EQUILONG MAPS OF THE ∞^3 CIRCLES

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1. **Conformal maps of the geodesic circles of a developable surface upon a plane.** If a surface is conformally mapped upon a plane with cartesian coordinates (X, Y), the linear element of the surface is given by $dS^2 = e^{\lambda(X,Y)}(dX^2+dY^2)$. The ∞^3 Minding geodesic circles, that is, the curves of constant geodesic curvature on the surface, are represented in the plane by a system of ∞^3 curves defined by the ordinary differential equation of third order

$$(\Omega 1)\qquad (1+Y'^2)Y''' - 3Y'Y''^2 = (1+Y'^2)^2[(\lambda_{XY}-\lambda_X\lambda_Y)(1-Y'^2) + (\lambda_{XX}-\lambda_{YY}+\lambda_X^2-\lambda_Y^2)Y'].$$

In minimal coordinates $U=X+iY$, $V=X-iY$, this family of ∞^3 curves is defined by the differential equation of third order

$$(\Omega 2)\qquad 2V'V''' - 3V''^2 = V'^2[(2\lambda_{UU}-\lambda_U^2) - V'^2(2\lambda_{VV}-\lambda_V^2)].$$

For any surface, λ is a perfectly general function of (X, Y), or (U, V). The surface is developable if and only if λ is a harmonic function, that is, $\lambda_{XX}+\lambda_{YY}=0$, or $\lambda_{UV}=0$. In this developable case, the ∞^3 curves in the plane defined by the differential equation of third order, $(\Omega 1)$ or $(\Omega 2)$, are equivalent under the conformal group to the totality of all ∞^3 circles and is called an Ω *family*(1).

Kasner and the writer have discussed various geometric properties of any Ω family of curves, not all circles. An Ω family may be characterized among all three-parameter families of curves in the plane by the following set of three properties:

Property I. The locus of the foci of the osculating parabolas of the ∞^1 curves of the family which pass through a lineal-element E is a lemniscate L with E as one of the two orthogonal tangent elements at the node of L.

Property II. As the direction of E is rotated about its point P, the locus of the centers of the orthogonal pairs of circles defining the ∞^1 focal lemniscates is an equilateral hyperbola H with the center of H at P.

Property III. The foci of the equilateral hyperbolas H are connected to the point P by a direct conformal transformation. That is, the correspondence between P and the foci is direct conformal.

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(1) Kasner and DeCicco, *Families of curves conformally equivalent to circles*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 378–391. *The geometry of velocity systems*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 236–245.

Another complete geometric characterization of an Ω family is the set:

Property I′. The envelope of the directrices of the osculating parabolas of the ∞^1 curves of the family which pass through a lineal-element E is an equilateral hyperbola H with the center of H at the point P of E and the line of E as one of the asymptotes of H.

Property II′. The locus of the foci of the directorial hyperbolas H is an equilateral hyperbola H' with its center at P.

Property III′. The foci of the equilateral hyperbolas H' of Property II′ are related to the point P by a direct conformal transformation(²).

2. **Equilong maps of the ∞^3 circles of a plane.** Conformal transformations are point-to-point correspondences of the plane such that the angle between the directions of any two curves passing through a given point is preserved; whereas equilong transformations, as defined by Scheffers, are line-to-line correspondences of the plane such that the distance between the two points of contact of any two curves along a common tangent line is preserved. Conformal transformations are defined by functions of $(X \pm iY)$, where $i^2 = -1$ and (X, Y) are cartesian coordinates of a point; and equilong transformations are given by functions of $(x \pm jy)$, where $j^2 = 0$ and (x, y) are equilong or hessian coordinates of a line. Thus conformal and equilong transformations are roughly dual.

Upon applying any equilong transformation to the totality of ∞^3 circles of the plane, there results a set of ∞^3 curves which is termed an ω *family*. Thus the Ω families and the ω families are dual geometric objects in a rough sense, the former being conformally, and the latter equilongly, equivalent to the ∞^3 circles of the plane. In equilong or hessian coordinates (x, y) of a line, any ω family is given by an ordinary differential equation of third order

$$(\omega 3) \qquad y''' = 2A(x) \cdot y' + A_x(x) \cdot y + B(x).$$

We shall prove that *the magnilong group is the totality of all line transformations of the plane carrying every ω family of curves into an ω family*. Any transformation of the magnilong group is the product of an equilong transformation followed by an ordinary magnification.

The line transformations of the plane may be divided into three distinct classes: (1) The group of *magnilong* transformations. Any correspondence of this group magnifies, by a constant $\gamma \neq 0$, the distance between the points of contact on the common tangent line of any two curves. If $\gamma = \pm 1$, the resulting correspondence is an *equilong* transformation as defined by Scheffers. (2) The set of *affinilong* transformations. Any transformation of this set preserves every parallel pencil of straight lines, but it is *not* a magnilong transformation. A nonmagnilong correspondence is an affinilong transformation if and only if the ratio into which one of the three points of contact on the common

(²) Kasner and DeCicco, *The conformal near-Moebius transformations*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 784–793.

tangent line of any three curves divides the segment determined by the other two points is preserved. (3) The set of *general* transformations. Any correspondence of this set is not of the types (1) or (2). Thus T is a general transformation if and only if it does *not* preserve every parallel pencil of straight lines(3).

It can be shown that *a general line transformation carries at most* $2\infty^2$ *curves of a given* ω *family into curves of the same family, an affinilong transformation converts at most* ∞^2 *curves of a given* ω *family into curves of the same family, and a magnilong correspondence sends at most* ∞^1 *curves of a given* ω *family into curves of the same family.*

An ω family may be characterized by the following set of three geometrical properties.

Property I. The locus of the foci of the osculating parabolas of the ∞^1 curves of the family which pass through a lineal-element E is a cissoid with the cusp at the lineal-element E.

Property II. For any parallel pencil of straight lines, there is induced by Property I a point transformation from the point P of the lineal-element E to the center of the base circle of the cissoid of Property I. The Property II states that this induced point transformation is an affine transformation. This affinity is not of the general type; it is such that any point (X, Y) is carried into a point (α, β) on the particular straight line of the fixed parallel pencil of straight lines determined by the point (X, Y). It also holds fixed every point of a certain straight line.

Property III. The resulting one-parameter family of affine transformations satisfies a single differential condition of first order.

Another complete geometric characterization of an ω family is:

Property I′. The envelope of the directrices of the osculating parabolas of the ∞^1 curves of the family which pass through a lineal-element E is a parabola with vertex at the point P of E and directrix perpendicular to the line L of E.

Property II′. For any parallel pencil of straight lines, there is induced by Property I′ a point transformation from the point P of the lineal-element E to the focus of the directorial parabola of Property I′. The Property II′ states that the induced transformation is an affine transformation similar to the one described in Property II.

Property III′. The resulting one-parameter family of affine transformations satisfies a single differential condition of first order.

Other properties may be described as follows. If the focal cissoids of Property I are constructed along a given line L, the envelope of the circles defining these ∞^1 cissoids is a pair of straight lines L_1 and L_2, symmetrical with respect to L. The induced line transformation from L into L_1 and L_2 is an affinilong

(3) DeCicco, *The magnilong near-Laguerre transformations*, National Mathematics Magazine vol. 19 (1945) pp. 1–7.

transformation satisfying a single differential condition of first order. In general, this is not a magnilong or equilong transformation.

Also the envelope of the ∞^1 directorial parabolas of the Property I′ consists of a pair of straight lines L_1' and L_2', symmetrical with respect to L. The induced line transformation from L to L_1 and L_2 is an affinilong transformation satisfying a single differential condition of first order(4).

3. **The transformation between the cartesian coordinates (X, Y, Y') and the equilong coordinates (x, y, y') of a lineal element.** Consider the equation

$$(1) \qquad (1 - x^2)X + 2xY = 2y,$$

where (X, Y) denote the cartesian coordinates of a point in the plane. Since fixed values of x and y define a straight line, the ordered number pair (x, y) may be called the *equilong coordinates* of a line. If u and v denote the normal angle and the normal distance of the line (1), it is seen that $u = 2 \arctan x$ and $v = 2y/(x^2+1)$. This gives the relationship between the hessian coordinates (u, v) and the equilong coordinates (x, y) of a straight line.

Since (1) is the directrix equation of a contact group of Lie, it is seen that the contact transformation is given by

$$(2) \qquad X = \frac{2(y - xy')}{x^2 + 1}, \qquad Y = \frac{2xy - y'(x^2 - 1)}{x^2 + 1}, \qquad Y' = \frac{x^2 - 1}{2x}.$$

This gives the relationship between the cartesian coordinates (X, Y, Y') and the equilong coordinates (x, y, y') of any lineal-element.

We shall need to know the second and third derivaives of this transformation (2). It is seen that

$$(3) \quad Y'' = -\frac{(x^2 + 1)^2}{4x^3(y'' + X)}, \quad Y''' = -\frac{(x^2 + 1)^3 y'''}{8x^4(y'' + X)^3} + \frac{3(x^2 - 1)(x^2 + 1)^2}{8x^5(y'' + X)^2}.$$

From (2) and (3), we find

$$(4) \qquad Y'Y''' - 3Y''^2 = -\frac{(x^2 - 1)(x^2 + 1)^3 y'''}{16x^5(y'' + X)^3} - \frac{3(x^2 + 1)^2}{4x^4(y'' + X)^2}.$$

This will be found useful for our later work.

This transformation (2) is important because the ∞^3 circles of the plane are represented in equilong coordinates by the ∞^3 vertical parabolas

$$(5) \qquad y = ax^2 + bx + c.$$

If (α, β) are the cartesian coordinates of the center of this circle and r its radius, then

$$(6) \qquad \alpha = -a + c, \qquad \beta = b, \qquad r = a + c.$$

(4) DeCicco, *General comparison of conformal and equilong geometries*, National Mathematics Magazine vol. 16 (1942) pp. 1–5.

Thus by this transformation (2), the differential condition for all ∞^3 circles, $(1+Y'^2)Y'''-3Y'Y''^2=0$, is transformed into the condition for all ∞^3 vertical parabolas, $y'''=0$. This is impossible under the group of all point transformations, but it is possible under the group of all contact transformations as indicated above.

4. **The enlarged seven-parameter Laguerre group** G_7. This is the complete group of all line transformations carrying the entire family of ∞^3 circles into itself. In equilong coordinates, this enlarged group of Laguerre transformations is the seven-parameter group G_7 defined by the equations

$$X = \frac{ax+b}{cx+d}, \qquad Y = \frac{ey+fx^2+gx+h}{(cx+d)^2}, \tag{7}$$

where (a, b, c, d, e, f, g, h) are constants such that $ad-bc\neq 0$ and $e\neq 0$. This is a subgroup of the magnilong group.

If in the preceding equations, we take e to be the quantity $ad-bc\neq 0$, the resulting *six-parameter Laguerre group* G_6 may be written in the compact form

$$Z = (\alpha z+\beta)/(\gamma z+\delta), \tag{8}$$

where each of the quantities $z, Z, \alpha, \beta, \gamma, \delta$ represents a *dual complex number* of the form $a+jb$, where $j^2=0$ and (a, b) are real or complex numbers. This is an equilong subgroup and is the one studied extensively by Laguerre. Our enlarged Laguerre group G_7 is the product of the Laguerre equilong group G_6, defined by (8), by the group of magnifications $X=x$, $Y=ey$.

5. **The differential equation of third order defining any ω family.** In this section, all coordinates mentioned are equilong line coordinates. In equilong coordinates, any (direct) equilong transformation is given by

$$X = \phi(x), \qquad Y = \phi_x(x)\cdot y + \psi(x), \tag{9}$$

where $\phi_x(x)\neq 0$. Extending this equilong transformation three times, we find

$$\begin{aligned}
Y' &= y' + \frac{\phi_{xx}}{\phi_x}y + \frac{\psi_x}{\phi_x},\\
Y'' &= \frac{1}{\phi_x}y'' + \frac{\phi_{xx}}{\phi_x^2}y' + \frac{1}{\phi_x^3}(\phi_x\phi_{xxx} - \phi_{xx}^2)y + \frac{1}{\phi_x^3}(\phi_x\psi_{xx} - \psi_x\phi_{xx}),\\
Y''' &= \frac{1}{\phi_x^2}y''' + \frac{1}{\phi_x^4}(2\phi_x\phi_{xxx} - 3\phi_{xx}^2)y'\\
&\quad + \frac{1}{\phi_x^5}(\phi_x^2\phi_{xxxx} - 4\phi_x\phi_{xx}\phi_{xxx} + 3\phi_{xx}^3)y\\
&\quad + \frac{1}{\phi_x^5}\left[\phi_x(\phi_x\psi_{xxx} - \psi_x\phi_{xxx}) - 3\phi_{xx}(\phi_x\psi_{xx} - \psi_x\phi_{xx})\right].
\end{aligned} \tag{10}$$

From (9) and (10), it is seen that the ∞^3 circles defined by the differential equation in equilong coordinates $Y'''=0$ correspond under any equilong transformation to the ∞^3 curves given by a differential equation of third order of the form

$$
\begin{aligned}
y''' = &-\frac{1}{\phi_x^2}(2\phi_x\phi_{xxx} - 3\phi_{xx}^2)y' - \frac{1}{\phi_x^3}(\phi_x^2\phi_{xxxx} - 4\phi_x\phi_{xx}\phi_{xxx} + 3\phi_{xx}^3)y \\
&-\frac{1}{\phi_x^3}[\phi_x(\phi_x\psi_{xxx} - \psi_x\phi_{xxx}) - 3\phi_{xx}(\phi_x\psi_{xx} - \psi_x\phi_{xx})].
\end{aligned} \tag{11}
$$

Now let

$$
\begin{aligned}
A &= -\frac{1}{2\phi_x^2}(2\phi_x\phi_{xxx} - 3\phi_{xx}^2), \\
B &= -\frac{1}{\phi_x^3}[\phi_x(\phi_x\psi_{xxx} - \psi_x\phi_{xxx}) - 3\phi_{xx}(\phi_x\psi_{xx} - \psi_x\phi_{xx})].
\end{aligned} \tag{12}
$$

From this it follows that A may be any function of x only and B may be any function of x only.

THEOREM 1. *A differential equation of third order represents an ω family of curves, defined as any family of curves equilongly equivalent to the totality of ∞^3 circles in the plane, if and only if it is of the form in equilong coordinates*

$$y''' = 2A(x)\cdot y' + A_x(x)\cdot y + B(x). \tag{13}$$

It is observed that any ω family is carried into an ω family, not only under the equilong group, but also under the magnilong group.

By the preceding differential equation (13), we observe that any ω family is uniquely determined by the functions (A, B). Therefore ω is a function of A and B only and we write $\omega=\omega(A, B)$.

If $\phi(x)$ and $\psi(x)$ are any two functions which satisfy (12), then the integral curves of (13) are the transforms under the equilong transformation (9) of the ∞^3 circles of the (X, Y)-plane, where (X, Y) are the equilong coordinates of any straight line. Therefore the curves of our family are

$$\phi_x(x)\cdot y + \psi(x) = a[\phi(x)]^2 + b\phi(x) + c. \tag{14}$$

Any ω family is thus a special type of linear families of curves. Of course, the differential equation (13) could have been obtained as a result of eliminating the arbitrary constants (a, b, c) from (14) by differentiation.

By (14), it is seen that under an equilong transformation or magnilong transformation, any ω family is converted into some other ω family. The group of line transformations which preserve a given $\omega(A, B)$ family of curves is a seven-parameter group G_7 $[\omega(A, B)]$, isomorphic with the extended La-

guerre group of circular transformations. Any transformation of this group is of the form TLT^{-1} where T is a definite magnilong transformation which converts the ∞^3 circles into our $\omega(A, B)$ family and L is any extended Laguerre transformation.

Because of this isomorphism with the extended Laguerre group G_7, we may make the following observations(5).

THEOREM 2. *A general line transformation carries at most $2\infty^2$ curves of a given $\omega(A, B)$ family into curves of the same family. An affinilong transformation sends at most ∞^2 curves of a given $\omega(A, B)$ family into curves of the same family. Finally a magnilong transformation converts at most ∞^1 curves of a given $\omega(A, B)$ family into curves of the same family.*

From this theorem, we obtain the following results. If a line transformation T carries $3\infty^2$ curves of a given $\omega(A, B)$ family into curves of the same family, then T belongs to the group G_7 $[\omega(A, B)]$. If a nongeneral line transformation T carries $2\infty^2$ curves of a given $\omega(A, B)$ family into curves of the same family, then T belongs to the group G_7 $[\omega(A, B)]$. Finally if a magnilong transformation T carries $2\infty^1$ curves of a given $\omega(A, B)$ family of curves into curves of the same family, then T belongs to the group G_7 $[\omega(A, B)]$.

Let T be a definite magnilong transformation and let R be any other transformation which carries the ∞^3 circles into a given $\omega(A, B)$ family of curves. Then obviously $R = TL$, where L is any Laguerre transformation.

6. **The osculating parabolas of a curve.** Just as a set of values for (x, y, y', y''), that is, a differential element of second order, is pictured most simply by means of the corresponding circle of curvature, so a differential element of third order, defined by (x, y, y', y'', y'''), may be pictured by the unique osculating parabola. We shall collect here the general formulas to be used in the subsequent discussion(6).

Let (α, β) denote the cartesian coordinates of the center of the circle of curvature and let R be the radius of curvature. Then

$$\begin{gathered} \alpha = y - xy' + y''(x^2 - 1)/2, \qquad \beta = y' - xy'', \\ R = y - xy' + y''(x^2 + 1)/2. \end{gathered} \tag{15}$$

If (α, β) denote the running cartesian coordinates of a point on a parabola, it is found that the unique osculating parabola of a third order differential element given in cartesian coordinates (X, Y, Y', Y'', Y''') is

$$\begin{aligned} &[(Y'Y''' - 3Y''^2)(\alpha - X) - Y'''(\beta - Y)]^2 \\ &\qquad + 18Y''^2[Y'(\alpha - X) - (\beta - Y)] = 0. \end{aligned} \tag{16}$$

(5) DeCicco, *The analogue of the Moebius group of circular transformations in the Kasner plane*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 936–943.

(6) Kasner and DeCicco, *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 23–38.

The focus is given by

$$\alpha + i\beta = X + iY + \frac{3Y''(1 + iY')^2}{2[Y'''(1 + iY') - 3iY''^2]}, \tag{17}$$

and the equation of the directrix is

$$2Y'''(\alpha - X) + 2(Y'Y''' - 3Y''^2)(\beta - Y) = 3Y''(1 + Y'^2). \tag{18}$$

By use of the formulas (2), (3), and (4), it is found that, if a third order differential element is given in equilong coordinates by (x, y, y', y'', y'''), the equation of the osculating parabola is

$$\begin{aligned}[\{(x^2 - 1)(x^2 + 1)y''' + 12x(y'' + X)\}(\alpha - X)\\ - 2\{x(x^2 + 1)y''' - 3(x^2 - 1)(y'' + X)\}(\beta - Y)]^2\\ + 144x^3(y'' + X)^4[(x^2 - 1)(\alpha - X) - 2x(\beta - Y)] = 0.\end{aligned} \tag{19}$$

The focus is given by

$$\alpha + i\beta = X + iY + \frac{3[2x + i(x^2 - 1)](y'' + X)^2}{2[(x^2 + 1)y''' + 6i(y'' + X)]}, \tag{20}$$

and the equation of the directrix is

$$\begin{aligned}4[- x(x^2 + 1)y''' + 3(x^2 - 1)(y'' + X)](\alpha - X)\\ - 2[(x^2 - 1)(x^2 + 1)y''' + 12x(y'' + X)](\beta - Y)\\ = - 3(x^2 + 1)^2(y'' + X)^2.\end{aligned} \tag{21}$$

7. **The focal cissoid.** Observe that (20) may be written in the form

$$\frac{1 + iY'}{(\alpha - X) + i(\beta - Y)} = \frac{(x^2 + 1)y''' + 6i(y'' + X)}{3x(y'' + X)^2}, \tag{22}$$

from which we find

$$\begin{aligned}\frac{(\alpha - X) + Y'(\beta - Y)}{(\alpha - X)^2 + (\beta - Y)^2} &= \frac{(x^2 + 1)y'''}{3x(y'' + X)^2},\\ \frac{Y'(\alpha - X) - (\beta - Y)}{(\alpha - X)^2 + (\beta - Y)^2} &= \frac{2}{x(y'' + X)}.\end{aligned} \tag{23}$$

Divide the first equation by the square of the second equation. We obtain

$$\begin{aligned}[(\alpha - X) + Y'(\beta - Y)][(\alpha - X)^2 + (\beta - Y)^2]\\ = (x/12)(x^2 + 1)y'''[Y'(\alpha - X) - (\beta - Y)]^2.\end{aligned} \tag{24}$$

THEOREM 3. *Property* I. *For the locus of the foci (the focal locus) of the osculating parabolas of the* ∞^1 *curves of a given family of* ∞^3 *curves which pass*

through a given lineal-element E, constructed at E, to be a cissoid with the cusp at E, it is necessary and sufficient that the differential equation of third order of the family be given in equilong coordinates by $y'''=f(x, y, y')$.

The center of the circle defining the focal cissoid of Property I is given in cartesian coordinates by

$$\alpha = X + x(x^2+1)y'''/24, \quad \beta = Y + (x^2-1)(x^2+1)y'''/48, \tag{25}$$

and the equation of the asymptote is

$$2x(\alpha - X) + (x^2-1)(\beta - Y) = (x^2+1)^3 y'''/24. \tag{26}$$

It is seen that $x=$const. defines a system of parallel lines. The ∞^2 points on these parallel lines may be defined by the equilong coordinates (y, y') which constitute a particular set of affine coordinates, defined by $(1-x^2)X+2xY=2y$, $-xX+Y=y'$.

THEOREM 4. *Property* II. *For the differential equation $y'''=f(x, y, y')$ to have the particular form $y'''=2A(x)\cdot y'+C(x)\cdot y+B(x)$, it is necessary and sufficient that the induced point transformation defined by* (25) *be affine.*

This affine transformation is not of the general type. It is such that any point (X, Y) is carried into a point (α, β), on the particular straight line of the given family of parallel lines determined by the point (X, Y). It also holds fixed every point of the straight line, defined by $2A(x)\cdot y'+C(x)\cdot y+B(x)=0$. This is a necessary and sufficient characterization of the affine transformation of Property II.

THEOREM 5. *Property* III. *The resulting one-parameter family of affine transformations, defined by Property* II, *satisfies the single differential condition of first order $C=A_x$.*

The Properties I, II, and III are a characteristic set of geometric conditions for a three-parameter family of curves to be an ω family.

8. **The directorial parabola.** From the equation (21), the following proposition may be obtained.

THEOREM 6. *Property* I′. *For the envelope of the directrices (the directorial locus) of the osculating parabolas of the ∞^1 curves of a given family of ∞^3 curves which pass through a given lineal element E, constructed at E, to be a parabola with vertex at the point of E and axis as the line of E, it is necessary and sufficient that the differential equation of third order of the family be given in equilong coordinates by $y'''=f(x, y, y')$.*

The equation of the directorial parabola is

$$\begin{aligned} &6[(x^2-1)(\alpha - X) - 2x(\beta - Y)]^2 \\ &\qquad + (x^2+1)^3 y'''[2x(\alpha - X) + (x^2-1)(\beta - Y)] = 0. \end{aligned} \tag{27}$$

The focus is

$$\alpha = X - x(x^2+1)y'''/6, \qquad \beta = Y - (x^2-1)(x^2+1)y'''/12, \tag{28}$$

and the equation of the directrix is

$$2x(\alpha - X) + (x^2-1)(\beta - Y) = (x^2+1)^2 y'''/12. \tag{29}$$

From (25) and (28), it is seen that the point (X, Y) divides the segment from the center of the circle defining the focal cissoid to the focus of the directorial parabola in the ratio 1:4. The distance of the directrix of this directorial parabola is twice that of the asymptote of the cissoid from the point of the lineal element E.

From the preceding remarks, it follows that the Properties II′ and III′ are very similar to the Properties II and III described in §7.

9. **Some additional properties of an ω family.** Let us consider a three-parameter family of curves with the Property I. The focal curve associated with any lineal element E is a cissoid with cusp at E. If we keep the line of E fixed and vary the point of E, there will result ∞^1 focal cissoids. The ∞^1 circles defining these cissoids will all have their centers on the fixed line. Therefore in order that these circles be tangent to two straight lines (which may be coincident), whose angle bisector is the line of E, it is only necessary that there exist a linear relationship between the abscissa of the center and the radius of the defining circle.

THEOREM 7. *A three-parameter family of curves with the Property* I *will possess the property that the ∞^1 circles defining the focal cissoids of the lineal elements of any fixed straight line be tangent to two other straight lines, if and only if its differential equation is of the form* $y''' = 2A(x, y)y' + C(x, y)$.

The induced line correspondence between any line (x, y) and the lines (X, Y) of this theorem is given by the equations

$$\begin{aligned} &(X+x)(xX-1) + (x^2+1)^2 A[(x-1)X + (x+1)]^2/48 = 0, \\ &Y = [(1 - X^2 + 2xX)/(x^2+1)]y \\ &\qquad\qquad + (x^2+1)C[(x-1)X + (x+1)]^2/96. \end{aligned} \tag{30}$$

For this induced transformation to be related to an ω family, it is seen that this must be an affinilong transformation ($A_y = 0$) which satisfies a certain differential condition of first order ($C_x = A$). In general, it is not an equilong or magnilong transformation.

A consideration of the directorial parabola (27) will establish the following result.

THEOREM 8. *A three-parameter family of curves with the Property* I′ *will possess the property that the ∞^1 directorial parabolas of the lineal elements of*

any fixed straight line will be tangent to two straight lines, symmetrical to the fixed line, if and only if its differential equation is of the form $y'''=2A(x, y)y' + C(x, y)$.

These two straight lines are given by

$$(31)\qquad \begin{aligned}(x^2+1)^2[2A\{2x\alpha+(x^2-1)\beta-2xy\}-C(x^2+1)]^2 \\ -48A[(x^2-1)\alpha-2x\beta+2y]^2=0.\end{aligned}$$

10. **The transformation theory of ω families.** Let us consider a general line transformation of the plane defined in equilong coordinates by

$$(32)\qquad X=\phi(x, y),\qquad Y=\psi(x, y),$$

where the jacobian $J=\phi_x\psi_y-\phi_y\psi_x\neq 0$, in a certain region of the (x, y)-plane. The first two extensions are

$$(33)\qquad Y'=\frac{\psi_x+\psi_y\cdot y'}{\phi_x+\phi_y\cdot y'},\qquad Y''=\frac{c_0+c_1y'+c_2y'^2+c_3y'^3+J\cdot y''}{(\phi_x+\phi_y\cdot y')^3},$$

where

$$(34)\qquad \begin{aligned} c_0&=\phi_x\psi_{xx}-\psi_x\phi_{xx}, & c_1&=-J_x+3(\phi_x\psi_{xy}-\psi_x\phi_{xy}),\\ c_2&=+J_y+3(\phi_y\psi_{xy}-\psi_y\phi_{xy}), & c_3&=\phi_y\psi_{yy}-\psi_y\phi_{yy}.\end{aligned}$$

The third extension is

$$(35)\qquad \begin{aligned} Y'''=&\frac{(d_0+d_1y'+d_2y'^2+d_3y'^3+d_4y'^4+d_5y'^5)}{(\phi_x+\phi_y\cdot y')^5}\\ &+\frac{(h_0+h_1y'+h_2y'^2)y''-3J\phi_yy''^2+(\phi_x+\phi_y\cdot y')Jy'''}{(\phi_x+\phi_y\cdot y')^5}\end{aligned}$$

where

$$(36)\qquad \begin{aligned} d_0&=\phi_xc_{0x}-3\phi_{xx}c_0, & d_1&=\phi_x(c_{0y}+c_{1x})+\phi_yc_{0x}-3\phi_{xx}c_1-6\phi_{xy}c_0,\\ d_2&=\phi_x(c_{1y}+c_{2x})+\phi_y(c_{0y}+c_{1x})-3\phi_{xx}c_2-6\phi_{xy}c_1-3\phi_{yy}c_0,\\ d_3&=\phi_x(c_{2y}+c_{3x})+\phi_y(c_{1y}+c_{2x})-3\phi_{xx}c_3-6\phi_{xy}c_2-3\phi_{yy}c_1,\\ d_4&=\phi_xc_{3y}+\phi_y(c_{2y}+c_{3x})-6\phi_{xy}c_3-3\phi_{yy}c_2, & d_5&=\phi_yc_{3y}-3\phi_{yy}c_3,\\ h_0&=\phi_x(c_1+J_x)-3J\phi_{xx}-3c_0\phi_y,\\ h_1&=\phi_x(J_y+2c_2)+\phi_y(J_x-2c_1)-6\phi_{xy}J,\\ h_2&=3\phi_xc_3+\phi_y(J_y-c_2)-3\phi_{yy}J.\end{aligned}$$

THEOREM 9. *The magnilong group consists of all line transformations in the plane converting every ω family into an ω family.*

In order to prove this result, it is necessary merely to prove that if a line

transformation T carries the family of ∞^3 circles into an ω family, then T is a magnilong transformation(7).

From (35), it is seen that if $Y'''=0$ corresponds to an ω family, then $\phi_y=h_0=h_1=h_2=0$. Since $\phi_y=0$, it follows automatically that $h_2=0$. From $h_1=0$, we get $\psi_{yy}=0$. From $h_0=0$, we find $\phi_x\psi_{xy}-\psi_y\phi_{xx}=0$. These prove immediately that our transformation must be magnilong. Thus the proof of our Theorem 9 is complete.

In conclusion, it may be observed that if we regard (32) as a point transformation of the plane, it follows by (35) that $Y'''=0$ can not possibly correspond to $(1+y'^2)y'''-3y'y''^2=0$, and hence the ∞^3 vertical parabolas cannot correspond to the ∞^3 circles of the plane under a point-to-point (or line-to-line) transformation.

(7) DeCicco, *Geometry of dual-velocity systems*, Revista, Universidad Nacional de Tucuman (Argentina) vol. 3 (1942) pp. 261–270. Also in this journal, a forthcoming paper, *The affinilong near-Laguerre transformations.*

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