A NEW CRITERION FOR COMPLETELY MONOTONIC FUNCTIONS

BY

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A function $f(x)$ is *completely monotonic* (*c.m.*) in $0<x<\infty$ if it belongs to C^∞ and

$$(-1)^k f^{(k)}(x) \geqq 0 \qquad (k \geqq 0,\ x > 0). \tag{1}$$

If $f(x)$ can be extended to be continuous at $x=0$ it is said to be c.m. in $0 \leqq x < \infty$.

Various conditions are known under which a function is c.m. [4](1). Bernstein proved that if

$$(-1)^k \Delta_h^k f(x) = \sum_0^k C_{k,n}(-1)^n f(x+nh) \geqq 0 \qquad (k \geqq 0,\ x > 0,\ h > 0) \tag{2}$$

then $f(x)$ is c.m. in $0<x<\infty$. It is known, though apparently not stated explicitly in the literature, that if we assume the continuity of $f(x)$, then we need require (2) only for some infinite sequence of integers k. (This may be obtained, for example, by use of the results of [1].) A fundamental theorem of Bernstein and Widder states that a function is c.m. in $0<x<\infty$ if and only if it admits the representation

$$f(x) = \int_0^\infty e^{-xt} dF(t), \qquad x > 0,\ F(t) \textit{ increasing}. \tag{3}$$

A new difference criterion which includes the above is suggested by the following considerations. If $h_k = o(1/k^2)$ then (3) is inverted by [3, Theorem 4.2],

$$F(t) - F(0) = f(\infty) + \lim_{k\to\infty} d_k \int_{k/t}^\infty x^{k-1} \Delta_{h_k}^k f(x)\,dx,$$

where

$$d_k = (-h_k)^{-k}/(k-1)!. \tag{4}$$

This suggests the following theorem, which is the principal result of this paper.

THEOREM. *Let $f(x)$ be continuous for $x \geqq 0$ and have a limit at infinity. Suppose it satisfies the inequalities*

Presented to the Society, September 13, 1943; received by the editors August 30, 1943, and, in revised form, November 1, 1943.

(1) Numbers in brackets refer to the references listed at the end of the paper.

$$(5) \qquad (-1)^k \Delta_{h_k}^k f(x) \geqq 0 \qquad (x > 0)$$

for an infinite sequence of integers k, where

$$h_k > 0, \qquad h_k = o(1/k^2) \qquad (k \to \infty).$$

Then $f(x)$ is c.m. in $0 \leqq x < \infty$.

This says essentially that in the difference criteria it is sufficient that the inequalities hold for *one* (suitable) value of h for each k, rather than *all* $h>0$.

Before proceeding to the proof we make some observations about the theorem.

(i) The conditions are trivially necessary.

(ii) The continuity of $f(x)$ is not a redundant condition. For let $\phi(x)$ be a discontinuous solution of the functional equation $\phi(x+y)=\phi(x)+\phi(y)$ [2, p. 96]. Both $\pm\phi(x)$ are convex and hence unbounded in every interval [2, pp. 91–92]. Then (5) is satisfied for arbitrarily small $\{h_k\}$ by the function $f(x)=e^{\phi(x)}$.

1. **Lemmas.** The following identity is known [4, p. 303].

LEMMA 1. *For $k>0$, $x \geqq 0$, $u>0$,*

$$\frac{\partial^k}{\partial u^k}\left[e^{-kx/u}u^{k-1}\right] = \frac{1}{u}\left(\frac{kx}{u}\right)^k e^{-kx/u}.$$

For fixed x, these functions are increasing in $0 \leqq u \leqq x/2$.

LEMMA 2. *If $g(x)$ and $r(x)$ are any functions of x then*

$$\Delta_h^k[g(x)r(x)] = \sum_0^k C_{k,n}\Delta_h^n g(x + \overline{k-n}\, h)\Delta_h^{k-n} r(x).$$

This is the analogue of Leibniz' rule for the differentiation of a product and can be established by induction.

LEMMA 3 (GENERALIZED ROLLE'S THEOREM). *If $f(x) \in C^k$ then*

$$\Delta_h^k f(x) = h^k f^{(k)}(X)$$

where X lies between x and $x+kh$.

LEMMA 4. *Suppose $k \geqq 1$, $x>0$, $h>0$ are fixed, and $f(u)$ is continuous for $u \geqq kh$, $f(\infty)=0$. Then*

$$\lim_{\epsilon \to 0} \int_{kh}^{\infty} \Delta_{-h}^k\left[e^{-\epsilon u} e^{-kx/u} u^{k-1}\right] f(u)\,du = \int_{kh}^{\infty} \Delta_{-h}^k\left[e^{-kx/u} u^{k-1}\right] f(u)\,du,$$

where we difference with respect to u.

Proof. The existence of the integral on the left is guaranteed by the presence of the factor $e^{-\epsilon u}$. The integral on the right converges since, by virtue of Lemmas 3 and 1, its integrand is dominated for large u by $(u-kh)^{-k-1}|f(u)|$.

If we subtract the right-hand side the problem is then to show that

$$H(\epsilon) = \int_{kh}^{\infty} \Delta_{-h}^{k}[g(u)r(u)]f(u)du = o(1) \qquad (\epsilon \to 0+), \tag{6}$$

where

$$g(u) = e^{-kx/u}u^{k-1}, \quad r(u) = 1 - e^{-\epsilon u}.$$

Using Lemma 2 with h replaced by $-h$ and separating out the term for which $n=k$ we obtain

$$\begin{aligned} H(\epsilon) &= \sum_{n=0}^{k-1} C_{k,n} \int_{kh}^{\infty} \Delta_{-h}^{n} g(u - \overline{k-n}\, h)\cdot \Delta_{-h}^{k-n} r(u)\cdot f(u)du \\ &\quad + \int_{kh}^{\infty} \Delta_{-h}^{k} g(u)\cdot r(u)\cdot f(u)du \\ &= K(\epsilon) + L(\epsilon). \end{aligned} \tag{7}$$

By Lemmas 3 and 1 the integral $\int_{kh}^{\infty}\Delta_{-h}^{k}g(u)\cdot f(u)du$ exists. Hence

$$L(\epsilon) = \int_{kh}^{\infty} (1 - e^{-\epsilon u})\Delta_{-h}^{k} g(u)f(u)du = o(1) \qquad (\epsilon \to 0+).$$

We turn now to the expression $K(\epsilon)$. First by Lemma 3

$$\Delta_{-h}^{k-n} r(u) = (-h)^{k-n} r^{(k-n)}(U) = (h\epsilon)^{k-n} e^{-\epsilon U}, \qquad u - (k-n)h \leqq U \leqq u,$$

so that for $0<\epsilon<1$

$$|\Delta_{-h}^{k-n} r(u)| \leqq K_1 \epsilon^{k-n} e^{-\epsilon u}, \tag{8}$$

where K_1 does not depend on u or ϵ. Also

$$\Delta_{-h}^{n} g(u - \overline{k-n}\, h) = (-h)^{n} g^{(n)}(U), \qquad u - kh \leqq U \leqq u - (k-n)h. \tag{9}$$

But

$$g^{(n)}(u) = \sum_{j=0}^{n} C_{n,j} \frac{\partial^j}{\partial u^j}[e^{-kx/u}][u^{k-1}]^{(n-j)}.$$

If $x>0$

$$\frac{\partial^j}{\partial u^j}[e^{-kx/u}] = O(u^{-j}) \qquad (u \to \infty),$$

so that

$$g^{(n)}(u) = \sum O(u^{-j}u^{k-1-n+j}) = O(u^{k-n-1}), \qquad 0 \leqq n \leqq k-1.$$

If $x=0$ this result is obvious. Then by (9)

$$(10) \qquad \left| \Delta_{-h}^{n} g(u - \overline{k-n}\, h) \right| \leqq K_2 u^{k-n-1} \qquad (0 \leqq n \leqq k-1),$$

where K_2 does not depend on u. Then by (7), (8), (10)

$$\left| K(\epsilon) \right| \leqq K_1 K_2 \sum_{n=0}^{k-1} C_{k,n} \int_{kh}^{\infty} \epsilon^{k-n} e^{-\epsilon u} u^{k-n-1} \left| f(u) \right| du \qquad (0 < \epsilon < 1).$$

Since $f(\infty)=0$ a simple Abelian argument proves that each term of the sum approaches zero with ϵ. Hence $K(0+)=0$, so $H(0+)=0$; this establishes (6).

LEMMA 5. *Let $k \geqq 1$, $h>0$, $x \geqq 0$ be fixed, $f(u)$ continuous, $0 \leqq u < \infty$, $f(\infty)=0$. If $\Delta_h^k f(u)$ does not change sign in $0 \leqq u < \infty$, then*

$$(11) \qquad \begin{aligned} &\int_0^{\infty} e^{-kx/u} u^{k-1} \Delta_h^k f(u) du \\ &= \sum_{n=0}^{k} C_{k,n} (-1)^{k-n} \int_{nh}^{kh} e^{-kx/(u-nh)} (u-nh)^{k-1} f(u) du \\ &\quad + \int_{kh}^{\infty} \Delta_{-h}^{k} [e^{-kx/u} u^{k-1}] \cdot f(u) du, \end{aligned}$$

where we difference with respect to u.

Clearly the integral

$$I(\epsilon) = \int_0^{\infty} e^{-\epsilon u} e^{-kx/u} u^{k-1} \Delta_h^k f(u) du$$

exists for all $\epsilon > 0$. We have

$$\begin{aligned} I(\epsilon) &= \sum_0^k C_{k,n} (-1)^{k-n} \int_0^{\infty} e^{-\epsilon u} e^{-kx/u} u^{k-1} f(u+nh) du \\ &= \sum_0^k C_{k,n} (-1)^{k-n} \left(\int_{nh}^{kh} + \int_{kh}^{\infty} \right) \exp\left[-\epsilon(u-nh) - kx/(u-nh) \right] \\ &\qquad \cdot (u-nh)^{k-1} f(u) du, \end{aligned}$$

obtained by a change of variable. Then

$$(12) \qquad I(\epsilon) = A(\epsilon) + B(\epsilon),$$

where

$$A(\epsilon) = \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh}^{kh} \exp\left[-\epsilon(u-nh) - kx/(u-nh) \right] \cdot (u-nh)^{k-1} f(u) du,$$

$$B(\epsilon) = \int_{kh}^{\infty} \Delta_{-h}^{k} [e^{-\epsilon u} e^{-kx/u} u^{k-1}] f(u) du.$$

By dominated convergence

$$A(0+) = \sum_0^k C_{k,n}(-1)^{k-n} \int_{nh}^{kh} e^{-kx/(u-nh)}(u-nh)^{k-1} f(u)du, \tag{13}$$

and by Lemma 4

$$B(0+) = \int_{kh}^{\infty} \Delta_{-h}^k [e^{-kx/u} u^{k-1}] \cdot f(u)du. \tag{14}$$

From (12), (13), (14) it follows that $I(0+)$ exists and equals the expression on the right-hand side of (11). Since $\Delta_h^k f(u)$ does not change sign in $0 \leqq u < \infty$, a Tauberian theorem enables us to conclude that $I(0+)$ is also equal to the left-hand side of (11) [4, p. 192].

For the remainder of the paper it is assumed that k belongs to some sequence S of non-negative integers; $k \to \infty$ means that k becomes infinite through the elements of S.

LEMMA 6. *Let $f(x)$ satisfy all the hypotheses of our theorem and suppose also that $f(\infty)=0$. Then for any fixed $x>0$ the quantities*

$$I_k = d_k \int_0^{\infty} e^{-kx/u} u^{k-1} \Delta_{h_k}^k f(u)du$$

approach $f(x)$ as $k \to \infty$. The d_k are defined as in (4).

Proof. By Lemma 5 we have $I_k = A_k + B_k$ where

$$A_k = d_k \sum_0^k C_{k,n}(-1)^{k-n} \int_{nh_k}^{kh_k} e^{-kx/(u-nh_k)}(u-nh_k)^{k-1} f(u)du,$$

$$B_k = d_k \int_{kh_k}^{\infty} \Delta_{-h_k}^k [e^{-kx/u} u^{k-1}] f(u)du.$$

We show first that A_k vanishes with $1/k$. Let M be the maximum of $|f(x)|$ in $(0, \infty)$. Then

$$\begin{aligned}
|A_k| &\leqq M|d_k| \sum_0^k C_{k,n} \int_{nh_k}^{kh_k} e^{-kx(u-nh_k)}(u-nh_k)^{k-1}du \\
&\leqq M|d_k| \sum C_{k,n} \int_0^{(k-n)h_k} e^{-kx/u} u^{k-1}du \\
&\leqq M|d_k| \sum C_{k,n} \int_0^{kh_k} e^{-kx/u} u^{k-1}du \\
&\leqq M|d_k| e^{-x/h_k} \sum_0^k C_{k,n} \int_0^{kh_k} u^{k-1}du \\
&= Me^{-x/h_k} 2^k k^k / k!.
\end{aligned}$$

Choose k_0 so that $kh_k < x/2$ for $k > k_0$. Then for $k > k_0$

$$|A_k| \leqq Me^{-2k}2^k k^k/k!.$$

By the test-ratio test this last is the general term of a convergent series, so that $A_k = o(1)$, $k \to \infty$.

We must prove then that

$$\lim_{k\to\infty} B_k = f(x).$$

By Lemmas 3 and 1, and equation (4),

$$\begin{aligned} B_k &= \frac{1}{(k-1)!}\left(\int_{x/2}^{\infty} + \int_{kh_k}^{x/2}\right)\left[\frac{1}{u}\left(\frac{kx}{u}\right)^k e^{-kx/u}\right]_{u=v-\phi_k} f(v)dv \\ &= C_k + D_k, \end{aligned}$$

where

$$0 \leqq \phi_k \leqq kh_k. \tag{15}$$

By the second part of Lemma 1

$$\begin{aligned} |D_k| &\leqq \frac{1}{(k-1)!}\left[\frac{1}{u}\left(\frac{kx}{u}\right)^k e^{-kx/u}\right]_{u=x/2} \int_0^{x/2} |f(v)|\, dv \\ &= o(1) \qquad (k \to \infty). \end{aligned}$$

Our problem then is to show that $\lim_{k\to\infty} C_k = f(x)$. But it is known that [4, p. 283]

$$J_k = \frac{1}{(k-1)!}\int_{x/2}^{\infty} \frac{1}{u}\left(\frac{kx}{u}\right)^k e^{-kx/u} f(u)du \to f(x), \qquad k \to \infty.$$

It therefore remains only to show that $\lim_{k\to\infty}(J_k - C_k) = 0$ and the proof of the lemma will be complete.

Now

$$J_k - C_k = \frac{1}{(k-1)!}\int_{x/2}^{\infty} \frac{1}{u}\left(\frac{kx}{u}\right)^k e^{-kx/u} f(u) P_k(u)du, \tag{16}$$

where

$$P_k(u) = 1 - \left(\frac{u}{u-\phi_k}\right)^{k+1} \exp\left(-\frac{kx\phi_k}{u(u-\phi_k)}\right) \qquad (u \geqq x/2).$$

We have

$$\log(1 - P_k(u)) = (k+1)\int_{u-\phi_k}^{u} \frac{dv}{v} - \frac{kx\phi_k}{u(u-\phi_k)},$$

$$|\log(1 - P_k(u))| \leqq \frac{(k+1)\phi_k}{u-\phi_k} + \frac{kx\phi_k}{u(u-\phi_k)}.$$

By (15) we have $0 \leqq \phi_k \leqq kh_k$. Since $h_k = o(1/k^2)$, $\phi_k < x/4$ for $k > k_0$. It follows that for $u \geqq x/2$, $k > k_0$,

$$\begin{aligned} |\log(1 - P_k(u))| &\leqq \frac{(k+1)kh_k}{x/2 - x/4} + \frac{kx \cdot kh_k}{(x/2)(x/2 - x/4)} \\ &= o(1) \qquad (k \to \infty) \end{aligned}$$

uniformly for $u \geqq x/2$.

Let $\epsilon > 0$ be arbitrary. Then for $k > k_1$,

$$|P_k(u)| < \epsilon \qquad (u \geqq x/2).$$

By (16)

$$|J_k - C_k| < \frac{\epsilon}{(k-1)!} \int_{x/2}^{\infty} \frac{1}{u} \left(\frac{kx}{u}\right)^k e^{-kx/u} |f(u)| \, du.$$

As $k \to \infty$ the right-hand side approaches $\epsilon |f(x)|$ [4, p. 283]. Hence

$$\limsup_{k \to \infty} |J_k - C_k| \leqq \epsilon |f(x)|,$$

and this completes the proof.

2. **Proof of the theorem.** We may assume $f(\infty) = 0$ (otherwise consider $f(x) - f(\infty)$). By hypothesis

$$L_{k,t}[f] = ((-h_k)^{-k}/k!)[x^{k+1} \Delta_{h_k}^k f(x)]_{x=k/t} \geqq 0$$

for an infinite sequence of integers k. By Lemma 6 with a change of variable we have

$$f(x) = \lim_{k \to \infty} \int_0^{\infty} e^{-xt} L_{k,t}[f] dt \qquad (x > 0).$$

It remains only to show that the integrals

$$L_k = \int_0^{\infty} L_{k,t}[f] dt \tag{17}$$

exist and are bounded. For then it will follow by a familiar argument that $f(x)$ has the representation (3) [4, p. 307]. From Lemma 5 with $x = 0$ it follows that

$$\begin{aligned} L_k &= d_k \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh_k}^{kh_k} (u - nh_k)^{k-1} f(u) du \\ &= (-h_k)^{-k} \Delta_{h_k}^k F_k(0) \end{aligned}$$

where

$$F_k(x) = \frac{1}{(k-1)!} \int_x^{kh_k} (u - x)^{k-1} f(u) du.$$

By Lemma 3

$$L_k = (-1)^k F_k^{(k)}(X_k), \qquad 0 \leqq X_k \leqq kh_k.$$

But $F_k^{(k)}(x) = (-1)^k f(x)$, so that $L_k = f(X_k)$ and $\lim_{k\to\infty} L_k = f(0)$. Then L_k is bounded.

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