NOTE ON THE STRONG SUMMABILITY OF FOURIER SERIES

BY

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1. Let $S_n(x)$ ($n = 0, 1, 2, \cdots$) be the partial sums of the Fourier series of a function $f(x) \in L^p (-\pi, \pi)$. Put

$$2\phi(t) = f(x + t) + f(x - t) - 2S,$$

and let

$$\Phi(t) = \int_0^t |\phi(u)|^p du = o(t).$$

Hardy and Littlewood have proved the following two theorems:

A. If $p > 1$, then (1.2) gives

$$\sum_{m=0}^n |S_m - S|^2 = o(n).$$

B. If $p = 1$, then (1.2) gives

$$\sum_{m=0}^n |S_m - S|^2 = o(n \log n).$$

In my previous paper, I have replaced $S_m$ in (1.4) by $S_{m_k}$ ($k = 2, 3, \cdots$). The object of the present paper is to show that the $S_m$ in (1.3) can also be replaced by the lacunary partial sums $S_{m_k}$, if (1.2) holds with $p > k$. In other words:

If $p$ is greater than a positive integer $k$, and

$$\int_0^t |\phi(u)|^p du = o(t),$$

then

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(3) C. T. Loo, Note on the strong summability of Fourier series, Science Record Academia Sinica vol. 1 (1942) pp. 76–83.
Without loss of generality, we may suppose that
\[ f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt \]
is an even function with zero mean value, and that \( x = 0, \ S = 0, \) so that \( \phi(t) = f(t), \quad S_m = a_1 + a_2 + \cdots + a_m. \)

We write
\[
\pi S_m = \int_{0}^{\pi} \frac{\sin (m^k + 1/2)t}{\sin (t/2)} f(t) dt
\]
\[
= \int_{0}^{\pi} \cos m^k t f(t) dt + \int_{0}^{1/n^k} \sin m^k t \cot (t/2) f(t) dt + \int_{1/n^k}^{1/n} \sin m^k t \cot (t/2) f(t) dt
\]
\[
= \alpha_m + \beta_m + \gamma_m + \delta_m.
\]
It suffices to prove the sum \( \sum_{m=0}^{n} (\alpha_m^2 + \beta_m^2 + \gamma_m^2 + \delta_m^2) \) is equal to \( o(n). \)

2. It is plain that \( \alpha_m = o(1), \) so that
\[
(2.1) \quad \sum_{0}^{n} \alpha_m^2 = o(n).
\]

Hereafter, we shall write briefly
\[
(2.2) \quad \xi = 1/n, \quad \eta = \pi/kn^{k-1}, \quad \xi = 1/n^k.
\]

If
\[
(2.3) \quad F(t) = \int_{0}^{t} |f(u)|^p du = o(t) \quad (\phi > 1),
\]
then the same relation holds true when \( \phi \) is replaced by any positive small index. Hence we have
\[
(2.4) \quad |\beta_m| = \left| \int_{0}^{1/n^k} \sin m^k t \cot (t/2) f(t) dt \right| \leq m^k \int_{0}^{1/n^k} |f(t)| dt = o(1),
\]
and
\[
(2.5) \quad \sum_{0}^{n} \beta_m^2 = o(n).
\]
Assuming
\[ r > 2, \quad r' = r/(r - 1) < p, \]
we are going to prove
\[ \sum_0^n |\delta_m|^{r'} = o(n). \]
This is stronger than
\[ \sum_1^n \delta_m^2 = o(n). \]

We denote by \( C_n(r) \) the \( n \)-th Fourier sine coefficient of the odd function \( \chi(t) \), which is equal to \( f(t) \) in \((0, \tau)\) and to zero in \((\tau, \pi)\). We have then
\[
\begin{align*}
\delta_m &= \int_\xi^\tau \sin m^k t \cot (t/2)f(t)dt \\
&= \int_\xi^\tau \cot (t/2) \left( \frac{d}{dt} \right) \int_0^t \sin m^k u f(u)du dt \\
&= - (\pi/2) \cot (\xi/2) C_m(\xi) + (\pi/4) \int_\xi^\tau \csc^2 (t/2) C_m(\xi) dt.
\end{align*}
\]

From \( r' < p \), it follows by Hausdorff's inequality that
\[
\left( \sum_1^n |C_m(\xi)|^r \right)^{1/r} \leq \left( \frac{1}{\pi} \int_\xi^\tau |\chi(u)|^{r'} du \right)^{1/r'} \\
= \left( \frac{1}{\pi} \int_{-\xi}^\tau |f(u)|^{r'} du \right)^{1/r'} = o(t^{1/r'}).
\]

Hence
\[
\left( \sum_1^n |\delta_m|^r \right)^{1/r} = o \left( \frac{1}{\xi} t^{1/r'} \right) + o \left( \int_\xi^\tau \frac{1}{t^2} t^{1/r'} dt \right) \\
= o(\xi^{-1/r}) = o(n^{1/r}).
\]

This establishes (2.6).

3. By the theorem of Riemann and Lebesgue, we have
\[
\gamma_m = \int_\xi^\tau \sin m^k t \cot (t/2)f(t)dt \\
= 2 \int_\xi^\tau \sin m^k \frac{f(t)dt}{t} + o(1).
\]
Hence
\[\sum_{1}^{n} \gamma_{m} = 4 \int_{\tau}^{\xi} \frac{f(u)}{u} \, du \int_{\tau}^{\xi} \frac{f(v)}{v} \, dv \sum_{1}^{n} \sin m^{k}u \sin m^{k}v \, dv + o(n)\]
\[= 2 \int_{\tau}^{\xi} \frac{f(u)}{u} \, du \int_{\tau}^{\xi} \frac{f(v)}{v} \, dv \sum_{1}^{n} \cos m^{k}(u - v) \, dv\]
\[= 2 \int_{\tau}^{\xi} \frac{f(u)}{u} \, du \int_{\tau}^{\xi} \frac{f(v)}{v} \, dv \sum_{1}^{n} \cos m^{k}(u + v) \, dv + o(n)\]
\[= I_{1} + I_{2} + o(n).\]

We shall use the fact that if
\[J_{n}(t) = \sum_{m=0}^{n} e^{it^{m}t},\]
then
\[\left| \sum_{1}^{n} \cos m^{k}t \right| \leq |J_{n}(t)|.\]  

We also require the following lemmas:

**Lemma 1.** Let \(a, b\) and \(k\) be three integers such that \(a < b, k \geq 2\). Let \(g(x)\) be a real function having differential coefficients of the first \(k\) orders. If \(\mathcal{R}g^{(k)}(x) \geq 1\) (or \(\leq -1\)) throughout the interval \((a, b)\), write
\[K = 2^{k}, \quad P = R \left| g^{(k-1)}(b) - g^{(k-1)}(a) \right|,\]
where \(R\) is a positive number. Then
\[\left| \sum_{m=a}^{b} e^{2\pi i \varphi(m)} \right| \leq 100P \left\{ \mathcal{R}^{-1/2} + \mathcal{R}^{2/3}P^{-2/3K} + P^{-2/3} \right\}.\]

This theorem is due to Van der Corput(4).

**Lemma 2.** If \(t > 0\), then
\[J_{n}(t) = O(n^{1/(K-2)}) + O(n^{1-2k/K}t^{-2/K}) + O(n^{1-2/3K}).\]

Put \(a = 0, b = n, g(x) = x^{k}t/2\pi (t > 0), R^{-1} = k!t/2\pi.\) We have
\[g^{(k-1)}(x) = k!xt/2\pi, \quad g^{(k)}(x) = k!t/2\pi = 1/R > 0, \quad P = n.\]

Lemma 2 is therefore an immediate corollary of Lemma 1.

**Lemma 3.** Let \(g(x)\) be a real differentiable function in the interval \((a, b)\), \(g'(x)\) be monotonic in this interval, and \(|g'(x)| < 1/2\). Then

\[(4) \text{ J. G. Van der Corput, Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunk-}
\text{probleme, Math. Zeit. vol. 28 (1928) p. 303.}\]
(3.7) \[ \sum_{a \leq n \leq b} e^{2\pi i \varphi(n)} = \int_{a}^{b} e^{2\pi i \varphi(x)} dx + O(1). \]

This theorem is due to Titchmarsh(6).

**Lemma 4.** If \( 0 < t \leq \pi/kn^{k-1} = \eta \), then

(3.8) \[ J_n(t) = O(1/t^{1/k}). \]

Put \( a = 0, b = n, g(x) = x^k t/2\pi \) \((t > 0)\). Evidently, \( g'(x) = kx^{k-1}t/2\pi \) is monotone in the interval \((0, \pi)\) and \( 0 < g'(x) < 1/2 \). Hence, by Lemma 3,

\[
J_n(t) = \sum_{m=0}^{n} e^{imk\eta} = \int_{0}^{n} e^{izk\eta} dx + O(1) \\
= \frac{1}{k^{1/k}} \int_{0}^{nn} e^{iu^{1/k} - 1} du + O(1) = O\left(\frac{1}{t^{1/k}}\right).
\]

4. We are now in a position to estimate \( \sum_{n} g_{m}^{2} \). Firstly, remembering \( \xi = 1/n, \zeta = 1/n^{k} \), we have

\[
I_{21} = n \int_{\xi}^{1} \frac{|f(u)|}{u} du \int_{\xi}^{1} \frac{|f(v)|}{v} (u + v)^{1/(K-2)} dv \\
\leq n \int_{\xi}^{1} u^{-1+1/(K-2)} |f(u)| du \int_{\xi}^{1} \frac{|f(v)|}{v} dv \\
+ n \int_{\xi}^{1} \frac{|f(u)|}{u} du \int_{\xi}^{1} v^{-1+1/(K-2)} |f(v)| dv \\
= o(n^{1-1/(K-2)} \log n) = o(n),
\]

\[
I_{22} = n^{1-k/K} \int_{\xi}^{1} \frac{|f(u)|}{u} du \int_{\xi}^{1} \frac{|f(v)|}{v} (u + v)^{-2/K} dv \\
\leq n^{1-k/K} \int_{\xi}^{1} u^{-1-1/K} \frac{|f(u)|}{u} du \int_{\xi}^{1} v^{-1-1/K} |f(v)| dv \\
= o(n^{1-k/K \cdot n^{k/K} \cdot n^{k/K}}) = o(n),
\]

\[
I_{23} = n^{1-k/K} \int_{\xi}^{1} \frac{|f(u)|}{u} du \int_{\xi}^{1} \frac{|f(v)|}{v} dv \\
= o(n^{1-k/K} \cdot (\log n)^{2}) = o(n).
\]

Observing the definition of \( I_{2} \) in (3.2) it follows from (3.3) and (3.6) that

(4.1) \[ I_2 = O(I_{21} + I_{22} + I_{23}) = o(n). \]

Secondly, observing (3.2) and (3.3) we have

\[
\begin{align*}
\left| I_1 \right| 
&\leq \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \left| f(u) \right| \left| f(v) \right| \sum_{k=0}^{n} \cos m^k(u - v) dv du \left| J_n(u - v) \right| dv \\
&\leq \left( \int_{-\xi}^{\xi} \left| f(u) \right| du \int_{-\xi}^{\xi} \left| f(v) \right| dv \right) S_{n, \xi}^\xi \int_{-\xi}^{\xi} \left| J_n(u - v) \right| dv \\
&= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{align*}
\]

Writing

\[
\begin{align*}
I_{11} &= \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \left| f(u) \right| \left| f(v) \right| \left| J_n(u - v) \right| dv du \\
&= I_{111} + I_{112},
\end{align*}
\]

we have

\[
I_{111} \leq n \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \left| f(u) \right| \left| f(v) \right| dv du,
\]

and

\[
\int_{-\xi}^{\xi} \left| f(v) \right| dv \leq \left( \int_{-\xi}^{\xi} \left| f(v) \right| dv \right)^{1/p} \left( \int_{-\xi}^{\xi} \left| f(v) \right| dv \right)^{1/q} = o(u^{1/p}(u - \xi)^{1/q}) = o(u^{1/p}u^{1/q}) = o(n^{-k/q}u^{1/p}).
\]

since \( u \leq 2\xi \). It follows that

\[
I_{111} \leq n^{1-k/q} \int_{-\xi}^{\xi} u^{-2+1/p} \left| f(u) \right| du = n^{1-k/q} \int_{-\xi}^{\xi} u^{-1-1/q} \left| f(u) \right| du
\]

\[
= o(n^{-k/q}u^{1/q}) = o(n^{1-k/q}u^{k/q}) = o(n).
\]

Again, from
\[
I_{112} \leq n^{1-k} \int_{\xi}^{2\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u} \frac{|f(v)|}{v} \, dv,
\]
and
\[
\int_{\xi}^{u} \frac{|f(v)|}{v} \, dv = o\left(\log \frac{u}{\xi}\right) = o(\log 2) = o(1),
\]
we obtain
\[
I_{112} \leq n^{1-k} \int_{\xi}^{2\xi} \frac{|f(u)|}{u^2} \, du = o(n^{1-k} n^k) = o(n).
\]
Therefore
\[
(4.3) \quad I_{11} = O(I_{111} + I_{112}) = o(n).
\]
The same method of estimation can be applied to \(J_{12}\), by assuming \(2\xi \leq u\). Thus we obtain
\[
(4.4) \quad I_{12} = o(n).
\]
Apply Lemma 2 to \(J_{15}\). We have
\[
I_{151} = n \int_{\omega + \xi}^{\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u-\eta} \frac{|f(v)|}{v} \, (u - v)^{1/(K-2)} \, dv
\]
\[
\leq n \int_{\omega + \xi}^{\xi} u^{-1+1/(K-2)} \, |f(u)| \, du \int_{\xi}^{u-\eta} \frac{|f(v)|}{v} \, dv
\]
\[
= o(n^{1-1/(K-2)} \log n) = o(n),
\]
\[
I_{152} = n^{1-2K/K} \int_{\omega + \xi}^{\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u-\eta} \frac{|f(v)|}{v} \, (u - v)^{-2/K} \, dv
\]
\[
\leq n^{1-2K/K - 1/K} \int_{\omega + \xi}^{\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u-\eta} \frac{|f(v)|}{v} \, dv
\]
\[
= o(n^{1-2K/K} n^{(k-1)2/K} (\log n)^2)
\]
\[
= o(n^{1-2/K} (\log n)^2) = o(n),
\]
\[
I_{153} = n^{1-2/K} \int_{\omega + \xi}^{\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u-\eta} \frac{|f(v)|}{v} \, dv
\]
\[
= o(n^{1-2/K} (\log n)^2) = o(n).
\]
It follows that
\[
(4.5) \quad I_{15} = O(I_{151} + I_{152} + I_{153}) = o(n).
\]
Observe that in \(I_{18}\)
\[
u \leq \eta + \xi \quad \text{and} \quad \xi \leq v \leq u - \xi,
\]
or

\[ 0 < \xi \leq u - v \leq u - \xi \leq \eta. \]

Using Lemma 4, we have

\[
I_{13} \leq \int_{2\xi}^{u+\xi} \frac{|f(u)|}{u} \, du \int_{\xi}^{u-\xi} \frac{|f(v)|}{v} (u - v)^{-1/k} \, dv
\]

\[
= \int_{2\xi}^{u+\xi} \frac{|f(u)|}{u^2} \, du \int_{\xi}^{u-\xi} \frac{|f(v)|}{v} (u - v)^{-1/k} \, dv
\]

\[
+ \int_{2\xi}^{u+\xi} \frac{|f(u)|}{u^2} \, du \int_{\xi}^{u-\xi} |f(v)| (u - v)^{-1/k} \, dv
\]

\[= I_{131} + I_{132}. \]

From

\[
I_{131} \leq \int_{2\xi}^{u+\xi} u^{-1-1/k} |f(u)| \, du \int_{\xi}^{u-\xi} \frac{|f(v)|}{v} \, dv,
\]

and

\[
\int_{\xi}^{u-\xi} \frac{|f(v)|}{v} \, dv \leq \left( \int_{\xi}^{u-\xi} |f(v)|^p \, dv \right)^{1/p} \left( \int_{\xi}^{u-\xi} v^{-q} \, dv \right)^{1/q}
\]

\[= o(u^{1/p_f-1+1/\eta}) = o(u^{1/p_f} u^{1/k-1}) = o(u^{1/p_f}), \]

we obtain

\[
I_{131} \leq n^{k/p} \int_{2\xi}^{u+\xi} u^{-1-1/k+1/p} |f(u)| \, du
\]

\[= o(n^{k/p_f-1+1/k+1/p}) = o(n^{k/p_f}) = o(n). \]

Again, from

\[
I_{132} = \int_{2\xi}^{u+\xi} \frac{|f(u)|}{u^2} \, du \int_{\xi}^{u-\xi} |f(v)| (u - v)^{-1/k} \, dv
\]

and

\[
\int_{\xi}^{u-\xi} |f(v)| (u - v)^{-1/k} \, dv \leq \left( \int_{\xi}^{u-\xi} |f(v)|^p \, dv \right)^{1/p} \left( \int_{\xi}^{u-\xi} (u - v)^{-q/k} \, dv \right)^{1/q}
\]

\[= o(u^{1/p_f}(u - \xi)^{-1/k+1/\eta}) = o(u^{-1/k}), \]

since \(-1/k+1/q>0\), we obtain

\[
I_{132} \leq \int_{2\xi}^{u+\xi} u^{-1-1/k} |f(u)| \, du
\]

\[= o(\xi^{-1/k}) = o(n). \]

Therefore

\[ (4.6) \quad I_{13} = O(I_{131} + I_{132}) = o(n). \]
Finally, by applying Lemma 4 to $I_{14}$ we easily deduce that

$$I_{14} = o(n).$$

Combining (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) we get

$$I_1 = O(I_{11} + I_{12} + I_{13} + I_{14} + I_{16}) = o(n).$$

It follows from (3.2), (4.1) and (4.8) that

$$\sum_{1}^{n} \gamma_m^2 = o(n).$$

Summarizing (2.1), (2.5), (2.7) and (4.9) we obtain

$$\sum_{1}^{n} (\alpha_m^2 + \beta_m^2 + \gamma_m^2 + \delta_m^2) = o(n).$$

The theorem is thus proved.

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