{18.5}$$

needed later may be verified now.

The number of absolute invariants containing g_{jk}, T_{jkl} and T_j is now determined by subtracting the number of independent defining equations from the number of independent arguments, namely, $6-4=2$, and hence *there must exist* 3 *relative invariants. But the invariant* θ *must be counted among these, leaving but* 2 *relative invariants to be discovered.* We postpone the determination of these until the next section and proceed now to extend the enumeration to the next higher order.

We annex now the set of 16 components $T_{jkl,m,n}$ which by (18.4) with T_{jk} written as $T_rT^r{}_{jk}$ are expressible in terms of the old arguments g_{jk}, T_{jkl}, T_j and the additional 4 new arguments $T_{j,k}$. Since the number of defining partial differential equations remains unaltered by the adjunction of more arguments, the increase in the number of absolute (and hence relative) invariants arising from any specific adjunction is equal to the number of independent arguments adjoined. Hence there are 4 additional relative invariants which contain $T_{ijk,l,m}$. We are ready now for generalization.

Adjunction of the 2^{s+2} components $T_{ijk,k_1,k_2,\cdots,k_s}$ adds the 2^s independent arguments $T_{k_1,k_2,\cdots,k_s}$. *Defining an invariant to be of order s if it contains* $T_{ijk,k_1,\cdots,k_s}$ *but does not contain T's of higher order, we conclude that the number of invariants of order s is* 2^s, $s=1, 2, \cdots$.

19. **Determination of the invariants.** If $T_{k_1\cdots k_{2r}}$ is a covariant tensor of rank $2r$ and weight w, complete contraction with combinations of g^{ij} and ϵ^{ij} as defined in §1, both of weight 1, will yield an invariant of weight $w+r$. All the invariants of a curved surface may be formed in this simple way.

The one invariant of order 0 is

$$R_0 = \theta = T_{rst}T^{rst}; \qquad \tilde{\theta} = 2\tilde{T}_{111}\tilde{T}_{222} \qquad (w = 1).$$

Two independent invariants of order 1 are

$$(19.1)\qquad \begin{aligned} &R_{1\pm} = (1/2)T_{rst}T^rT^s(g^{tu} \pm \epsilon^{tu})T_u;\ \tilde{R}_{1+} = \tilde{T}_{111}(\tilde{T}_2)^3,\ \tilde{R}_{1-} = \tilde{T}_{222}(\tilde{T}_1)^3 \\ &\qquad\qquad (w = 2). \end{aligned}$$

Four independent invariants of order 2 are

$$(19.2)\qquad \begin{aligned} &R_{2\pm} = (1/2)T_{r,s}(g^{rs} \pm \epsilon^{rs}); \qquad \tilde{R}_{2+} = \tilde{T}_{1,2}, \qquad \tilde{R}_{2-} = \tilde{T}_{2,1} \qquad (w = 1), \\ &R'_{2\pm} = (1/2)T^r{}_{st}T^s(g^{tu} \pm \epsilon^{tu})T_{r,u}; \\ &\tilde{R}'_{2+} = \tilde{T}_{111}\tilde{T}_2\tilde{T}_{2,2}, \qquad \tilde{R}'_{2-} = \tilde{T}_{222}\tilde{T}_1\tilde{T}_{1,1} \qquad (w = 2). \end{aligned}$$

Eight independent invariants of order 3 are

$$(19.3)\qquad \begin{aligned} &R_{3\pm} = (1/2)T_{r,s,u}T^{rs}{}_t(g^{tu} \pm \epsilon^{tu});\ \tilde{R}_{3+} = \tilde{T}_{111}\tilde{T}_{2,2,2},\ \tilde{R}_{3-} = \tilde{T}_{222}\tilde{T}_{1,1,1} \\ &\qquad\qquad (w = 2), \\ &R'_{3\pm} = \theta^{-1}T_{r,s,t}T_wT^{wrv}T^s{}_{vu}(g^{ut}\pm\epsilon^{ut});\ \tilde{R}'_{3+}= \tilde{T}_1\tilde{T}_{1,2,2},\ \tilde{R}'_{3-} = \tilde{T}_2\tilde{T}_{2,1,1} \\ &\qquad\qquad (w = 2), \\ &R''_{3\pm} = \theta^{-1}T_{s,t,r}T_wT^{wrv}T^s{}_{vu}(g^{ut} \pm \epsilon^{ut}); \\ &\tilde{R}''_{3+} = \tilde{T}_1\tilde{T}_{2,2,1}, \qquad \tilde{R}''_{3-} = \tilde{T}_2\tilde{T}_{1,1,2} \qquad (w = 2), \\ &\tilde{R}'''_{3\pm} = \theta^{-1}T_{t,r,s}T_wT^{wrv}T^s{}_{vu}(g^{ut} \pm \epsilon^{ut}); \\ &\tilde{R}'''_{3+} = \tilde{T}_1\tilde{T}_{2,1,2}, \qquad \tilde{R}'''_{3-} = \tilde{T}_2\tilde{T}_{1,2,1} \qquad (w = 2). \end{aligned}$$

In this way we might continue to the computation of the set of 2^s *invariants of*

order s, the canonical form of each such invariant of the set containing linearly one of the 2^s distinct components of $T_{k_1,\ldots,k_s}$. It is possible, however, to select another equivalent system of basic invariants which admits extension very readily to any desired order. This will be done in the next section.

Any invariant of order not greater than 3 in the T's is expressible in terms of the basic invariants of this section. As an illustration which will be useful later, let it be desired to express the "curvature" scalar $\lambda = g^{st}\lambda^r{}_{str}$ $(w=1)$ in terms of the basic invariants. From (15.5),

$$-(1/2)\tilde{\lambda} = \partial_{jk}\log\tilde{\theta} = (\tilde{T}_{iii}\partial_{jk}\tilde{T}_{iii} - \partial_j\tilde{T}_{iii}\partial_k\tilde{T}_{iii})/(\tilde{T}_{iii})^2 + (\tilde{T}_{kkk}\partial_{jk}\tilde{T}_{kkk} - \partial_j\tilde{T}_{kkk}\partial_k\tilde{T}_{kkk})/(\tilde{T}_{kkk})^2,$$

while from (18.2), (18.3) and (15.6),

$$\tilde{T}_k = 2\partial_k\tilde{T}_{iii}/\tilde{T}_{iii} + \partial_k\tilde{T}_{kkk}/\tilde{T}_{kkk},$$
$$\tilde{T}_{k,j} = \partial_j\tilde{T}_k = 2(\tilde{T}_{iii}\partial_{jk}\tilde{T}_{iii} - \partial_j\tilde{T}_{iii}\partial_k\tilde{T}_{iii})/(\tilde{T}_{iii})^2 + (\tilde{T}_{kkk}\partial_{jk}\tilde{T}_{kkk} - \partial_j\tilde{T}_{kkk}\partial_k\tilde{T}_{kkk})/(\tilde{T}_{kkk})^2,$$

and hence $-(1/2)\tilde{\lambda} = (1/3)(\tilde{T}_{j,k} + \tilde{T}_{k,j})$ so that $\lambda = -(2/3)[(R_{2+}) + (R_{2-})]$. It follows immediately that $\lambda = -(2/3)T^r{}_{,r}$ and hence

$$\lambda_{ijkl} = (1/2)(g_{il}g_{jk} - g_{ik}g_{jl})\lambda = -(1/3)(g_{il}g_{jk} - g_{ik}g_{jl})T^r{}_{,r}; \qquad g^{rs}\lambda_{jrsk} = -(1/3)g_{jk}T^r{}_{,r}, \tag{19.4}$$

as may be verified in canonical coordinates from (15.4).

It should be remarked at this point that we have not considered those invariants which might be functions of $\lambda^i{}_{jk}$, derivatives of $\lambda^i{}_{jk}$ and derivatives of g_{jk}. If we define the tensor G_{jk} of weight 0 by $G_{jk} = \theta g_{jk}$, then $\lambda^i{}_{jk}$ are the Christoffel symbols formed from G_{jk} and $\lambda^i{}_{jkl}$ are components of the related curvature tensor. T. Y. Thomas and A. D. Michal [5] have shown that any invariant function of G_{jk}, $\lambda^i{}_{jk}$ and their derivatives is expressible in terms of G_{jk}, $\lambda^i{}_{jkl}$ and successive covariant derivatives of $\lambda^i{}_{jkl}$. But since G_{jk} is expressible in terms of g_{jk} and θ and since $\lambda^i{}_{jkl}$ is expressible in terms of $T^j{}_{,k}$, it is unnecessary to consider explicitly $\lambda^i{}_{jk}$ and its derivatives or derivatives of g_{jk} when seeking invariants of the system.

20. **An alternative basic system of invariants.** If I is any relative invariant of weight w and order s, then

$$I_\pm = (1/2)(g^{rs} \pm \epsilon^{rs})T_rI_{,s}; \qquad I_+ = \tilde{T}_1\tilde{I}_{,2}, \qquad \tilde{I}_- = \tilde{T}_2\tilde{I}_{,1}$$

will be two additional invariants of weight $w+1$ and order $s+1$. Hence on starting with the two invariants $R_{1\pm}$ of (19.1) of weight 2 and order 1, all the 2^s invariants of order s may be formed by successive applications of this process. For example, write

$$I = R_{1+}, \qquad \tilde{I} = \tilde{T}_{111}(\tilde{T}_2)^3 \qquad (w = 2).$$

Since g^{jk} and ϵ^{jk} both behave as constants under covariant differentiation, an invariant in its canonical form may be differentiated covariantly by the familiar product rule of ordinary differentiation. Thus, on referring to (18.3),

$$\tilde{J} = \tilde{T}_2\tilde{I}_{,1} = \tilde{T}_{111}\tilde{T}_1(\tilde{T}_2)^2[3\tilde{T}_{2,2} + (\tilde{T}_2)^2] \qquad (w = 3)$$

is an invariant of order 2. A second application of this process gives one of the 8 basic invariants of order 3 and weight 4 of our new system,

$$\begin{aligned}\tilde{K} = \tilde{T}_2\tilde{J}_{,1} = {} & 3[\tilde{T}_1\tilde{T}_{2,2,1}][\tilde{T}_{111}(\tilde{T}_2)^3] + 6[\tilde{T}_{111}\tilde{T}_2\tilde{T}_{2,2}][\tilde{T}_{2,1}][\tilde{T}_1\tilde{T}_2] \\ & + 3[\tilde{T}_{222}\tilde{T}_1\tilde{T}_{1,1}][\tilde{T}_{111}\tilde{T}_2\tilde{T}_{2,2}][\tilde{T}_1\tilde{T}_2]^2[\tilde{T}_{222}(\tilde{T}_1)^3]^{-1} \\ & - 3[\tilde{T}_{111}\tilde{T}_2\tilde{T}_{2,2}][\tilde{T}_1\tilde{T}_2]^2 + 4[\tilde{T}_{111}(\tilde{T}_2)^3][\tilde{T}_1\tilde{T}_2][\tilde{T}_{2,1}] \\ & + 2[\tilde{T}_{222}\tilde{T}_1\tilde{T}_{1,1}][\tilde{T}_{111}(\tilde{T}_2)^3]^2[2\tilde{T}_{111}\tilde{T}_{222}]^{-1}[\tilde{T}_1\tilde{T}_2]^{-1} \\ & - [\tilde{T}_{111}(\tilde{T}_2)^3][\tilde{T}_1\tilde{T}_2]^2.\end{aligned}$$

Since $\tilde{T}_1\tilde{T}_2 = [2(\tilde{R}_{1+})(\tilde{R}_{1-})/\tilde{\theta}]^{1/3}$, $w=1$, comparison with the canonical forms of the invariants of the preceding section shows how $\tilde{K}$ is expressed in terms of the invariants of the first basic system. Finally, the non-canonical K will be this same function of the non-canonical R's since an equivalence of two invariants of the same weight holds in all coordinate systems.

21. **Enumeration of the invariants containing γ_{jk} and its covariant derivatives.** Inspection of the definitions (14.4) of γ_{jk} shows that these quantities contain second derivatives of θ and therefore are of order 2 in the seminvariants $B^i{}_{jk}$. When asking for the invariants of order 2 containing γ_{jk}, we must add to the arguments g_{jk}, T_{jkl}, T_j, $T_{j,k}$ the 3 components γ_{jk} satisfying the single identity $g^{rs}\gamma_{rs}=0$. Since the number of independent differential equations yielding the invariants is unaffected by the addition of new arguments, *the number of invariants of order* 2 *involving* γ_{jk} *is precisely* 2.

Proceeding to the invariants of order 3 which contain $\gamma_{jk,l}$, we must add to the arguments $g_{jk}, \cdots, T_{j,k,l}$; γ_{jk} the 6 quantities $\gamma_{jk,l}$ satisfying the 2 identities $g^{rs}\gamma_{rs,j}=0$ and the 2 integrability conditions (16.4) which stipulate symmetry of $I^i{}_{jk}$ in j and k. To write the conditions in a more explicit form, we note from (19.4) and (18.1) that the definitions (14.12) of $E^0{}_{j\infty}$ and $E^i{}_{j\infty}$ become

$$E^0{}_{j\infty} = \gamma^r{}_{j,r} + \gamma_{rs}T^{rs}{}_j, \qquad E^i{}_{j\infty} = (1/3)\delta^i{}_jT^r{}_{,r} + T_rT^{ri}{}_j + (1/2)\theta\delta^i{}_j + \gamma^i{}_j.$$

Substitution into (16.5) gives

$$\begin{aligned}I'^i{}_{jk} = {} & (1/3)\delta^i{}_jT^r{}_{,r,k} + T_rT^{ri}{}_{j,k} + T_{r,k}T^{ri}{}_j + \gamma^i{}_{j,k} \\ & + (T_sT^{sr}{}_j + \gamma^r{}_j)T^i{}_{rk} + E^0{}_{j\infty}\delta^i{}_k,\end{aligned}$$

where $I'^i{}_{jk}$ results from $I^i{}_{jk}$ on discarding terms which are known to be symmetric in j and k. The 2 integrability conditions obtained by stipulating symmetry of $I'^i{}_{jk}$ in j and k are equivalent to the 2 conditions of equality of the

right sides of

$$I'^r{}_{jr} = (1/3)T^r{}_{,r,j} + (T_rT_s + T_{r,s})T^{rs}{}_j + \gamma^r{}_{j,r} + 2E^0{}_{j\infty},$$
$$I'^r{}_{rj} = (2/3)T^r{}_{,r,j} + (1/2)\theta T_j + \gamma_{rs}T^{rs}{}_j + E^0{}_{j\infty},$$

namely

$$I_j = 2\gamma^r{}_{j,r} - (1/3)T^r{}_{,r,j} + (T_{r,s} + T_rT_s)T^{rs}{}_j - (1/2)\theta T_j = 0.$$

Let the 4 components c_{jk} be defined by the equations

$$\gamma_{jk,l} = T^r{}_{jk}c_{rl}, \qquad (1/2)\theta c_{jk} = T^{rs}{}_j\gamma_{rs,k},$$

then the integrability conditions $I_j=0$ are

$$(2c_{rs} - (2/3)\theta^{-1}T^t{}_{,t,u}T^u{}_{rs} + (T_{r,s} - T_{rs} + T_rT_s))T^{rs}{}_j = 0.$$

We define a tensor Γ_{jk} by

$$\Gamma_{jk} = 2c_{jk} - (2/3)\theta^{-1}T^t{}_{,t,u}T^u{}_{jk} + (T_{j,k} - T_rT^r{}_{jk} + T_jT_k) \qquad (w = 0),$$
$$\Gamma_{rs}T^{rs}{}_j = 0,$$

and use the 4 components Γ_{jk} satisfying the 2 integrability conditions $\Gamma_{rs}T^{rs}{}_j=0$ as arguments equivalent to $\gamma_{jk,l}$, since $\gamma_{jk,l}$ is expressible in terms of Γ_{jk} and the T's. *There will thus be* 2 *invariants of order* 3 *which contain* Γ_{jk}.

The invariants of order 4 will be found by annexing to the set $g_{jk}, \cdots, T_{j,k,l,m}; \gamma_{jk}, \Gamma_{jk}$ the 8 components $\Gamma_{jk,l}$ satisfying the 4 differentiated integrability conditions $(\Gamma_{rs}T^{rs}{}_j)_{,k}=0$. But there is the additional integrability condition (16.4) which stipulates the symmetry of $I_{jk}=E^0{}_{j\infty,k} + E^r{}_{j\infty}\gamma_{rk}$. Now $E^0{}_{j\infty}=(c_{rs}+\gamma_{rs})T^{rs}{}_j$ and hence on discarding terms which are known to be symmetrical in j and k the new integrability conditions stipulate the symmetry in j and k of

$$(21.1) \qquad I'_{jk} = \theta\Gamma_{jk} + (2/3)T^r{}_{,r,j,k} - 2[(T_{r,s} + T_rT_s)T^{rs}{}_j]_{,k} + 4\gamma_{rs}T^{rs}{}_{j,k},$$

that is, $I=\epsilon^{rs}I'_{rs}=0$. Since the addition of the 8 components $\Gamma_{jk,l}$ satisfying the 4 identities $(\Gamma_{rs}T^{rs}{}_j)_{,k}=0$ gives rise to 1 further identity $I=0$, the total number of independent arguments $T_j, \cdots, T_{j,k,l,m}; \gamma_{jk}, \Gamma_{jk}$ is increased by 3 and hence *there exists* 3 *invariants of order* 4 *containing* $\Gamma_{jk,l}$.

We are ready now for generalization. Asking for the number of invariants of order s, $s>4$, we must annex to the set $g_{jk}, \cdots, T_{k_1,\ldots,k_s}; \gamma_{jk}, \Gamma_{jk}, \cdots, \Gamma_{k_1k_2,k_3,\ldots,k_{s-2}}$ the 2^{s-1} components $\Gamma_{k_1k_2,k_3,\ldots,k_{s-1}}$ satisfying the 2^{s-2} differentiated integrability conditions $(\Gamma_{r_1r_2}T^{r_1r_2}{}_{k_3})_{,k_4,\ldots,k_s}=0$ and giving rise to the 2^{s-4} integrability relations $I_{,k_1,\ldots,k_{s-4}}=0$ among the Γ's of order $s-1$ and the T's of order s. Since the total set of independent arguments has been increased by $3\cdot 2^{s-4}$ we conclude that *there exist* $3\cdot 2^{s-4}$ *invariants of order* s, $s\geqq 4$, *containing the* $(s-2)$*th covariant derivatives of* γ_{jk}.

22. **Determination of the invariants containing γ_{jk} and its derivatives.** In

forming invariants of order 2, because of the identity $g^{rs}\gamma_{rs}=0$ $(\tilde{\gamma}_{12}=0)$ the components γ_{11} and γ_{22} will serve as independent arguments. Similarly for order 3 the identities $\Gamma_{rs}T^{rs}{}_j=0$ become $\tilde{\Gamma}_{jj}=0$ so that the components $\tilde{\Gamma}_{12}$ and $\tilde{\Gamma}_{21}$ are independent. For order 4 the 4 identities $(\Gamma_{rs}T^{rs}{}_j)_{,k}=0$ become $\tilde{\Gamma}_{ii,l}=0$ and the identity $I=0$ becomes $\tilde{\Gamma}_{12}-\tilde{\Gamma}_{21}=F(\tilde{T}_j, \cdots, \tilde{T}_{j,k,l,m})$. We choose as independent arguments $\tilde{\Gamma}_{12,1}$, $\tilde{\Gamma}_{12,2}$, and $\tilde{\Gamma}_{21,1}$. For order 5, the 8 identities $(\Gamma_{rs}T^{rs}{}_i)_{,l,m}=0$ become $\Gamma_{ii,l,m}=0$ and $I_{,j}=0$ becomes $(\tilde{\Gamma}_{12}-\tilde{\Gamma}_{21})_{,j}=F_j(\tilde{T}_{(5)})$, where $F_j(\tilde{T}_{(5)})$ denote functions of the $\tilde{T}$'s of order 5. We choose as independent arguments the 4 components $\tilde{\Gamma}_{12,i,l}$ and the 2 components $\tilde{\Gamma}_{21,1,i}$. Proceeding in this way the independent arguments of order 6 would be taken as the 8 components $\tilde{\Gamma}_{12,i,l,m}$ and the 4 components $\tilde{\Gamma}_{21,1,i,l}$ and similarly for the successive orders.

The 2 invariants of order 2 containing γ_{jk} may be taken as

$$(22.1)\qquad S_{2\pm}=(1/2)T_uT^{ur}{}_s\gamma_{rt}(g^{st}\pm\epsilon^{st}),\qquad \tilde{S}_{2+}=\tilde{T}_2\tilde{T}_{111}\tilde{\gamma}_{22},\qquad \tilde{S}_{2-}=\tilde{T}_1\tilde{T}_{222}\tilde{\gamma}_{11}\qquad (w=2),$$

the 2 of order 3 containing Γ_{jk} as

$$(22.2)\qquad S_{3\pm}=(1/2)\Gamma_{rs}(g^{rs}\pm\epsilon^{rs}),\qquad \tilde{S}_{3+}=\tilde{\Gamma}_{12},\qquad \tilde{S}_{3-}=\tilde{\Gamma}_{21}\qquad (w=1),$$

the 3 of order 4 containing $\Gamma_{jk,l}$ as

$$(22.3)\qquad \begin{aligned} &S_4=(1/2)T^r\Gamma_{rs,t}(g^{st}-\epsilon^{st}),\qquad \tilde{S}_4=\tilde{T}_2\tilde{\Gamma}_{12,1} && (w=2),\\ &S_{4\pm}=(1/2)T^r\Gamma_{sr,t}(g^{st}\pm\epsilon^{st}),\quad \tilde{S}_{4+}=\tilde{T}_1\tilde{\Gamma}_{12,2},\quad \tilde{S}_{4-}=\tilde{T}_2\tilde{\Gamma}_{21,1} && (w=2),\end{aligned}$$

and the 6 of order 5 containing $\Gamma_{jk,l,m}$ as

$$(22.4)\qquad \begin{aligned} &S_{5\pm}=(1/4)\Gamma_{rs,t,u}(g^{rs}+\epsilon^{rs})(g^{tu}\pm\epsilon^{tu}),\\ &S_5=(1/4)\Gamma_{rs,t,u}(g^{rs}-\epsilon^{rs})(g^{tu}+\epsilon^{tu}),\\ &\tilde{S}_{5+}=\tilde{\Gamma}_{12,1,2},\qquad \tilde{S}_{5-}=\tilde{\Gamma}_{12,2,1},\qquad \tilde{S}_5=\tilde{\Gamma}_{21,1,2} && (w=2),\\ &S'_{5\pm}=(1/2)T^{rtv}T_v\Gamma_{sr,t,u}(g^{su}\pm\epsilon^{su}),\quad S'_5=(1/2)T^{rtv}T_v\Gamma_{rs,t,u}(g^{su}-\epsilon^{su}),\\ &\tilde{S}'_{5+}=\tilde{T}_{111}\tilde{T}_2\tilde{\Gamma}_{12,2,2},\quad \tilde{S}'_{5-}=\tilde{T}_{222}\tilde{T}_1\tilde{\Gamma}_{21,1,1},\quad \tilde{S}'_5=\tilde{T}_{222}\tilde{T}_1\tilde{\Gamma}_{12,1,1} && (w=3).\end{aligned}$$

In this way we might continue to form all the basic invariants to any desired order.

As an alternative basic system, we choose the invariants of order 4 and less as above, renaming the set (22.3) by the scheme

$$\tilde{I}=\tilde{T}_2\tilde{\Gamma}_{12,1},\qquad \tilde{J}=\tilde{T}_1\tilde{\Gamma}_{12,2},\qquad \tilde{K}=\tilde{T}_2\tilde{\Gamma}_{21,1}\qquad (w=2).$$

Then as in §20, the operations $I_\pm=(1/2)(g^{rs}\pm\epsilon^{rs})T_rI_{,s}$, $\tilde{I}_+=\tilde{T}_1\tilde{I}_{,2}$, $\tilde{I}_-=\tilde{T}_2\tilde{I}_{,1}$ build from any invariant I two derived invariants of one higher order and one greater weight. This process yields as the set of 6 basic invariants of order 5,

$$\begin{aligned} &\tilde{I}_+=\tilde{T}_1(\tilde{T}_2\tilde{\Gamma}_{12,1})_{,2},\qquad \tilde{J}_+=\tilde{T}_1(\tilde{T}_1\tilde{\Gamma}_{12,2})_{,2},\qquad \tilde{K}_+=\tilde{T}_1(\tilde{T}_2\tilde{\Gamma}_{21,1})_{,2},\\ &\tilde{I}_-=\tilde{T}_2(\tilde{T}_2\tilde{\Gamma}_{12,1})_{,1},\qquad \tilde{J}_-=\tilde{T}_2(\tilde{T}_1\tilde{\Gamma}_{12,2})_{,1},\qquad \tilde{K}_-=\tilde{T}_2(\tilde{T}_2\tilde{\Gamma}_{21,1})_{,1}.\end{aligned}$$

A repetition of this process applied to the set of 6 invariants just derived would

generate the 12 *invariants of order* 6 *and so on to any desired order.*

Regarding the invariants of §19 and those of the first part of this section as comparising a basic system, while looking upon those of §20 and the latter part of this section, all derived from a certain stage on by the T_j contraction process, as forming a second but equivalent basic system, it may perhaps be interesting to select one invariant from this second system and explicitly express it in terms of the invariants of the first. Take then for example

$$\tilde{I}_+ = [\tilde{T}_1\tilde{T}_2]([\tilde{T}_{111}\tilde{T}_2\tilde{T}_{2,2}][\tilde{T}_2\tilde{\Gamma}_{12,1}][\tilde{T}_{111}(\tilde{T}_2)^3]^{-1} + [\tilde{\Gamma}_{12,1,2}]) \qquad (w = 3),$$

which is readily expressed in terms of invariants of the first basic system by referring to §19, (22.3) and (22.4). A second example is

$$\tilde{I}_- = [\tilde{T}_1\tilde{T}_2]^2[\tilde{T}_{222}\tilde{T}_1\tilde{\Gamma}_{12,1,1}][\tilde{T}_{222}(\tilde{T}_1)^3]^{-1} + [\tilde{T}_{2,1}][\tilde{T}_2\tilde{\Gamma}_{12,1}] \qquad (w = 3).$$

In order to furnish a basis of comparison of the effectiveness of tensor analysis in the discovery of complete systems of invariants with non-tensor methods, we comment that Wilczynski [1, p. 197] went no farther than asking for a complete set of functionally independent invariants which in our notation would contain T_{ijk}, g_{jk} and their derivatives to the second order and γ_{jk} but not derivatives thereof. Now θ is of order 0 in the T's and g's, from (19.1) the two relative invariants $R_{1\pm}$ are of order 1, from (19.2) the four invariants $R_{2\pm}$ and $R'_{2\pm}$ are of order 2, and finally from (22.1) the two invariants $S_{2\pm}$ are of order 0 in γ_{jk} and of order 1 in the T's and g's. Thus there are 9 relative invariants or 8 absolute invariants of the stipulated orders. Wilczynski concluded his paper with a proof that there were only 8 such invariants and he exhibited a functionally independent set of these. He had a process for deriving new invariants of higher order from known ones, but he was not in a position to answer immediately the questions of their completeness and functional independence, at least not without labor increasing in complexity with each successive order. Careful reading will show, however, that Wilczynski was systematically employing the method of elimination of second derivatives of the coordinate transformations under infinitesimal transformations for the formation of covariant derivatives and hence he was essentially using a tensor analysis of infinitesimal transformations without the benefit of modern notation.

23. **A basic system of covariants.** Equations (13.1) show the quantities Z_0, Z_j, Z_∞ to be of orders 0, 1, 2 respectively in the seminvariants $B^i{}_{jk}$. Since the semi-covariants Y_A are linear forms in Z_A, any covariant of the defining system of differential equations is expressible in terms of Z_A and the coefficients of the system. Furthermore, it is unnecessary to consider derivatives of Z_A for these are expressible in terms of the coefficients and Z_A by the system itself. Asking first for all covariants of order 0, we must add to the arguments g_{jk}, T_{jkl} the one quantity Z_0 and hence *there is just one covariant of order* 0.

Asking next for the covariants of order 1 depending on Z_j, we must augment the set g_{jk}, T_{jkl}, T_j; Z_0 by the 2 arguments Z_j, *giving rise to* 2 *covariants of order* 1. Finally for the covariants of order 2 we must increase the set $g_{jk}, \cdots, T_{j,k}$; Z_0, Z_j by Z_∞ and hence *there is but* 1 *covariant of order* 2. We choose as the basic system of 4 covariants,

$$C_0 = Z_0 \quad (w = 0), \qquad C_2 = Z_\infty \quad (w = 1),$$
$$C_{1\pm} = (1/2)(g^{rs} \pm \epsilon^{rs})T_rZ_s, \qquad \tilde{C}_{1+} = \tilde{T}_1\tilde{Z}_2, \qquad \tilde{C}_{1-} = \tilde{T}_2\tilde{Z}_1 \qquad (w = 1).$$

Any covariant of orders s is now expressible in terms of the 4 *basic covariants of this section and basic invariants of orders from* 1 *to s. This completes the determination of all invariants and covariants of the fundamental system.*

24. **The principle of duality and the adjoined system.** The tangent plane to the integrating surface S has the equation $|Z, Z_0, Z_1, Z_2| = 0$ and the four cofactors $\zeta_0 = |Z_0, Z_1, Z_2|$ are the coordinates of the tangent plane. We seek the system of partial differential equations in the dependent variable ζ_0 whose solutions will be the four coordinates of the tangent plane to S. Using the tensor form (16.2) of the differential equations and writing $\zeta_0 = (1/2)\epsilon^{rs}|Z_0, Z_r, Z_s|$ we find

$$\zeta_{0,j} = \epsilon^{rs}(|Z_0, Z_r, Z_t|\, T^t{}_{sj} + |Z_0, Z_r, Z_\infty|\, g_{sj}).$$

But

$$|Z_0, Z_r, Z_t| = \epsilon_{rt}|Z_0, Z_1, Z_2|, \qquad \epsilon^{rs}\epsilon_{rt} = \delta^s{}_t \quad \text{and} \quad T^r{}_{rj} = 0$$

so that $\zeta_{0,j} = \epsilon^{rs}|Z_0, Z_r, Z_\infty|g_{sj}$ or $|Z_0, Z_k, Z_\infty| = \zeta_{0,r}g^{rs}\epsilon_{ks}$. We define $\zeta_{0,j} = \zeta_j$ and form

$$\begin{aligned}\zeta_{j,k} &= g_{js}\epsilon^{rs}(|Z_k, Z_r, Z_\infty| + |Z_0, Z_t, Z_\infty|\, T^t{}_{rk} + |Z_0, Z_r, Z_t|\, E^t{}_{k\infty}) \\ &= g_{js}\epsilon^{rs}(\epsilon_{kr}|Z_1, Z_2, Z_\infty| + \zeta_u g^{uv}\epsilon_{tv}T^t{}_{rk} + \epsilon_{rt}|Z_0, Z_1, Z_2|\, E^t{}_{k\infty}).\end{aligned}$$

But $g_{js}\epsilon^{rs}g^{uv}\epsilon_{tv}T^t{}_{rk} = -T^u{}_{jk}$ as may be verified by expressing both sides in canonical coordinates so that

$$\zeta_{j,k} = E_{jk\infty}\zeta_0 - T^r{}_{jk}\zeta_r - g_{jk}|Z_1, Z_2, Z_\infty|.$$

Finally we define $\zeta_\infty = (1/2)g^{rs}\zeta_{r,s} = -|Z_1, Z_2, Z_\infty| = -(1/2)\epsilon^{rs}|Z_r, Z_s, Z_\infty|$ and form

$$\begin{aligned}\zeta_{\infty,k} &= -\epsilon^{rs}(|Z_0, Z_s, Z_\infty|\, \gamma_{rk} + |Z_t, Z_s, Z_\infty|\, T^t{}_{rk}) - (1/2)\epsilon^{rs}|Z_r, Z_s, Z_0|\, E^0{}_{k\infty} \\ &= -\zeta_0 E^0{}_{k\infty} + \zeta_r\gamma^r{}_k.\end{aligned}$$

The system

$$(24.1)\quad \zeta_{0,j} = \zeta_j, \quad \zeta_{j,k} = E_{jk\infty}\zeta_0 - T^r{}_{jk}\zeta_r + g_{jk}\zeta_\infty, \quad \zeta_{\infty,k} = -E^0{}_{k\infty}\zeta_0 + \gamma^r{}_k\zeta_r,$$

will be called the system *adjoined* to (16.2), or more briefly, the *adjoint*. The adjoint relationship is reciprocal, for on forming the adjoint of the adjoint,

we recover the original system (16.2). System (24.1) represents the same surface S in plane coordinates as (16.2) in point coordinates. But we may also regard $\zeta^A{}_0$ to be point coordinates of a surface S' which is dual to S. The principle of duality associates any plane $\zeta^A{}_0$ of S with the point $\zeta^A{}_0$ of S' and any property P of the planes of S characterized by an invariant system Σ' of equations in the coefficients of (24.1) has as its dual a property P' of the points of S' characterized by the same system Σ'. If we now replace the point $\zeta^A{}_0$ of S' by its tangent plane $Z^A{}_0$ and the coefficients of Σ' by the corresponding coefficients of (16.2) we arrive at a property P of the points of S which is characterized by a system of equations Σ in the coefficients of (16.2). This property P of S will in general be different from the property P' of S', the necessary and sufficient condition that these two properties be the same being that the system Σ should be identical with the system Σ', that is, Σ must remain invariant under the replacement of $T^i{}_{jk}$ by $-T^i{}_{jk}$, γ_{jk} by $E_{jk\infty}$, $E_{jk\infty}$ by γ_{jk} and $E^0{}_{k\infty}$ by $-E^0{}_{k\infty}$. When these conditions hold the property P is called *self-dual*. In particular, the dual of a surface S swept out by a two-parameter family of points $Z^A{}_0$ is the envelope S' of the planes $Z^A{}_0$ and S will be self-dual only when S' coincides with S, the conditions being that (24.1) and (16.2) be identical,

$$T^i{}_{jk} = 0, \qquad \gamma_{jk} = E_{jk\infty}, \qquad E^0{}_{k\infty} = 0.$$

But from §21, $E_{jk\infty}=\gamma_{jk}+(1/3)g_{jk}T^r{}_{,r}+T_rT^r{}_{jk}+(1/2)g_{jk}\theta$, and hence $T^i{}_{jk}=0$ implies $\gamma_{jk}=E_{jk\infty}$. Furthermore, the integrability conditions (16.4) yield $\gamma^i{}_{k,l}-\gamma^i{}_{l,k}+E^0{}_{k\infty}\delta^i{}_l-E^0{}_{l\infty}\delta^i{}_k=0$, and on contracting, $-\gamma^r{}_{k,r}=E^0{}_{k\infty}$, but at the same time, from the definitions (14.12), $E^0{}_{k\infty}=\gamma^r{}_{k,r}$. Hence $E^0{}_{k\infty}=0$ is also a consequence of $T^i{}_{jk}=0$ so that *the necessary and sufficient conditions that S be self-dual are* $T^i{}_{jk}=0$. But in §11 it was shown that such a surface is necessarily a ruled quadric. This constitutes a proof that a ruled quadric is the only self-dual surface.

25. **The relative coordinate system and the fixed point conditions.** With the points Z_A taken as vertices of the relative reference tetrahedron, any point X may be expressed as a linear combination of Z_A, $X=x^AZ_A$. Under the parameter transformation $u^i\rightarrow\bar{u}^i$ the Z_A transform by $\bar{Z}_B=H^Q{}_BZ_Q$ with $H^A{}_B$ defined by (14.9). If we stipulate invariance of X under $u^i\rightarrow\bar{u}^i$, the relative coordinates x^A of X must transform by $\bar{x}^A=\eta^A{}_Qx^Q$; $\bar{x}^0=x^0$, $\bar{x}^i=x^r\partial_r\bar{x}^i$, $\bar{x}^\infty=|\partial u/\partial\bar{u}|^{-1}x^\infty$. The tensor ϵ_{ABCD} introduced in §1 transforms by $\bar{\epsilon}_{QRST}\eta^Q{}_A\eta^R{}_B\eta^S{}_C\eta^T{}_D=|\partial u/\partial\bar{u}|^{-2}\epsilon_{ABCD}$ reducing to $\epsilon_{0rs\infty}\partial_j\bar{u}^r\partial_k\bar{u}^s=|\partial u/\partial\bar{u}|^{-1}\epsilon_{0jk\infty}$ or simply $\bar{\epsilon}_{rs}\partial_j\bar{u}^r\partial_k\bar{u}^s=|\partial u/\partial\bar{u}|^{-1}\epsilon_{jk}$. By means of this tensor the covariant components ω_{AB} of the line joining x^A and y^A are given by $\omega_{AB}=\epsilon_{ABQR}x^Qy^R$ and its contravariant components by $\omega^{AB}=(1/2)\epsilon^{ABQR}\omega_{QR}$. This latter relationship is reciprocal, $\omega_{AB}=(1/2)\epsilon_{ABQR}\omega^{QR}$, and is the analytic characterization of the self-duality of a line in three dimensions. Expansion gives

$$\omega^{0i} = \epsilon^{ir}\omega_{r\infty}, \qquad \omega^{0\infty} = (1/2)\epsilon^{rs}\omega_{rs}, \qquad \omega^{ij} = \epsilon^{ij}\omega_{0\infty}, \qquad \omega^{i\infty} = \epsilon^{ir}\omega_{r0},$$

and their duals as obtained by raising and lowering all indices. Of course the line determined by the planes u_A and v_A has the contravariant components $\omega^{AB}=\epsilon^{ABQR}u_Q v_R$. The plane determined by the three non-collinear points x^A, y^A, z^A has the components $a_A=\epsilon_{AQRS}x^Q y^R z^S$ as evidenced by the fact that any point $w^A=\lambda x^A+\mu y^A+\nu z^A$ of this plane must necessarily satisfy $a_Q w^Q=0$ in virtue of the skew-symmetry of ϵ_{ABCD}. Expansion gives

$$
\begin{aligned}
&a_0=\epsilon_{rs}(x^r y^s z^\infty+y^r z^s x^\infty+z^r x^s y^\infty) && (w=-2),\\
(25.1)\qquad &a_i=\epsilon_{ir}\left[x^r(y^0 z^\infty-y^\infty z^0)+y^r(z^0 x^\infty-z^\infty x^0)+z^r(x^0 y^\infty-x^\infty y^0)\right] && (w=-2),\\
&a_\infty=-\epsilon_{rs}(y^r z^s x^0+z^r x^s y^0+x^r y^s z^0) && (w=-1).
\end{aligned}
$$

The polar plane of the point y^A as to the quadric $G_{QR}x^Q x^R=0$ is $u_A=G_{AQ}y^Q$ and dually the pole of the plane u_A is the point $y^A=G^{AQ}u_Q$. The polar line of the line joining y^A and z^A is the line $\omega^{AB}=\epsilon^{ABQR}G_{QS}y^S G_{RT}z^T$, formed by the polar planes of y^A and z^A. These same components ω^{AB} may also be formed by the scheme of raising indices of the covariant components of the line L_{yz} by means of the quadric tensor, $\omega^{AB}=|G|\,\epsilon_{QRST}y^S z^T G^{QA}G^{RB}$. The equivalence of these two expressions for ω^{AB} may be recognized by referring the quadric to a self-polar reference tetrahedron so that it will be of the form $G_{QQ}(x^Q)^2=0$. By the first formula, $\omega^{ij}=\epsilon^{ij}(G_{00}G_{\infty\infty})(y^0 z^\infty-z^0 y^\infty)$ and by the second, $\omega^{ij}=(G_{00}G_{11}G_{22}G_{\infty\infty})\epsilon_{ij}(y^0 z^\infty-z^0 y^\infty)/G_{11}G_{22}$, and similarly for the other components.

The point $X=x^A(u)Z_A$ will in general move about in the enveloping space S_3 as the vertex $Z_0(u)$ of the relative reference tetrahedron moves over the surface S. We seek the conditions on the relative coordinates $x^A(u)$ of the point X that it be fixed in S_3. These will be found by stipulating the existence of some scalar $e^{-\phi(u)}$ such that $e^{-\phi(u)}x^A(u)Z_A{}^{(B)}=k^{(B)}$, where the k's are constants since they are the absolute coordinates of the stationary X in S_3. Differentiation and substitution by (14.1), namely $\partial_j Z_A=E^Q{}_{jA}Z_Q$, gives

$$
\begin{aligned}
\partial_j x^A+x^Q E^A{}_{jQ}&=\partial_j\phi x^A \quad\text{or}\quad \partial_j x^0=\partial_j\phi x^0-x^r\gamma_{rj}-x^\infty E^0{}_{j\infty},\\
\partial_j x^i+x^r\lambda^i{}_{rj}&=\partial_j\phi x^i-x^0\delta^i{}_j-x^r T^i{}_{jr}-x^\infty E^i{}_{j\infty},\\
\partial_j x^\infty+x^\infty\partial_j\log\theta&=\partial_j\phi x^\infty-x^r g_{rj},
\end{aligned}
$$

the left sides of the expansions being the covariant derivatives of x^0, x^i, x^∞ of weights 0, 0, -1 respectively as per (12.4). The proportionality factor ϕ may be eliminated by considering nonhomogeneous coordinates $X^i=x^i/x^0$ $X^\infty=x^\infty/x^0$ of weights 0, -1 respectively:

$$
\begin{aligned}
\partial_j X^i+X^r\lambda^i{}_{rj}&=X^i X^r\gamma_{jr}+X^i X^\infty E^0{}_{j\infty}-X^r T^i{}_{jr}-X^\infty E^i{}_{j\infty}-\delta^i{}_j,\\
\partial_j X^\infty+X^\infty\partial_j\log\theta&=-X^r g_{rj}+X^\infty X^r\gamma_{jr}+X^\infty X^\infty E^0{}_{j\infty},
\end{aligned}
$$

where again the left sides are the covariant derivatives of X^i and X^∞. Conditions of this type on the nonhomogeneous coordinates (X^i, X^∞) of a point X

that X be fixed in S_3 have been called in all generality by G. Kowalewski "Identitätsbedingungen" for the geometry of a configuration under a Lie group, in this case a surface S under the projective group. By the methods of Kowalewski the "Identitätsbedingungen" are taken as the fundamental point of departure for the study of the geometry of a configuration under a Lie group.

26. **Power series expansions.** To obtain a fourth order power series expansion

$$
\begin{aligned}
Z = Z_0 &+ \partial_j Z_0 du^j + (1/2)\partial_{jk} Z_0 du^j du^k + (1/6)\partial_{jkl} Z_0 du^j du^k du^l \\
&+ (1/24)\partial_{jklm} Z_0 du^j du^k du^l du^m + \cdots
\end{aligned}
\tag{26.1}
$$

of the surface S at a point Z in the neighborhood of Z_0 it will suffice to compute nonsymmetrical coefficients whose complete contraction with the differentials du^i will produce the desired summations. We shall write $A=0$ (mod du) to mean that the k index quantity A when completely contracted with the differentials du^i forms a vanishing sum. For example, $\phi_j\psi_k+\psi_j\phi_k = 2\phi_j\psi_k = 2\phi_k\psi_j$ (mod du). From (16.2),

$$
\begin{aligned}
Z_j &= \partial_j Z_0, \qquad Z_{j,k} = \partial_{jk} Z_0 - Z_r \lambda^r{}_{jk}, \\
Z_{j,k,l} &= \partial_{jkl} Z_0 - Z_r(\partial_j \lambda^r{}_{kl} + \lambda^r{}_{js}\lambda^s{}_{kl}) - 3Z_{j,r}\lambda^r{}_{kl} \ (\text{mod } du) \\
&= (T^r{}_{jk}\gamma_{rl} + g_{jk}E^0{}_{l\infty} + \gamma_{jk,l})Z_0 \\
&\quad + (T^r{}_{jk}T^i{}_{rl} + \gamma_{jk}\delta^i{}_l + g_{jk}E^i{}_{l\infty} + T^i{}_{jk,l})Z_i \\
&\quad + T_{jkl}Z_\infty \ (\text{mod } du).
\end{aligned}
$$

From (14.12) and (11.5), $E^i{}_{k\infty} = \gamma^i{}_k + T^i{}_k - (1/2)(\lambda - \theta)\delta^i{}_k$, so that

$$
\begin{aligned}
\partial_j Z_0 &= Z_j, \qquad \partial_{jk} Z_0 = \gamma_{jk} Z_0 + (T^i{}_{jk} + \lambda^i{}_{jk})Z_i + g_{jk}Z_\infty, \\
\partial_{jkl} Z_0 &= [T^r{}_{jk}\gamma_{rl} + g_{jk}E^0{}_{l\infty} + \gamma_{jk,l} + (3\gamma_{jr}\lambda^r{}_{kl})]Z_0 \\
&\quad + [T^r{}_{jk}T^i{}_{rl} + 2\gamma_{jk}\delta^i{}_l + g_{jk}T^i{}_l - (1/2)g_{jk}\delta^i{}_l(\lambda - \theta) \\
&\quad + T^i{}_{jk,l} + (\partial_j \lambda^i{}_{kl} + \lambda^i{}_{jr}\lambda^r{}_{kl} + 3T^i{}_{jr}\lambda^r{}_{kl})]Z_i \\
&\quad + [T_{jkl} + (3g_{jr}\lambda^r{}_{kl})]Z_\infty \ (\text{mod } du).
\end{aligned}
$$

Only the coefficient of Z_∞ in $\partial_{jklm} Z_0$ will be needed;

$$
\begin{aligned}
\partial_{jklm} Z_0 = \cdots &+ [T^r{}_{jk}T^i{}_{rl} + 2g_{jk}\gamma_{lm} + g_{jk}T_{lm} \\
&- (1/2)g_{jk}g_{lm}(\lambda - (1/2)\theta) + 2T_{jkl,m} + 4g_{jr}\partial_k \lambda^r{}_{lm} \\
&+ 4g_{jr}\lambda^r{}_{ks}\lambda^s{}_{lm} + 3g_{rs}\lambda^r{}_{jk}\lambda^s{}_{lm} \\
&+ 6T_{jkr}\lambda^r{}_{lm}]Z_\infty \ (\text{mod } du).
\end{aligned}
$$

If the neighboring point Z given by the power series (26.1) is written $Z = x^A Z_A$, then the relative coordinates x^A are

$$
\begin{aligned}
x^0 &= 1 + (1/2)\gamma_{jk}du^j du^k + \cdots, \\
x^i &= du^i + (1/2)(T^i{}_{jk} + \lambda^i{}_{jk})du^j du^k + (1/6)[T^i{}_{jr}T^r{}_{kl} + 2\gamma_{jk}\delta^i{}_l \\
&\quad + g_{jk}T^i{}_l - (1/2)g_{jk}\delta^i{}_l(\lambda - \theta) + T^i{}_{jk,l} \\
&\quad + (\partial_j\lambda^i{}_{kl} + \lambda^i{}_{jr}\lambda^r{}_{kl} + 3T^i{}_{jr}\lambda^r{}_{kl})]du^j du^k du^l + \cdots, \\
x^\infty &= (1/2)g_{jk}du^j du^k + (1/6)(T_{jkl} + 3g_{jr}\lambda^r{}_{kl})du^j du^k du^l \\
&\quad + (1/24)[T_{jkr}T^r{}_{lm} + 2g_{jk}\gamma_{lm} + g_{jk}T_{lm} - (1/2)g_{jk}g_{lm}(\lambda - \theta) \\
&\quad + 2T_{jkl,m} + (4g_{jr}\partial_k\lambda^r{}_{lm} + 4g_{jr}\lambda^r{}_{ks}\lambda^s{}_{lm} \\
&\quad + 3g_{rs}\lambda^r{}_{jk}\lambda^s{}_{lm} + 6T_{jkr}\lambda^r{}_{lm})]du^j du^k du^l du^m + \cdots.
\end{aligned}
$$

These may be simplified by means of (18.4) and (18.5) which become

$$
\begin{aligned}
T_{jkl,m} &= g_{jk}T_{lm} - T_jT_{klm} \pmod{du}, \\
T_{jkr}T^r{}_{lm} &= (1/4)g_{jk}g_{lm}\theta \pmod{du}.
\end{aligned}
$$

The nonhomogeneous coordinates $X^i = x^i/x^0$, $x^\infty = x^\infty/x^0$ now have the expansions

$$
\begin{aligned}
X^i &= du^i + (1/2)(T^i{}_{jk} + \lambda^i{}_{jk})du^j du^k + (1/6)[-\gamma_{jk}\delta^i{}_l + g_{jk}T^i{}_l \\
&\quad - (1/2)g_{jk}\delta^i{}_l\lambda + (3/4)\delta^i{}_l g_{jk}\theta + \delta^i{}_l T_{jk} - T^i T_{jkl} \\
&\quad + (\partial_j\lambda^i{}_{kl} + \lambda^i{}_{jr}\lambda^r{}_{kl} + 3T^i{}_{jr}\lambda^r{}_{kl})]du^j du^k du^l + \cdots, \\
X^\infty &= (1/2)g_{jk}du^j du^k + (1/6)(T_{jkl} + 3\lambda_{jkl})du^j du^k du^l \\
&\quad + (1/24)[-4g_{jk}\gamma_{lm} + 3g_{jk}T_{lm} - 2T_jT_{klm} - (1/2)g_{jk}g_{lm}\lambda \\
&\quad + (3/4)g_{jk}g_{lm}\theta + (4g_{jr}\partial_k\lambda^r{}_{lm} + 4g_{jr}\lambda^r{}_{ks}\lambda^s{}_{lm} \\
&\quad + 3g_{rs}\lambda^r{}_{jk}\lambda^s{}_{lm} + 6T_{jkr}\lambda^r{}_{lm})]du^j du^k du^l du^m + \cdots.
\end{aligned}
\tag{26.2}
$$

Obviously X^∞ is of the form

$$
X^\infty = (1/2)g_{jk}X^jX^k - (1/3)T_{jkl}X^jX^kX^l + (1/12)a_{jklm}X^jX^kX^lX^m + \cdots
$$

and the unknown coefficients a_{jklm} may be evaluated by comparison of the expanded right side,

$$
\begin{aligned}
&(1/2)g_{jk}du^j du^k + (1/6)(T_{jkl} + 3g_{jr}\lambda^r{}_{kl})du^j du^k du^l \\
&\quad + (1/12)(-2g_{jk}\gamma_{lm} + 4g_{jk}T_{lm} - g_{jk}g_{lm}\lambda + (3/8)g_{jk}g_{lm}\theta - 2T_jT_{klm} \\
&\qquad + a_{jklm})du^j du^k du^l du^m + \cdots,
\end{aligned}
$$

with the computed expansion of X^∞. *This yields*

$$
\begin{aligned}
X^\infty &= (1/2)g_{jk}X^jX^k - (1/3)T_{jkl}X^jX^kX^l + (1/12)(T_jT_{klm} \\
&\quad + (3/4)g_{jk}g_{lm}\lambda - (5/2)g_{jk}T_rT^r{}_{lm})X^jX^kX^lX^m + \cdots
\end{aligned}
\tag{26.3}
$$

as the nonhomogeneous expansion of the surface S about Z_0.

It will be informative to compare some of our results with those given by Lane. Recalling that $\tilde{T}_{jj}=\tilde{T}_k\tilde{T}_{jjj}$ and noting from (18.2), (18.3) and (15.6) that

$$\begin{aligned}\tilde{T}_k = \tilde{T}_{jj}/\tilde{T}_{jjj} &= [\partial_k\tilde{T}_{jjj} + \tilde{T}_{jjj}(\partial_k\tilde{T}_{kkk}/\tilde{T}_{kkk} + \partial_k\tilde{T}_{jjj}/\tilde{T}_{jjj})]/\tilde{T}_{jjj}\\ &= \partial_k \log [(\tilde{T}_{jjj})^2\tilde{T}_{kkk}],\end{aligned}$$

we recognize on referring to (15.2) that (T_1, T_2) is the vector with components (ϕ, ψ) defined by Lane (p. 64) according to $\phi=(\log \beta\gamma^2)_u$, $\psi=(\log \beta^2\gamma)_v$. The expansion (26.3) has the canonical form

$$\begin{aligned}Z = XY - (1/3)(\beta X^3 + \gamma Y^3) + (1/12)[\beta\phi X^4 - 4\beta\psi X^3Y - 6(\log\theta)_{uv}X^2Y^2\\ - 4\gamma\phi XY^3 + \gamma\psi Y^4] + \cdots\end{aligned}$$

as given by Lane (p. 73).

27. **Algebraic contact surfaces.** When referred to the relative reference frame the coefficients of an algebraic q-ic surface will be the $(q+1)(q+2)(q+3)/6$ components of a symmetric covariant tensor $G_{A_1\cdots A_q}$. Those q-ic surfaces which have contact of the kth order with S at Z_0 will be found by insisting that when the power series expansions of the coordinates x^A of a point in the neighborhood of Z_0 on S are substituted into the form $G_{A_1\cdots A_q}x^{A_1}\cdots x^{A_q}$ the terms of degree less than or equal to k shall vanish identically in the increments du^i. This imposes $(k+1)(k+2)/2$ conditions so that there is a 3-parameter family of second order contact quadrics, a 9-parameter family of third order contact cubic surfaces and of these latter a 4-parameter subfamily having fourth order contact. These families will now be found.

The quadric $G_{00}+G_{0j}X^j+G_{jk}X^jX^k+G_{0\infty}X^\infty+G_{j\infty}X^jX^\infty+G_{\infty\infty}X^\infty X^\infty=0$ will have second order contact if substitution of the expansion (26.3) produces terms which vanish identically in the two variables X^j when of less than third degree. The conditions are $G_{00}=G_{0j}=0$, $(1/2)G_{0\infty}g_{jk}+G_{jk}=0$, so that *the three-parameter family of second order contact quadrics is*

$$G_{0\infty}(X^\infty - (1/2)g_{jk}X^jX^k) + G_{j\infty}X^jX^\infty + G_{\infty\infty}X^\infty X^\infty = 0. \tag{27.1}$$

Similarly, the third order contact cubic surfaces are found by insisting that substitution of (26.3) into

$$\begin{aligned}G_{000} + 3G_{00j}X^j + 3G_{00\infty}X^\infty + 3G_{0jk}X^jX^k + 6G_{0j\infty}X^jX^\infty + 3G_{0\infty\infty}X^\infty X^\infty\\ + G_{jkl}X^jX^kX^l + 3G_{jk\infty}X^jX^kX^\infty + 3G_{j\infty\infty}X^jX^\infty X^\infty + G_{\infty\infty\infty}X^\infty X^\infty X^\infty = 0\end{aligned}$$

shall produce terms which vanish identically in the variables X^j when of less than fourth degree. The conditions are

$$G_{0jk} + (1/2)G_{00\infty}g_{jk} = 0,$$

$$(G_{0\infty j}g_{kl} + G_{0\infty k}g_{lj} + G_{0\infty l}g_{jk}) - G_{00\infty}T_{jkl} + G_{jkl} = 0,$$

so that *the 9-parameter family of third order contact cubic surfaces is represented by*

$$
\begin{aligned}
&3G_{00\infty}(X^\infty - (1/2)g_{jk}X^jX^k + (1/3)T_{jkl}X^jX^kX^l) \\
&\qquad + 3G_{0\infty l}(2X^\infty X^l - g_{jk}X^jX^kX^l) + 3G_{\infty\infty j}X^\infty X^\infty X^j \\
&\qquad + 3G_{\infty jk}X^\infty X^jX^k + 3G_{0\infty\infty}X^\infty X^\infty + G_{\infty\infty\infty}X^\infty X^\infty X^\infty = 0.
\end{aligned}
\tag{27.2}
$$

The subset of fourth order contact cubics will be found by insisting that the substitution of the expansion (26.3) into the cubic expression above shall produce fourth degree terms which vanish identically in the variables X^i. On equating to zero the symmetrized coefficient of $X^iX^kX^lX^m$ we obtain the 5 conditions

$$
\begin{aligned}
&(1/4)(T_jT_{klm} + T_kT_{lmj} + T_lT_{mjk} + T_mT_{jkl}) + (1/4)(g_{jk}g_{lm} + g_{jl}g_{mk} + g_{jm}g_{lk})\lambda \\
&\quad - (5/12)(g_{jk}T_{lm} + g_{jl}T_{mk} + g_{jm}T_{lk} + g_{lm}T_{jk} + g_{mk}T_{jl} + g_{lk}T_{jm})G_{00\infty} \\
&\quad - 2(G_{0\infty j}T_{klm} + G_{0\infty k}T_{lmj} + G_{0\infty l}T_{mjk} + G_{0\infty m}T_{jkl}) \\
&\quad + (g_{jk}g_{lm} + g_{jl}g_{mk} + g_{jm}g_{lk})G_{0\infty\infty} \\
&\quad + (g_{jk}G_{\infty lm} + g_{jl}G_{\infty mk} + g_{jm}G_{\infty lk} + g_{lm}G_{\infty jk} + g_{mk}G_{\infty jl} + g_{lk}G_{\infty im}) = 0
\end{aligned}
$$

homogeneous in the 7 quantities $G_{00\infty}$, $G_{0\infty j}$, $G_{0\infty\infty}$, $G_{\infty jk}$, which we now proceed to solve for $G_{0\infty j}$ and $G_{\infty jk}$ in terms of $G_{00\infty}$ and $G_{0\infty\infty}$. Contracting with T^{jkl},

$$-5\theta G_{0\infty m} + 3T_m{}^{rs}G_{\infty rs} = 0,$$

with g^{jk},

$$(-2T_{lm} + g_{lm}\lambda)G_{00} - 4G_{0\infty r}T^r{}_{lm} + 4g_{lm}G_{0\infty\infty} + 6G_{\infty lm} + g_{lm}g^{rs}G_{\infty rs} = 0,$$

and contracting this with g^{lm} and $T^{lm}{}_k$ respectively,

$$\lambda G_{00\infty} + 4G_{0\infty\infty} + 4g^{rs}G_{\infty rs} = 0, \qquad -\theta T_kG_{00\infty} - 2\theta G_{0\infty k} + 6G_{\infty rs}T^{rs}{}_k = 0,$$

we find on combining,

$$8G_{0\infty k} = G_{00\infty}T_k, \qquad 24G_{\infty jk} = (10T_{jk} - 3g_{jk}\lambda)G_{00\infty} - 12g_{jk}G_{0\infty\infty}.$$

The 4-parameter family of fourth order contact cubic surfaces is then represented by

$$
\begin{aligned}
&G_{00\infty}[24X^\infty + 6T_jX^jX^\infty + (8T_{jkl} - 3g_{jk}T_l)X^jX^kX^l \\
&\quad + (10T_rT^r{}_{jk} - 3g_{jk}\lambda)X^jX^kX^\infty - 12g_{jk}X^jX^k] + 12G_{0\infty\infty}[2X^\infty X^\infty - g_{jk}X^jX^kX^\infty] \\
&\qquad + 24G_{j\infty\infty}X^jX^\infty X^\infty + 8G_{\infty\infty\infty}X^\infty X^\infty X^\infty = 0.
\end{aligned}
$$

28. **A pair of reciprocal congruences; the directrices of Wilczynski.** The line l_1 joining Z_0 to the point $Y = -\phi^rZ_r + Z_\infty$, ϕ^i any given contravariant vector of weight 1, will generate a congruence as the coordinates u^i vary.

This has been called by Lane the congruence Γ_1 and we now adapt to the methods of this paper Lane's demonstration that *the lines of* Γ_1 *can be assembled into a one-parameter family of developable surfaces in two ways so that there is one developable of each family through each* l_1. Let C_1 be a curve on S defined by $u^i = u^i(t)$. We seek the differential equation to be satisfied by $u^i(t)$ if C_1 is to be a curve of intersection of a developable surface D of Γ_1 with S. Let $Z = \rho Z_0 + Y$, ρ a scalar of weight 1, define the edge of regression of the developable D, then l_1 is tangent to the locus of Z as t varies along C_1 so that

$$dZ/dt = Z_{,j}(du^j/dt) = [(E^0{}_{j\infty} + \rho_{,j} - \phi^r\gamma_{rj})Z_0 + (E^i{}_{j\infty} - \phi^r T^i{}_{rj} - \phi^i{}_{,j} - \phi_j\phi^i + \rho\delta^i{}_j)Z_i - \phi_j Y](du^j/dt)$$

is a linear combination of Z_0 and Y only, the conditions being

$$(E^i{}_{j\infty} - \phi^r T^i{}_{rj} - \phi^i{}_{,j} - \phi_j\phi^i + \rho\delta^i{}_j)(du^j/dt) = 0.$$

Using $E^i{}_{j\infty} = \gamma^i{}_j + T^r T^i{}_{rj} - (1/2)\delta^i{}_j(\lambda - \theta)$ we eliminate ρ by contraction with $\epsilon_{ik}(du^k/dt)$ *to obtain*

$$\epsilon_{sk}(\gamma^s{}_j + T^r T^s{}_{rj} - \phi^r T^s{}_{rj} - \phi^s{}_{,j} - \phi_j\phi^s)du^j du^k = 0$$

as the differential equation of the net of curves C_1 *on* S *in which the two families of developables of* Γ_1 *cut* S. Writing $(\phi^1, \phi^2) = (a, b)$ we obtain the canonical form

$$[p - b_u + b(\log\theta)_u - b^2 - a\beta + \beta\psi]du^2 - [b_v - a_u]dudv - [q - a_v + a(\log\theta)_v - a^2 - b\gamma + \gamma\phi]dv^2 = 0$$

of the net C_1 as given by Lane (p. 84).

To define a congruence Γ_2 which will be called reciprocal to Γ_1 with respect to the quadrics of Darboux (that is, the pencil of second order contact quadrics (27.1) for which $G_{j\infty} = 0$), consider the line l_2 which is the polar line of l_1 through Z_0 as to the quadrics of Darboux. The polar plane of $(1, 0, 0, 0)$ is $x^\infty = 0$ and of $(0, -\phi^i, 1)$ is $-G_{0\infty}(\phi_j x^j + x^0) + 2G_{\infty\infty}x^\infty = 0$ so that l_2 is the intersection of $x^\infty = 0$ and $x^0 + \phi_j x^j = 0$. The two points $\rho_j = -\phi_j Z_0 + Z_j$ determine l_2 and the congruence Γ_2 reciprocal to Γ_1 is defined as the congruence swept out by l_2 as l_1 varies over Γ_1. To find the differential equations of the curves C_2 cut out from S by the developables D of Γ_2 we let $Z = \mu^j\rho_j$, μ^j of weight 0, trace the edge of regression of D as l_2 varies along C_2. Then

$$dZ/dt = (\mu^r\rho_r)_{,j}(du^j/dt) = [\mu_r(\gamma^r{}_j + \phi^s T^r{}_{sj} - \phi^r\phi_j - \phi^r{}_{,j})Z_0 + (\mu^r T^i{}_{rj} - \phi_r\mu^r\delta^i{}_j + \mu^i{}_{,j})\rho_i + \mu_j Z_\infty](du^j/dt)$$

must be a linear combination of ρ_i so that

$$\mu_r(\gamma^r{}_j + \phi^s T^r{}_{sj} - \phi^r\phi_j - \phi^r{}_{,j})du^j = 0, \qquad \mu_r du^r = 0,$$

and the elimination of μ_i *gives*

$$(\gamma^r{}_j + \phi^s T^r{}_{sj} - \phi^r \phi_j - \phi^r{}_{,j})\epsilon_{rk} du^j du^k = 0$$

as the differential equation of the net C_2 cut out from S by the developables of Γ_2. Its canonical form is

$$[p - b_u + b(\log\theta)_u - b^2 + a\beta]du^2 - [b_v - a_u]dudv \\ - [q - a_v + a(\log\theta)_v - a^2 + b\gamma]dv^2 = 0$$

as given by Lane (p. 86). When the pair of directions defined by a net $A_{jk}du^j du^k = 0$ separates the asymptotic tangents with the directions $g_{jk}du^j du^k = 0$ harmonically, the net is called *conjugate*, the condition being the vanishing of the apolarity invariant, $-\epsilon^{jk}\epsilon^{lm}g_{jl}A_{km} = g^{rs}A_{rs} = 0$. For the nets of Γ_1 and Γ_2 curves this becomes $\epsilon^{rs}\phi_{r,s} = 0$ with the canonical form $b_v - a_u = 0$.

Comparison of the differential equations of the Γ_1 and Γ_2 curves shows that $\phi^j = (1/2)T^j$ are sufficient conditions that these curves coincide. The lines l_1 and l_2 which generate Γ_1 and Γ_2 in this case are denoted by d_1 and d_2 and are called the *directrices of Wilczynski* which were found by him in quite a different way.

29. **Equivalence of surfaces.** Let there be given a surface S parameterized by $y^A(u^1, u^2)$ and a second surface $\bar{S}$ parameterized by $\bar{y}^A(\bar{u}^1, \bar{u}^2)$. S and $\bar{S}$ will be called *equivalent* if there exist transformations of the groups G_1, G_2, G_3 of §1 sending the functions y^A into $\bar{y}^A$. *We seek the conditions for equivalence.* First, after the manner of §2 and its sequel, form the tensor equations defining S whose solutions are $y^A(u)$ and whose coefficients are unaffected by the transformations of G_3 and G_1. Do likewise for the functions $\bar{y}^A(\bar{u})$. Then the conditions for equivalence are that there exists a transformation $u^i = u^i(\bar{u})$ for which the respective coefficients are related by the tensor laws

$$|\partial u/\partial\bar{u}|\,\bar{g}_{jk} = g_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s, \qquad \bar{\gamma}_{jk} = \gamma_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s, \qquad \bar{T}^r{}_{jk}\bar{\partial}_r u^i = T^i{}_{rs}\bar{\partial}_j u^r \bar{\partial}_k u^s.$$

The law $\bar{\theta} = |\partial u/\partial\bar{u}|\,\theta$ is a consequence of the first and third of these. Represent $\bar{\partial}_j u^i$ by $H^i{}_j(\bar{u})$, then from (12.2) we are asking for the conditions that the system of first order partial differential equations

$$\bar{\partial}_j u^i = H^i{}_j, \qquad \partial_k H^i{}_j = \bar{\lambda}^r{}_{jk}H^i{}_r - \lambda^i{}_{rs}H^r{}_j H^s{}_k, \tag{29.1}$$

in the 6 dependent variables u^i and $H^i{}_j$ and the 2 independent variables $\bar{u}^i$ satisfying the finite conditional equations

$$\begin{aligned} \bar{\theta}(\bar{u})\bar{g}_{jk}(\bar{u}) - \theta(u)g_{rs}(u)H^r{}_j H^s{}_k &= 0, \\ \bar{\gamma}_{jk}(\bar{u}) - \gamma_{rs}(u)H^r{}_j H^s{}_k &= 0, \qquad \text{set } F_0 = 0; \\ \bar{T}^r{}_{jk}(\bar{u})H^i{}_r - T^i{}_{rs}(u)H^r{}_j H^s{}_k &= 0, \end{aligned} \tag{29.2}$$

admit solutions $u^i(\bar{u})$ and $H^i{}_j(\bar{u})$ for which $|H^i{}_j| \neq 0$.

Suppose the set $F_0=0$ to be differentiated with substitutions from (29.1) and $F_0=0$ itself. For example, differentiation of the first of (29.2) with substitutions from these equations themselves and from (29.1) results in the transformation law of the identically vanishing covariant derivative $(\theta g_{jk})_{,l}$. Hence no additional conditional equations result from this operation. If one or more new conditions arise when similarly operating on the full set $F_0=0$, designate this new set by $F_1=0$. Continuing with $F_1=0$ in this way we arrive at a sequence of sets of finite conditional equations $F_0=0, \cdots, F_N=0$, among the 6 dependent and 2 independent variables which must necessarily terminate for some N for which $N\leqq 5$ if the variables $\bar{u}^i$ are to be independent. From a theorem of partial differential equations [6, p. 3] it then follows that *the necessary and sufficient conditions that S and $\bar{S}$ be equivalent are that the sets $F_0=0, \cdots, F_N=0$ be algebraically consistent when regarded as equations for the determination of the dependent variables as functions of the independent variables $\bar{u}^i$ and that the set $F_{N+1}=0$ be satisfied identically as a consequence of the preceding sets. Moreover, if p is the number of independent equations in the first N sets, the transformation $\bar{u}^i\rightarrow u^i$ involves exactly $6-p$ arbitrary constants.* For example, the maximum possible number of independent equations in the set $F_0=0$ is $2+2+2=6$ in virtue of the identities $|g|=1$, $g^{rs}\gamma_{rs}=0$, $T^r{}_{jr}=0$ and the symmetry of T_{jkl}. In this case, the theorem states that the set $F_1=0$ must be satisfied identically as a consequence of $F_0=0$ and that the functions $u^i(\bar{u})$ are found by solving algebraically the set $F_0=0$ for the 6 dependent variables as functions of the 2 independent variables $\bar{u}^i$. The solution does not involve any arbitrary constants.

References

1. E. J. Wilczynski, *Projective differential geometry of curved surfaces* (*first memoir*), Trans. Amer. Math. Soc. vol. 8 (1907) pp. 233–260; *fourth memoir*, ibid. vol. 10 (1909) pp. 176–200.

2. E. P. Lane, *Projective differential geometry of curves and surfaces*, University of Chicago Press, 1932.

3. W. Blaschke, *Vorlesungen über Differentialgeometrie* II, Berlin, Springer, 1923.

4. O. Veblen, *Projektive Relativitätstheorie*, Ergebnisse der Mathematik, vol. 2.

5. T. Y. Thomas and A. D. Michal, *Differential invariants of relative quadratic differential forms*, Ann. of Math. (2) vol. 28 (1926–1927) pp. 631–688.

6. L. P. Eisenhart, *Continuous groups of transformations*, Princeton University Press, 1933.

Stanford University,
Stanford University, Calif.