NOTE ON THE STRONG SUMMABILITY OF FOURIER SERIES

BY

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1. Let $S_n(x)$ $(n=0, 1, 2, \cdots)$ be the partial sums of the Fourier series of a function $f(x)\in L^p$ $(-\pi, \pi)$. Put

$$2\phi(t) = f(x+t) + f(x-t) - 2S, \tag{1.1}$$

and let

$$\Phi(t) = \int_0^t |\phi(u)|^p du = o(t). \tag{1.2}$$

Hardy and Littlewood have proved the following two theorems:

A(1). *If* $p>1$, *then* (1.2) *gives*

$$\sum_{m=0}^{n} |S_m - S|^2 = o(n). \tag{1.3}$$

B(2). *If* $p=1$, *then* (1.2) *gives*

$$\sum_{m=0}^{n} |S_m - S|^2 = o(n \log n). \tag{1.4}$$

In my previous paper(3), I have replaced S_m in (1.4) by S_{m^k} $(k=2, 3, \cdots)$. The object of the present paper is to show that the S_m in (1.3) can also be replaced by the lacunary partial sums S_{m^k}, if (1.2) holds with $p>k$. In other words:

If p is greater than a positive integer k, and

$$\int_0^t |\phi(u)|^p du = o(t), \tag{1.5}$$

then

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(1) G. H. Hardy and J. E. Littlewood, *Notes on the theory of series*. IV. *On the strong summability of Fourier series*, Proc. London Math. Soc. (2) vol. 26 (1926) pp. 273–286.

(2) G. H. Hardy and J. E. Littlewood, *The strong summability of Fourier series*, Fund. Math. vol. 25 (1935) pp. 162–189.

(3) C. T. Loo, *Note on the strong summability of Fourier series*, Science Record Academia Sinica vol. 1 (1942) pp. 76–83.

$$\sum_{m=0}^{n} | S_{m^k} - S |^2 = o(n). \tag{1.6}$$

Without loss of generality, we may suppose that

$$f(t) \sim \sum_{1}^{\infty} a_n \cos nt$$

is an even function with zero mean value, and that $x=0$, $S=0$, so that

$$\phi(t) = f(t), \qquad S_m = a_1 + a_2 + \cdots + a_m.$$

We write

$$\begin{aligned} \pi S_{m^k} &= \int_0^\pi \frac{\sin (m^k + 1/2)t}{\sin (t/2)} f(t)dt \\ &= \int_0^\pi \cos m^k t f(t)dt + \int_0^{1/n^k} \sin m^k t \cot (t/2) f(t)dt \\ &\quad + \int_{1/n^k}^{1/n} \sin m^k t \cot (t/2) f(t)dt + \int_{1/n}^{\pi} \sin m^k t \cot (t/2) f(t)dt \\ &= \alpha_m + \beta_m + \gamma_m + \delta_m. \end{aligned} \tag{1.7}$$

It suffices to prove the sum $\sum_{m=0}^{n}(\alpha_m^2+\beta_m^2+\gamma_m^2+\delta_m^2)$ is equal to $o(n)$.

2. It is plain that $\alpha_m = o(1)$, so that

$$\sum_{0}^{n} \alpha_m^2 = o(n). \tag{2.1}$$

Hereafter, we shall write briefly

$$\xi = 1/n, \qquad \eta = \pi/kn^{k-1}, \qquad \zeta = 1/n^k. \tag{2.2}$$

If

$$F(t) = \int_0^t | f(u) |^p du = o(t) \qquad (p > 1), \tag{2.3}$$

then the same relation holds true when p is replaced by any positive small index. Hence we have

$$| \beta_m | = \left| \int_0^{1/n^k} \sin m^k t \cot (t/2) f(t)dt \right| \leqq m^k \int_0^{1/n^k} | f(t) | \, dt = o(1), \tag{2.4}$$

and

$$\sum_{0}^{n} \beta_m^2 = o(n). \tag{2.5}$$

Assuming

$$r > 2, \qquad r' = r/(r-1) < p,$$

we are going to prove

$$\sum_0^n |\delta_m|^r = o(n). \tag{2.6}$$

This is stronger than

$$\sum_1^n \delta_m^2 = o(n). \tag{2.7}$$

We denote by $C_n(\tau)$ the nth Fourier sine coefficient of the odd function $\chi(t)$, which is equal to $f(t)$ in $(0, \tau)$ and to zero in (τ, π). We have then

$$\begin{aligned} \delta_m &= \int_\xi^\pi \sin m^k t \cot (t/2) f(t) dt \\ &= \int_\xi^\pi \cot (t/2) \left(\frac{d}{dt} \int_0^t \sin m^k u f(u) du \right) dt \\ &= -(\pi/2) \cot (\xi/2) C_{mk}(\xi) + (\pi/4) \int_\xi^\pi \csc^2 (t/2) C_{mk}(t) dt. \end{aligned} \tag{2.8}$$

From $r' < p$, it follows by Hausdorff's inequality that

$$\begin{aligned} \left(\sum_1^n |C_{mk}(t)|^r \right)^{1/r} &\leqq \left(\frac{1}{\pi} \int_{-\pi}^\pi |\chi(u)|^{r'} du \right)^{1/r'} \\ &= \left(\frac{1}{\pi} \int_{-t}^t |f(u)|^{r'} du \right)^{1/r'} = o(t^{1/r'}). \end{aligned} \tag{2.9}$$

Hence

$$\begin{aligned} \left(\sum_1^n |\delta_m|^r \right)^{1/r} &= o\left(\frac{1}{\xi} \xi^{1/r'} \right) + o\left(\int_\xi^\pi \frac{1}{t^2} t^{1/r'} dt \right) \\ &= o(\xi^{-1/r}) = o(n^{1/r}). \end{aligned} \tag{2.10}$$

This establishes (2.6).

3. By the theorem of Riemann and Lebesgue, we have

$$\begin{aligned} \gamma_m &= \int_\zeta^\xi \sin m^k t \cot (t/2) f(t) dt \\ &= 2 \int_\zeta^\xi \frac{\sin m^k t}{t} f(t) dt + o(1). \end{aligned} \tag{3.1}$$

Hence

$$
\begin{aligned}
\sum_1^n \gamma_m^2 &= 4\int_\zeta^\xi \frac{f(u)}{u}\,du \int_\zeta^\xi \frac{f(v)}{v} \sum_1^n \sin m^k u \sin m^k v dv + o(n) \\
&= 2\int_\zeta^\xi \frac{f(u)}{u}\,du \int_\zeta^\xi \frac{f(v)}{v} \sum_1^n \cos m^k(u-v)dv \\
&\quad - 2\int_\zeta^\xi \frac{f(u)}{u}\,du \int_\zeta^\xi \frac{f(v)}{v} \sum_1^n \cos m^k(u+v)dv + o(n) \\
&= I_1 + I_2 + o(n).
\end{aligned}
\tag{3.2}
$$

We shall use the fact that if

$$J_n(t) = \sum_{m=0}^n e^{imkt},$$

then

$$\left|\sum_1^n \cos m^k t\right| \leqq |J_n(t)|. \tag{3.3}$$

We also require the following lemmas:

LEMMA 1. *Let a, b and k be three integers such that $a<b$, $k\geqq 2$. Let $g(x)$ be a real function having differential coefficients of the first k orders. If $Rg^{(k)}(x)\geqq 1$ (or $\leqq -1$) throughout the interval (a, b), write*

$$K = 2^k, \qquad P = R\left|g^{(k-1)}(b) - g^{(k-1)}(a)\right|, \tag{3.4}$$

where R is a positive number. Then

$$\left|\sum_{m=a}^b e^{2\pi i g(m)}\right| \leqq 100P\left\{R^{-1/(K-2)} + R^{2/K}P^{-2k/K} + P^{-2/K}\right\}. \tag{3.5}$$

This theorem is due to Van der Corput(4).

LEMMA 2. *If $t>0$, then*

$$J_n(t) = O(nt^{1/(K-2)}) + O(n^{1-2k/K}t^{-2/K}) + O(n^{1-2/K}). \tag{3.6}$$

Put $a=0$, $b=n$, $g(x)=x^k t/2\pi$ $(t>0)$, $R^{-1}=k!t/2\pi$. We have

$$g^{(k-1)}(x) = k!xt/2\pi, \qquad g^{(k)}(x) = k!t/2\pi = 1/R > 0, \qquad P = n.$$

Lemma 2 is therefore an immediate corollary of Lemma 1.

LEMMA 3. *Let $g(x)$ be a real differentiable function in the interval (a, b), $g'(x)$ be montonic in this interval, and $|g'(x)|<1/2$. Then*

(4) J. G. Van der Corput, *Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunktprobleme*, Math. Zeit. vol. 28 (1928) p. 303.

$$\sum_{a\leqq n\leqq b} e^{2\pi i g(n)} = \int_a^b e^{2\pi i g(x)}dx + O(1). \tag{3.7}$$

This theorem is due to Titchmarsh(5).

LEMMA 4. *If* $0<t\leqq \pi/kn^{k-1}=\eta$, *then*

$$J_n(t) = O(1/t^{1/k}). \tag{3.8}$$

Put $a=0$, $b=n$, $g(x)=x^k t/2\pi$ $(t>0)$. Evidently, $g'(x)=kx^{k-1}t/2\pi$ is monotone in the interval $(0, \pi)$ and $0<g'(x)<1/2$. Hence, by Lemma 3,

$$\begin{aligned} J_n(t) &= \sum_0^n e^{imkt} = \int_0^n e^{ixkt}dx + O(1) \\ &= \frac{1}{kt^{1/k}}\int_0^{nkt} e^{iu}u^{1/k-1}du + O(1) = O\left(\frac{1}{t^{1/k}}\right). \end{aligned}$$

4. We are now in a position to estimate $\sum_1^n \gamma_m^2$. Firstly, remembering $\xi=1/n$, $\zeta=1/n^k$, we have

$$\begin{aligned} I_{21} &= n\int_\zeta^\xi \frac{|f(u)|}{u}du\int_\zeta^\xi \frac{|f(v)|}{v}(u+v)^{1/(K-2)}dv \\ &\leqq n\int_\zeta^\xi u^{-1+1/(K-2)}|f(u)|\,du\int_\zeta^\xi \frac{|f(v)|}{v}dv \\ &\quad + n\int_\zeta^\xi \frac{|f(u)|}{u}du\int_\zeta^\xi v^{-1+1/(K-2)}|f(v)|\,dv \\ &= o(n^{1-1/(K-2)}\log n) = o(n), \\ I_{22} &= n^{1-2k/K}\int_\zeta^\xi \frac{|f(u)|}{u}du\int_\zeta^\xi \frac{|f(v)|}{v}(u+v)^{-2/K}dv \\ &\leqq n^{1-2k/K}\int_\zeta^\xi u^{-1-1/K}|f(u)|\,du\int_\zeta^\xi v^{-1-1/K}|f(v)|\,dv \\ &= o(n^{1-2k/K}\cdot n^{k/K}\cdot n^{k/K}) = o(n), \\ I_{23} &= n^{1-2/K}\int_\zeta^\xi \frac{|f(u)|}{u}du\int_\zeta^\xi \frac{|f(v)|}{v}dv \\ &= o(n^{1-2/K}\cdot(\log n)^2) = o(n). \end{aligned}$$

Observing the definition of I_2 in (3.2) it follows from (3.3) and (3.6) that

(5) E. C. Titchmarsh, *On Van der Corput's method and the zeta-function of Riemann*, Quart. J. Math. vol. 2 (1931) pp. 161–173.

$$I_2 = O(I_{21} + I_{22} + I_{23}) = o(n). \tag{4.1}$$

Secondly, observing (3.2) and (3.3) we have

$$\begin{aligned}
\frac{|I_1|}{4} &\leqq \int_\zeta^\xi \frac{|f(u)|}{u}\,du \int_\zeta^u \frac{|f(v)|}{v} \left| \sum_0^n \cos m^k(u-v) \right| dv \\
&\leqq \int_\zeta^{2\zeta} \frac{|f(u)|}{u}\,du \int_\zeta^u \frac{|f(v)|}{v} |J_n(u-v)|\,dv \\
&\quad + \int_{2\zeta}^\xi \frac{|f(u)|}{u}\,du \int_{u-\zeta}^u \frac{|f(v)|}{v} |J_n(u-v)|\,dv \\
&\quad + \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u}\,du \int_\zeta^{u-\zeta} \frac{|f(v)|}{v} |J_n(u-v)|\,dv \\
&\quad + \int_{\eta+\zeta}^\xi \frac{|f(u)|}{u}\,du \int_{u-\eta}^{u-\zeta} \frac{|f(v)|}{v} |J_n(u-v)|\,dv \\
&\quad + \int_{\eta+\zeta}^\xi \frac{|f(u)|}{u}\,du \int_\zeta^{u-\eta} \frac{|f(v)|}{v} |J_n(u-v)|\,dv \\
&= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned} \tag{4.2}$$

Writing

$$\begin{aligned}
I_{11} &= \int_\zeta^{2\zeta} \frac{|f(u)|}{u^2}\,du \int_\zeta^u |f(v)|\,|J_n(u-v)|\,dv \\
&\quad + \int_\zeta^{2\zeta} \frac{|f(u)|}{u^2}\,du \int_\zeta^u \frac{|f(v)|}{v}(u-v)\,|J_n(u-v)|\,dv \\
&= I_{111} + I_{112},
\end{aligned}$$

we have

$$I_{111} \leqq n \int_\zeta^{2\zeta} \frac{|f(u)|}{u}\,du \int_\zeta^u |f(v)|\,dv,$$

and

$$\begin{aligned}
\int_\zeta^u |f(v)|\,dv &\leqq \left(\int_\zeta^u |f(v)|^p dv\right)^{1/p} \left(\int_\zeta^u dv\right)^{1/q} \\
&= o(u^{1/p}(u-\zeta)^{1/q}) = o(u^{1/p}\zeta^{1/q}) = o(n^{-k/q}u^{1/p}).
\end{aligned}$$

since $u \leqq 2\zeta$. It follows that

$$\begin{aligned}
I_{111} &\leqq n^{1-k/q} \int_\zeta^{2\zeta} u^{-2+1/p} |f(u)|\,du = n^{1-k/q} \int_\zeta^{2\zeta} u^{-1-1/q} |f(u)|\,du \\
&= o(n^{1-k/q}\zeta^{-1/q}) = o(n^{1-k/q}n^{k/q}) = o(n).
\end{aligned}$$

Again, from

$$I_{112} \leqq n^{1-k} \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^{u} \frac{|f(v)|}{v} dv,$$

and

$$\int_{\zeta}^{u} \frac{|f(v)|}{v} dv = o\left(\log \frac{u}{\zeta}\right) = o(\log 2) = o(1),$$

we obtain

$$I_{112} \leqq n^{1-k} \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u^2} du = o(n^{1-k}n^k) = o(n).$$

Therefore

$$I_{11} = O(I_{111} + I_{112}) = o(n). \tag{4.3}$$

The same method of estimation can be applied to I_{12}, by assuming $2\zeta \leqq u$. Thus we obtain

$$I_{12} = o(n). \tag{4.4}$$

Apply Lemma 2 to I_{15}. We have

$$\begin{aligned}
I_{151} &= n \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} (u-v)^{1/(K-2)} dv \\
&\leqq n \int_{\eta+\zeta}^{\xi} u^{-1+1/(K-2)} |f(u)| du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\
&= o(n^{1-1/(K-2)} \log n) = o(n), \\
I_{152} &= n^{1-2k/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} (u-v)^{-2/K} dv \\
&\leqq n^{1-2k/K} \eta^{-2/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\
&= o(n^{1-2k/K} \cdot n^{(k-1)2/K} \cdot (\log n)^2) \\
&= o(n^{1-2/K} (\log n)^2) = o(n), \\
I_{153} &= n^{1-2/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\
&= o(n^{1-2/K} (\log n)^2) = o(n).
\end{aligned}$$

It follows that

$$I_{15} = O(I_{151} + I_{152} + I_{153}) = o(n). \tag{4.5}$$

Observe that in I_{13}

$$u \leqq \eta + \zeta \quad \text{and} \quad \zeta \leqq v \leqq u - \zeta,$$

or

$$0 < \zeta \leqq u - v \leqq u - \zeta \leqq \eta.$$

Using Lemma 4, we have

$$\begin{aligned} I_{13} &\leqq \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} (u-v)^{-1/k} dv \\ &= \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} (u-v)^{1-1/k} dv \\ &\quad + \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv \\ &= I_{131} + I_{132}. \end{aligned}$$

From

$$I_{131} \leqq \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k} |f(u)| du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} dv,$$

and

$$\begin{aligned} \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} dv &\leqq \left(\int_{\zeta}^{u-\zeta} |f(v)|^p dv \right)^{1/p} \left(\int_{\zeta}^{u-\zeta} v^{-q} dv \right)^{1/q} \\ &= o(u^{1/p} \zeta^{-1+1/q}) = o(u^{1/p} n^{k-k/q}) = o(u^{1/p} n^{k/p}), \end{aligned}$$

we obtain

$$\begin{aligned} I_{131} &\leqq n^{k/p} \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k+1/p} |f(u)| du \\ &= o(n^{k/p} \zeta^{-1/k+1/p}) = o(n^{k/p} n^{1-k/p}) = o(n). \end{aligned}$$

Again, from

$$I_{132} = \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv$$

and

$$\begin{aligned} \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv &\leqq \left(\int_{\zeta}^{u-\zeta} |f(v)|^p dv \right)^{1/p} \left(\int_{\zeta}^{u-\zeta} (u-v)^{-q/k} \right)^{1/q} \\ &= o(u^{1/p} (u-\zeta)^{-1/k+1/q}) = o(u^{1-1/k}), \end{aligned}$$

since $-1/k+1/q>0$, we obtain

$$\begin{aligned} I_{132} &\leqq \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k} |f(u)| du \\ &= o(\zeta^{-1/k}) = o(n). \end{aligned}$$

Therefore

$$I_{13} = O(I_{131} + I_{132}) = o(n). \tag{4.6}$$

Finally, by applying Lemma 4 to I_{14} we easily deduce that

$$I_{14} = o(n). \tag{4.7}$$

Combining (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) we get

$$I_1 = O(I_{11} + I_{12} + I_{13} + I_{14} + I_{15}) = o(n). \tag{4.8}$$

It follows from (3.2), (4.1) and (4.8) that

$$\sum_1^n \gamma_m^2 = o(n). \tag{4.9}$$

Summarizing (2.1), (2.5), (2.7) and (4.9) we obtain

$$\sum_1^n (\alpha_m^2 + \beta_m^2 + \gamma_m^2 + \delta_m^2) = o(n).$$

The theorem is thus proved.

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