VALUATION IDEALS IN POLYNOMIAL RINGS

BY

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Introduction. Among the valuation ideals in a polynomial ring $\mathfrak{O} = K[x, y]$ in two indeterminates, the ones of central importance are the simple valuation ideals, that is, the valuation ideals which are not products of two ideals different from $\mathfrak{O}$, since every valuation ideal has a unique factorization into simple valuation ideals.

The problem of the characterization of simple valuation ideals has been dealt with by Zariski, in the case that the field K is algebraically closed and of characteristic 0, in his paper, *Polynomial ideals defined by infinitely near base points*. There the problem is referred to the ring of holomorphic functions in x, y: $\mathfrak{O}^* = K\{x, y\}$, and a valuation ideal q in $\mathfrak{O}^*$ is simple if and only if its general element is absolutely irreducible. If, however, only the characterization of the simple valuation ideals is desired, the notions of the general element of an ideal in $\mathfrak{O}^*$ and its absolute irreducibility are somewhat too strong for the problem set; although it should be stated that these notions are applied by Zariski to other topics not touched upon here.

In this paper we treat the theory of simple valuation ideals by a more explicit and direct method and we also extend the theory to algebraically closed fields of arbitrary characteristic $p \neq 0$. We characterize the simple v-ideals q in the sequence of zero-dimensional valuation ideals in $\mathfrak{O}$, for a given valuation v, in terms of the value $v(q)$ (under v) of q (that is, the least value assumed by elements of q) and of the least value greater than $v(q)$ assumed by elements of $\mathfrak{O}$. If q is not simple, an explicit factorization for q in terms of the two mentioned values is given

Since in our treatment the field K is of arbitrary characteristic, Puiseux series expansions for valuations are not available. A corresponding tool is found in Theorem 6. There for a given valuation v, a certain (finite or infinite) sequence of polynomials $f_i(x, y)$ is introduced. In the case that K is of characteristic 0, if v is given[1], for example, by $y = c_1 x^{r(1)} + c_2 x^{r(2)} + \cdots$, $c_i \in K$, $r(i) = r_i$ rational, with $0 < r_i < r_{i+1}$, then the polynomials $f_i(x, y)$ correspond roughly to the irreducible polynomials $g_1(x, y)$, $g_2(x, y)$, $\cdots$ which have $y = c_1 x^{r(1)}$, $y = c_1 x^{r(1)} + c_2 x^{r(2)}$, $\cdots$ respectively as roots.

We next reduce our considerations to valuations of rational rank 2. This reduction serves two purposes. First it unifies the discussion; but much more

important is the following. If v is of rational rank 2 then the value group contains irrational numbers, and if τ is the least irrational value assumed by elements of $\mathfrak{O}$, then the description of the valuation ideals in $\mathfrak{O}$ for the valuation v is intimately connected with the approximants and quasi approximants to a certain integral multiple of τ.

Our treatment also provides a proof for the main theorem proved by Zariski in the case $p=0$, to the effect that under a quadratic transformation simple v-ideals are transformed into simple v-ideals.

1. Valuation-ideals and 0-dimensional valuations. Let Σ be a field containing a subfield K and let v be a valuation of Σ over K, that is, a valuation in which the elements of K other than zero have value zero. Let $\mathfrak{B}$ be the valuation ring of v. Let $\mathfrak{O}$ be an integral domain contained in $\mathfrak{B}$. By the *valuation ideals, or v-ideals, in $\mathfrak{O}$ belonging to, or for, the valuation v we mean the contracted ideals in $\mathfrak{O}$ of the ideals in $\mathfrak{B}$*, that is, the ideal $\mathfrak{A}$ in $\mathfrak{O}$ is a v-ideal for the valuation v if $\mathfrak{A} = \mathfrak{a} \cap \mathfrak{O}$, where $\mathfrak{a}$ is an ideal in $\mathfrak{B}$. It is clear that the v-ideals in $\mathfrak{O}$ belonging to v may be characterized as the sets of elements in $\mathfrak{O}$ which contain together with any element all elements of $\mathfrak{O}$ of equal or greater value. Another simple property of valuation ideals in $\mathfrak{O}$ dependent only on the fact that they are contracted ideals of ideals in an overlying valuation ring is the following. If v' is a second valuation of Σ over K with valuation ring $\mathfrak{B}'$ such that $\mathfrak{O} \subset \mathfrak{B}' \subset \mathfrak{B}$ then the valuation ideals in $\mathfrak{O}$ belonging to the valuation v are among the valuation ideals in $\mathfrak{O}$ belonging to the valuation v'. In fact, if $\mathfrak{A} = \mathfrak{a} \cap \mathfrak{O}$, $\mathfrak{a}$ an ideal in $\mathfrak{B}$, then also $\mathfrak{A} = (\mathfrak{a} \cap \mathfrak{B}') \cap \mathfrak{O}$, and $\mathfrak{a} \cap \mathfrak{B}'$ is an ideal in $\mathfrak{B}'$.

Let now $\mathfrak{O}$ be a finite integral domain over K, $\mathfrak{O} = K[x_1, \cdots, x_k]$, contained in $\mathfrak{B}$ (that is, $\mathfrak{O}$ consists of the polynomials in a finite number of elements of $\mathfrak{B}$ with coefficients in K) and let Σ be of finite degree of transcendency $n \geq 1$ over K. By the dimension of v is meant the degree of transcendency, over K, of the residue field of v. We assert that the valuation ideals in $\mathfrak{O}$ belonging to v are among the valuation ideals in $\mathfrak{O}$ belonging to a 0-dimensional valuation v' of Σ over K. In fact, if v is not already 0-dimensional, let v_0 be a 0-dimensional valuation of $\mathfrak{B}/\mathfrak{P}$ over K, where $\mathfrak{P}$ is the ideal of non-units in $\mathfrak{B}$. It is well known[2] that the two valuations v and v_0 together determine a valuation v' of Σ over K such that $v'(\alpha/\beta) = 0$ if and only if $v(\alpha/\beta) = 0$ and $v_0([\alpha/\beta]) = 0$; and such that $v'(\alpha/\beta) > 0$ if and only if $v(\alpha/\beta) \geq 0$ and $v_0([\alpha/\beta]) > 0$, where $[\alpha/\beta]$ is the residue, or image, of α/β under the homomorphism $\mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{P}$. Clearly the valuation ring $\mathfrak{B}'$ of v' is contained in $\mathfrak{B}$ and v' is 0-dimensional. Moreover v_0 can be taken such that $v'(x_i) \geq 0$, $i = 1, \cdots, k$. In fact, since $v(x_i) \geq 0$ it follows that the v-residues $\bar{x}_i$ of the x_i are finite, and hence there exists a v_0 such that $v_0(\bar{x}_i) \geq 0$, whence $v'(x_i) \geq 0$. $\mathfrak{O}$ is then contained in $\mathfrak{B}'$, and this completes the proof of our assertion.

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[2] This has reference to the process of composing v and v_0. See Krull, *Allgemeine Bewertungstheorie*, §5, Journal für Mathematik vol. 167 (1932).

Below, Σ is taken to be a pure transcendental extension of K of degree 2, and $\mathfrak{O}$ to be a polynomial ring over K in two indeterminates. As our purpose is a study of the v-ideals in $\mathfrak{O}$ belonging to valuations v of Σ over K, these valuations are restricted without loss of generality to be 0-dimensional.

2. **Valuation ideals in a ring with chain theorem.** Let v be a valuation of Σ over K with v-ring $\mathfrak{B}$, and let $\mathfrak{O}$ be an integral domain with chain theorem contained in $\mathfrak{B}$, and let the quotient field of $\mathfrak{O}$ be Σ. Then *every v-ideal in $\mathfrak{O}$ belonging to v has an immediate successor in the ordered set of v-ideals belonging to v.* In fact, let $(\alpha_1, \cdots, \alpha_k)$ be a basis for an arbitrary ideal $\mathfrak{A}$ in $\mathfrak{O}$, and place $v(\mathfrak{A}) = \min(v(\alpha_1), \cdots, v(\alpha_k))$; it is immediate from the valuation axioms that $v(\mathfrak{A})$ is independent of the basis. If $v(\mathfrak{A}) = v(\mathfrak{A}')$ for two ideals $\mathfrak{A}$ and $\mathfrak{A}'$ in $\mathfrak{O}$, we also write $\mathfrak{A} \sim \mathfrak{A}'$. Since v is non-trivial (by assumption), there exists an element α of Σ with $v(\alpha) > 0$. Let $\alpha = f/g, f, g \in \mathfrak{O}$; then $v(f) = v(\alpha) + v(g) > 0$, since $g \in \mathfrak{O} \subset \mathfrak{B}$ implies $v(g) \geqq 0$. Thus there exist elements of value greater than $v(\mathfrak{A})$, for example, the elements of $\mathfrak{A}f$. Let $\mathfrak{A}$ be a v-ideal for v. The set of elements $\mathfrak{A}'$ in $\mathfrak{O}$ of value greater than $v(\mathfrak{A})$ has the property that together with any element it contains all the elements of $\mathfrak{O}$ of equal or greater value. Hence $\mathfrak{A}'$ is a v-ideal for v, and clearly $\mathfrak{A}'$ is the immediate successor of $\mathfrak{A}$.

Let v be 0-dimensional, and consider the well-ordered set of v-ideals in $\mathfrak{O}$ belonging to v. The first v-ideal q_0 in $\mathfrak{O}$ is $\mathfrak{O}$ itself. The immediate successor q_1 of q_0 is the contraction to $\mathfrak{O}$ of the ideal of non-units $\mathfrak{P}$ in $\mathfrak{B}$, and is a prime ideal in $\mathfrak{O}$. Moreover, it is 0-dimensional. In fact, since $\mathfrak{P} \cap \mathfrak{O} = q_1$, $\mathfrak{O}/q_1 \subseteq \mathfrak{B}/\mathfrak{P}$, whence $\mathfrak{O}/q_1$ is algebraic over K, or in other words, q_1 is 0-dimensional. The well-ordered set of v-ideals in $\mathfrak{O}$ belonging to v starts with a simple sequence: $q_0, q_1, q_2, \cdots$. *Each q_i, $i \geqq 1$, is a primary ideal with q_1 as associated prime ideal.* In fact, $v(q_1) < v(q_1^2) < \cdots < v(q_1^i)$, whence $v(q_1^i) \geqq v(q_i)$; therefore $q_1^i \subseteq q_i \subseteq q_1$, whence q_i is a primary ideal with q_1 as associated prime ideal.

The intersection Δq_i of all the ideals q_i is a prime ideal. In fact, if $ab \equiv 0(\Delta q_i)$, $a \not\equiv 0(\Delta q_i)$, $b \not\equiv 0(\Delta q_i)$, then let q_r be the last ideal in the sequence $\{q_i\}$ containing a, and q_s the last ideal in the sequence $\{q_i\}$ containing b. Then $v(a) = v(q_r)$, $v(b) = v(q_s)$, whence $v(ab) = v(q_r q_s)$. Since every v-ideal for v contains together with any element all elements of equal or greater value, we have $q_r q_s \equiv 0(\Delta q_i)$. Since some power of q_1 is in q_r, and similarly for q_s, also some power of q_1 is in Δq_i, say $q_1^n \equiv 0(\Delta q_i)$. This is a contradiction, for if $q_1^n \equiv 0(q_i)$ then i cannot exceed the length of q_1^n.

If K is algebraically closed (and v is 0-dimensional) the sequence $q_0, q_1, \cdots$ is called a *Jordan sequence*, in view of the fact that in this case *each q_{i+1} is a maximal subideal of*[3] q_i.

We are particularly concerned below with all the above results in the case that K is algebraically closed, of arbitrary characteristic, that Σ is a pure

[3] O. Zariski, *Polynomial ideals defined by infinitely near base points*, Theorem 1, Amer. J. Math. vol. 60 (1938).

transcendental extension of degree 2, and that $\mathfrak{O}$ is a polynomial ring in two indeterminates (generating elements of Σ over K). These assumptions on K and Σ are assumed hereafter. Also any considered valuation v of Σ over K is understood to be 0-dimensional.

In general, $\Delta\mathfrak{q}_i$ is prime. In the case that $\mathfrak{O}$ is a polynomial ring in two indeterminates, $\Delta\mathfrak{q}_i$ is either the ideal (0), or it is 1-dimensional, whence principal, say $\Delta\mathfrak{q}_i = (f)$. In the former case every v-ideal belonging to v is in the sequence $\{\mathfrak{q}_i\}$. In the latter case every v-ideal belonging to v is of the form $(f)^n\mathfrak{q}_m$, $n = 0, 1, \cdots$; $m = 0, 1, \cdots$. As a consequence, the study of v-ideals in $\mathfrak{O}$ reduces essentially to the study of the 0-dimensional v-ideals.

3. **v-ideals and quadratic transformations.** Let v be a 0-dimensional valuation of Σ over K. Since any element or its reciprocal has non-negative v-value, Σ has two generating elements x, y with non-negative value. Let $\mathfrak{O} = K[x, y]$. The center of the valuation v in $\mathfrak{O}$, that is, the contraction to $\mathfrak{O}$ of the ideal of non-units in the v-ring $\mathfrak{B}$ of v, is the prime ideal $\mathfrak{q}_1$. Since $\mathfrak{q}_1$ is 0-dimensional and K is algebraically closed there exist constants c, d such that $x \equiv c(\mathfrak{q}_1)$, $y \equiv d(\mathfrak{q}_1)$. Clearly $\mathfrak{q}_1 = (x - c, y - d)$. Replacing, if necessary, $x - c$ by x and $y - d$ by y, we may without loss of generality assume $\mathfrak{q}_1 = (x, y)$. At least one of the elements x, y is not in $\mathfrak{q}_2$, say $x \not\equiv 0(\mathfrak{q}_2)$, or equivalently, $v(\mathfrak{q}_1) = v(x)$. Since v is 0-dimensional and K is algebraically closed there exists a constant b such that $v(y - bx) > 0$. Replacing, if necessary, $y - bx$ by y, we may without loss of generality assume $v(y) > v(x)$.

With these assumptions on $v(x)$ and $v(y)$, we consider the quadratic transformation T:

$$x' = x, \quad y' = y/x; \qquad x = x', \quad y = x'y',$$

having at $x = y = 0$ a fundamental point, and we denote by $\mathfrak{O}'$ the ring of polynomials in x', y': $\mathfrak{O}' = K[x', y']$. $\mathfrak{O}$ is a subring of $\mathfrak{O}'$, and moreover $\mathfrak{O}' \subset \mathfrak{B}$ since $v(y') = v(y/x) > 0$. Let $\mathfrak{q}_0', \mathfrak{q}_1', \cdots$ be the Jordan sequence of v-ideals in $\mathfrak{O}'$ belonging to v.

If $\mathfrak{A}$ is any ideal in $\mathfrak{O}$ and if $\mathfrak{O}'\mathfrak{A} = x'^h\mathfrak{A}'$, $\mathfrak{A}' \not\equiv 0(x')$, the ideal $\mathfrak{A}'$ is called the *transformed ideal* of $\mathfrak{A}$ (under the quadratic transformation T), in symbols: $\mathfrak{A}' = T(\mathfrak{A})$. *By a theorem of Zariski*[4], *the transform of a v-ideal in $\mathfrak{O}$ for v is a v-ideal in $\mathfrak{O}'$ for v.* We shall have occasion to give another proof of this theorem.

Our principal goal below is to prove that the transform of a simple v-ideal is simple. By a *simple ideal* we mean the following. An *arbitrary ideal $\mathfrak{A}$ in $\mathfrak{O}$ is said to be composite if it is the product of two ideals $\mathfrak{B}$, $\mathfrak{C}$ in $\mathfrak{O}$ different from the unit ideal*: $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$, $\mathfrak{B} \neq \mathfrak{O}$, $\mathfrak{C} \neq \mathfrak{O}$. $\mathfrak{A}$ *is said to be simple if it is not composite.* It is true that *if $\mathfrak{A}$ is a v-ideal for v and is composite, $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$, then $\mathfrak{A}$ is also the product of two v-ideals for v different from the unit ideal.* In fact, let $\mathfrak{B}'$, $\mathfrak{C}'$ be the v-ideals in $\mathfrak{O}$ for v such that $v(\mathfrak{B}) = v(\mathfrak{B}')$, $v(\mathfrak{C}) = v(\mathfrak{C}')$. Since $\mathfrak{B}$, $\mathfrak{C}$ differ

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(4) O. Zariski, ibid., Theorem 4.1.

from $\mathfrak{O}$ and therefore $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{A} \subset \mathfrak{C}$ properly by a well known theorem, we have $v(\mathfrak{B}) > 0$, $v(\mathfrak{C}) > 0$, whence $v(\mathfrak{B}') > 0$, $v(\mathfrak{C}') > 0$, and $\mathfrak{B}'$, $\mathfrak{C}'$ differ from $\mathfrak{O}$. Since $\mathfrak{B}' \supseteq \mathfrak{B}$, $\mathfrak{C}' \supseteq \mathfrak{C}$ we have $\mathfrak{B}'\mathfrak{C}' \supseteq \mathfrak{B}\mathfrak{C} = \mathfrak{A}$. On the other hand, $v(\mathfrak{A}) = v(\mathfrak{B}\mathfrak{C})$ $= v(\mathfrak{B}'\mathfrak{C}')$, and since $\mathfrak{A}$ is a v-ideal for v, $\mathfrak{A} \supseteq \mathfrak{B}'\mathfrak{C}'$. Thus $\mathfrak{A} = \mathfrak{B}'\mathfrak{C}'$. This completes the proof. This remark permits us in proving the simplicity of a given v-ideal $\mathfrak{A}$ to restrict the discussion, first, to any valuation v for which $\mathfrak{A}$ is a v-ideal, and second, to the ideals in $\mathfrak{O}$ which are v-ideals for v.

4. **The valuation given by** $y = x^\tau$. Let $\mathfrak{O} = K[x, y]$ be a polynomial ring in two indeterminates x, y over the field[5] K, and let Σ be the quotient field of $\mathfrak{O}$. Let τ be an arbitrary irrational number greater than zero.

There exists one and only one valuation v *of* Σ *over* K *such that* $v(x) = 1$ *and* $v(y) = \tau$. v *is necessarily* 0-*dimensional*. The value group of this valuation is the additive group of real numbers of the form $a + b\tau$ where a and b are rational integers. (See, for example, Saunders MacLane and O. F. G. Schilling, *Zero-dimensional branches of rank one on algebraic varieties, construction* I, Ann. of Math. vol. 40 (1939).) We *shall say that the valuation* v *is given by* $y = x^\tau$, *or that* $y = x^\tau$ *is the valuation* v.

Consider the valuation v given by $y = x^\tau$, τ an arbitrary positive irrational number. Since $v(x) > 0$ and $v(y) > 0$, the ring $\mathfrak{O} = K[x, y]$ is contained in the valuation ring $\mathfrak{B}$ of v. Let $\mathfrak{q}_0$, $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\cdots$ be the Jordan sequence of v-ideals for v. Here $\mathfrak{q}_0 = \mathfrak{O}$, $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$.

From the definition of v every v-ideal $\mathfrak{q}_i$ has value $m_i + n_i\tau$, m_i, n_i non-negative integers. Conversely, every number $m + n\tau$, m, n non-negative integers, is the value of one of the ideals $\mathfrak{q}_i$, namely that one which contains all the elements in $\mathfrak{O}$ of value equal to or greater than $v(x^m y^n)$. Hence the sequence $v(\mathfrak{q}_0)$, $v(\mathfrak{q}_1)$, $v(\mathfrak{q}_2)$, $\cdots$ coincides with the set of numbers $\{g + h\tau\}$, g and h non-negative integers, ordered according to magnitude.

THEOREM 1. *Every v-ideal for the valuation given by $y = x^\tau$ has a basis consisting of monomials $x^m y^n$.*

Proof. Let $\mathfrak{q}$ be a v-ideal for v, and let $g(x, y) \in \mathfrak{q}$. If $g(x, y) = \sum c_{mn} x^m y^n$, $c_{mn} \in K$, $c_{mn} \neq 0$, then $v(g) = \min (v(x^m y^n))$, by definition; that is, $v(x^m y^n) \geq v(g)$ $\geq v(\mathfrak{q})$ for every term $c_{mn} x^m y^n$ of $g(x, y)$, whence $x^m y^n \in \mathfrak{q}$ for every such term. Thus a basis for $\mathfrak{q}$ can be replaced by all the various terms in the elements of the basis. This completes the proof.

COROLLARY. *Let* $m_1 + n_1\tau$, $m_2 + n_2\tau$, $\cdots$ *be the set of positive numbers* $\{g + h\tau\}$, *g, h non-negative integers, ordered according to magnitude. The ideals* $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\cdots$ *can be described in the following manner:* $\mathfrak{q}_i$ *has a basis consisting of the elements* $x^{m(k)} y^{n(k)}$, $k \geq i$, *where* $m(k) = m_k$, *a notation used later also.*

THEOREM 2.1. *If* $v(\mathfrak{q}_i) = m + n\tau$, $v(\mathfrak{q}_j) = m$, $v(\mathfrak{q}_k) = n\tau$ *then* $\mathfrak{q}_i = \mathfrak{q}_j\mathfrak{q}_k$.

Theorem 2.2. *If $v(q_i) = s\tau$, $v(q_{i+1}) = m + n\tau$ and if $mn \neq 0$, then $s > n$ and $q_i = q_j q_k$, where $v(q_j) = n\tau$, $v(q_k) = (s - n)\tau$.*

Theorem 2.3. *If $v(q_i) = r$, $v(q_{i+1}) = m + n\tau$ and if $mn \neq 0$, then $r > m$ and $q_i = q_j q_k$, where $v(q_j) = m$, $v(q_k) = r - m$.*

Proof. 2.1. Clearly $q_j q_k \subseteq q_i$. Conversely, let $x^{m'} y^{n'} \in q_i$. Since $m' + n'\tau \geq m + n\tau$, either $n' \geq n$ or $m' \geq m$. In the first case,

$$x^{m'} y^{n'} = x^{m'} y^{n'-n} \cdot y^n \in q_j q_k$$

since $x^{m'} y^{n'-n} \in q_j$ (in view of $m' + (n' - n)\tau \geq m$) and $y^n \in q_k$. In the second case

$$x^{m'} y^{n'} = x^m \cdot x^{m'-m} y^{n'} \in q_j q_k$$

since $x^m \in q_j$, $x^{m'-m} y^{n'} \in q_k$. This completes the proof.

2.2. We cannot have $n \geq s$, for $n \geq s$ together with $m \neq 0$ implies $m + n\tau - s\tau \geq 1$, whence there exists an integer between the irrational numbers $s\tau$ and $m + n\tau$. (Note that $s \neq 0$, for if $s = 0$ then $v(q_{i+1}) = 1$ or τ.) This contradicts the fact that $s\tau$ and $m + n\tau$ are successive in the ordered set of numbers $\{g + h\tau\}$, g, h non-negative integers. Hence $s > n$, and the ideals q_j, q_k exist. Clearly $q_j q_k \subseteq q_i$. Conversely, let $q_i = (y^s, x^m y^n, x^{m'} y^{n'}, \cdots)$. Then clearly $y^s = y^n y^{s-n} \in q_j q_k$ and $x^m y^n \in q_j q_k$ since $m > (s - n)\tau$, so that $x^m \in q_k$. As for $x^{m'} y^{n'}$, either $n' \geq m$ or $n' \geq n$, since $m' + n'\tau \geq m + n\tau$. Hence if we write $x^{m'} y^{n'}$ in the form $x^m \cdot x^{m'-m} y^{n'}$ or in the form $x^{m'} y^{n'-n} \cdot y^n$, we conclude in either case that $x^{m'} y^{n'} \in q_j q_k$. This completes the proof.

2.3. This case is really not different from the case 2.2. In fact, consider the order preserving automorphism of the additive group of real numbers given by: $x \rightarrow x/\tau$. If we replace the value group of v by the isomorphic group obtained under this automorphism, the statement and proof of 2.3 are the same as those of 2.2. This completes the proof.

Theorem 3. *Let h be the greatest integer less than τ if $\tau > 1$, or the greatest integer less than $1/\tau$ if $\tau < 1$. If $i \leq h$ then q_i is simple. If $i \geq h$ and if q_i is simple, then of the two numbers $v(q_i)$, $v(q_{i+1})$ one is an integer, the other an integral multiple of τ.*

Proof. If $\tau > 1$ then $v(q_1) = 1$, $v(q_2) = 2$, $\cdots$, $v(q_h) = h$, $v(q_{h+1}) = \tau$. If $\tau < 1$ then $v(q_1) = \tau$, $v(q_2) = 2\tau$, $\cdots$, $v(q_h) = h\tau$, $v(q_{h+1}) = 1$. In the first case $y \in q_i$, $i = 1, 2, \cdots, h$; in the second case $x \in q_i$, $i = 1, \cdots, h$. In either case the ideals $q_1, q_2, \cdots, q_h$ must be simple because each element of the product of two ideals which are different from the unit ideal must have a leading form of degree at least 2.

If, now, $i \geq h$ and $v(q_i) = m + n\tau$, $m \neq 0$, $n \neq 0$, then Theorem 2.1 proves that q_i is not simple.

If $i \geq h$, $v(q_i) = s\tau$ and $v(q_{i+1}) = m + n\tau$ and $n \neq 0$, then also $m \neq 0$. In fact, if $m = 0$ then clearly $n = s + 1$. If $\tau > 1$, then $v(q_{i+1}) - v(q_i) = \tau > 1$, whence there

is an integer between $v(q_i)$ and $v(q_{i+1})$: this is a contradiction. If $\tau < 1$ then $s\tau > 1$, since $i \geq h$ and $i = h$ is excluded because $v(q_{i+1}) = (s+1)\tau \neq 1$. If k is the greatest integer less than $(s\tau - 1)/\tau + 1$, that is,

$$(s\tau - 1)/\tau < k < (s\tau - 1)/\tau + 1, \quad k \text{ an integer},$$

then $s\tau < 1 + k\tau < (s+1)\tau; k > 0$ since $s\tau - 1 > 0$. This is a contradiction. Hence $m \neq 0$. By Theorem 3.2, then, q_i is not simple.

If $i \geq h$, $v(q_i) = r$ and $v(q_{i+1}) = m + n\tau$ and $m \neq 0$ then similarly q_i is not simple. In fact, this case is not really different from the one immediately above, as may be seen by the argument given in 2.3. This completes the proof.

We wish now to prove the converse of Theorem 3. For this we need two lemmas.

Let $m_1 + 1/(m_2 + 1/(m_3 + \cdots$ be the continued fraction expansion of τ. Consider the sequence of numbers: m_1, $m_1 + 1/1$, $m_1 + 1/2$, $\cdots$, $m_1 + 1/m_2$, $m_1 + 1/(m_2 + 1/1$, $\cdots$, $m_1 + 1/(m_2 + 1/m_3$, $\cdots$. Of these, the numbers m_1, $m_1 + 1/m_2$, $m_1 + 1/(m_2 + 1/m_3$, $\cdots$ are known as *approximants, the others as quasi approximants*[6], to τ.

LEMMA 1. *If the two numbers r, $s\tau$ (or $s\tau$, r), r, s positive integers, are successive in the set of numbers $\{g + h\tau\}$, g, h non-negative integers, ordered according to magnitude, then $(r, s) = 1$ and r/s is an approximant or quasi approximant to τ (and therefore s/r is also an approximant or quasi approximant to $1/\tau$). Conversely, if r/s, r, s positive integers with $(r, s) = 1$, is an approximant or quasi approximant to τ and s/r is an approximant or quasi approximant to $1/\tau$ then r, $s\tau$ (or $s\tau$, r) are successive in the set of numbers $\{g + h\tau\}$.*

Proof. Let r, $s\tau$ (or $s\tau$, r) be successive in the ordered set of numbers $\{g + h\tau\}$. *From the symmetry of the lemma*, we need consider only one case, say the case $r < s\tau$. A theorem of Lagrange and its converse[7] state that r/s is an approximant or quasi approximant to τ if and only if $n > s$ for every rational number m/n, m, n positive integers such that $r/s < m/n < \tau$. Now, if $r/s < m/n < \tau$, m, n positive integers and $n \leq s$, then

$$s\tau - r > (s/n)(n\tau - m) \geq n\tau - m > 0,$$

whence $r < m + (s - n)\tau < s\tau$, which contradicts the assumption that r and $s\tau$ are successive in the set $\{g + h\tau\}$. Hence $n > s$ and r/s is an approximant or quasi approximant to τ.

Conversely, suppose $(r, s) = 1$, r, s positive integers, and that $r < u + v\tau < s\tau$ for some non-negative integers u, v. If $v \neq 0$ then either $(r - u)/v > r/s$ or $u(s - v) > r/s$, for $(r - u)/v \leq r/s$ and $u/(s - v) \leq r/s$ imply $rs - us \leq rv$ and $us \leq rs - rv$, whence $us = rs - rv$, and $r/s = u/(s - v)$ which contradicts the assumption $(r, s) = 1$. Moreover $(r - u)/v$ and $u/(s - v)$ are both less than τ.

[6] Perron, *Nebennäherungsbrüche*.

[7] See O. Perron, *Die Lehre von den Kettenbrüchen*, p. 38, Theorems 20 and 21.

Hence there exists a rational number m/n (either $(r-u)/v$ or $u/(s-v)$) with $0<n\leqq s$ such that $r/s<m/n<\tau$. The same is true even if $v=0$; namely $m/n=u/s$. By the theorem mentioned above, r/s is therefore neither an approximant nor a quasi approximant to τ. This completes the proof.

LEMMA 2. *If* $m_1+n_1\tau, m_2+n_2\tau, \cdots, m_k+n_k\tau, r, s\tau$ *is an initial segment of the ordered set of positive numbers* $\{g+h\tau\}$, g, h *non-negative integers, then there exists an irrational number* $\tau'>0$ *such that* $m_1+n_1\tau', m_2+n_2\tau', \cdots, m_k +n_k\tau', s\tau', r$ *is an initial segment of the ordered set of positive numbers* $\{g+h\tau'\}$, g, h *non-negative integers. A similar statement holds for an initial segment* $m_1+n_1\tau, m_2+n_2\tau, \cdots, m_k+n_k\tau, s\tau, r$.

Proof. If $s=1$, then $n_1=n_2=\cdots=n_k=0$ and any irrational number τ' between $r-1/2$ and r satisfies the lemma. If $r=1$, then any irrational number τ' between $1/(s+1)$ and $1/s$ satisfies the lemma. If $s\neq 1$, $r\neq 1$ then r/s and s/r are not integers, since $(r, s)=1$. Let r/s have a continued fraction expansion $b_1+1/(b_2+1/(b_3+\cdots+1/(b_l$ with $b_l>1$. It is well known that if σ is an arbitrary irrational number greater than 1 then $b_1+1/(b_2+\cdots+1/((b_l-1)+1/(1+1/(\sigma$ and $b_1+1/(b_2+\cdots+1/(b_l+1/(\sigma$ are irrational numbers having r/s as an approximant, of which one is larger, the other smaller than r/s. Thus there exist irrational numbers $\tau'<r/s$ having r/s as an approximant: s/r will also be an approximant to $1/\tau'$. Let τ' ($<r/s$) be any irrational number having r/s as an approximant or quasi approximant. We assert that τ' satisfies the lemma. In fact, by Lemma 1, $s\tau'$ and r are successive in the ordered set of numbers $\{g+h\tau'\}$. Moreover, if $u+v\tau'<s\tau'$, u, v non-negative integers, then $v<s$, whence $u+v\tau<s\tau$ since $\tau'<\tau$. Hence $u+v\tau\leqq r$; equality is excluded since $u+v\tau=r$ implies $v=0$, $u=r>s\tau'$. Thus $u+v\tau<r$. Similarly, if $u+v\tau<r$, u, v non-negative integers, then $u+v\tau'<r$ since $\tau'<\tau$. Hence $u+v\tau'\leqq s\tau'$; equality is excluded, since $u+v\tau'=s\tau'$ implies $u=0$, $v\tau=s\tau<r$. It remains to prove that $m_i+n_i\tau'<m_{i+1}+n_{i+1}\tau'$, $i=1, \cdots, k-1$. If $n_{i+1}=n_i$ this is clear; if $n_i\neq n_{i+1}$, then $(m_i-m_{i+1})/(n_{i+1}-n_i) >\tau$ or $<\tau$ according as $n_{i+1}<n_i$ or $n_{i+1}>n_i$. If $(m_i-m_{i+1})/(n_{i+1}-n_i)>\tau(>\tau')$ then $n_{i+1}-n_i<0$ and $m_i+n_i\tau'<m_{i+1}+n_{i+1}\tau'$. If $(m_i-m_{i+1})/(n_{i+1}-n_i)<\tau$ then $(m_i-m_{i+1})/(n_{i+1}-n_i)<r/s$ also since $0<n_{i+1}-n_i<s$ and r/s is an approximant or quasi approximant to τ. But then, again because $0<n_{i+1}-n_i<s$ and r/s is an approximant or quasi approximant to τ', $(m_i-m_{i+1})/(n_{i+1}-n_i)>\tau'$ is impossible, that is, $(m_i-m_{i+1})/(n_{i+1}-n_i)<\tau'$ and $m_i+n_i\tau'<m_{i+1}+n_{i+1}\tau'$ follows. This completes the proof.

THEOREM 4. *If of the numbers* $v(q_i)$, $v(q_{i+1})$ *one is an integer, the other an integral multiple of* τ, *then* q_i *is simple.*

Proof. The case $i=0$ is trivial. If $i>0$ then $v(q_i)=r$, $v(q_{i+1})=s\tau$ or $v(q_i)=s\tau$, $v(q_{i+1})=r$, r, s positive integers. As both cases are similar, we consider for the sake of brevity only the case $v(q_i)=s\tau$. Consider the sequence

$v(\mathfrak{q}_1)$, $v(\mathfrak{q}_2)$, $\cdots$, that is, the sequence:

$$m_1 + n_1\tau, \; m_2 + n_2\tau, \; \cdots, \; m_{i-1} + n_{i-1}\tau, \; s\tau, \; r, \; \cdots,$$

where $v(\mathfrak{q}_j) = m_j + n_j\tau$. By Lemma 2 there exists an irrational number τ' such that

$$m_1 + n_1\tau', \; m_2 + n_2\tau', \; \cdots, \; m_{i-1} + n_{i-1}\tau', \; r, \; s\tau'$$

is an initial segment of the set of positive numbers $\{g+h\tau'\}$, g, h non-negative integers, ordered according to magnitude. Consider the valuation v' given by $y = x^{\tau'}$, and let $\mathfrak{q}_1' = (x, y)$, $\mathfrak{q}_2'$, $\cdots$ be the Jordan sequence of (0-dimensional) valuation ideals in $\mathfrak{O} = K[x, y]$ for the valuation v'. By the corollary to Theorem 1, $\mathfrak{q}_1 = \mathfrak{q}_1'$, $\mathfrak{q}_2 = \mathfrak{q}_2'$, $\cdots$, $\mathfrak{q}_i = \mathfrak{q}_i'$. The assumption $\mathfrak{q}_i = \mathfrak{q}_j\mathfrak{q}_k$, $j < i$, $k < i$, now leads to a contradiction. In fact, $(m_j + n_j\tau) + (m_k + n_k\tau) = s\tau$, whence $m_j + m_k = 0$. Since m_j, m_k are non-negative, $m_j = m_k = 0$, whence $v(\mathfrak{q}_j) = n_j\tau$, $v(\mathfrak{q}_k) = n_k\tau$. But the jth and kth numbers in the sequence $m_1 + n_1\tau'$, $m_2 + n_2\tau'$, $\cdots$, $m_{i-1} + n_{i-1}\tau'$, r, $s\tau'$ are $n_j\tau'$, $n_k\tau'$, whence $v'(\mathfrak{q}_j) = n_j\tau'$, $v'(\mathfrak{q}_k) = n_k\tau'$, whence $v'(\mathfrak{q}_i) = v'(\mathfrak{q}_j\mathfrak{q}_k) = (n_j + n_k)\tau' = s\tau'$; this is a contradiction. This completes the proof.

Let $m_1 + 1/(m_1 + 1/(m_3 + \cdots$ be the continued fraction expansion for τ; and consider the sequence: $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $\cdots$, $a_{m(1)} = m_1$, $a_{m(1)+1} = m_1 + 1/1$, $a_{m(1)+2} = m_1 + 1/2$, $\cdots$, $a_{m(1)+m(2)} = m_1 + 1/m_2$, $\cdots$. Let $\mathfrak{P}_0$, $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\cdots$ be the sequence of simple v-ideals for the valuation $y = x^\tau$ in $K[x; y]$, $\mathfrak{P}_i \supset \mathfrak{P}_{i+1}$, $i \geq 0$.

COROLLARY. *There is a 1-1 correspondence $\mathfrak{P}_i \rightleftarrows a_i$ between the simple v-ideals $\mathfrak{P}_i$ and the a_i, $i \geq 0$. The correspondence is such that if $a_i = r/s$, r, s non-negative integers with $(r, s) = 1$, then $v(\mathfrak{P}_i) = \min(r, s\tau)$.*

Proof. Consider first the case $\tau > 1$, or equivalently, $m_1 \geq 1$. By Theorem 3, $v(\mathfrak{P}_i) = i = \min(i, \tau)$, $i = 0, 1, \cdots, m_1$. By Theorems 3 and 4, for $k \geq m_1$, $\mathfrak{q}_k$ is simple if and only if either $v(\mathfrak{q}_k) = r$, $v(\mathfrak{q}_{k+1}) = s\tau$ or $v(\mathfrak{q}_k) = s\tau$, $v(\mathfrak{q}_{k+1}) = r$, r, s positive integers. By Lemma 1, r/s is an approximant or quasi approximant to τ. Conversely, let r/s, r, s positive integers with $(r, s) = 1$, be an approximant or quasi approximant to τ. Since $1/\tau < 1$, s/r is certainly also an approximant or quasi approximant to $1/\tau$. Hence r, $s\tau$ or $s\tau$, r are successive in the ordered set $\{g + h\tau\}$, g, h non-negative integers; that is, if $v(\mathfrak{q}_k) = \min(r, s\tau)$ then $v(\mathfrak{q}_{k+1}) = \max(r, s\tau)$ and $\mathfrak{q}_k$ is simple by Theorem 4. We make $\mathfrak{q}_k$ and r/s correspond, and it remains to show that $\mathfrak{P}_i \rightleftarrows a_i$. It is well known that if $a_i = r_i/s_i$, r_i, s_i positive integers with $(r_i, s_i) = 1$, then $r_i < r_{i+1}$. Since the set $\{g + h\tau\}$ is ordered according to magnitude, it is clear that the correspondence could not be other than $\mathfrak{P}_i \rightleftarrows a_i$.

Consider now the case $\tau < 1$, or equivalently, $m_1 = 0$. Here $a_0 = 0$, $a_1 = 0 + 1/1$, $\cdots$. The valuation $y = x^\tau$ is also given by $x = y^{1/\tau}$. Let $b_0 = 0$, b_1, b_2, $\cdots$ be the sequence for $1/\tau$ corresponding to the sequence $\{a_i\}$ for τ. Clearly $b_i = 1/a_i$ for $i > 0$; if $a_i = r_i/s_i$, r_i, s_i positive integers,

$(r_i, s_i) = 1$, then $b_i = s_i/r_i$. By the case $\tau > 1$, $\mathfrak{P}_i \rightleftarrows b_i$ and the correspondence is such that $v(\mathfrak{P}_i) = \min (s_i v(y), (r_i/\tau)v(y))$. Since $v(y) = \tau$, $v(\mathfrak{P}_i) = \min (s_i\tau, r_i)$. This completes the proof.

Theorems 3 and 4 enable us to prove our principal theorem for the valuation $y = x^\tau$; that is, that the transform of a simple v-ideal under the transformation T (see §3) is simple. Here we assume that $\tau > 1$, whence $v(y) > v(x)$. T is then the transformation:

$$T: \quad x' = x, \quad y' = y/x; \qquad x = x', \quad y = x'y'.$$

As above we denote the ring $K[x', y'] = K[x, y/x]$ by $\mathfrak{O}'$, and the v-ideals in $\mathfrak{O}'$ for v by $\mathfrak{q}_0'$, $\mathfrak{q}_1'$, $\mathfrak{q}_2'$, $\cdots$. Consider the valuations given by $y = x^\tau$ and $y' = x'^{\tau-1}$. Since in both the value of x is 1 and the value of y' is $\tau - 1$, they are, in fact, the same valuation: that is, the valuation $y = x^\tau$ is also given by $y' = x'^{\tau-1}$.

Let $\mathfrak{P}_0 = \mathfrak{q}_0$, $\mathfrak{P}_1 = \mathfrak{q}_1$, $\mathfrak{P}_2$, $\cdots$ be the simple v-ideals for v in $\mathfrak{O}$ and let $\mathfrak{P}_0' = \mathfrak{q}_0'$, $\mathfrak{P}_1' = \mathfrak{q}_1'$, $\mathfrak{P}_2'$, $\cdots$ be the simple v-ideals for v in $\mathfrak{O}'$.

THEOREM 5. *The transform of the ith simple v-ideal $\mathfrak{P}_i$, $i > 0$, is $\mathfrak{P}_{i-1}'$.*

Proof. Let $m_1 + 1/(m_2 + 1/(m_3 + \cdots$ be the continued fraction expansion for τ; $m_1 \geq 1$ since $\tau > 1$. Let $a_1 = 1$, $a_2 = 2$, $\cdots$, $a_{m(1)} = m_1$, $a_{m(1)+1} = m_1 + 1/1$, $a_{m(1)+2} = m_1 + 1/2$, $\cdots$, $a_{m(1)+m(2)} = m_1 + 1/m_2$, $\cdots$. By the preceding corollary there is a 1-1 correspondence between the numbers a_i and the ideals $\mathfrak{P}_i$; the correspondence is such that if $a_i = r/s$, r, s positive integers with $(r, s) = 1$, then $v(\mathfrak{P}_i) = \min (r, s\tau)$. Let $\{a_i'\}$ represent the corresponding numbers for $\tau - 1$ (>0). Clearly $a_i' + 1 = a_{i+1}$.

For $i = 1, 2, \cdots, m_1$ the ideal $\mathfrak{P}_i = \mathfrak{q}_i$ is simple and contains the element y. Clearly $\mathfrak{O}\mathfrak{q}_i \equiv 0(x')$. Moreover $y = y'x' \in \mathfrak{q}_i$ so that $\mathfrak{O}\mathfrak{q}_i \not\equiv 0(x'^2)$. Hence, by the definition of T, $v(T(\mathfrak{q}_i)) = v(\mathfrak{q}_i) - 1 = i - 1$, whence $T(\mathfrak{q}_i) \subseteq \mathfrak{P}_{i-1}'$. By the theorem of Zariski mentioned in §3, $T(\mathfrak{q}_i)$ is a v-ideal for v, and since $v(T(\mathfrak{q}_i)) = v(\mathfrak{P}_{i-1}')$ we have $T(\mathfrak{q}_i) = \mathfrak{P}_{i-1}'$.

In the special case with which we are dealing here it is also possible to prove this result by a direct and simple calculation. In fact, clearly $\mathfrak{P}_{i-1}' = \mathfrak{q}_{i-1}' = (y', x'^{i-1})$. The element y' arises from $y \in \mathfrak{q}_i$, and the element x'^{i-1} arises from $x^i \in \mathfrak{q}_i$, whence, together with $T(\mathfrak{q}_i) \subseteq \mathfrak{P}_{i-1}'$, we have $T(\mathfrak{q}_i) = \mathfrak{P}_{i-1}'$. Thus Theorem 5 is true at least for $i = 1, 2, \cdots, m_1$.

For $i > m_1$, $v(\mathfrak{P}_i) = r$, $v(\mathfrak{q}) = s\tau$ or $v(\mathfrak{P}_i) = s\tau$, $v(\mathfrak{q}) = r$, r, s positive integers, where $\mathfrak{q}$ is the element of the sequence $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\cdots$ immediately following $\mathfrak{P}_i$. The two cases are not really different: in fact, let $m_1 + n_1\tau$, $m_2 + n_2\tau$, $\cdots$, $m_k + n_k\tau$, r, $s\tau$ be an initial segment of the ordered set of positive numbers $\{g + h\tau\}$, g, h non-negative integers. By Lemma 2 of Theorem 4, there exists an irrational number $\tau' > 0$ such that $m_1 + n_1\tau'$, $m_2 + n_2\tau'$, $\cdots$, $m_k + n_k\tau'$, $s\tau'$, r is an initial segment of the ordered set $\{g + h\tau'\}$, g, h non-negative integers. By the description of the v-ideals for $y = x^\tau$ given in the corollary to

Theorem 1, we see that $q_1, q_2, \cdots, \mathfrak{P}_i$ are v-ideals for the valuation $y = x^{\tau'}$: and in this valuation the value of $\mathfrak{P}_i$ is $s\tau'$. Moreover, since $i > m_1$ the numbers m_1, τ are among the first k numbers of the set $\{g + h\tau\}$, whence also m_1, τ' are among the first k of $\{g + h\tau'\}$, whence $m_1 < \tau'$. Thus by changing from τ to τ' we retain the assumption $\tau > 1$. Therefore without loss of generality we need consider only the case $v(\mathfrak{P}_i) = s\tau$, $v(q) = r$.

Now, $a_i = r/s$, whence $a'_{i-1} = (r - s)/s$ (and clearly $(r - s, s) = 1$). Since $s\tau < r$, we have $s\tau' = s(\tau - 1) < r - s$; whence $v(\mathfrak{P}'_{i-1}) = s\tau$. On the other hand, $\mathfrak{P}_i$ has a basis consisting of monomials $x^m y^n$ with $m + n\tau \geqq s\tau$; since $\tau > 1$, $m + n \geqq s$, whence $\mathfrak{O}\mathfrak{P}_i \equiv 0(x'^s)$ and $T(\mathfrak{P}_i) \not\equiv 0(x'^{s+1})$ since $\mathfrak{P}_i \supset y^s = y'^s x'^s \not\equiv 0(x'^{s+1})$. By definition, then, $v(T(\mathfrak{P}_i)) = v(\mathfrak{P}_i) - s = s(\tau - 1) = s\tau' = v(\mathfrak{P}'_{i-1})$. Applying the fact that $T(\mathfrak{P}_i)$ is a v-ideal for v, we have $T(\mathfrak{P}_i) = \mathfrak{P}'_{i-1}$. This completes the proof.

5. The general case; reduction to rational rank 2 valuations.

Let $\gamma_1, \gamma_2, \cdots$ be a sequence of positive rational numbers. Let Δ_0 be the additive group of integers. For a given $i \geqq 1$, we consider the additive abelian group G_i generated by $1, \gamma_1, \cdots, \gamma_i$, that is, the set of numbers of the form $m + n_1\gamma_1 + \cdots + n_i\gamma_i$, m, n_j integers. In this sequence of groups let Δ_j be the jth group G containing its predecessor properly. Let Γ_0 be the set of non-negative integers, Γ_j the set of numbers in Δ_j which are of the form $m + n_1\gamma_{i(1)} + \cdots + n_j\gamma_{i(j)}$, where m, n_i are non-negative integers. We denote the numbers $\gamma_{i(1)}, \gamma_{i(2)}, \cdots$ by $\delta_1, \delta_2, \cdots$.

Let $\delta_k = r_k/s_k$, where r_k, s_k are positive integers and $(r_k, s_k) = 1$; let $t_0 = 1$, $t_k = \text{l.c.m.} (s_1, \cdots, s_k)$; and let $d_k = t_k/t_{k-1}$. Since Δ_k consists of all the numbers m/t_k, m an integer, it is clear that d_k *is the smallest positive integer such that* $d_k\delta_k \in \Delta_{k-1}$.

LEMMA 3. *If for a given j, $\delta_l > d_{l-1}\delta_{l-1}$ for all $l \leqq j$, then*
(a$_j$) $k/t_j \geqq d_j\delta_j$, k *an integer*, $j \geqq 1$, *implies* $k/t_j \in \Gamma_j$,
(b$_j$) $d_j\delta_j \in \Gamma_{j-1}$, $j \geqq 1$.

Proof. The lemma may be vacuously true, in fact, is vacuous if and only if the γ_i are integers. We suppose this not to be the case: $\delta_1 = \gamma_{i(1)} = r_1/s_1$ is then the first non-integer in the sequence $\gamma_1, \gamma_2, \cdots$.

For $j = 1$ we have $t_j = t_1 = s_1$, $t_{j-1} = t_0 = 1$, so that $d_1 = s_1$; whence (b) holds for $j = 1$. For $j = 1$, (a$_1$) becomes: if $k/s_1 \geqq s_1 \cdot r_1/s_1 = r_1$ then $k = ms_1 + nr_1$, m, n non-negative integers. Since $(r_1, s_1) = 1$, we can at least get the equality $k = ms_1 + nr_1$, m, n integers. If now n is taken such that $0 \leqq n < s_1$, then m must be positive since $k \geqq s_1 r_1 > n r_1$: whence (a$_j$) is true for $j = 1$. Now (b$_j$) follows from (a$_{j-1}$). In fact, since $t_j \equiv 0(s_j)$ we have $d_j\delta_j = t_j/t_{j-1} \cdot r_j/s_j = k/t_{j-1}$, k an integer, and since $t_j > t_{j-1}$ and, by hypothesis, $r_j/s_j > d_{j-1}\delta_{j-1}$, we conclude that $k/t_{j-1} > d_{j-1}\delta_{j-1}$; thus by (a$_{j-1}$), $t_j/t_{j-1} \cdot r_j/s_j \in \Gamma_{j-1}$.

Let $s'_j = \text{g.c.d.} (t_{j-1}, s_j)$; we have also $t_j = \text{l.c.m.} (t_{j-1}, s_j)$. Then $s'_j t_j = t_{j-1} s_j$. Now g.c.d. $(s_j^2, s'_j t_j) = \text{g.c.d.} (s_j^2, t_{j-1}, s_j) = s_j \cdot \text{g.c.d.} (s_j, t_{j-1}) = s_j s'_j$, whence

g.c.d. $(s_j/s_{j-1}, t_j/t_{j-1}) = 1$. $s_j/s_j' = t_j/t_{j-1} = d_j$. Let $t_j/s_j = s_j''$. Then $(r_js_j'', d_j) = 1$, whence there exists a non-negative integer $m < d_j$ such that $k - mr_js_j'' \equiv 0(d_j)$, whence

$$\frac{k}{t_j} - \frac{mr_j}{s_j} = \frac{k}{t_j} - \frac{mr_js_i''}{d_js_j's_j''} = \frac{k'}{s_j's_j''} = \frac{k'}{t_{j-1}} \geqq \frac{r_j}{s_j} > d_{j-1}\frac{r_{j-1}}{s_{j-1}}.$$

Statement (a_j) now follows by induction.

Let K be an algebraically closed field of arbitrary characteristic, $\Sigma = K(x, y)$ a pure transcendental extension field of K. Let v be a 0-dimensional valuation of Σ over K with $v(x) > 0$, $v(y) > 0$. In the case that v is of rank 1, we may normalize the value group of v by placing $v(x) = 1$; the meaning of the phrase "$v(z)$ is a rational multiple of $v(x)$" is then clear. In the case that v is of rank 2, the phrase "$v(z)$ is a rational multiple of $v(x)$" has the following precise meaning: there exist integers r, s, $s > 0$, such that $v(z^s) = v(x^r)$.

Let q_1, q_2, $\cdots$ be the Jordan sequence of 0-dimensional v-ideals in $K[x, y]$; $\gamma_i = v(q_i)$. Let Δ_0 be the group of values $\{mv(x)\}$, m an integer. In case v is of rank 1, $v(x)$ is taken to be 1, and Δ_0 is simply the group of integers. Let D_i be the additive abelian group generated by $v(x)$, γ_1, γ_2, $\cdots$, γ_i. We have $\Delta_0 = D_0 \subseteq D_1 \subseteq \cdots$. Let Δ_j be the jth group $D_{i(j)}$ containing its predecessor properly; we also place $\Delta_0 = D_{i(0)}$. Let Γ_0 be the set of values $\{m(v(x))\}$, m a non-negative integer; Γ_j the set of values $\{m + n_1\gamma_{i(1)} + \cdots + n_j\gamma_{i(j)}\}$, where m, n_i are non-negative integers. We denote by δ_1, δ_2, $\cdots$ the integers $\gamma_{i(1)}$, $\gamma_{i(2)}$, $\cdots$.

THEOREM 6. *Let $\Delta_1 \subset \Delta_2 \subset \cdots$ be the finite or infinite sequence of groups introduced above. There exists a corresponding sequence of polynomials, $f_1(x, y)$, $f_2(x, y)$, $\cdots$, elements of $K[x, y]$, such that:*

(1_j) $f_j(x, y)$ *is monic in y, that is, the coefficient of the highest power of y is 1.*

(2_j) $v(f_j) = \delta_j$, $j \geqq 1$.

(3_j) *Degree in y of f_j is equal to $d_{j-1} \cdot$ degree in y of f_{j-1}, $j > 1$, where d_{j-1} is an integer; degree in y of f_1 is 1.*

(4_j) $d_{j-1}v(f_{j-1}) \in \Delta_{j-2}$; $kv(f_{j-1}) \notin \Delta_{j-2}$, $0 < k < d_{j-1}$, $j > 1$, *where k is an integer.*

(5_j) $v(f_j) > d_{j-1}v(f_{j-1})$, $j > 1$.

Proof. If every γ_i is an integral multiple of $v(x)$, we have $D_0 = D_1 = \cdots$, and there is nothing to prove; that is, the theorem is vacuously true. Suppose then that some γ_i, say γ_k, is not an integral multiple of $v(x)$. If $v(y)$ is not an integral multiple of $v(x)$ then $v(y) = \gamma_{i(1)} = \delta_1$, that is, any element $g(x, y) \in K[x, y]$ of value less than $v(y)$ has an integral multiple of $v(x)$ as value. Thus Theorem 6 is satisfied, in this case, at least for $j = 1$ ($f_1 = y$). Suppose $v(y) = k_1v(x)$, k_1 an integer. Since v is 0-dimensional and K is algebraically closed, there exists an element $c_1 \in K$ such that $v(y - c_kx^{k(1)}) > v(y)$; if $v(y - c_1x^{k(1)}) = k_2v(x)$, k_2 an integer, then there exists an element $c_2 \in K$ such that $v(y - c_1x^{k(1)} - c_2x^{k(2)}) > v(y - c_1x^{k(1)})$. Suppose this process can be

repeated i times, so that we have integers $k_1, k_2, \cdots, k_{i+1}$ and elements $c_1, c_2, \cdots, c_i \in K$ such that $v(y - c_1 x^{k(1)} - \cdots - c_i x^{k(i)}) = k_{i+1} v(x)$. Setting $y_1 = y - c_1 x^{k(1)} - \cdots - c_i x^{k(i)})$, we assert that $v(y_1) < \gamma_k$. In fact, let $g(x, y)$ be an element of $K[x, y]$ with $v(g) = \gamma_k$, and suppose $v(y_1) > v(g(x, y))$ (equality is not possible since $v(g)$ is not an integral multiple of $v(x)$). Then $g(x, y) = g(x, y_1 + c_1 x^{k(1)} + \cdots + c_i x^{k(i)}) = h(x, y_1) = h'(x) + y_1 h''(x, y_1)$, where $h'(x) \in K[x]$, $h''(x, y_1) \in K[x, y_1] = K[x, y]$. Thus $v(y_1 h'') > v(g)$, whence $v(g) = v(h'(x))$, which is a contradiction, since $v(h')$ is an integer. Only a finite number of values less than $v(g)$ are assumed by elements of $K[x, y]$. Therefore, since $k_1 < k_2 < \cdots$, there exists an integer m such that $y - c_1 x^{k(1)} - \cdots - c_m x^{k(m)}$ has a value not an integral multiple of $v(x)$. As above, $v(y - c_1 x^{k(1)} - \cdots - c_m x^{k(m)}) > v(g)$ is impossible, whence $v(y - c_1 x^{k(1)} - \cdots - c_m x^{k(m)}) = \gamma_{i(1)} = \delta_1$. Thus assertion 2 of the theorem is satisfied by $f_1(x, y) = y - c_1 x^{k(1)} - \cdots - c_m x^{k(m)} = y'$. Also $f_1(x, y)$ is monic in y. Thus Theorem 6 is satisfied at least for $j = 1, 3$, parts 4, 5 being so far vacuously true. Note that we may without loss of generality take $f_1(x, y) = y$, since $K[x, y] = K[x, y']$ and any monic polynomial $h(x, y')$ is transformed into a monic polynomial $h_1(x, y) = h(x, y + c_1 x^{k(1)} + \cdots + c_m x^{k(m)})$ with $\deg_{y'} h(x, y') = \deg_y h(x, y)$.

The proof now proceeds by induction on j. Suppose, then, that we have polynomials $f_1, f_2, \cdots, f_j$ satisfying conditions 1, 2, 3, 4, 5. We have either to show that $D_{i(j)} = D_{i(j)+1} = \cdots$, or produce an $f_{j+1}(x, y) \in K[x, y]$ such that the five conditions are satisfied by $f_1, f_2, \cdots, f_j, f_{j+1}$.

Note that $v(f_1), v(f_2), \cdots, v(f_{j-1})$ are rational multiples of $v(x)$. In fact, suppose $v(f_1), v(f_2), \cdots, v(f_k)$, $k < j - 1$, are rational multiples of $v(x)$; these values generate Δ_k. Since $d_{k+1} v(f_{k+1}) \in \Delta_k$ by 4, $v(f_{k+1})$ is a rational multiple of $v(x)$. Thus $v(f_1), \cdots, v(f_{j-1})$ are rational multiples of $v(x)$. Let $v(f_k) = \delta_k = r_k/s_k \cdot v(x)$, r_k, s_k positive integers, $(r_k, s_k) = 1$, $k \leq j - 1$. Let $t_0 = 1$, $t_k = $ l.c.m. $(s_1, \cdots, s_k)$. By Lemma 3, in view of condition 5, we have:

(a$_{j-1}$) If $k/t_{j-1} \cdot v(x) \geq d_{j-1} v(f_{j-1})$, k an integer, then $k/t_{j-1} \cdot v(x) \in \Gamma_{j-1}$.

(b$_{j-1}$) $d_{j-1} = t_{j-1}/t_{j-2}$.

Note that if $v(f_j)$ is a rational multiple of $v(x)$, and d_j is defined by (4$_{j+1}$) that is, d_j is the least positive integer such that $d_j v(f_j) \in \Delta_{j-1}$, then (a$_j$) and (b$_j$) hold.

We now prove:

(c$_j$) If $\deg_y g(x, y) < \deg_y f_j$, $g(x, y) \in K[x, y]$, then $g(x, y)$ is a sum of monomials in $x, f_1, \cdots, f_{j-1}$,

$$g(x, y) = \sum c(\alpha_0, \cdots, \alpha_{j-1}) F(\alpha_0, \cdots, \alpha_{j-1}), \quad c(\alpha_0, \cdots, \alpha_{j-1}) \in K,$$

with $0 \leq \alpha_k < d_k$, $k = 1, 2, \cdots, j-1$, in which no two monomials have equal values, and hence $v(g) \in \Delta_{j-1}$ where $F(\alpha_0, \cdots, \alpha_{j-1})$ is the monomial $x^{\alpha(0)} f_1^{\alpha(1)} \cdots f_{j-1}^{\alpha(j-1)}$, a notation used throughout.

The proof is again by induction. For $j = 1$, $\deg_y f_1 = 1$, whence $g(x, y) = g'(x) \in K[x]$, and the statement is trivial. For the general case, since $f_{j-1}(x, y)$

is monic in y we may write

$$g(x, y) = a_0(x, y) + a_1(x, y)f_{j-1}(x, y) + \cdots + a_l(x, y)f_{j-1}^l(x, y),$$

$a_i(x, y) \in K[x, y]$, with $\deg_y a_i(x, y) < \deg_y f_{j-1}(x, y)$, and $l < d_{j-1}$ since $\deg_y g(x, y) < \deg_y f_j(x, y) = d_{j-1} \cdot \deg_y f_{j-1}(x, y)$. By induction, $v(a_k) \in \Delta_{j-2}$, $k = 0, 1, \cdots, l$, whence

$$v(a_k f_{j-1}^k) \neq v(a_{k'} f_{j-1}^{k'}) \quad \text{for} \quad k \neq k'$$

by 4, that is, since $v(f_{j-1}^{k-k'}) \notin \Delta_{j-2}$. (c) now follows by induction. Note that if $v(f_j)$ is a rational of $v(x)$, and d_j is defined by (4_{j+1}), then (c_{j+1}) holds when we write $d_j \deg_y f_j$ instead of $\deg_y f_{j+1}$.

If, now, $v(f_j)$ is not a rational multiple of $v(x)$ then $\Delta_j = D_m$, $m \geq i_j$. In fact, if $g(x, y) \in K[x, y]$ then, since $f_j(x, y)$ is monic in y, we may write

$$g(x, y) = a_0(x, y) + a_1(x, y)f_j(x, y) + \cdots + a_l(x, y)f_j^l(x, y),$$

where $a_i(x, y) \in K[x, y]$, and $\deg_y a_i(x, y) < \deg_y f_j(x, y)$. By (c), $v(a_k) \in \Delta_{j-1}$ whence $v(a_k f_j^k) \neq v(a_{k'} f_j^{k'})$ for $k \neq k'$ since $v(f_j^{k-k'})$ is not a rational multiple of $v(x)$, whereas $v(a_k/a_{k'})$ is. Thus $v(g(x, y)) = v(a_k(x, y)f_j^k(x, y))$, some k, $0 \leq k \leq l$, whence $v(g(x, y)) \in \Delta_j$. In this case, as remarked before, there is nothing more to prove.

If $v(f_j)$ is a rational multiple of $v(x)$, we define d_j by means of condition 4. Since $d_j v(f_j) = k/t_{j-1} > d_{j-1} v(f_{j-1})$, we have, by (a), $d_j v(f_j) = \alpha_0 + \alpha_1 \delta_1 + \cdots + \alpha_{j-1} \delta_{j-1}$, with $0 \leq \alpha_i < d_i$, $i = 1, 2, \cdots, j-1$. Since v is 0-dimensional and K is algebraically closed, there exists an element $c \in K$ such that $v(f_j^{d(j)} - cF(\alpha_0, \cdots, \alpha_{j-1})) > (f_j^{d(j)})$. Let $f_{j+1}^{(1)}(x, y) = f_j^{d(j)} - cF(\alpha_0, \cdots, \alpha_{j-1})$. Now

$$\deg_y F(\alpha_0, \alpha_1, \cdots, \alpha_{j-1}) < \deg_y F(0, d_1, d_2 - 1, d_3 - 1, \cdots, d_{j-1} - 1) = \deg_y f_j$$

whence $f_{j+1}^{(1)}$ is monic in y. If $v(f_{j+1}^{(1)}) \in \Delta_j$ then by (a_j), one has

$$v(f_{j+1}^{(1)}) = \alpha_0^{(2)} + \alpha_1^{(2)} \delta_1 + \cdots + \alpha_j^{(2)} \delta_j, \quad 0 \leq \alpha_i^{(2)} < d_i, \quad i = 1, 2, \cdots, j.$$

Then there exists an element $c_2 \in K$ such that $v(f_{j+1}^{(1)} - c_2 F(\alpha_0^{(2)}, \cdots, \alpha_j^{(2)})) > v(f_{j+1}^{(1)})$. Let $f_{j+1}^{(2)} = f_{j+1}^{(1)} - c_2 F(\alpha_0^{(2)}, \cdots, \alpha_j^{(2)})$. Note, as above, that $f_{j+1}^{(2)}$ is monic in y. Repeating the above process, we obtain a sequence $f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots$, which is finite if, for some integer k, $v(f_{j+1}^{(k)}) \notin \Delta_j$, and which is infinite otherwise.

In case the sequence is infinite, all groups D_m, $m \geq i_j$, coincide with Δ_j. In fact, if $g(x, y) \in K[x, y]$ and $v(g(x, y)) = \gamma_k$ then there are only a finite number of values less than $v(g)$ assumed by elements in $K[x, y]$, whence for some k, $v(f_{j+1}^{(k)}) > v(g)$. Since $f_{j+1}^{(k)}$ is monic in y, $g(x, y)$ can be written:

$$g(x, y) = a(x, y) + g'(x, y)f_{j+1}^{(k)}(x, y),$$

with $a(x, y)$, $g'(x, y) \in K[x, y]$ and $\deg_y a(x, y) < \deg_y f_{j+1}^{(k)} = \deg_y f_j^{d(i)}$, whence $v(g(x, y)) = v(a(x, y)) \in \Delta_j$, by (c_{j+1}).

In case the sequence is not infinite, that is, in case for some integer k we have $v(f_{j+1}^{(1)}), v(f_{j+1}^{(2)}), \cdots, v(f_{j+1}^{(k-1)}) \in \Delta_j$, while $v(f_{j+1}^{(k)}) \notin \Delta_j$, then either $v(f_{j+1}^{(k)})$ is infinitely large with respect to $v(x)$ in which case $\Delta_j = D_m$, $m \geq i_j$, or $f_{j+1} = f_{j+1}^{(k)}$ together with $f_1, \cdots, f_j$ satisfy $(1, 2, 3, 4, 5, j+1)$. In fact, in either case 1, 3, 4, 5, are clearly satisfied. It thus remains to prove in either case that if $v(g(x, y)) < v(f_{j+1})$, $g(x, y) \in K[x, y]$, then $v(g) \in \Delta_j$. Since f_{j+1} is monic in y we may write $g(x, y) = g'(x, y) + h(x, y)f_{j+1}(x, y)$, where $g'(x, y)$, $h(x, y) \in K[x, y]$ and $\deg_y g'(x, y) < \deg_y f_{j+1} = \deg_y f_j^{d(i)}$, whence $v(g(x, y)) = v(g'(x, y))$. The statement now follows from (c_{j+1}). This completes the proof.

COROLLARY 1. *The sequence* $f_1, f_2, \cdots$ *is infinite if and only if v is non-discrete, and of rational rank 1.*

Proof. Let the sequence $f_1, f_2, \cdots$ be infinite. Since $\deg_y f_i \to \infty$ as $i \to \infty$, we have, by (c) of the above theorem, that $\lim \Delta_j$ is the value group. Since the value group contains an infinite ascending sequence of subgroups it must be non-discrete; since $v(f_i)$ is rationally dependent on $v(x)$, it is rational rank 1. The converse is trivial. (Actually, the corollary does not depend on any special properties of the f_j, but solely on the fact that the f_j are in 1-1 correspondence with the Δ_j; in fact, the value group is non-discrete, rational rank 1 if and only if the value group contains an infinite ascending sequence of subgroups).

COROLLARY 2. *If v is of rank 2, in which case the sequence $f_1, f_2, \cdots$ is finite, containing j terms say, then, except for the valuation in which $v(y)$ is infinitely small with respect to $v(x)$, the possibilities* (a) $\Delta q_i \neq 0$ *and* (b) $\Delta q_i = 0$ *correspond respectively to the possibilities* (a') $v(f_j)$ *is a rational multiple of $v(x)$, and* (b') $v(f_j)$ *is not a rational multiple of $v(x)$.*

Proof. Also this corollary may be proved without recourse to any special properties of the f_i. In fact, in all cases of rank 2 valuations, if $\Delta q_i \neq 0$ then the γ_i generate a group of the type of all integers (in fact, an isolated subgroup of the value group). Hence any two γ's are rationally dependent. In particular, then, if $v(y)$ is not infinitely small with respect to $v(x)$, then $x \notin \Delta q_i$ and the statement follows.

If $\Delta q_i = 0$, the γ_i generate the entire value group, which does not possess a rational base of one element. This completes the proof.

We wish to remark that if v is of rank 1 then "$v(f_j)$ not a rational multiple of $v(x)$" means $v(f_j)$ is an irrational number: if, however, v is of rank 2, then "$v(f_j)$ not a rational multiple of $v(x)$" can be distinguished into two cases. Namely, either $v(f_j)$ may be infinitely small with respect to $v(x)$; or, although $mv(x) > v(f_j)$ for some integer m, $v(x^n) \neq v(f_j^{n'})$ for any integers n, n' different

from 0. The former case can arise, however, only for $j=1$ and if $v(y)$ is infinitely small with respect to $v(x)$. (The case $v(f_j)$ infinitely large with respect to $v(x)$ cannot arise here, since $i_jv(x) > \delta_j = v(f_j)$).

Let v be a rational rank 2 valuation of Σ over K with $v(x) > 0$, $v(y) > 0$ and let $f_1, f_2, \cdots, f_j$ be the polynomials for v introduced in this theorem.

COROLLARY 3. *If v_1 is a rank 1 valuation of Σ over K (a subgroup of the additive group of real numbers being taken as a value group) and if $v_1(x) = v(x)$, $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j$, then $v_1 = v$.*

Proof. We have shown that if $g(x, y) \in K[x, y]$ then $g(x, y) = a_0(x, y) + a_1(x, y)f_j + \cdots + a_l(x, y)f_j^l$, where $a_i(x, y) \in K[x, y]$ and $\deg_y a_i(x, y) < \deg_y f_j(x, y)$; moreover, that $v(a_k f_j^k) \neq v(a_{k'} f_j^{k'})$ for $k \neq k'$. Hence by (c), $g(x, y)$ is a sum of monomials in $x, f_1, \cdots, f_j$,

$$g(x, y) = \sum c(\alpha_0, \cdots, \alpha_j) F(\alpha_0, \cdots, \alpha_j), \quad c(\alpha_0, \cdots, \alpha_j) \in K,$$

α_i, $i = 0, 1, \cdots, j$, non-negative integers, with

$$v(F(\alpha_0, \cdots, \alpha_j)) \neq v(F(\alpha_0', \cdots, \alpha_j'))$$

unless $\alpha_i = \alpha_i'$, $i = 0, 1, \cdots, j$. Hence the value of g is the minimum of the values of these monomials. The corollary is an immediate consequence.

We insert here a table displaying the polynomials f introduced in Theorem 6 for a given 0-dimensional valuation. The valuation in which $v(y)$ is infinitely small with respect to $v(x)$ forms an exception, and is omitted.

Rank 1

Discrete: $f_1, f_2, \cdots, f_j, f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots$ (infinite).
Non-discrete:

Rational Rank 1. $f_1, f_2, \cdots, f_j, f_{j+1}, \cdots$,
Rational Rank 2. $f_1, f_2, \cdots, f_j, f_{j+1}$.

Rank 2

$\Delta q_i = 0$. $f_1, f_2, \cdots, f_j, f_{j+1}$.
$\Delta q_i \neq 0$. $f_1, f_2, \cdots, f_j, f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots$ (finite or infinite).

Let v be a rational rank 2 valuation of $K(x, y)$ with $v(x) = 1$, $v(y) > 0$. Let Δ_{j+1} be the value group, and $f_1, f_2, \cdots, f_j, f_{j+1}$, $j \geq 0$, the polynomials constructed in Theorem 6. $v(f_{j+1}) = \tau$ is an irrational number.

THEOREM 7. *For any irrational number $\sigma > \tau = v(f_{j+1})$ there exists a rational rank 2 valuation v_1 of $K(x, y)$ with $v_1(x) = 1$, $v_1(y) > 0$, and such that $v_1(f_i) = v(f_i)$, $i = 1, 2, \cdots, j$, $v_1(f_{j+1}) = \sigma$.*

Proof. Since f_{j+1} is monic in y, every polynomial $g(x, y) \in K[x, y]$ may be written

(1) $$g(x, y) = a_0(x, y) + a_1(x, y)f_{j+1} + \cdots + a_k(x, y)f_{j+1}^k$$

with $a_i(x, y) \in K[x, y]$ and $\deg_y a_i < \deg_y f_{j+1}$. The expansion (1) for $g(x, y)$ is unique; in fact, let $g(x, y) = a_0'(x, y) + a_1'(x, y)f_{j+1} + \cdots + a_{k'}'(x, y)f_{j+1}^{k'}$ with $a_i'(x, y) \in K[x, y]$ and $\deg_y a_i' < \deg_y f_{j+1}$. Since $a_0 - a_0' \equiv 0(f_{j+1})$ and $\deg_y f_{j+1} > \deg_y(a_0 - a_0')$ we have $a_0 = a_0'$. Similarly $k = k'$ and $a_i = a_i'$. By (c) of Theorem 6, $v(a_i) \in \Delta_j$. We now place

$$v_1(g) = \min (v(a_i) + i\sigma),$$

where i runs from 1 to k. Clearly $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j$, $v_1(f_{j+1}) = \sigma$, $v_1(x) = 1$ and $v_1(y) > 0$. If $h(x, y) = b_0(x, y) + b_1(x, y)f_{j+1} + \cdots + b_{k'}(x, y)f_{j+1}^{k'}$ is the expansion (1) for a polynomial $h(x, y) \in K[x, y]$ then

$$g + h = (a_0 + b_0) + (a_1 + b_1)f_{j+1} + \cdots + (a_{k''} + b_{k''})f_{j+1}^{k''}$$

is the expansion (1) for $g + h$, where $k'' = \max(k, k')$ and $a_i = 0$ for $i > k$, $b_i = 0$ for $i > k'$. Then

$$v_1(g + h) = \min_i (v(a_i + b_i) + i\sigma) \geqq \min_i (\min (v(a_i), v(b_i)) + i\sigma)$$

$$= \min_i (v(a_i) + i\sigma, v(b_i) + i\sigma) = \min (\min_i (v(a_i) + i\sigma)),$$

$$\min_i (v(b_i) + i\sigma)) = \min (v_1(g), v_1(h)),$$

that is,

(2) $$v_1(g + h) \geqq \min (v_1(g), v_1(h)).$$

Moreover, since $v_1(1) = v_1(-1) = 0$ we also have

(2') $$v_1(g + h) = \min (v_1(g), v_1(h)) \quad \text{if} \quad v_1(g) \neq v_1(h).$$

We wish now to prove

(3) $$v_1(gh) = v_1(g) + v_1(h).$$

Because of the uniqueness of (1) and because of (2) and (2') it is sufficient to prove (3) for $g = a_i(x, y)f_{j+1}^i$, $h = b_{i'}(x, y)f_{j+1}^{i'}$. Since the expansion (1) for $k(x, y)f_{j+1}^m$, $k(x, y) \in K[x, y]$, is simply obtained from the expansion for $k(x, y)$ by multiplication by f_{j+1}^m, we may assume, without loss of generality, that $i = i' = 0$, that is, $g = a_i(x, y)$, $h = b_i(x, y)$. Since $\deg_y a_i < \deg_y f_{j+1}$ and $\deg_y b_i < \deg_y f_{j+1}$, the expansion (1) for gh is

$$gh = a(x, y) + b(x, y)f_{j+1}.$$

Since $v(gh)$ and $v(a)$ are elements of Δ_j while $v(bf_{j+1}) \in \Delta_j$, we have $v(gh) = \min (v(a), v(bf_{j+1}))$, whence $v(gh) = v(a) < v(bf_{j+1}) = v(b) + \tau < (bv) + \sigma$. Hence $v_1(gh) = v_1(a) = v(gh) = v(g) + v(h) = v_1(g) + v_1(h)$, which proves (3). If we now place

$$v_1(g(x, y)/h(x, y)) = v_1(g) - v_1(h)$$

where $g(x, y)$, $h(x, y) \in K[x, y]$, v_1 becomes a valuation of $K(x, y)$. Since v_1 is of rational rank 2, it is necessarily 0-dimensional. This completes the proof.

Let v be a 0-dimensional valuation of Σ over K with $v(x) > 0$, $v(y) > 0$.

1. If v is non-discrete, rational rank 1, place $v(x) = 1$, and let $f_1, f_2, \cdots$ be the infinite sequence of polynomials constructed in Theorem 6.

2. If v is rank 1, discrete, or of rank 2 with $\Delta q_i \neq 0$ and $v(y)$ is not infinitely small with respect to $v(x)$, let $f_1, \cdots, f_j; f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots, j \geq 0$, be the sequence of polynomials constructed in Theorem 6. If v is of rank 1 then the sequence $f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots$ is infinite: this is also possible for v of rank 2, but the case $v(f_{j+1}^{(k)})$ infinitely great with respect to $v(x)$ can arise. To conserve notation, we place $f_{j+1}^{(k)} = f_{j+1}^{(k+1)} = \cdots$, in the latter case.

3. If v is rational rank 2 or of rank 2 with $\Delta q_i = 0$, let $f_1, f_2, \cdots, f_k, f_{k+1}$, $k \geq 0$, be the sequence of polynomials constructed in Theorem 6.

In all three cases we write $\pi(v(g)) = r/s$, $g \in \Sigma$, if $sv(g) = rv(x)$, r, s integers, $s \neq 0$.

THEOREM 8.1. *For any integer $j > 0$ and any irrational number $\tau > v(f_j^{d(j)})$ there exists a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$ such that $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j$, and $v_1(f_{j+1}) = \tau$.*

THEOREM 8.2. *For any integer $k > 0$ and any irrational number $\tau > \pi(v(f_j^{d(j)}))$, there exists a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$, $v_1(f_i) = \pi(v(f_i))$, $i = 1, \cdots, j$, and $v_1(f_{j+1}^{(k)}) = \tau$. (For $j = 0$, place $f_j^{d(j)} = y$.)*

THEOREM 8.3. *For any j, $0 \leq j \leq k$, and any irrational number $\tau > \pi(v(f_j^{d(j)}))$, there exists a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$ such that $v_1(f_i) = \pi(v(f_i))$, $i = 1, \cdots, j$, $v_1(f_{j+1}) = \tau$. (For $j = 0$, place $f_j^{d(j)} = y$; for $k = 0$, the theorem is trivial, where τ is any positive irrational number.)*

Proof. By Theorem 6 there exists a monomial $cF(\gamma_0, \cdots, \gamma_{j-1})$, $c \in K$, $0 \leq \gamma_i < d_i$, $i = 1, \cdots, j-1$, such that $v(F(0, \cdots, 0, d_j) - cF(\gamma_0, \cdots, \gamma_{j-1}))$ $> v(F(0, \cdots, 0, d_j))$. Let $\bar{f}_{j+1} = F(0, \cdots, 0, d_j) - cF(\gamma_0, \cdots, \gamma_{j-1})$. If v is non-discrete, rational rank 1, or rank 1, discrete, then $v(\bar{f}_{j+1})$ is certainly a rational multiple of $v(x)$; if v is rank 2 with $\Delta q_i \neq 0$ (as in Theorem 8.2) then $v(\bar{f}_{j+1})$ is possibly infinitely large with respect to $v(x)$ (in which case $\bar{f}_{j+1} = f_{j+1}^{(1)}$ $= f_{j+1}^{(2)} = \cdots$) or $v(\bar{f}_{j+1})$ is a rational multiple of $v(x)$; in Theorem 8.3, $v(\bar{f}_{j+1})$ will be a rational multiple of $v(x)$, unless $j = k$, and $\bar{f}_{j+1} = f_{k+1}$. In case $v(\bar{f}_{j+1})$ is a rational multiple of $v(x)$, it is sufficient, by Theorem 7, to prove Theorem 8 for an irrational number τ such that

$$\pi v(F(0, \cdots, 0, d_j)) < \tau < \pi v(F(0, \cdots, 0, d_j) - cF(\gamma_0, \cdots, \gamma_{j-1})),$$

since any number greater than $\pi v(f_j^{d(j)})$ is also greater than some irrational number satisfying this inequality. Since $\bar{f}_{j+1}$ is monic in y, any polynomial $g(x, y) \in K[x, y]$ can be written as

(1) $g(x, y) = a_0(x, y) + a_1(x, y)\bar{f}_{i+1} + \cdots + a_k(x, y)\bar{f}_{i+1}^k,$

where $a_i(x, y) \in K[x, y]$ and $\deg_y a_i(x, y) < \deg_y \bar{f}_{i+1}$. Just as in Theorem 7, one proves that the expansion (1) is unique. Placing

$$v_1(g) = \min_i (\pi(v(a_i)) + i\tau),$$

we can prove exactly as in Theorem 7 that

(2) $v_1(g + h) \geqq \min (v_1(g), v_1(h))$

and

(2') $v_1(g + h) = \min (v_1(g), v_1(h)) \quad \text{if} \quad v_1(g) \neq v_1(h)$

where $g, h \in K[x, y]$. Also as before, in proving

(3) $v_1(gh) = v_1(g) + v_1(h)$

we may assume without loss of generality that

$\deg_y g(x, y) < \deg_y \bar{f}_{i+1} = \deg_y f_j^{d(j)}$ and $\deg_y h(x, y) < \deg_y \bar{f}_{i+1} = \deg_y f_j^{d(j)}.$

By (c) of Theorem 6, $g(x, y)$, $h(x, y)$ are sums of monomials $cF(\alpha_0, \cdots, \alpha_j)$, $c \in K$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j$, of distinct value. Hence by the uniqueness of (1), and because of (2) and (2'), we may further assume without loss in generality that

$g(x, y) = F(\alpha_0, \cdots, \alpha_j),$ $0 \leq \alpha_0, 0 \leq \alpha_i < d_i, i = 1, \cdots, j$

and

$h(x, y) = F(\beta_0, \cdots, \beta_j),$ $0 \leq \beta_0, 0 \leq \beta_i < d_i, i = 1, \cdots, j,$

or, more generally, that

$g(x, y) = g'(x, y)f_j^{\alpha(j)},$ where $g'(x, y) \in K[x, y]$, $\deg_y g' < \deg_y f_j$, and $\alpha_j < d_j,$

and

$h(x, y) = h'(x, y)f_j^{\beta(j)},$ where $h'(x, y) \in K[x, y]$ $\deg_y h' < \deg_y f_j$, and $\beta_j < d_j.$

Consider first the case $\alpha_j = 0$. If $\beta_j < d_j - 1$ then $\deg_y gh = \deg_y g' h' f_j^{\beta(j)}$ $< \deg_y f_j^2 f_j^{d(j)-2} = \deg_y f_j^{d(j)}$, whence $v_1(gh) = \pi v(gh) = \pi v(g) + \pi v(h) = v_1(g) + v_1(h)$. If $\beta_j = d_j - 1$, then

$$gh = g'h'f_j^{d(j)-1} = (a(x, y) + b(x, y)f_j)f_j^{d(j)-1}$$

$$= a(x, y)f_j^{d(j)-1} + b(x, y)(\bar{f}_{i+1} + cF(\gamma_0, \cdots, \gamma_{j-1}))$$

where $a, b \in K[x, y]$ and $\deg_y a < \deg_y f_j$, $\deg_y b < \deg_y f_j$. Also, $v(g'h') = \min (v(a),$

$v(bf_j)) = v(a)$, that is,

$$v(a) < v(bf_j)$$

since $v(g'h')$, $v(a) \in \Delta_{j-1}$ while $v(bf_j) \notin \Delta_{j-1}$. Because of the limited degree in y of $a(x, y)$ and $b(x, y)$,

$$v_1(af_j^{d(j)-1}) = \pi(v(af_j^{d(j)-1})),$$

$$v_1(bf_j^{d(j)}) = \min\,(\pi(v(b)) + \tau, \quad \pi(v(b \cdot F(\gamma_0, \cdots, \gamma_{j-1}))))$$

$$= \pi(v(b \cdot F(\gamma_0, \cdots, \gamma_{j-1}))) = \pi(v(b \cdot f_j^{d(j)})) > \pi(v(af_j^{d(j)-1}))$$

$$= v_1(af_j^{d(j)-1})$$

whence

$$v_1(gh) = v_1(af_j^{d(j)-1}) = \pi(v(af_j^{d(j)-1})) = \pi(v(gh)) = \pi v(g) + \pi v(h) = v_1(g) + v_1(h).$$

Consider now the case $\alpha_j > 0$, the general case. Here we can write, for reasons given in the case $\alpha_j = 0$,

$$gh = g'h'f_j^{\alpha(j)+\beta(j)} = (a(x, y) + b(x, y)f_j)f_j^{\alpha(j)+\beta(j)},$$

with the same conditions on $a(x, y)$ and $b(x, y)$, and again deduce that $v(a) < v(bf_j)$. We assert that

$$v_1(af_j^{\alpha(j)+\beta(j)}) = \pi v(af_j^{\alpha(j)+\beta(j)})$$

and that

$$v_1(bf_j^{\alpha(j)+\beta(j)+1}) = \pi v(bf_j^{\alpha(j)+\beta(j)+1}),$$

whence will follow,

$$v_1(gh) = \min\,(v_1(af_j^{\alpha(j)+\beta(j)})), \; v_1(bf_j^{\alpha(j)+\beta(j)+1})) = \pi(v(af_j^{\alpha(j)+\beta(j)}))$$

$$= \pi(v(gh)) = \pi(v(g)) + \pi(v(h)) = v_1(g) + v_1(h).$$

Since $\alpha_j < d_j$ and $\beta_j < d_j$ we have $\alpha_j + \beta_j + 1 \leqq 2d_j - 1$. If $\alpha_j + \beta_j + 1 < d_j$ then $\deg_y bf_j^{\alpha(j)+\beta(j)+1} < \deg_y f_j^{(j)}$, whence

$$v_1(bf_j^{\alpha(j)+\beta(j)+1}) = \pi(v(bf_j^{\alpha(j)+\beta(j)+1})).$$

If $\alpha_j + \beta_j + 1 \geqq d_j$ we can write

$$bf_j^{\alpha(j)+\beta(j)+1} = bf_j^{\alpha(j)+\beta(j)+1-d(j)}(\bar{f}_{j+1} + cF(\gamma_0, \cdots, \gamma_{j-1})).$$

Since $\deg_y bf_j^{\alpha(j)+\beta(j)+1-d(j)} < \deg_y f_j^{d(j)}$,

$$v_1(bf_j^{\alpha(j)+\beta(j)+1-d(j)}\bar{f}_{j+1}) = \pi v(bf_j^{\alpha(j)+\beta(j)+1-d(j)}) + \tau,$$

and since $\alpha_j + \beta_j + 1 - d_j < d_j$, we have by the case $\alpha_j = 0$,

$$v_1(bf_j^{\alpha(j)+\beta(j)+1-d(j)} \cdot F(\gamma_0, \cdots, \gamma_{j-1}))$$

$$= v_1(bf_j^{\alpha(j)+\beta(j)+1-d(j)}) + v_1(F(\gamma_0, \cdots, \gamma_{j-1}))$$

$$= \pi(v(bf_j^{\alpha(j)+\beta(j)+1-d(j)}) + \pi(v(f_j^{d(j)})),$$

whence

$$v_1(bf_j^{\alpha(j)+\beta(j)+1}) = v_1(bf_j^{\alpha(j)+\beta(j)+1-d(j)} \cdot F(\gamma_0, \cdots, \gamma_{j-1})) = \pi(v(bf_j^{\alpha(j)+\beta(j)+1})).$$

Similarly $v_1(af_j^{\alpha(j)+\beta(j)}) = \pi(v(af_j^{\alpha(j)+\beta(j)}))$.

Thus (3) is proved. Placing $v_1(g/h) = v_1(g) - v_1(h)$, g/h any element of Σ, $g, h \in K[x, y]$, we obtain that v_1 is a valuation of Σ over K. Since v_1 is of rational rank 2, it is 0-dimensional. That $v_1(x) = 1$, $v_1(f_i) = \pi(v(f_i))$, $i = 1, \cdots, j$, is immediate from the definition of v_1. f_1 is monic in y of degree 1, whence $f_1 - y \equiv 0(x)$, since $v(f_1 - y) > 0$. Hence $v_1(y) > 0$, since $v_1(f_1) > 0$ and $v_1(x) > 0$.

It remains to prove the conditions on $v_1(f_{j+1})$, $v_1(f_{j+1}^{(k)})$, respectively. In the case $v(\bar{f}_{j+1})$ is not a rational multiple of $v(x)$, $\bar{f}_{j+1} = f_{k+1}$ or $\bar{f}_{j+1} = f_{j+1}^{(k)}$ and the condition has been verified. In the other cases we assume $\tau < \pi v(\bar{f}_{j+1})$. Recalling the construction of $f_{j+1}^{(k)}$, note that $f_{j+1}^{(1)} = \bar{f}_{j+1}$. Thus for $k = 1$, the required statement holds. Now unless $f_{j+1}^{(k)} = f_{j+1}^{(1)}$, we have $\deg_y(f_{j+1}^{(k)} - f_{j+1}^{(1)}) < \deg_y f_j^{d(j)}$, whence $v_1(f_{j+1}^{(k)} - f_{j+1}^{(1)}) = \pi(v(f_{j+1}^{(k)} - f_{j+1}^{(1)}))$. Hence,

$$v_1(f_{j+1}^{(k)}) = v_1(f_{j+1}^{(1)} + (f_{j+1}^{(k)} - f_{j+1}^{(1)})) = \min(\tau, \pi(v(f_{j+1}^{(k)} - f_{j+1}^{(1)}))) = \tau,$$

since $\tau < \pi v(f_{j+1}^{(1)}) = \pi(v(f_{j+1}^{(k)} - f_{j+1}^{(1)}))$. The statement for $v_1(f_{j+1})$ follows similarly if we consider f_{j+1} constructed from f_j as in Theorem 6. Note also, however, that since f_{j+1} and $\bar{f}_{j+1}$ are both monic in y of the same degree it is true that $v(f_{j+1} - \bar{f}_{j+1}) \in \Delta_j$ by (c) of Theorem 6. Hence $v(f_{j+1} - \bar{f}_{j+1}) \geq v(\bar{f}_{j+1})$, since $v(f_{j+1}) \bar{\in} \Delta_j$. The statement for f_{j+1} now follows exactly as above for $f_{j+1}^{(k)}$. This completes the proof.

Remark. For $j = 0$, the restriction $\tau > v(f_j^{d(j)}) = v(y)$ is superfluous with $\tau > 0$ only required. In fact, for $j = 0$, and the restriction on τ lifted, Theorems 8.2 and 8.3 become: For any monic polynomial $f_1 \in K[x, y]$ of degree 1 in y, and any irrational number $\tau > 0$, there exists a rational rank 2 valuation v_1 such that $v_1(x) = 1$, $v_1(y) > 0$, and $v_1(f_1) = \tau$. This is clear directly, without reference to the theorem.

Let v be a 0-dimensional valuation of Σ over K with $v(x) > 0$, $v(y) > 0$ and let $q_1, q_2, \cdots$ be the Jordan sequence of 0-dimensional v-ideals in $K[x, y]$ for v.

THEOREM 9. *For any integer $m > 0$ there exists a valuation v_1 of Σ over K of rational rank 2 with $v_1(x) = 1$, $v_1(y) > 0$ such that $q_1, q_2, \cdots, q_m$ are the first m valuation-ideals for v_1 in $K[x, y]$.*

Proof. Consider first the case that v is non-discrete and rational rank 1; let $v(x) = 1$, and let $f_1, f_2, \cdots$ be the (infinite) sequence of polynomials intro-

duced in Theorem 6. We assert that if $\mathfrak{q}$ is a v-ideal with $v(\mathfrak{q}) < v(f_{j+1})$ then $\mathfrak{q}$ has a basis consisting of f_{j+1} and monomials $F(\alpha_0, \cdots, \alpha_j)$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j$. In fact, if $g(x, y) \in \mathfrak{q}$ then, since f_{j+1} is monic in y,

$$g(x, y) = g'(x, y) + g''(x, y) f_{j+1}(x, y),$$

where $g'(x, y)$, $g''(x, y) \in K[x, y]$ and $\deg_v g'(x, y) < \deg_v f_{j+1}(x, y)$. Since $f_{j+1} \in \mathfrak{q}$, also $g' = g - g'' f_{j+1} \in \mathfrak{q}$, whence the statement follows by (c) of Theorem 6. Since $\mathfrak{q}_m$ is preceded by (contained in) only a finite number of v-ideals for v, there exists an integer $j = j(m)$ such that $v(\mathfrak{q}_m) < v(f_{j+1})$. Consider the set of monomials $F(\alpha_0, \cdots, \alpha_j)$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j = j(m)$, and order them according to their value: $F(\alpha_0, \cdots, \alpha_j)$ precedes $F(\alpha_0', \cdots, \alpha_j')$ if $v(F(\alpha_0, \cdots, \alpha_j)) < v(F(\alpha_0', \cdots, \alpha_j'))$. Since every $F(\alpha_0, \cdots, \alpha_j)$ has only a finite number of predecessors, the ordered set $\{F(\alpha_0, \cdots, \alpha_j)\}$ is a simple sequence:

(1) $u_1, u_2, \cdots$.

For some k, $v(u_k) < v(f_{j+1}) < v(u_{k+1})$, where clearly $k \geq m$; consider also the sequence:

(2) $u_1, u_2, \cdots, u_k, f_{j+1}, u_{k+1}, u_{k+2}, \cdots$.

Every v-ideal $\mathfrak{q}$ with $v(\mathfrak{q}) < v(f_{j+1})$ has for value the value of some u_i, $i \leq k$, and conversely, whence $v(\mathfrak{q}_i) = v(u_i)$, $i = 1, \cdots, k$. Since $\mathfrak{q}_i$, $i = 1, \cdots, k$, has a basis consisting of elements in (2), the $\mathfrak{q}_i$ may be described in the following manner: $\mathfrak{q}_i$ has a basis consisting of u_i and the elements of (2) following u_i, $i = 1, \cdots, k$. Since $v(f_j^{d(j)}) = v(u_i)$, some i, and $v(f_j^{d(j)}) < v(f_{j+1})$, we have $i \leq k$, whence $v(u_k) \geq v(f_j^{d(j)})$. Thus if τ is an irrational number with $v(u_k) < \tau < v(u_{k+1})$ then there exists, by Theorem 8, a rational rank 2 valuation of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$ such that $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j$, and $v_1(f_{j+1}) = \tau$. Since $v_1(u_i) = v(u_i)$, every i, and $v_1(u_k) < v_1(f_{j+1}) < v_1(u_{k+1})$, the set of monomials $\{F(\alpha_0, \cdots, \alpha_j)\}$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j$, together with f_{j+1} ordered according to their value v_1, again yields the sequence (2). The v_1-ideals of v_1-value less than $v_1(f_{j+1})$ can be described again as above: the ith v_1-ideal has for basis u_i and the elements of (2) following u_i, $i = 1, \cdots, k$. Since $m \leq k$, $\mathfrak{q}_1, \cdots, \mathfrak{q}_m$ are the first m v_1-ideals.

Next, consider the case that v is discrete, rank 1 or of rank 2 with $\Delta \mathfrak{q}_i \neq 0$ (leaving aside the case $v(y)$ is infinitely small with respect to $v(x)$, for which case the theorem is readily verified). In this case, the sequence $f_1, f_2, \cdots$ of Theorem 6 is finite, containing j terms, say. If v is of rank 1 then the sequence $f_{j+1}^{(1)}, f_{j+1}^{(2)}, \cdots$ is infinite: this is also possible for v of rank 2, but the case $v(f_{j+1}^{(k)})$ infinitely great with respect to $v(x)$ can arise. To conserve notation, we place $f_{j+1}^{(k)} = f_{j+1}^{(k+1)} = \cdots$, in the latter case. Now, since $\mathfrak{q}_m$ is contained in only a finite number of v-ideals, there exists an integer l such that $v(\mathfrak{q}_m) < v(f_{j+1}^{(l)})$. The ideal $\mathfrak{q}_i$, $i = 1, \cdots, m$, then has a basis consisting of $f_{j+1}^{(l)}$

and monomials $F(\alpha_0, \cdots, \alpha_j)$, $0 \leqq \alpha_0$, $0 \leqq \alpha_i < d_i$, $i = 1, \cdots, j$. We order the set $\{F(\alpha_0, \cdots, \alpha_j)\}$ by means of v, again obtaining the sequence (1); and if $v(f_{j+1}^{(l)}) = v(u_k)$ we also introduce the sequence:

$$(2') \qquad\qquad u_1, u_2, \cdots, u_k, f_{j+1}^{(l)}, u_{k+1}, \cdots .$$

Introducing a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$ such that $v_1(f_i) = \pi(v(f_i))$, $i = 1, \cdots, j$, where $\pi(v(f_i)) = r/s$ if $v(f_i^s) = v(x^r)$, and $v_1(f_{j+1}^{(0)}) = \tau$, where τ is an irrational number satisfying $\pi(u_k) < \tau < \pi(u_{k+1})$, we can conclude as above that $q_1, \cdots, q_m$ are the first m v_1-ideals.

Finally, we have the case that v is of rank 2 with $\Delta q_i = 0$. The sequence $f_1, f_2, \cdots$ is finite, containing $j + 1$ terms, say. Here every v-ideal has a basis consisting of monomials $F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})$, $0 \leqq \alpha_0$, $0 \leqq \alpha_{j+1}$, $0 \leqq \alpha_i < d_i$, $i = 1, 2, \cdots, j$, and two distinct such monomials have distinct value. These may be ordered according to their value v, and again we obtain the sequence (1). We introduce a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$, $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j$, $v_1(f_{j+1}) = \tau > v(f_j^{d(j)})$, τ irrational. The valuation v_1 also orders the set $\{F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})\}$, $0 \leqq \alpha_0$, $0 \leqq \alpha_{j+1}$, $0 \leqq \alpha_i < d_i$, $i = 1, \cdots, j$, yielding the simple sequence

$$(1') \qquad\qquad w_0, w_1, \cdots$$

corresponding to (1). We wish to determine the extent to which (1') coincides with (1); or rather, how large an initial segment of (1) may be made to coincide with an initial segment of (1') by a proper choice of τ. To this end, consider two terms of (1):

$$(3) \qquad\qquad F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1}), \qquad F(0, \beta \cdots, \beta_j, \beta_{j+1})$$

with $v(F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})) < v(F(\beta_0, \cdots, \beta_j, \beta_{j+1}))$. If $\alpha_{j+1} = \beta_{j+1}$ then these two terms will have the same order in (1') as in (1), no matter what τ, since $v_1(F(\alpha_0, \cdots, \alpha_j)) = v(F(\alpha_0, \cdots, \alpha_j))$, $v_1(F(\beta_0, \cdots, \beta_j)) = v(F(\beta_0, \cdots, \beta_j))$. Thus we need consider only the cases $\alpha_{j+1} < \beta_{j+1}$ and $\alpha_{j+1} > \beta_{j+1}$. In case $\alpha_{j+1} > \beta_{j+1}$ we shall use the letters γ, δ instead of α, β; that is, consider two terms of (1):

$$(3') \qquad\qquad F(\gamma_{0j} \cdots, \gamma_j, \gamma_{j+1}), \qquad F(\delta_0, \cdots, \delta_j, \delta_{j+1})$$

with $v(F(\gamma_0, \cdots, \gamma_j, \gamma_{j+1})) < v(F(\delta_0, \cdots, \delta_j, \delta_{j+1}))$, where $\gamma_{j+1} > \delta_{j+1}$ and $\alpha_{j+1} < \beta_{j+1}$ is now understood in (3). The two terms of (3) will have the same order in (1') if and only if τ is such that

$$\pi(v(F(\alpha_0, \cdots, \alpha_j))) + \alpha_{j+1}\tau < \pi(v(F(\beta_0, \cdots, \beta_j))) + \beta_{j+1}\tau,$$

that is, if and only if,

$$(4) \qquad\qquad \tau > \frac{\pi(v(F(\alpha_0, \cdots, \alpha_j))) - \pi(v(F(\beta_0, \cdots, \beta_j)))}{\beta_{j+1} - \alpha_{j+1}} .$$

For the appropriate terms, condition (4) includes the condition $\tau > d_j v(f_j)$, since $d_j v(f_j)$ is the value of some u_i with $v(u_i) < v(f_{j+1})$. Similarly, the two terms of $(3')$ will have the same order in $(1')$ if and only if

(4')
$$\tau < \frac{\pi(v(F(\delta_0, \cdots, \delta_j))) - \pi(v(F(\gamma_0, \cdots, \gamma_j)))}{\gamma_{j+1} - \delta_{j+1}}.$$

Now

$$(\gamma_{j+1} - \delta_{j+1})v(F(\alpha_0, \cdots, \alpha_j)/F(\beta_0, \cdots, \beta_j))$$
$$< (\beta_{j+1} - \alpha_{j+1})(\gamma_{j+1} - \delta_{j+1})v(f_{j+1})$$
$$< (\beta_{j+1} - \alpha_{j+1})v(F(\delta_0, \cdots, \delta_j)/F(\gamma_0, \cdots, \gamma_j))$$

by (3) and $(3')$, whence

$$\frac{\pi(v(F(\alpha_0, \cdots, \alpha_j))) - \pi(v(F(\beta_0, \cdots, \beta_j)))}{\beta_{j+1} - \alpha_{j+1}}$$
$$< \frac{\pi(v(F(\delta_0, \cdots, \delta_j))) - \pi(v(F(\gamma_0, \cdots, \gamma_j)))}{\gamma_{j+1} - \delta_{j+1}}.$$

Thus τ can be taken such that (4) and $(4')$ be satisfied. Similarly, any finite number of conditions (4) and $(4')$ are satisfied by some irrational number τ. Thus for any initial segment of (1) there exists a valuation v_1 such that the elements of this segment retain the same relative order in $(1')$; that is, for any integer $n > 0$ there exists a valuation $v_1 = v_1(n)$ such that if $u_i = w_{k(i)}$ then $k_i < k_{i+1}$ provided $i < n$. For a given i, k_i is a function of v_1: $k_i = k_i(v_1)$. Let $k^{(i)} = \min \{k_i(v_1)\}$ over all v_1. We assert that $k^{(i)} \to \infty$ as $i \to \infty$. In fact, consider the monomials $\{F(\alpha_0, \cdots, \alpha_j)\}$, $0 \le \alpha_0$, $0 \le \alpha_i < d_i$, $i = 1, \cdots, j$, and order them according to their v-values, which are also their v_1-values, obtaining the sequence:

$$z_1, z_2, \cdots .$$

Clearly the sequence (1) may also be written:

$$z_{l(1)}f_{j+1}^{m(1)}, \; z_{l(2)}f_{j+1}^{m(2)}, \cdots .$$

We have

$$k^{(i)} \geqq \max (l_i, m_i),$$

since $v_1(w_{k(i)}) = v_1(u_i) = v_1(z_{l(i)}f_{j+1}^{m(i)}) \geqq \max (v_1(z_{l(i)}), v_1(f_{j+1}^{m(i)}))$. Since not both l_i, m_i are bounded, $k^{(i)} \to \infty$ as $i \to \infty$. Hence for any integer $N > 0$ there exists an integer $n = n(N)$ such that if $u_i = w_k$, where $k = k_i(v_1)$, then $k_i(v_1) > N$ for any v_1, provided $i > n$. Thus if the elements $u_1, u_2, \cdots, u_n$ retain their order in $(1')$ then the initial segment $u_1, \cdots, u_N$ of (1) coincides with the initial segment $w_1, \cdots, w_N$ of $(1')$.

Let s be an integer such that $v(q_m) < v(f_{j+1}^s)$. Then every ideal $q_1, \cdots, q_m$ has a basis consisting of f_{j+1}^s and monomials $F(\alpha_0, \alpha_1, \cdots, \alpha_j, \alpha_{j+1})$, $0 \le \alpha_0$, $0 \le \alpha_i < d_i$, $i = 1, \cdots, j$, $\alpha_{j+1} < s$. In fact, if $g(x, y) \in q_i$, $i \le m$, then, since f_{j+1} and f_{j+1}^s are monic in y one can write:

$$g(x, y) = a_0(x, y) + a_1(x, y)f_{j+1} + \cdots + a_{s-1}(x, y)f_{j+1}^{s-1} + a_s(x, y)f_{j+1}^s$$

where $a_i(x, y) \in K[x, y]$, $i = 0, 1, \cdots, s$, and $\deg_y a_i < \deg_y f_{j+1}$, $i = 0, 1, \cdots, s-1$. The statement now follows from (c) of Theorem 6. We can therefore describe the ideals $q_1, \cdots, q_m$ in terms of the sequence (1) in the following manner: q_k has a basis consisting of the elements u_i with $i \ge k$. If, now, v_1 is such that the initial segment of (1) preceding f_{j+1}^s coincides with an initial segment of (1'), one concludes, in a manner employed above, that $q_1, \cdots, q_m$ are the first m v-ideals of v_1. This completes the proof.

The significance of Theorem 9 is that it allows us in the study of 0-dimensional v-ideals of 0-dimensional valuations to restrict ourselves to rational rank 2 valuations, at least if this study is confined to the ring $K[x, y]$.

COROLLARY 1. *In the case v is non-discrete, rational rank 1, or rank 1 and discrete, or of rank 2 with $\Delta q_i \ne 0$, the v_1-ideals of value less than τ are the first v-ideals for v. In the case v is of rank 2 with $\Delta q_i = 0$, the v_1-ideals of value less than $s\tau$ are the first v-ideals for v (where s is as in the theorem).*

COROLLARY 2. *If the v_1-ideals of v_1-value less than $s\tau$ are the first v-ideals for v and if $\tau > 1$ then:*
  (a) *in the case that v is non-discrete, rational rank 1, $s \le d_{j+1}$,*
  (b) *in the case that v is rank 1 and discrete or of rank 2 with $\Delta q_i \ne 0$, $s = 1$.*

Proof. Suppose $s \ge d_{j+1}$. Then the v_1-ideals of v_1-value less than $v_1(f_{j+1}^{d(j+1)})$ are the first v-ideals for v. We order the elements $F(\alpha_0, \cdots, \alpha_{j+1})$, $0 \le \alpha_0$, $0 \le \alpha_i < d_i$, $i = 1, \cdots, j+1$, according to their v- and v_1-value, obtaining the sequences

(1) $u_1, u_2, \cdots,$

(2) $w_1, w_2, \cdots.$

If $v_1(w_k) < v_1(f_{j+1}^{d(j+1)}) < v_1(w_{k+1})$, then as in the theorem it follows that $u_i = w_i$, $i = 1, \cdots, k$. Let $q_1, q_2, \cdots$; $q_1^*, q_2^*, \cdots$ be the Jordan sequences of 0-dimensional valuation ideals for v and v_1 respectively. We know that $v(f_{j+1}^{d(j+1)}) = v(u_m)$, for some u_m, and since $q_k = q_k^*$, we have $m \ge k$. If $m = k$ then $q_{k+1} \ne q_{k+1}^*$ since $f_{j+1}^{d(j+1)} \in q_{k+1}^*$ but $f_{j+1}^{d(j+1)} \in q_{k+1}$, whence $s > d_{j+1}$ is impossible. If $m > k$ then $q_{k+1} = q_{k+1}^*$ but $q_{k+2} \ne q_{k+2}^*$, since $u_{k+1} \in q_{k+2}^*$ but $u_{k+1} \notin q_{k+2}$. Now if $s > d_{j+1}$ then $s\tau > d_{j+1}\tau + 1$, whence $v_1(q_{k+2}) < v_1(f_{j+1}^s)$. But then $q_{k+2} = q_{k+2}^*$, a contradiction. This completes the proof of (a). Part (b) follows in the same way if we place $d_{j+1} = 1$ (and write $f_{j+1}^{(l)}$ for f_{j+1}). This completes the proof.

6. **Characterization of simple v-ideals.** Let v be a 0-dimensional valuation

of Σ over K with $v(x) > 0$, $v(y) > 0$. If we further assume that $v(y) \geq v(x)$ then the groups D_0, D_1, $\cdots$ introduced in Theorem 6 no longer depend on the element x, but solely on the ring $K[x, y]$; in fact, $v(y) \geq v(x) > 0$ implies that $v(q_1) = v(x)$, and we may write $v(q_1)$ instead of $v(x)$ in Theorem 6. If v does not belong to the set of non-discrete rational rank 1 valuations, then for some integer j we have $\Delta_j = D_m$, $m \geq i_j$. This integer j depends only on v and on the ring $K[x, y]$. We shall call j the *type* of v with respect to the ring $K[x, y]$.

THEOREM 10. *Let v be a rational rank 2 valuation of Σ over K with $0 < v(x) < v(y)$ and of type j with respect to $K[x, y]$, $j > 1$, and let $f_1, f_2, \cdots, f_j$ be the polynomials introduced in Theorem 6. Then the v-ideals for v of value equal to or less than $d_{j-1}v(f_{j-1})$ are also the first v-ideals for a rational rank 2 valuation v_1 of Σ over K with $0 < v_1(x) < v_1(y)$ and of type $j-1$ with respect to $K[x, y]$. The polynomials $f_1, \cdots, f_{j-1}$ satisfy Theorem 6 for the valuation v_1.*

Proof. If q is a v-ideal with $v(q) \leq d_{j-1}v(f_{j-1})$ then q has a basis consisting of $f_{j-1}^{d(j-1)}$ together with monomials $\{F(\alpha_0, \cdots, \alpha_{j-1})\}$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j-1$. As in Theorem 9, we order the set $\{F(\alpha_0, \cdots, \alpha_{j-1})\}$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j-1$, by means of v, obtaining the simple sequence:

$$(1) \qquad u_1, u_2, \cdots .$$

If $v(f_{j-1}^{d(j-1)}) = v(u_k)$ then any v-ideal q_i with $i \leq k$ can be described in the following manner: q_i has a basis consisting of $f_{j-1}^{d(j-1)}$ together with the elements $u_{i'}$ of (1) with $i' \geq i$. Normalizing v so as to have $v(x) = 1$, we introduce for every irrational $\tau > d_{j-2}v(f_{j-2})$ (if $j = 2$, for every irrational $\tau > 1$) a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$, $v_1(f_i) = v(f_i)$, $i = 1, \cdots, j-2$, and $v_1(f_{j-1}) = \tau$. The valuation v_1 also orders the set $\{F(\alpha_0, \cdots, \alpha_{j-1})\}$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j-1$, yielding the simple sequence:

$$(1') \qquad w_1, w_2, \cdots$$

corresponding to (1). Now $|v(u_i) - v_1(u_i)| < d_{j-1}|\tau - v(f_{j-1})|$, since $\alpha_{j-1} < d_{j-1}$ in any term u_i, whence, if

$$(2) \qquad d_{j-1}|\tau - v(f_{j-1})| < \min_{i=1, \cdots, k} |v(u_i) - v(u_{i+1})|$$

then the initial segment $u_1, \cdots, u_k$ of (1) coincides with an initial segment of (1'). (We may note that if $v_1(u_i) > v(u_i)$ for one u_i then $v_1(u_i) \geq v(u_i)$ for every i.) Let V_1 denote the set of valuations $\{v_1\}$ for which (2) holds. For every valuation $v_1 \in V_1$ such that $\tau > v(f_{j-1})$, the ideals $q_1, \cdots, q_k$ are the first k v_1-ideals; in fact $d_{j-1}v_1(f_{j-1}) > d_{j-1}v(f_{j-1}) = v(u_k) = v_1(u_k) = v_1(w_k)$ ($v(u_k) = v_1(u_k)$, since by condition 4 of Theorem 6, u_k does not involve f_{j-1}), whence we can draw the required conclusion in a manner several times employed in Theo-

rem 9. The polynomials $f_1, f_2, \cdots, f_{j-1}$ clearly satisfy Theorem 6_{j-1}. Since $v_1(f_{j-1})$ is irrational, the sequence $f_1, f_2, \cdots$ for v_1 terminates, as is shown in Theorem 6, with f_{j-1}; that is, v_1 is of type $j-1$ with respect to $K[x, y]$. It remains to prove $v_1(y) > v_1(x)$. If $f_1 = y$, this is clear. If $f_1 \neq y$, then $v(f_1 - y)$ is an integer, and since $v(f_1)$ is not an integer, $v(f_1 - y) = v(y) \geq 2$. Hence also $v_1(f_1 - y) \geq 2$, since $v_1(f_1 - y) = v(f_1 - y)$. Since $v_1(f_1) > 1$, $v_1(y) > 1 = v_1(x)$ follows. This completes the proof.

COROLLARY 1. *The ideals* $\mathfrak{q}_1, \cdots, \mathfrak{q}_k$ *are the first k v_1-ideals for any valuation* $v_1 \in V_1$.

Proof. For $\tau > v(f_{j-1})$ the proof has been given above. The proof there depends on the fact that, since $d_{j-1}v(f_{j-1}) > v(u_k) = v_1(u_k) = v_1(w_k)$, the kth v_1-ideal has a basis consisting of $f_{j-1}^{d(j-1)}$ together with the monomials w_i with $i \geq k$. If $\tau < v(f_{j-1})$ then $d_{j-1}v(f_{j-1}) < v(u_k) = v_1(u_k) = v_1(w_k)$. However, if $d_{j-1}v(f_{j-1}) > v_1(w_{k-1})$ then the kth v_1-ideal can still be described as before: it has a basis consisting of $f_{j-1}^{d(j-1)}$ together with the monomials w_i with $i \geq k$, whence, as before, $\mathfrak{q}_k$ would be the kth v_1-ideal. And, in fact, $v(u_{k-1}) \geq v_1(u_{k-1})$, whence

$$d_{j-1}v(f_{j-1}) - d_{j-1}v_1(f_{j-1}) < v(u_k) - v(u_{k-1}) \leq v(u_k) - v_1(u_{k-1}),$$

by (2), and $d_{j-1}v_1(f_{j-1}) > v_1(u_{k-1})$. This completes the proof.

COROLLARY 2. *The v-ideal $\mathfrak{q}_k$ with $v(\mathfrak{q}_k) = d_{j-1}v(f_{j-1})$ is a simple v-ideal.*

Proof. Since $v(f_{j-1}) > d_{j-2}v(f_{j-2})$, $\tau = v_1(f_{j-1})$, $v_1 \in V_1$, may be taken greater than or less than $v(f_{j-1})$. If $\tau > v(f_{j-1})$ then $d_{j-1}v_1(f_{j-1}) > d_{j-1}v(f_{j-1}) = v(u_k) = v_1(u_k) = v_1(\mathfrak{q}_k)$, whence $v_1(\mathfrak{q}_k)$ is rational, while if $\tau < v(f_{j-1})$ then $d_{j-1}v_1(f_{j-1}) < v_1(u_k)$, whence $v_1(\mathfrak{q}_k) = d_{j-1}v_1(f_{j-1})$, that is, $v_1(\mathfrak{q}_k)$ is irrational. Suppose now that $\mathfrak{q}_k$ is not simple: $\mathfrak{q}_k = \mathfrak{q}_{k(1)}\mathfrak{q}_{k(2)} \cdots \mathfrak{q}_{k(m)}$, $m > 1$. Since $u_{k(l)}, l = 1, 2, \cdots, m$, belong to the initial segment $u_1, u_2, \cdots, u_{k-1}$, we have

$$v_1(\mathfrak{q}_{k(l)}) = v_1(u_{k(l)}), \qquad\qquad l = 1, \cdots, m,$$

and $v_1(\mathfrak{q}_k) = v_1(u_{k(1)}) + \cdots + v_1(u_{k(m)})$. Let $v_1(u_{k(l)}) = r_{k(l)} + s_{k(l)}v_1(f_{j-1})$, where $r_{k(l)}$ is a non-negative rational number, and $s_{k(l)}$ is a non-negative integer, $l = 1, \cdots, m$. The numbers $r_{k(l)}, s_{k(l)}$ are independent of the valuation $v_1 : s_{k(l)}$ is the exponent of f_{j-1} in $u_{k(l)}$, while $r_{k(l)} = v_1(u_{k(l)}f_{j-1}\exp s_{k(l)})$ is the same for any $v_1 \in V_1$, since $u_{k(l)}/f_{j-1}\exp s_{k(l)}$ is a monomial in $x, f_1, f_2, \cdots, f_{j-2}$. If now $v_1 \in V_1$ is taken such that $v_1(\mathfrak{q}_k)$ is irrational $(=d_{j-1}v_1(f_{j-1}))$, we have that $r_{k(1)} + \cdots + r_{k(m)} = 0$, whence $r_{k(1)} = \cdots = r_{k(l)} = 0$. But then $v_1(u_{k(l)}) = s_{k(m)}v_1(f_{j-1})$ for every $v_1 \in V_1$, $l = 1, \cdots, m$, whence $v_1(\mathfrak{q}_k) = (s_{k(1)} + \cdots + s_{k(m)})v_1(f_{j-1})$, that is, $v_1(\mathfrak{q}_k)$ is irrational for every $v_1 \in V_1$. This is a contradiction.

COROLLARY 3. *All the v-ideals $\mathfrak{q}_i$ with $d_{j-1}v(f_{j-1}) \leq v(\mathfrak{q}_i) \leq v(f_j)$ are simple.*

Proof. For $v(\mathfrak{q}_i) = d_{j-1}v(f_{j-1})$, this is Corollary 2. If $v(\mathfrak{q}_i) = v(f_j)$, then $\mathfrak{q}_i$ is

certainly simple, since $v(q_i)$ is irrational, while $v(q_1)$, $v(q_2)$, $\cdots$, $v(q_{i-1})$ are rational. Suppose $v(q_i) = u_i < v(f_j)$, $i > k$, where $v(u_k) = d_{j-1}v(f_{j-1})$. Since $v(u_{i-1}) \geq d_{j-1}v(f_{j-1})$, there exists by Theorem 8.3 for any irrational number τ, $v(u_{i-1}) < \tau < v(u_i)$, a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$ such that $v_1(f_i) = v(f_i)$, $i = 1$, $\cdots$, $j-1$ and $v_1(f_j) = \tau$. The first i v_1-ideals (in $K[x, y]$) may clearly be described in the following manner: the lth v_1-ideal, $l \leq i$, has a basis consisting of f_j and the monomials u_k with $k \geq l$. Hence, q_1, q_2, $\cdots$, q_i are the first i v_1-ideals. Moreover, since $v_1(f_j) < v_1(u_i)$, we have that $v(q_i) = v_1(f_j) = \tau$. This case, however, has already been considered. This completes the proof.

COROLLARY 4. *Every v-ideal q_i with $d_{j-1}v(f_{j-1}) < v(q_i) \leq v(f_j)$ belongs to a rational rank 2 valuation v_1 of Σ over K with $v_1(x) = 1$, $v_1(y) > 0$, with f_1, $\cdots$, f_i as associated polynomials, and with q_1, $\cdots$, q_i as first v-ideals, such that $v_1(q_i) = v_1(f_j) = \tau'$.*

Proof. This has just been proved in Corollary 3.

Let v be a rational rank 2 valuation of Σ over K with $v(x) = 1$ and $v(y) > 0$, and let f_1, f_2, $\cdots$, f_n be the sequence of polynomials for v introduced in Theorem 6. We have $v(f_i) = r_i/s_i$, r_i, s_i positive integers, $(r_i, s_i) = 1$, $i = 1$, $\cdots$, $n-1$, and $v(f_n) = \tau$ is irrational. Every v-ideal has a basis consisting of monomials in x, f_1, $\cdots$, f_n, thus every ideal has a value $u/t + v\tau$, u, v non-negative integers, where $t = t_{n-1} = \text{l.c.m.}$ (s_1, $\cdots$, s_{n-1}).

LEMMA. *If $v(F(\alpha_0, \cdots, \alpha_k)) > v(F(\beta_0, \cdots, \beta_k)) + v(f_{k+1}) - 1$, $0 \leq \alpha_i$, $0 \leq \beta_i$, $i = 0, 1, \cdots, k$, then there exist non-negative integers α_i', β_i' such that $\alpha_i' \geq \beta_i'$, $v(F(\alpha_0', \cdots, \alpha_k')) = v(F(\alpha_0, \cdots, \alpha_k))$, $v(F(\beta_0', \cdots, \beta_k')) = v(F(\beta_0, \cdots, \beta_k))$.*

Proof. The proof is by induction on k, being trivial for $k = 0$. (That is, $\alpha_0 > \beta_0 - 1$, whence $\alpha_0 \geq \beta_0$.)

First, if $\alpha_k < \beta_k$ then $v(F(\alpha_0, \cdots, \alpha_{k-1})) > v(F(\beta_0, \cdots, \beta_{k-1})) + v(f_k) - 1$, so that by induction we may assume $\alpha_i \geq \beta_i$, $i = 0, 1, \cdots, k-1$. Factoring out $F(\beta_0, \cdots, \beta_{k-1})$, we may assume $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$. Assuming $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$, let α_k be as large as possible, that is, let $\alpha_k = \max \alpha_k'$ over the set of monomials $F(\alpha_0', \cdots, \alpha_k')$ of value $= v(F(\alpha_0, \cdots, \alpha_k))$, where $0 \leq \alpha_i'$, $i = 0, 1, \cdots, k$; that is, we may without loss of generality suppose that $\alpha_k' \leq \alpha_k$ if $v(F(\alpha_0', \cdots, \alpha_k')) = v(F(\alpha_0, \cdots, \alpha_k))$, where $0 \leq \alpha_i'$, $i = 0, 1, \cdots, k$. Then $\alpha_k \geq \beta_k$. In fact, supposing $\alpha_k < \beta_k$, then $v(F(\alpha_0, \cdots, \alpha_{k-1})) > v(f_k) + v(f_{k+1}) - 1 > d_k v(f_k) + v(f_k) - 1$. Now $d_k v(f_k) = v(F(\gamma_0, \cdots, \gamma_{k-1}))$. By induction we may assume that $\alpha_i \geq \gamma_i$, $i = 0, 1, \cdots, k-1$. But then

$$v(F(\alpha_0, \cdots, \alpha_{k-1}, \alpha_k)) = v(F(\alpha_0 - \gamma_0, \cdots, \alpha_{k-1} - \gamma_{k-1}, \alpha_k + d_k))$$

which is in contradiction with the maximality of α_k.

Second, if $\alpha_k \geq \beta_k$, we may assume, factoring out $f_k \exp \beta_k$, that $\beta_k = 0$. Assuming $\beta_k = 0$, let α_k be as small as possible, that is, let $\alpha_k = \min \alpha_k'$ over the

set of monomials $F(\alpha_0', \cdots, \alpha_k')$ of value $= v(F(\alpha_0, \cdots, \alpha_k))$, where $0 \leq \alpha_i'$, $i = 0, 1, \cdots, k$. Then certainly $\alpha_k < d_k$, whence $v(F(\alpha_0, \cdots, \alpha_{k-1}))$ $> v(F(\beta_0, \cdots, \beta_{k-1})) + v(f_k) - 1$, since $v(f_{k+1}) > v(f_k^{d(k)}) > v(f_k^{\alpha(k)})$. The proof is thus complete by induction.

The essential facts of the following theorem have already been proved for $n = 1$. Principally on the grounds of simplicity, therefore, we assume $n > 1$.

THEOREM 11. *Let v be as in the lemma, and let $n > 1$.*

1. *If $v(\mathfrak{q}_i) = u/t + v\tau$, then $\mathfrak{q}_i = \mathfrak{q}_j\mathfrak{q}_k$, where $v(\mathfrak{q}_j) = u/t$, $v(\mathfrak{q}_k) = v\tau$; $\mathfrak{q}_i$, $\mathfrak{q}_j$, $\mathfrak{q}_k$, v-ideals for v.*

2. *If $v(\mathfrak{q}_i) = m\tau$ $(i > 0)$ and $v(\mathfrak{q}_{i+1}) = u/t + v\tau$ then $v < m$ and $\mathfrak{q}_i = \mathfrak{q}_j\mathfrak{q}_k$ where $v(\mathfrak{q}_j) = (m - v)\tau$, $v(\mathfrak{q}_k) = v\tau$; $\mathfrak{q}_i$, $\mathfrak{q}_j$, $\mathfrak{q}_k$, v-ideals for v.*

3. *If $v(\mathfrak{q}_i) = m/t$ $(m$ an integer$)$, $v(\mathfrak{q}_{i+1}) = u/t + v\tau$ and $v \neq 0$, then $u < m$ and $\mathfrak{q}_i = \mathfrak{q}_j\mathfrak{q}_k$ where $v(\mathfrak{q}_j) = (m - u)/t$, $v(\mathfrak{q}_k) = u/t$, $\mathfrak{q}_i$, $\mathfrak{q}_j$, $\mathfrak{q}_k$, v-ideals for v.*

Proof. 1. We have to show that if $v(F(\alpha_0, \cdots, \alpha_n)) \geq u/t + v\tau$ then $F(\alpha_0, \cdots, \alpha_n) = \sum a_i(x, y)b_i(x, y)$, a_i, $b_i \in K[x, y]$, with $v(a_i) \geq u/t$, $v(b_i) \geq v\tau$, where $\sum$ is the summation of a finite number of products a_ib_i. This is clear for $v(F(\alpha_0, \cdots, \alpha_n))$ sufficiently large; in fact, let $k_0, k_1, \cdots, k_n, l_0, l_1, \cdots, l_n$ be integers such that $k_0v(x) \geq u/t$, $k_1v(f_1) \geq u/t$, $\cdots$, $k_nv(f_n) \geq u/t$, $l_0v(x) \geq v\tau$, $l_1v(f_1) \geq v\tau$, $\cdots$, $l_nv(f_n) \geq v\tau$. If, now, $v(F(\alpha_0, \cdots, \alpha_n)) \geq v(F(k_0+l_0, \cdots, k_n+l_n))$ then $\alpha_i \geq k_i + l_i$ for at least one i, $i = 0, 1, \cdots, n$; if $\alpha_m \geq k_m + l_m$, say, then $F(\alpha_0, \cdots, \alpha_n) \equiv 0(f_m \exp k_m + l_m) \equiv 0(\mathfrak{q}_j\mathfrak{q}_k)$. Hence in the proof we may assume that all monomials of value greater than $v(F(\alpha_0, \cdots, \alpha_n))$ are in $\mathfrak{q}_j\mathfrak{q}_k$. Moreover, if $v(F(\alpha_0, \cdots, \alpha_n)) = v(F(\beta_0, \cdots, \beta_n))$ then $v(F(\alpha_0, \cdots, \alpha_n) - cF(\beta_0, \cdots, \beta_n)) > v(F(\alpha_0, \cdots, \alpha_n))$ for some $c \in K$, so that $F(\alpha_0, \cdots, \alpha_n) - cF(\beta_0, \cdots, \beta_n)$ is a sum of monomials of value greater than $v(F(\alpha_0, \cdots, \alpha_n))$, since every v-ideal $\mathfrak{q}$ has a basis consisting of monomials in $x, f_1, \cdots, f_n$. Hence it is sufficient to prove $F(\alpha_0, \cdots, \alpha_n) = \sum a_ib_i$ for some monomial of value equal to $v(F(\alpha_0, \cdots, \alpha_n))$. If $u/t + v\tau = v(F(\beta_0, \cdots, \beta_{n-1}, v))$ and $\alpha_n \geq v$, then this is clear: $F(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n) = F(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n - v) \cdot f_n^v \equiv 0(\mathfrak{q}_j\mathfrak{q}_k)$; if $\alpha_n < v$, then by the lemma we may assume $\alpha_i \geq \beta_i$, $i = 0, 1, \cdots, n-1$, whence $F(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n) = F(\alpha_0 - \beta_0, \cdots, \alpha_{n-1} - \beta_{n-1}, \alpha_n) \cdot F(\beta_0, \cdots, \beta_{n-1}) \equiv 0(\mathfrak{q}_k\mathfrak{q}_j)$. This completes the proof.

2. Let $[r]$ indicate the greatest integer equal to or less than the real number r. If $\mathfrak{q}$ is a v-ideal then $1 + v(\mathfrak{q})$ is the value of some v-ideal, whence $v(\mathfrak{q}_{i+1}) - v(\mathfrak{q}_i) \leq 1$. Since $\tau = v(f_n) \geq v(f_2) > d_1v(f_1) \geq v(x) = 1$, we must have $v \leq m$. Moreover, $v = m$ is impossible; in fact, $v = m$ implies $u > 0$ and $m\tau < [u + mt\tau]/t$ $< u/t + m\tau$. By (a) of Theorem 6, $[u + mt\tau]/t$ is the value of a v-ideal; this is a contradiction. Hence $v < m$. The only monomial of value $m\tau$ in $\mathfrak{q}_i$ is f_n^m $(= f_n^{m-v} \cdot f_n^v)$, which is clearly in $\mathfrak{q}_j\mathfrak{q}_k$. For monomials $F(\alpha_0, \cdots, \alpha_n)$ of value $\geq u/t + v\tau$, we have, as in 1, $F(\alpha_0, \cdots, \alpha_n) = \sum a_ib_i$, with $v(a_i) \geq u/t$, $v(b_i)$ $\geq v\tau$; since $u/t > (m - v)\tau$, $a_i \in \mathfrak{q}_j$, whence $F(\alpha_0, \cdots, \alpha_n) \equiv 0(\mathfrak{q}_j\mathfrak{q}_k)$. This completes the proof.

3. Here $u/t+v\tau-1<m/t<u/t+v\tau$, since $v(q_{i+1})-v(q_i)<1$ (equalities are excluded since τ is irrational and $v\neq0$), so that not only $m>u$, but $(m-u)/t>\tau-1=v(f_n)-1$. Hence it follows from the lemma that there exist non-negative integers α_i, β_i with $\alpha_i\geq\beta_i$ such that $v(F(\alpha_0,\cdots,\alpha_{n-1}))=m/t>u/t=v(F(\beta_0,\cdots,\beta_{n-1}))$, whence at least one monomial of value m/t is in q_jq_k. For monomials of value greater than m/t, the proof goes as in 2.

THEOREM 12. *Let $v(q_i)\geq\tau$, v as in Theorem 11. The v-ideal q_i is simple only if of the numbers $v(q_i)$, $v(q_{i+1})$, one is rational and the other an integral multiple of τ.*

Proof. If $v(q_i)=u/t+v\tau$, $u\neq0$, $v\neq0$, then q_i is not simple by Theorem 11.1. If $v(q_i)=m\tau$, $v(q_{i+1})=u/t+v\tau$, and $v\neq0$ then $u\neq0$ since $v<m$. Hence q_i is not simple by Theorem 11.2. If $v(q_i)=m/t$, m an integer, $t=t_{n-1}$, $v(q_{i+1})=u/t+v\tau$, u, v integers, and if, first, $v\neq0$, then either $u\neq0$ and q_i is not simple by Theorem 11.3, or $u=0$ (and there is nothing to prove). If $v=0$ then by (a) of Theorem 6, $v(q_{i+1})=(m+1)/t$. Let q_k be the first v-ideal after q_i of value $u/t+v\tau$ such that $v\neq0$. Then also $u\neq0$. In fact, if $u=0$ then $v>1$. Moreover, $\tau>d_{n-1}v(f_{n-1})=k/t$, k an integer, whence $[t\tau]/t\geq k/t=d_{n-1}v(f_{n-1})$. Hence, by (a) of Theorem 6, there exists a v-ideal q_j with $v(q_j)=(v-1)\tau+[t\tau]/t$. $i<j<k$ since $0<\tau-[t\tau]/t<1/t$. This is a contradiction, since $v-1\neq0$. Hence $u\neq0$. Then $u<m$ and $q_i=q_jq_l$, with $v(q_j)=(m-u)/t$, $v(q_l)=u/t$. In fact, proceeding as in Theorem 11, we need only prove that some monomial of value $=v(q_i)=m/t$ is in q_jq_l (because at the same time this proves that some monomial of value $=v(q_{i'})$, $i\leq i'<k$, is in q_jq_l). To prove this it is only necessary, by (a) of Theorem 6, to show that $m/t-u/t>\tau-1$ (the proof then proceeding, by means of the lemma to Theorem 10, as in Theorem 10.3). If $v>1$ then $u/t+(v-1)\tau<m/t$, whence $m/t-u/t>(v-1)\tau\geq\tau$. If $v=1$ then $u/t<\tau$. In fact, if $u/t>\tau$ then $u/t>d_{n-1}v(f_{n-1})=k/t$, k an integer, whence $(u-1)/t\geq d_{n-1}v(f_{n-1})$; by (a) of Theorem 6, $(u-1)/t+\tau$ is then the value of a v-ideal, whence $(u-1)/t+\tau<m/t$, or $u/t+\tau<(m+1)/t$: this is a contradiction since we are in the case $v(q_{i+1})=(m+1)/t<u/t+v\tau=u/t+\tau$. Since $v(q_l)=u/t<\tau$, we have $v(q_{l-1})=u'/t$, u' an integer ($l-1$ possibly $=0$), and $u/t-u'/t\leq1$. From $u'/t+\tau<m/t$ follows: $m/t-u/t>\tau-1$. This completes the proof.

Let v be as in Theorem 11, q_1, q_2, $\cdots$ the Jordan sequence of 0-dimensional v-ideals.

THEOREM 13. *If of the numbers $v(q_i)$, $v(q_{i+1})$ one is rational and the other an integral multiple of τ, then q_i is simple.*

Proof. a. Consider the case $v(q_i)=v\tau$, $v(q_{i+1})=u/t$, u an integer. Consider the ordered set of numbers, $v(q_1)$, $v(q_2)$, $\cdots v(q_{i-1})$, $v(q_i)$, $v(q_{i+1})$, $\cdots$, that is, the numbers

(1) $$\cdots,\ m/t+n\tau,\ v\tau,\ u/t,\ \cdots.$$

We shall show that there exists a rational rank 2 valuation v_2 of Σ over K with $v_2(x) = 1$, $v_2(y) > 0$, $v_2(f_i) = v(f_i)$, $i = 1, \cdots, n-1$, $v_2(f_n) = \tau'$ such that the set of numbers: $v_2(q_1')$, $v_2(q_2')$, $\cdots$, $v_2(q_{i-1}')$, $v_2(q_i')$, $v_2(q_{i+1}')$, $\cdots$, where $q_1', q_2', \cdots$ is the Jordan sequence of 0-dimensional v_2-ideals, is the sequence:

$$(2) \qquad\qquad \cdots, m/t + n\tau', u/t, v\tau', \cdots .$$

This implies that q_i is simple. In fact, every v_2-ideal q' has a basis consisting of monomials $F(\alpha_0, \cdots, \alpha_n)$, α_i non-negative integers, $i = 0, 1, \cdots, n-1$. This follows for the v_2-ideals exactly as for the v-ideals. We can now assert that $q_1 = q_1', \cdots, q_i = q_i'$. For example, to prove $q_i = q_i'$, let $F(\alpha_0, \cdots, \alpha_n) \in q_i$, whence $v(F(\alpha_0, \cdots, \alpha_{n-1})) + \alpha_n \tau \geq v\tau$. Hence $v(F(\alpha_0, \cdots, \alpha_{n-1})) + \alpha_n \tau$ is not among the first $i-1$ numbers of (1), hence $v_2(F(\alpha_0, \cdots, \alpha_{n-1})) + \alpha_n \tau'$ ($= v(F(\alpha_0, \cdots, \alpha_{n-1})) + \alpha_n \tau'$) is not among the first $i-1$ numbers of (2). Hence $v_2(F(\alpha_0, \cdots, \alpha_{n-1})) + \alpha_n \tau' \geq u/t$, whence $F(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n) \in q_i'$. Thus $q_i \subseteq q_i'$. Similarly $q_i' \subseteq q_i$, whence $q_i = q_i'$. Similarly, $q_1 = q_1', \cdots, q_{i-1} = q_{i-1}'$. Now suppose $q_i = q_j q_k$. Since $v(q_i) = v\tau$, we have $v(q_j)$, $v(q_k)$ are integral multiples of τ, say $v(q_j) = p\tau$, $v(q_k) = (v-p)\tau$. Since $j < i$, $k < i$, the numbers $p\tau'$, $(v-p)\tau'$ are the jth and kth numbers in the sequence (2). Hence $v_2(q_j) = p\tau'$, $v_2(q_k) = (v-p)\tau'$, and $v_2(q_i) = v\tau'$. This is a contradiction.

Thus it remains to prove the existence of the valuation v_2. Suppose that there exists an irrational number τ' such that the set of positive numbers $\{\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{n-1} v(f_{n-1}) + \alpha_n \tau'\}$, α_i non-negative integers, ordered according to magnitude, is the sequence (2); then $\tau' > \tau$ is immediately implied by (2) and (1): $\tau' > u/vt > \tau$. The existence of the valuation v_2 is then given by Theorem 7. Thus it remains to find τ'.

Consider the sequence obtained by multiplying (1) by $t = t_{n-1}$, that is, the sequence

$$(1') \qquad\qquad \cdots, m + n(t\tau), v(t\tau), u, \cdots .$$

This set of numbers will be contained, in the same order, in the set of positive numbers $\{g + h(t\tau)\}$, g, h non-negative integers, ordered according to magnitude, and, in fact, (1) can be obtained from that sequence by first dropping the numbers $k + l(t\tau)$ in which k is not of the form $t(\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{n-1} v(f_{n-1}))$, $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ non-negative integers, and then dividing the sequence by t.

Now we assert that the numbers $v(t\tau)$, u which are successive in (1') are also successive in the ordered set of numbers $\{g + h(t\tau)\}$. In fact, let $k + l(t\tau)$ be the number immediately following $v(t\tau)$ in the sequence $\{g + h(t\tau)\}$. Since $v(t\tau) < [v(t\tau) + 1] < v(t\tau) + 1$, necessarily $0 < k + l(t\tau) - v(t\tau) < 1$, whence $l < v$, and $k > (v-l)t\tau \geq t\tau = tv(f_n)$. Hence by (a) of Theorem 6, k is of the form $t(\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{n-1} v(f_{n-1}))$. Hence $k/t + l\tau$ succeeds $v\tau$ in (1), whence $l = 0$ and $k = u$.

Let then the sequence of numbers $\{g + h(t\tau)\}$ be

$$1, \cdots, p + q(t\tau), v(t\tau), u, \cdots.$$

From our consideration of valuations given by $y = x^r$, there exists an irrational number τ' such that the ordered set of numbers $\{g + h(t\tau')\}$ is the sequence

$$1, \cdots, p + q(t\tau'), u, v(t\tau'), \cdots.$$

Dropping from this sequence the numbers $k + l(t\tau')$ in which k is not of the form $t(\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{n-1} v(f_{n-1}))$, and dividing by t, we obtain the sequence (2). This completes the proof in case a.

  b. This is the case: $v(q_i) = u/t$, $v(q_{i+1}) = v\tau$. The proof is almost the same. Here the sequence $v(q_1), v(q_2), \cdots, v(q_{i-1}), v(q_i), v(q_{i+1}), \cdots$ is the sequence:

(1) $\cdots, m/t + n\tau, u/t, v\tau, \cdots.$

Multiplying (1) by t, we get the sequence

(1') $\cdots, m + n(t\tau), u, v(t\tau), \cdots$

which we wish to compare with the ordered set of numbers $\{g + h(t\tau)\}$, g, h non-negative integers; namely, we wish to show that u and $v(t\tau)$ are successive in the set $\{g + h(t\tau)\}$, ordered according to magnitude. In fact, let $k + l(t\tau)$ be the number immediately preceding $v(t\tau)$ in the set $\{g + h(t\tau)\}$. Then $l < v$ and $k + 1 + l(t\tau) > v(t\tau)$ (clearly $k + 1 + l(t\tau) \geq v(t\tau)$; equality is excluded since $k + 1$ is an integer, while $(v - l)t\tau$ is not), so that $k + 1 > (v - l)t\tau \geq t\tau > td_{n-1}v(f_{n-1})$, whence $k \geq td_{n-1}v(f_{n-1})$ since k, $td_{n-1}v(f_{n-1})$ are integers. Hence, as above, $l = 0$, $k = u$, and we can find an irrational number τ' such that the ordered set of numbers $\{\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{n-1} v(f_{n-1}) + \alpha_n \tau'\}$ is the sequence

(2) $\cdots, m/t + n\tau', v\tau', u/t, \cdots.$

However, it does not follow as in (a) that there exists the required valuation v_2. For $v > 1$, however, it does follow. In fact, $(v - 1)\tau + d_{n-1}v(f_{n-1}) \leq u/t$, since u/t and $v\tau$ are successive in (1), and if $v > 1$ then $(v - 1)\tau + d_{n-1}v(f_{n-1}) < u/t$, equality being excluded since u/t is rational while $(v - 1)\tau + d_{n-1}v(f_{n-1})$ is not. Hence, by (2), $(v - 1)\tau' + d_{n-1}v(f_{n-1}) < v\tau'$; hence $\tau' > d_{n-1}v(f_{n-1})$. Thus for $v > 1$ the required valuation v_2 exists by Theorem 8.3.

  For $v = 1$, the fact that q_i is simple is given by Corollary 3 of Theorem 10. This completes the proof.

  COROLLARY. *If* $v(q_i) = u/t$, $v(q_{i+1}) = v\tau$ *or if* $v(q_i) = v\tau$, $v(q_{i+1}) = u/t$ *then* $(u, v) = 1$ *and* u/v *is an approximant or quasi approximant to* $t\tau$; *and conversely.*

  Proof. The direct statement follows by Lemma 1 of Theorem 4 from the fact that u, $v(t\tau)$ (or $v(t\tau)$, u) are successive in the ordered set of numbers $\{g + h(t\tau)\}$, g, h non-negative integers. Conversely, let u/v be an approximant or quasi approximant to $t\tau$. Since v is of type $n > 1$, $\tau = v(f_n) \geq v(f_2) > d_1 v(f_1) \geq 1$, since $d_1 v(f_1)$ is a positive integer. Hence v/u is certainly an approximant or

quasi approximant to $1/t\tau$ (<1). Hence by Lemma 1 of Theorem 4, u, $v(t\tau)$ (or $v(t\tau)$, u) are successive in the set $\{g+h(t\tau)\}$. Now $\tau > d_{n-1}v(f_{n-1}) = k/t$, k an integer, whence $u \geq [t\tau] \geq k = td_{n-1}v(f_{n-1})$. Hence by (a) of Theorem 6, u/t is an element of (1); that is, u/t, $v\tau$ (or $v\tau$, u/t) are successive in (1). This completes the proof.

7. **The quadratic transformation T and rational rank 2 valuations.** As already indicated, Theorem 9 allows us in the study of valuation ideals to restrict ourselves to rational rank 2 valuations, at least if the study is confined to the ring $\mathfrak{O} = K[x, y]$. We wish now to see to what extent the same restriction can be made when the transformation T and the ring $\mathfrak{O}' = K[x', y']$ are also involved.

Let, then, v be a 0-dimensional valuation in which, without loss of generality, we assume $v(y) > v(x) > 0$. T is the transformation:

$$T: \quad x' = x, \quad y' = y/x; \qquad x = x', \quad y = x'y'.$$

Let $f_1(x, y)$, $f_2(x, y)$, $\cdots$ be the (finite or infinite) sequence of polynomials introduced in Theorem 6. From these polynomials, we wish to derive polynomials $f_1'(x', y')$, $f_2'(x', y')$, $\cdots$, in the ring $K[x', y']$, satisfying Theorem 6 for v. In accordance with (b) of Theorem 6, let $t_{i-1} = \deg_y f_i$.

LEMMA. $f_i(x, y) = f_i(x', x'y') \equiv 0(x' \exp t_{i-1})$, $f_i(x', x'y') \not\equiv 0(x' \exp t_{i-1}+1)$.

Proof. The proof is by induction on i. For $i=1$, $\deg_y f_1 = 1$, and f_1 is monic in y, whence $f_1(x, y) = f_1(x', x'y') \not\equiv 0(x'^2)$; since $v(f_1) > 0$, $f_1(x, y) = f_1(x', x'y') \equiv 0(x')$. For $i > 1$, $\deg_y(f_i(x, y) - f_{i-1}^{d(i-1)}(x, y)) < \deg_y f_i(x, y)$, whence, by (c) of Theorem 6, $f_i - f_{i-1}^{d(i-1)}$ is a sum of monomials in x, f_1, $\cdots$, f_{i-1} with distinct values:

$$f_i - f_{i-1}^{d(i-1)} = \sum c(\alpha_0, \cdots, \alpha_{i-1}) F(\alpha_0, \cdots, \alpha_{i-1}), \quad c(\alpha_0, \cdots, \alpha_{i-1}) \in K,$$

$0 \leq \alpha_0$, $0 \leq \alpha_k < d_k$, $k = 1, \cdots, i-1$. Hence

$$v(F(\alpha_0, \cdots, \alpha_{i-1})) \geq v(f_i - f_{i-1}^{d(i-1)}) = v(f_{i-1}^{d(i-1)})$$

for each term of the sum $\sum$; that is,

(1) $$\alpha_0 + \alpha_1 v(f_1) + \cdots + \alpha_{i-1} v(f_{i-1}) \geq d_{i-1} v(f_{i-1}).$$

By (3) and (5) of Theorem 6,

$$d_{i-1} \deg_y f_{i-1} = d_1 \deg_y f_1 + (d_2 - 1) \deg_y f_2 + \cdots + (d_{i-1} - 1) \deg_y f_{i-1}$$

and

$$d_{i-1} v(f_{i-1}) > d_1 v(f_1) + (d_2 - 1) v(f_2) + \cdots + (d_{i-1} - 1) v(f_{i-1}).$$

A fortiori,

$$d_{i-1}(v(f_{i-1}) - \deg_y f_{i-1}) > \alpha_1(v(f_1) - \deg_y f_1) \cdots + \alpha_{i-1}(v(f_{i-1}) - \deg_y f_{i-1}).$$

Combining this with (1) we have,

$$\alpha_0 + \alpha_1 \deg_y f_1 + \cdots + \alpha_{i-1} \deg_y f_{i-1} > d_{i-1} \deg_y f_{i-1} = \deg_y f_i.$$

Hence $f_i(x, y) - f_{i-1}^{d(i-1)}(x, y) = f_i(x', x'y') - f_{i-1}^{d(i-1)}(x', x'y') \equiv 0(x' \exp t_{i-1}+1)$, by induction. Also by induction,

$$f_{i-1}^{d(i-1)}(x, y) = f_{i-1}^{d(i-1)}(x', x'y') \equiv 0(x' \exp t_{i-1})$$

but

$$f_{i-1}^{d(i-1)}(x', x'y') \not\equiv 0(x' \exp t_{i-1} + 1),$$

whence

$$f_i(x, y) = f_i(x', x'y') \equiv 0(x' \exp t_{i-1}),$$

but

$$f_i(x', x'y') \not\equiv 0(x' \exp t_{i-1} + 1).$$

This completes the proof. We also make the remark (already proved) that if $0 \leq \alpha_k < d_k$, $k = 1, \cdots, i-2$, then $v(f_{i-1}) - \deg_y f_{i-1} > \alpha_1(v(f_1) - \deg_y f_1) + \cdots + \alpha_{i-2}(v(f_{i-2}) - \deg_y f_{i-2})$. Note also the following: $\deg_y f_i = $ subdegree $f_i (= $degree of the leading form of f_i).

THEOREM 14. *We place*

$$f_i'(x', y') = f_i(x', x'y')/x' \exp t_{i-1}.$$

The elements $f_i'(x', y')$ are polynomials in $K[x', y']$ and satisfy Theorem 6 for v.

Proof. That the $f_i(x', y')$ are polynomials, and, moreover, monic in y', is the content of the lemma. We proceed now by induction on i. Note that $v(f_i) - v(f_i')$ is an integral multiple of $v(x)$, in fact equals $t_{i-1}v(x)$. Since f_1' is monic, of degree 1 in y', it follows as in Theorem 6, that any polynomial $g(x', y') \in K[x', y']$ with $v(g) < v(f_1')$ has an integral multiple of $v(x)$ as value; that is, (2) of Theorem 6 is satisfied for $i = 1$. For $i > 1$, (3) is trivial since $\deg_y f_i = \deg_{y'} f_i'$; (4) follows directly from the remark that $v(f_i') - v(f_i)$ is an integral multiple of $v(x)$, and the fact that all the groups D_i involved contain the group of integral multiples of $v(x)$. (We assume by induction that $\Delta_{i-1} = \Delta_{i-1}'$.) As for (5),

$$v(f_i') = v(f_i) - t_{i-1}v(x) > d_{i-1}v(f_{i-1}) - t_{i-1}v(x)$$
$$= d_{i-1}(v(f_{i-1}') + t_{i-2}v(x)) - t_{i-1}v(x) = d_{i-1}v(f_{i-1}').$$

Thus it remains to prove (2). Explicitly stated, we have to prove that if $g(x', y') \in K[x', y']$ has value less than the value of f_i' then $v(g) = v(F'(\alpha_0, \cdots, \alpha_{i-1}))$ for some integers $\alpha_0, \alpha_1, \cdots, \alpha_{i-1}$, where F' is a monomial in the f_i'. Since f_i' is monic in y and $v(g) < v(f_i')$, reducing $g(x', y')$ mod $f_i'(x', y')$, we may assume $m = \deg_{y'} g(x', y') < \deg_{y'} f_i'$. But then

$$g(x', y') = g(x, y/x) = g'(x, y)/x^m, \ g'(x, y) \in K[x, y].$$

Hence $v(g') = mv(x) + v(g) < t_{i-1}v(x) + v(f_i') = v(f_i)$, whence $v(g') \in \Delta_{i-1}$. Since $v(g') - v(g)$ is an integral multiple of $v(x)$, (2) now follows. Note finally that if the sequence $f_1, f_2, \cdots$ is finite, then the sequence $f_1', f_2', \cdots$ is finite, and of the same length, as follows at once by considering the 1-1 correspondence with the groups Δ_i. This completes the proof.

We now derive a theorem which is a continuation, in a sense, to Theorem 9. Let v_1 be the rational rank 2 valuation introduced for v in Theorem 9. If τ is the least irrational v_1-value assumed by elements in $\mathfrak{O}$ then we have seen by Corollary 1 to Theorem 9 that for certain integers s it is true that the v_1-ideals of value less than $s\tau$ are the first v-ideals for v. Note that in Theorem 9 we now have $\tau > 1$, since we are assuming $v(y) > v(x) > 0$.

THEOREM 15. *Let v and v_1 be as in Theorem 9; let τ be the least irrational v_1-value assumed by elements in $\mathfrak{O}$, and let τ' be the least irrational v_1-value assumed by elements in $\mathfrak{O}'$. If for some integer s the v_1-ideals in $\mathfrak{O}$ of v_1-value less than $s\tau$ are the first v-ideals for v, then also in $\mathfrak{O}'$ the v_1-ideals of v_1-value less than $s\tau'$ are the first v-ideals for v.*

Proof. Let $f_1, \cdots, f_{j+1}$ be the polynomials of Theorem 9, and consider the elements $F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})$, $0 \le \alpha_0$, $0 \le \alpha_i < d_i$, $i = 1, \cdots, j$, $0 \le \alpha_{j+1} < s$. Any two of these have distinct v_1-value and, by Corollary 2 of Theorem 9, the integer s is bounded in such manner that they also have distinct v-value. We may therefore order these elements according to their v- and v_1-values, obtaining the simple sequences

(1) $u_1, u_2, \cdots,$

(2) $w_1, w_2, \cdots,$

respectively. Let $q_1, q_2, \cdots; q_1^*, q_2^*, \cdots$ be the Jordan sequences of 0-dimensional valuation ideals for v and v_1 respectively. If $v_1(w_k) < v_1(f_{j+1}^s) < v_1(w_{k+1})$ then, as in Corollary 2 of Theorem 9, $u_i = w_i$, $i = 1, \cdots, k$. In the ring $\mathfrak{O}' = K[x', y']$ consider the elements $F'(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})$, with the same limitations on the α_i. Now if

(3) $v(F'(\alpha_0, \cdots, \alpha_{j+1})) < v(F'(\beta_0, \cdots, \beta_{j+1})) < sv(f_{j+1}')$

then

(4) $v(F(\alpha_0 + m - \sigma_1, \alpha_1, \cdots, \alpha_{j+1})) < v(F(\beta_0 + m - \sigma_2, \beta_1, \cdots, \beta_{j+1}))$
$$< sv(f_{j+1})$$

where $m = s \cdot \deg_y f_{j+1}$, $\sigma_1 = \deg_y F(0, \alpha_1, \cdots, \alpha_{j+1})$, $\sigma_2 = \deg_y F(0, \beta_1, \cdots, \beta_{j+1})$. Since $\sigma_1 < m$ and $\sigma_2 < m$, the elements evaluated in (4) are in $K[x, y]$, whence the inequality (4) holds with v replaced by v_1, whence it follows that (3) holds with v replaced by v_1. Likewise, any two distinct $F'(\alpha_0, \cdots, \alpha_{j+1})$ have dis-

tinct v-value. Hence the elements $F'(\alpha_0, \cdots, \alpha_{j+1})$ can be ordered according to their v and v_1-values to yield the sequences

(1') $u_1', u_2', \cdots,$

(2') $w_1', w_2', \cdots,$

respectively, and if $v_1(w_i') < v_1(f_{j+1}^s) < v(w_{i+1}')$ then $u_i' = w_i'$, $i = 1, \cdots, l$. It follows now in a manner several times employed that the first l valuation ideals in $\mathfrak{O}'$ for v_1 coincide with the first l for v. This completes the proof.

Theorem 15, like Theorem 9, will be used to replace a given valuation by a rational rank 2 valuation.

COROLLARY. *Theorem 15 also holds for a rational rank 2 valuation v (of type not less than $j+1$) having $f_1, \cdots, f_{j+1}$ as the first $j+1$ associated polynomials, and such that $v(f_i) = v_1(f_i)$, $i = 1, \cdots, j$.*

8. **The transform of a simple v-ideal, general case.** We come now to the proof that the transform of a simple v-ideal is simple. As usual we assume $v(y) > v(x) > 0$; and T is the transformation

$$T: \quad x' = x, \quad y' = y/x; \quad x = x', \quad y = x'y'.$$

THEOREM 16. *The transform of a v-ideal $\mathfrak{q}$ is a valuation ideal for v.*

Proof. If v is not already of rational rank 2, let v_1 be a rational rank 2 valuation, as given in Theorem 9, such that $v_1(\mathfrak{q}) < sv(f_{j+1})$ and such that the v_1-ideals of v_1-value less than $sv_1(f_{j+1})$ are also the first v-ideals for v. We assert that $v_1(T(\mathfrak{q})) < sv_1(f_{j+1}')$. In fact, the ideal $\mathfrak{q}$ has a basis consisting of f_{j+1}^s and of elements $F(\alpha_0, \cdots, \alpha_j, \alpha_{j+1})$, $0 \leq \alpha_0$, $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j$, $0 \leq \alpha_{j+1} < s$. Placing

$$h = h(\mathfrak{q}) = \min \left\{ \alpha_0 + \sum \alpha_i \deg_v f_i \right\}$$

over the elements of this basis, we have $v_1(T(\mathfrak{q})) = v_1(\mathfrak{q}) - h$. Clearly $h \leq \deg_v f_{j+1}^s$. If equality holds, the assertion follows at once. Supposing $h < \deg_v f_{j+1}^s$, then for at least one of the basis elements $F(\alpha_0, \cdots, \alpha_{j+1})$, it is true that

$$v(F(\alpha_0, \cdots, \alpha_{j+1})) - (\alpha_0 + \sum \alpha_i \deg_v f_i) \geq v(T(\mathfrak{q})),$$

whence the assertion follows from the remark at the end of the lemma to Theorem 14. Hence, by Theorem 15, if $T(\mathfrak{q})$ is a v_1-ideal it is also a v-ideal, and we may pass from v to v_1, that is we may assume without loss of generality that v is rational rank 2.

We now prove a lemma.

LEMMA. *Let $0 \leq \alpha_i < d_i$, $i = 1, \cdots, j$; $0 \leq \alpha_{j+1}$. If the subdegree k of $F(0, \alpha_1, \cdots, \alpha_{j+1})$ is equal to or greater than $h = h(\mathfrak{q})$ then*

$$v(F(0, \alpha_1, \cdots, \alpha_{j+1})) - k \geq v(\mathfrak{q}) - h.$$

Proof. The proof is by induction on k. In fact, let $k > h$. Since subd $f_1 = 1$ and $v(f_1) > 1 = v(x)$ we have, if $\alpha_1 \neq 0$,

$$v(F(0, \alpha_1, \cdots, \alpha_{j+1})) - k > v(F(0, \alpha_1 - 1, \alpha_2, \cdots, \alpha_{j+1})) - (k-1) \geqq v(\mathfrak{q}) - h,$$

by induction. If $\alpha_1 = 0$, let $\alpha_1 = \cdots = \alpha_{i-1} = 0$, $\alpha_i \neq 0$; then

$$v(F(0, \alpha_1, \cdots, \alpha_{j+1})) - k$$
$$> v(F(0, d_1 - 1, \cdots, d_{i-1} - 1, \alpha_i - 1, \alpha_{i+1}, \cdots, \alpha_{j+1})) - (k-1)$$
$$\geqq v(\mathfrak{q}) - h,$$

by induction. Thus it remains to prove the lemma for $k = h$. In the case $k = h$, let $F(\beta_0, \cdots, \beta_{j+1})$ be an element of $\mathfrak{q}$ of subdegree h. Clearly $\alpha_{j+1} \geqq \beta_{j+1}$. If $\alpha_i = \beta_i$, $i = 1, \cdots, j+1$, then $\beta_0 = 0$ and the lemma is trivial. Otherwise, let $\alpha_{i+1} = \beta_{i+1}, \cdots, \alpha_{j+1} = \beta_{j+1}$ but $\alpha_i > \beta_i$. Then clearly,

$$v(F(0, \alpha_1 + d_1 - \beta_1, \alpha_2 + d_2 - \beta_2 - 1, \cdots, \alpha_i + d_i - \beta_i - 1, \alpha_{i+1} - \beta_{i+1} - 1)$$
$$> \beta_0 v(x),$$

since both terms evaluated have like subdegree, and $v(y) > v(x)$. The lemma is an immediate consequence.

To proceed with the proof of the theorem, let $g(x', y') \in \mathfrak{O}'$, with $v(g) \geqq v(T(\mathfrak{q}))$: to prove $g \in T(\mathfrak{q})$. We can write $g(x', y') = \sum c(\alpha_0, \cdots, \alpha_{j+1}) \cdot F'(\alpha_0, \cdots, \alpha_{j+1})$, where $\sum$ is a finite sum of terms with distinct value. Thus it suffices to assume $g = F'(\alpha_0, \cdots, \alpha_{j+1})$, where $0 \leqq \alpha_0$, $0 \leqq \alpha_i < d_i$, $i = 1, \cdots, j$, $0 \leqq \alpha_{j+1}$. Clearly, if α_0 is sufficiently large then $g \in T(\mathfrak{q})$; in fact, if $v(x^{\alpha(0)}) \geqq v(\mathfrak{q})$ then $x'^{\alpha(0)}$ arises from $x^{\alpha(0)+h}$, and g is a multiple of $x'^{\alpha(0)}$. Thus we may assume that if $h(x', y') \equiv 0(x'^{\alpha(0)+1})$ and $v(h) \geqq v(T(\mathfrak{q}))$ then $h(x', y') \in T(\mathfrak{q})$. We now make an induction on $k = \text{subd } F'(0, \alpha_1, \cdots, \alpha_{j+1})$. If $k \leqq h$ then $g(x', y')$ arises from $F(\alpha_0 + h - k, \alpha_1, \cdots, \alpha_{j+1})$. Let, now, $k > h$. If $\alpha_1 > 0$ then by the lemma $F'(\alpha_0, \alpha_1 - 1, \alpha_2, \cdots, \alpha_{j+1})$ has value not less than $T(\mathfrak{q})$, and hence is in $T(\mathfrak{q})$ by induction, whence also $g \in T(\mathfrak{q})$. If $\alpha_1 = 0$, let $\alpha_1 = \cdots = \alpha_{i-1} = 0$, $\alpha_i \neq 0$. Then

$$F'(\alpha_0, 0, \cdots, 0, \alpha_i, \cdots, \alpha_{j+1})$$
$$= f_1' \cdot F'(\alpha_0, d_1 - 1, \cdots, d_{i-1} - 1, \alpha_i - 1, \alpha_{i+1}, \cdots, \alpha_{j+1})$$
$$+ x'^{\alpha(0)+1} h(x', y'),$$

where $h(x', y') \in \mathfrak{O}'$. The first term on the right is a multiple of $F'(\alpha_0, d_1 - 1, \cdots, d_{i-1} - 1, \alpha_i - 1, \alpha_{i+1}, \cdots, \alpha_{j+1})$, which by the lemma has value not less than $T(\mathfrak{q})$, and hence is in $T(\mathfrak{q})$ by induction. Hence also the second term has value not less than $T(\mathfrak{q})$, and since it is divisible by $x'^{\alpha(0)+1}$ it also is in $T(\mathfrak{q})$. Thus $g \in T(\mathfrak{q})$. This completes the proof.

THEOREM 17. *The transform by T of the ith simple v-ideal $\mathfrak{P}_i$ is the $(i-1)$th simple v-ideal $\mathfrak{P}'_{i-1}$.*

Proof. As in the previous theorem, if v is not already of rational rank 2 then there exists a rational rank 2 valuation v_1 such that the v-ideals up to and including $\mathfrak{P}_i$ coincide with the v_1-ideals up to and including $\mathfrak{P}_i$ and such that the v-ideals in $\mathfrak{O}'$ up to and including $T(\mathfrak{P}_i)$ coincide with the v_1-ideals up to and including $T(\mathfrak{P}_i)$. Hence $\mathfrak{P}_i$ is also the ith simple v_1-ideal, and if $T(\mathfrak{P}_i)$ is the $(i-1)$th simple v_1-ideal then it is also the $(i-1)$th simple v-ideal for v. Thus we may pass from v to v_1; that is, without loss of generality we may assume v to be of rational rank 2.

Let n be the type of v with respect to the ring $K[x, y]$; $f_1(x, y), \cdots, f_n(x, y)$, the polynomials introduced in Theorem 6 for v, and let $v(f_n) = \tau$. The proof is by induction on n. For $n = 1$, the proof has already been given in Theorem 5. For $n > 1$, every v-ideal of value equal to or less than $d_{n-1}v(f_{n-1})$ belongs also, by Theorem 10, to a rational rank 2 valuation v' of type $n-1$ with respect to $K[x, y]$: and moreover v' exists such that $\mathfrak{P}_i$ is the ith simple v'-ideal. (Here also, by corollary, Theorem 15, if $T(\mathfrak{P}_i)$ is the $(i-1)$th simple v'-ideal, then it is the $(i-1)$th simple v-ideal.) Thus we may assume $v(\mathfrak{P}_i) > d_{n-1}v(f_{n-1})$ by induction. If $v(\mathfrak{P}_i) < \tau = v(f_n)$ then, by Corollary 4 of Theorem 10, $\mathfrak{P}_i$ is also the ith simple valuation ideal for a rational rank 2 valuation v', of type n with respect to $K[x, y]$ and with $f_1, \cdots, f_n$ as associated polynomials, such that $v'(\mathfrak{P}_i) = \tau' = v'(f_n)$. In short, we may assume without loss of generality that $v(\mathfrak{P}_i) \geqq \tau$.

Let $\mathfrak{q}$ be the immediate successor of $\mathfrak{P}_i$ in the sequence $\mathfrak{q}_1, \mathfrak{q}_2, \cdots$. By Theorem 12, either $v(\mathfrak{P}_i) = s\tau$, $v(\mathfrak{q}) = r/t$ or $v(\mathfrak{P}_i) = r/t$, $v(\mathfrak{q}) = s\tau$, s, r positive integers, $t = t_{n-1} = \deg_y f_n$. The two cases are not really different, since, supposing $v(\mathfrak{P}_i) = r/t$, there exists a second valuation v' with $f_1, \cdots, f_n$ as associate polynomials such that $\mathfrak{q}_1, \mathfrak{q}_2, \cdots$, $\mathfrak{P}_i$ are the first v'-ideals in $K[x, y]$ and such that $v'(\mathfrak{P}_i) = s\tau'$, where $\tau' = v'(f_n)$. We suppose therefore that $v(\mathfrak{P}_i) = s\tau$, $v(\mathfrak{q}) = r/t$. If $F(\alpha_0, \cdots, \alpha_n) \in \mathfrak{P}_i$ then, just as in the lemma to Theorem 14, since $v(F(\alpha_0, \cdots, \alpha_n)) \geqq v(f_n^s)$, we have

$$x^{\alpha(0)} f_1^{\alpha(1)}(x, y) \cdots f_n^{\alpha(n)}(x, y) = x'^{\alpha(0)} f_1^{\alpha(1)}(x', x'y') \cdots f_n^{\alpha(n)}(x', x'y') \equiv 0(x'^{st}).$$

Also $f_n^s(x, y) = f_n^s(x', x'y') \not\equiv 0(x'^{st+1})$, whence, from the definition of T, $v(T(\mathfrak{P}_i)) = v(\mathfrak{P}_i) - st$. Now $v(f_n) - v(f_n') = t$, whence, placing $v(f_n') = \tau'$, $v(T(\mathfrak{P}_i)) = s\tau'$. By the corollary to Theorem 13, r/s is an approximant or quasi approximant to $t\tau$. Since $\tau' = \tau - t$, clearly $(r - st^2)/s$ is an approximant or quasi approximant to $t\tau'$. By the corollary to Theorem 13, then, $s\tau'(= \min(s\tau', (r - st^2)/s))$ is the value of a simple v-ideal for v in $K[x, y]$. Since[8] $T(\mathfrak{P}_i)$ is a v-ideal, it is therefore a simple v-ideal. It remains to prove that $T(\mathfrak{P}_i)$ is the $(i-1)$th v-ideal in $K[x', y']$.

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[8] In the case of a simple v-ideal $\mathfrak{P}_i$, the proof that $T(\mathfrak{P}_i)$ is a valuation ideal for v is quite immediate. In fact, let $\mathfrak{q}'$ be the v-ideal in $K[x', y']$ having value $s\tau'$. We have $T(\mathfrak{P}_i) \subseteq \mathfrak{q}'$ and we wish to prove $\mathfrak{q}' \subseteq T(\mathfrak{P}_i)$. Let $g'(x', y') \in \mathfrak{q}'$. Since $v(\mathfrak{q}') = v(f_n')$ and since f_n' is monic in y', reducing $g'(x', y')$ mod f_n', we may assume $m = \deg_{y'} g'(x', y') < \deg_{y'} f_n' = st$. There follows: $g'(x', y')$

We have shown above that if $v(\mathfrak{P}_i)=v(f_n^s)$ then $v(T(\mathfrak{P}_i))=v(f_n'^s)$; similarly one shows that if $v(\mathfrak{P}_k)=d_{n-1}v(f_{n-1})$ then $v(T(\mathfrak{P}_k))=d_{n-1}v(f_{n-1}')$ whence $v(\mathfrak{P}_k)-v(T(\mathfrak{P}_k))=d_{n-1}v(f_{n-1})-d_{n-1}v(f_{n-1}')=d_{n-1}t_{n-2}=t_{n-1}=t$. (The ideal of value $d_{n-1}v(f_{n-1})$ is simple by Corollary 2 of Theorem 10.) Also, if $v(\mathfrak{P}_i)=\tau$ then $v(\mathfrak{P}_i)-v(T(\mathfrak{P}_i))=t$. Hence if q is a v-ideal for v with $\mathfrak{P}_k\supset q\supset \mathfrak{P}_i$, the inclusion being proper, then $v(q)-v(T(q))=t$ also, $T(\mathfrak{P}_k)\supset T(q)\supset T(\mathfrak{P}_i)$, and each inclusion is proper. Hence there are at least as many v-ideals between $T(\mathfrak{P}_k)$ and $T(\mathfrak{P}_i)$ as there are between $\mathfrak{P}_k$ and $\mathfrak{P}_i$. Since q is simple by Corollary 3 of Theorem 10, $T(q)$ is simple, and there are at least as many simple v-ideals between $T(\mathfrak{P}_k)$ and $T(\mathfrak{P}_i)$ as there are between $\mathfrak{P}_k$ and $\mathfrak{P}_i$. If, in addition, there are not more v-ideals between $T(\mathfrak{P}_k)$ and $T(\mathfrak{P}_i)$ than there are between $\mathfrak{P}_k$ and $\mathfrak{P}_i$ then the present theorem certainly holds up to $\mathfrak{P}_i$, since by induction it holds up to $\mathfrak{P}_k$. This is the case. In fact, as we have seen, $v(T(\mathfrak{P}_i))=\tau'$, and every v-ideal in $K[x', y']$ preceding $T(\mathfrak{P}_i)$ has rational value u'/t, u' a non-negative integer. For every integer u' such that $v(T(\mathfrak{P}_k))<u'/t<v(T(\mathfrak{P}_i))$ there exists, by (a) of Theorem 6, a v-ideal for v in $K[x', y']$ with value u'/t. Thus there are as many v-ideals between $T(\mathfrak{P}_k)$ and $T(\mathfrak{P}_i)$ as there are integers in the interval $(tv(T(\mathfrak{P}_k)), tv(T(\mathfrak{P}_i)))$. Since $v(\mathfrak{P}_k)-v(T(\mathfrak{P}_k))=v(\mathfrak{P}_i)-v(T(\mathfrak{P}_i))=t$, an integer, there are the same number of integers in the interval $(tv(\mathfrak{P}_k), tv(\mathfrak{P}_i))$. By the reasoning just used, this is the number of v-ideals between $\mathfrak{P}_k$ and $\mathfrak{P}_i$, which was to be proved. Hence $T(\mathfrak{P}_i)=\mathfrak{P}_{i-1}'$ if $v(\mathfrak{P}_i)=\tau$. Placing $v(\mathfrak{P}_m)=\tau$, we have $T(\mathfrak{P}_m)=\mathfrak{P}_{m-1}'$.

Returning to the general case: $v(\mathfrak{P}_i)=s\tau$, $v(q)=r/t$, we know that r/s is an approximant or quasi approximant to $t\tau$, say the kth. The correspondence between the simple v-ideals and the approximants and quasi approximants to $t\tau$ is such that $\mathfrak{P}_i$ is the $(k-2)$th simple v-ideal after $\mathfrak{P}_m$: that is, $i=m+k-2$. Since $(r-st^2)/s$ is the kth approximant or quasi approximant to $t\tau'(=t(\tau-t))$, $T(\mathfrak{P}_i)$ is the $(k-2)$th simple v-ideal after the simple v-ideal in $K[x', y']$ of value τ'. Hence $T(\mathfrak{P}_i)=\mathfrak{P}_{m-1+k-2}'=\mathfrak{P}_{i-1}'$. This completes the proof.

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$=g'(x, y/x)=g(x, y)/x^m$, $g(x, y)\in\mathfrak{O}$, whence $g'(x', y')$ arises under T from $g(x, y)x^{st-m}$. This last element is in $\mathfrak{P}_i$ since it has value $=v(g')+st\geq v(f_n'^s)+st=v(f_n'^s)$.