THE WEIERSTRASS $E$-FUNCTION IN THE CALCULUS OF VARIATIONS

BY
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1. Introduction. The present paper is the first of a set of three papers concerned primarily with the isoperimetric problem of Bolza. This problem is one of the most general problems in the calculus of variations and can be described as follows: Consider a class of arcs

C: \( a^h, y^i(t) \) \( (t^1 \leq t \leq t^2; h = 1, \ldots, r; i = 0, 1, \ldots, n) \)

in \( ay \)-space satisfying a set of conditions of the form

\begin{align}
(1.1) & \quad \phi^\beta(a, y, \dot{y}) = 0 \quad (\beta = 1, \ldots, m < n), \\
(1.2) & \quad y^i(t^1) = T_{t^1}(a), \quad y^i(t^2) = T_{t^2}(a), \\
(1.3) & \quad I^\sigma(C) = g^\sigma(a) + \int_C f^\sigma(a, y, \dot{y}) \, dt = 0 \quad (\sigma = 1, \ldots, s).
\end{align}

The components \( a^h \) of \( C \) are constants. We seek to find in this class of arcs one that minimizes a function

\begin{equation}
(1.4) \quad I(C) = g(a) + \int_C f(a, y, \dot{y}) \, dt.
\end{equation}

The functions \( \phi^\beta, f^\sigma, f \) are positively homogeneous in the variables \( \dot{y}^i \). The problem here formulated contains as special cases most of the interesting problems in the calculus of variations involving simple integrals\(^1\).

In the present paper we shall develop certain interesting properties of the Weierstrass $E$-function. These properties are of interest apart from their applications to be found in the papers that follow. We shall be concerned particularly with the concept of $E$-dominance. This concept can be described briefly as follows: Let \( \mathcal{D} \) be the set of all admissible elements \( (a, y, \rho) \) satisfying the conditions \( \phi^\beta(a, y, \rho) = 0 \) and let \( C_0 \) be an arc whose elements \( (a, y, \rho) = (a, y, \dot{y}) \) are in \( \mathcal{D} \). A function \( F(a, y, \rho) \) will be said to $E$-dominate a second function \( H(a, y, \rho) \) near \( C_0 \) on \( \mathcal{D} \) if there is a neighborhood \( \mathcal{D}_0 \) relative to \( \mathcal{D} \) of the elements \( (a, y, \rho) \) on \( C_0 \) and a constant \( b > 0 \) such that the inequality

\[ E_F(a, y, \rho, q) \geq b | E_H(a, y, \rho, q) | \]

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\(^1\) Cf. M. R. Hestenes, Generalized problem of Bolza, Duke Math. J. vol. 5 (1939) pp. 309–324. A discussion of several different formulations of the problem of Bolza can be found in this paper.
holds whenever \((a, y, p)\) is in \(\mathcal{D}_0\) and \((a, y, q)\) is in \(\mathcal{D}\). Here \(E_F\) and \(E_H\) are the \(E\)-functions of \(F\) and \(H\) respectively. We are especially interested in finding necessary and sufficient conditions that a function \(F(a, y, p)\) shall \(E\)-dominate the integrand \(L(p) = (p^i p^j)^{1/2}\) of the length integral. It is shown below that \(L(p)\) is \(E\)-dominated by a function of the form

\[
F = l^0 f + l^r f^r + m^\theta(a, y) \phi^\theta
\]

if and only if the arc \(C_0\) satisfies, with the multipliers \(l^0, l^r, m^\theta(a, y)\), the strengthened condition of Weierstrass and the condition of nonsingularity. That the strengthened condition of Weierstrass plus nonsingularity implies that \(L\) is \(E\)-dominated by a function \(F\) of this type has also been established by W. T. Reid in connection with an expansion proof of a sufficiency theorem for parametric problems. His results have not been published as yet. In the present paper we not only show the equivalence of these relations but show further that a function \(FE\)-dominates \(L\) near \(C_0\) on \(\mathcal{D}\) if and only if there is a function of the form

\[
F^* = F + \theta(a, y, p) \phi^\theta \phi^\theta
\]

that \(E\)-dominates \(L\) near \(C_0\) on the class of all admissible elements \((a, y, p)\). The results here given indicate that for a function \(F\) possessing only first derivatives the condition that \(F\) shall \(E\)-dominate \(L\) appears to be a natural extension of the strengthened condition of Weierstrass and the condition of nonsingularity.

The concept of \(E\)-dominance will be used freely in the two papers that follow. In the first of these it will be used in connection with a theorem of Lindeberg analogous to one given by Reid\(^{(2)}\) for the nonparametric case. In the second paper it will be shown that the sufficiency theorems for the problem of Bolza can be obtained from those of the problem of Mayer, a result that does not appear to have been completely justified. Following the method used by Reid we shall show that sufficiency theorems for the isoperimetric problem of Bolza can be obtained from those of the problem of Bolza without isoperimetric conditions. Moreover it will be seen that sufficiency theorems for parametric problems can be obtained from those for nonparametric problems. Interesting results will also be obtained in regard to the regions in which the sufficiency theorems are valid.

The third and final paper of the present series will be devoted to the proof of a sufficiency theorem for a proper strong relative minimum for the isoperimetric problem of Bolza. This sufficiency theorem is essentially one conjectured by McShane\(^{(3)}\) with the usual inequality \(I(C) > I(C_0)\) \((C \neq C_0)\) replaced by an inequality of the form


where \(|C, C_0|\) is a suitably defined metric for the class of arcs under consideration. This new inequality enables one to obtain an analogue of Osgood’s theorem as a corollary to our sufficiency theorem. One of the interesting features of the method here used is that it is applicable without modification to isoperimetric problems, that is, the method is the same for a problem with isoperimetric side conditions as for one without isoperimetric side conditions. The method used is essentially the one used by McShane in order to establish a sufficiency theorem for a weak relative minimum and later extended by Myers(4) in order to establish a sufficiency theorem for a semistrong relative minimum for the nonparametric problem of Lagrange.

The results given in these three papers are applicable to the nonparametric case as well as to the parametric case.

2. Preliminary definitions and lemmas. The present section will be devoted to a description of some of the hypotheses, definitions and notations that will be used in the three papers. We shall use the following notations

\[ a = (a_1, \ldots, a_r), \quad y = (y_0, y_1, \ldots, y_n), \quad \dot{y} = (\dot{y}_0, \dot{y}_1, \ldots, \dot{y}_n), \]

\[ p = (p^0, p^1, \ldots, p^n), \quad q = (q^0, q^1, \ldots, q^n). \]

If \(k\) is a real number, then \(kp = (kp^0, kp^1, \ldots, kp^n)\). Repeated indices in a term denote summation with respect to that index. The length \(|p|^{1/2}\) of the vector \(p\) will be denoted by \(|p|\). We distinguish between the symbols \(|p|, |p^i|\). The latter denotes the absolute value of the \(i\)th component \(p^i\) of \(p\). Similar remarks hold for the symbols \(|a|, |a^k|, |y|, |y'|\), and so on.

We suppose that we have given an open set \(\mathcal{R}\) of \((r+2n+2)\)-dimensional points \((a, y, p) \neq (a, y, 0)\) with the property that if \((a, y, p)\) is in \(\mathcal{R}\) so also is \((a, y, kp)\) for every positive number \(k\). An element \((a, y, p)\) will be called admissible if it is in \(\mathcal{R}\). By an admissible subregion \(\mathcal{R}_0\) of \(\mathcal{R}\) will be meant one such that if \((a, y, p)\) is in \(\mathcal{R}_0\) so also is the element \((a, y, kp)\) \((k > 0)\).

By an admissible function \(H(a, y, p)\) will be meant a real single-valued function on \(\mathcal{R}\) that satisfies the homogeneity condition

\[ H(a, y, kp) = kH(a, y, p) \quad (k > 0) \]

on \(\mathcal{R}\), is continuous, and has continuous first and second derivatives with respect to the variables \(p^0, \ldots, p^n\). As a consequence of the relation (2.1) one has on \(\mathcal{R}\) the well known identities

\[ H = p^iH_{p^i}, \quad p^iH_{p^ip^j} = 0 \quad (i, j = 0, 1, \ldots, n), \]

and the homogeneity relations

\[ H_{pi}(a, y, kp) = H_{pi}(a, y, p), \]
\[ H_{pi+pi}(a, y, kp) = k^{-1}H_{pi+pi}(a, y, p), \]

where \( k > 0 \). These relations will be used freely.

It will be understood throughout that the functions \( f(a, y, p), f^\varepsilon(a, y, p), \phi^\theta(a, y, p) \) appearing in the formulation of our problem are admissible functions of class \( C'' \) on \( \mathcal{R} \). The functions \( g(a), g^\varepsilon(a), T^{ii}(a), T^{12}(a) \) are assumed to be of class \( C'' \) on \( \mathcal{R} \).

An element \( (a, y, p) \) in \( \mathcal{R} \) will be said to be differentially admissible if \( \phi^\theta(a, y, p) = 0 \) \( (\beta = 1, \ldots, m) \). It is clear from (2.1) that the class of all differentially admissible elements form an admissible subregion of \( \mathcal{R} \). This subregion will be denoted by \( \mathcal{D} \).

Consider now a rectifiable curve \( C \) in \( ay \)-space having an absolutely continuous representation

\[ a, y(t) \quad (t^1 \leq t \leq t^2), \]
the components \( a = (a^1, \ldots, a^r) \) being constants. By virtue of our conventions \( y(t) \) represents the set \( y^0(t), y^1(t), \ldots, y^n(t) \). Their derivatives \( y^0(t), y^1(t), \ldots, y^n(t) \) exist almost everywhere on \( t^1t^2 \) and define a vector \( \dot{y}(t) \). At the points where \( \dot{y}(t) \) is not defined we set \( \dot{y}(t) = (0, \ldots, 0) \). We shall consider throughout only the rectifiable arcs (2.4) for which the element \( (a, y(t), \dot{y}(t)) \) is in \( \mathcal{R} \) for almost all values of \( t \) on \( t^1t^2 \). By an admissible arc \( C \) will be meant an arc (2.4) of this type satisfying the differential equations (1.1), the isoperimetric conditions (1.3), and the end conditions (1.2).

We shall center our attention on a particular admissible arc \( C_0: a_0, y_0(t) \quad (t^1 \leq t \leq t^2) \) of class \( C'' \). It will be assumed throughout that \( C_0 \) does not intersect itself and that the matrix

\[ \|\phi^\theta(a_0, y_0(t), \dot{y}_0(t))\| \quad (\beta = 1, \ldots, m; i = 0, 1, \ldots, n), \]

has rank \( m \) on \( t^1t^2 \). An element \( (a, y, p) \) will be said to be on \( C_0 \) if there is a constant \( k > 0 \) and a value \( t \) on \( t^1t^2 \) such that

\[ a = a_0, \quad y = y_0(t), \quad p = ky_0(t). \]

By a neighborhood of the elements \( (a, y, p) \) on \( C_0 \) will be meant an admissible subregion \( \mathcal{R}_0 \) of \( \mathcal{R} \) containing the elements \( (a, y, p) \) on \( C_0 \) in its interior. It will be convenient to designate such a neighborhood by the phrase "a neighborhood \( \mathcal{R}_0 \) of \( C_0 \)." Similarly by a neighborhood \( \mathcal{D}_0 \) of \( C_0 \) relative to \( \mathcal{D} \) will be meant the set of all differentially admissible elements in a neighborhood \( \mathcal{R}_0 \) of \( C_0 \).

The assumption that the arc \( C_0 \) is of class \( C'' \) is made only for convenience. In the first two papers it would be sufficient to assume that \( C_0 \) is of class \( C' \).
In the third paper the conditions imposed on \( C_0 \) imply that it is of class \( C''' \). It is for this reason that we make our initial assumption that \( C_0 \) is of class \( C''' \). Similar remarks hold regarding our assumptions concerning the functions \( f, f', \phi^\beta, g, g^\alpha, T^{\alpha_1}, T^{\alpha_2} \).

The following lemma will be useful.

**Lemma 2.1.** There exist \( n - m \) admissible functions \( \phi^\gamma(a, y, r) \) \((\gamma = m + 1, \ldots, n)\) of class \( C''' \) such that the determinant

\[
\left| \begin{array}{c}
\phi^\alpha_j \\
\phi^\alpha_{p_i}
\end{array} \right| \quad (i = 0, 1, \ldots, n; \alpha = 1, \ldots, n)
\]

is different from zero on \( C_0 \). Moreover the equations

\[
|r| = c|p|, \quad \phi^\alpha(a, y, r) = 0, \quad \phi^\gamma(a, y, r) = \phi^\gamma(a, y, p) + v^\gamma
\]

have solutions \( r^i(a, y, p, v, c) \) of class \( C''' \) on a neighborhood of the values \((a, y, p, v, c) = (a, y, p, 0, 1) \) on \( C_0 \). The solutions satisfy the homogeneity conditions

\[
r^i(a, y, kp, kv, c) = kr^i(a, y, p, v, c) \quad (k > 0).
\]

The first statement in the lemma has been established by Bliss(6). The second statement follows from implicit function theorems.

3. **Weak E-dominance.** The present paper is concerned primarily with the properties of the Weierstrass E-function. In this section will be found a description of certain properties of this function that will be useful later. Most of these properties are well known.

By the **Weierstrass E-function** \( E_F \) associated with an admissible function \( F \) will be meant the function

\[
E_F(a, y, p, q) = F(a, y, q) - qF_{p^i}(a, y, p).
\]

In view of the relations (2.1) and (2.3) we have

\[
E_F(a, y, kp, k'q) = k'E_F(a, y, p, q) \quad (k > 0, k' > 0).
\]

As a consequence of these relations it is clear that we can at will restrict ourselves to normed sets \((a, y, p)\) and \((a, y, q)\), that is, to sets for which \(|p| = 1\) and \(|q| = 1\).

The \( E \)-function

\[
E_L(p, q) = L(q) - q^i\frac{p^i}{L(p)}.
\]

associated with the integrand

\[
L(p) = \left| \frac{p^i}{L(p)} \right| = (p^i)^{1/2}
\]

of the length integral will play a dominant role in this paper. It is easily seen that

\( 0 < E_L(p, q) \leq 2L(q) \quad (q \neq kp, k > 0). \)

Moreover we have

\( L_{pi^i z^i} \geq 0 \quad (z \neq pp), \)

as can be seen from the identity

\[ L_{pi^i z^i} = \frac{1}{2L^3}(z^i p^i - z^i p^i)(z^i p^i - z^i p^i). \]

Let \( F \) and \( H \) be admissible functions and let \( E_F \) and \( E_H \) be the corresponding \( E \)-functions. The function \( F \) will be said to weakly \( E \)-dominate \( H \) near \( C_0 \) on \( \mathcal{D} \) if there is a neighborhood \( \mathcal{D}_0 \) of \( C_0 \) relative to \( \mathcal{D} \) and a constant \( b > 0 \) such that the inequality

\( \| E_H(a, y, p, q) \| \leq bE_F(a, y, p, q) \)

holds for every pair of elements \( (a, y, p) \) and \( (a, y, q) \) in \( \mathcal{D}_0 \). Similarly if an inequality of the form (3.7) holds for every pair of elements \( (a, y, p) \) and \( (a, y, q) \) on a neighborhood \( \mathcal{R}_0 \) of \( C_0 \), then \( F \) will be said to weakly \( E \)-dominate \( H \) near \( C_0 \) on \( \mathcal{R} \). It is clear that \( F \) weakly \( E \)-dominates \( H \equiv 0 \) near \( C_0 \) on \( \mathcal{D} \) if and only if the inequality

\( E_F(a, y, p, q) \geq 0 \)

holds for every pair of elements \( (a, y, p), (a, y, q) \) on a neighborhood \( \mathcal{D}_0 \) of \( C_0 \) relative to \( \mathcal{D} \).

In order to prove certain consequences of weak \( E \)-dominance we shall make use of the following well known result.

**Lemma 3.1.** If the inequality

\( E_F(a, y, p, q) \geq 0 \)

holds whenever \( (a, y, p) \) is on \( C_0 \) and \( (a, y, q) \) is a differentially admissible element on a neighborhood \( \mathcal{R}_0 \) of those on \( C_0 \), then the inequality

\( F_{pi^i z^i} \geq 0 \)

holds on \( C_0 \), subject to the conditions

\( z^i \phi^i = 0, \quad z^i \neq \rho \phi^i. \)

The inequality (3.9) subject to the conditions (3.10) in general does not imply the inequality (3.8). These inequalities are however equivalent in case \( F \) is nonsingular on \( C_0 \) relative to \( \phi^1, \ldots, \phi^m \), that is, in case the determinant
Theorem 3.1. Given an admissible function $F$ the following conditions are equivalent.

I. The inequality

$$F(p) > 0$$

holds on $C_0$ subject to the conditions (3.10).

II. The function $F$ is nonsingular on $C_0$ relative to $\phi_1, \cdots, \phi_m$ and the inequality (3.9) holds on $C_0$ subject to the conditions (3.10).

III. The function $F$ is nonsingular on $C_0$ relative to $\phi_1, \cdots, \phi_m$ and weakly $E$-dominates $H=0$ near $C_0$ on $D$.

IV. The function $F$ weakly $E$-dominates $L=|p|$ near $C_0$ on $D$.

V. Every admissible function $H$ is weakly $E$-dominated by $F$ near $C_0$ on $D$.

VI. There is a constant $c$ such that the inequality

$$F^* + z^p > 0$$

holds on $C_0$ for every set $z \not= \rho p$, where $F^*$ is the admissible function$^4$

$$F^* = F + (c/L)\phi^p\phi^p.$$  

VII. There is an admissible function $F^*$ of the form (3.13) that weakly $E$-dominates $L$ near $C_0$ on $R$.

The equivalence of the first three conditions is well known. The equivalence of the last two follows from the equivalence of the first and fourth for the case when there are no side conditions $\phi^\beta = 0$. We shall accordingly restrict our attention to conditions IV, V, VI. Because of the homogeneity properties (2.3) we can suppose throughout that the vectors $p$ and $z$ occurring in (3.9), (3.11) and (3.12) satisfy the relations

$$p^i p^j = 0, \quad |z| = 1, \quad |p| = 1.$$  

In this case the condition $z \not= \rho p$ is automatically satisfied and hence need not be considered in the arguments given below.

Suppose now that IV holds. Setting $G = F - bL$ ($b > 0$) it follows from IV that $b$ can be chosen so that the inequality

$$E_0(a, y, p, q) = E_F(a, y, p, q) - bE_L(p, q) \geq 0$$

holds whenever $(a, y, p)$ and $(a, y, q)$ are on a neighborhood $D_0$ of $C_0$ relative to $D$. Let $Q_F$ be the Legendre form

\( Q_F = Q_F(a, y, p, z) = F_{\mu\nu\rho\sigma} z^i \)

for \( F \) and denote the corresponding forms for \( G \) and \( L \) by \( Q_G \) and \( Q_L \) respectively. Using Lemma 3.1 we see that the inequality

\[
0 \leq Q_G(a, y, p, z) = Q_F(a, y, p, z) - bQ_L(p, z)
\]

is satisfied whenever \((a, y, p)\) is on \( C_0 \) and the relations \((3.10)\) and \((3.14)\) hold. Since \( Q_L > 0 \) on this set it follows that \( Q_F > 0 \) also. Consequently condition IV (and hence also V) implies condition I.

We shall show next that condition I implies condition V and hence also condition IV. Let \( S \) be the set of points \((a, y, p, z)\) having \((a, y, p)\) on \( C_0 \) and satisfying the conditions \((3.10)\) and \((3.14)\). By I we have \( Q_F > 0 \) on \( S \). Consequently if \( H \) is an admissible function there is a constant \( b_0 \geq 0 \) such that \( Q_F - bQ_H > 0 \) on \( S \) provided \(|b| \leq b_0\). Consequently if we set \( G = F - bH \) we have \( Q_G > 0 \) on \( S \) whenever \(|b| \leq b_0\). Since I implies III, the inequality

\[
E_0(a, y, p, q) = Q_F(a, y, p, q) - bE_H(a, y, p, q) \geq 0
\]

holds whenever \(|b| \leq b_0\) and \((a, y, p), (a, y, q)\) lie in a suitably chosen neighborhood \( \mathcal{D}_0 \) of \( C_0 \) relative to \( \mathcal{D} \). It follows that condition I implies condition V and hence IV.

It remains to show that condition I is equivalent to condition VI. To this end let \( T \) be the set of points \((a, y, p, z)\) with \((a, y, p)\) on \( C_0 \) and satisfying \((3.14)\). Set \( H = \phi^\beta \phi^\beta / L \). Since \( \phi^\beta = 0 \) on \( T \) we have

\[
Q_H = \phi^\beta \phi^\beta z^i z^j
\]

on \( T \). Consequently \( Q_H = 0 \) if and only if \( \phi^\beta \phi^\beta \phi^i = 0 \), that is, if and only if \((a, y, p, z)\) is on the set \( S \) described in the last paragraph. It follows that on \( S \) the Legendre form \( Q_F^* = Q_F + cQ_H \) for the function \( F^* = F + cH \) is equal to \( Q_F \). Hence condition VI implies condition I. Conversely if condition I holds, then \( Q_F > 0 \) on \( S \), that is, on the subset of \( T \) on which \( Q_H = 0 \). Since \( Q_F \) and \( Q_H \) are continuous functions on \( T \) with \( Q_H \geq 0 \) there is a constant \( h > 0 \) such that \( Q_F > 0 \) at all points of \( T \) having \( Q_H < h \). Choose \( c \) so that \( ch > m \), where \(-m\) is the minimum value of \( Q_F \) on \( T \). For this value of \( c \) we have \( Q_F^* = Q_F + cQ_H > 0 \), as desired. Condition VI is therefore implied by condition I. This completes the proof of Theorem 3.1.

4. Dominance and \( E \)-dominance. An admissible function \( F \) will be said to dominate a second admissible function \( G \) near \( C_0 \) on \( \mathcal{D} \) if there is a neighborhood \( \mathcal{G} \) of \( C_0 \) in \( ay \)-space and a constant \( b > 0 \) such that the inequality

\[
|H(a, y, p)| \leq bF(a, y, p)
\]

holds for every set \((a, y, p)\) in \( \mathcal{D} \) with \((a, y)\) in \( \mathcal{G} \). If \( b \) and \( \mathcal{G} \) can be chosen so that the inequality \((4.1)\) holds for every set \((a, y, p)\) in \( \mathcal{R} \) having \((a, y)\) in \( \mathcal{G} \), then \( F \) will be said to dominate \( H \) near \( C_0 \) on \( \mathcal{R} \).
An admissible function $F$ will be said to $E$-dominate an admissible function $H$ near $C_0$ on $\mathcal{D}$ if there is a neighborhood $\mathcal{D}_0$ of $C_0$ relative to $\mathcal{D}$ and a constant $b > 0$ such that the inequality

$$|E_H(a, y, p, q)| \leq bE_F(a, y, p, q)$$

holds whenever $(a, y, p)$ is in $\mathcal{D}_0$ and $(a, y, q)$ is in $\mathcal{D}$. If this inequality holds whenever $(a, y, q)$ is in $\mathcal{R}$ and $(a, y, p)$ is in a neighborhood $\mathcal{R}_0$ of $C_0$, then $F$ will be said to $E$-dominate $H$ near $C_0$ on $\mathcal{R}$.

Relations between dominance and $E$-dominance are given in Theorem 4.1.

**Theorem 4.1.** Let $F$ and $H$ be admissible functions. If $F$ dominates $L = |p|$ and $E$-dominates $H$ near $C_0$ on $\mathcal{D}$, then $F$ dominates $H$ near $C_0$ on $\mathcal{D}$. Conversely, if $F$ dominates $H$ and $E$-dominates $L$ near $C_0$ on $\mathcal{D}$, then $H$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{D}$.

The proof of this result will be given in the next section. Taking $F = L$ one obtains the following corollaries.

**Corollary 1.** An admissible function $H$ is $E$-dominated by $L$ near $C_0$ on $\mathcal{D}$ if and only if it is dominated by $L$ near $C_0$ on $\mathcal{D}$. If $H \equiv 0$ on $\mathcal{D}$, then $H$ is $E$-dominated by $L$ near $C_0$ on $\mathcal{D}$. Every admissible function $H$ of the form

$$H(a, y, p) = \lambda(a, y, p)\phi(a, y, p)$$

is $E$-dominated by $L$ near $C_0$ on $\mathcal{D}$.

**Corollary 2.** Suppose that $\mathcal{R}$ is the set of all elements $(a, y, p) \neq (a, y, 0)$ whose components $(a, y)$ are in a region in ay-space. Then every admissible function $H$ is $E$-dominated by $L$ near $C_0$ on $\mathcal{R}$.

**Corollary 3.** Suppose $F$ dominates and $E$-dominates $L$ near $C_0$ on $\mathcal{D}$. Then an admissible function $H$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{D}$ if and only if $F$ dominates $H$ near $C_0$ on $\mathcal{D}$.

A further result of this type is given in the following theorem.

**Theorem 4.2.** If $F$ is an admissible function that $E$-dominates $L$ near $C_0$ on $\mathcal{D}$, there is an admissible function of the form

$$G(a, y, p) = D^i(a, y)\phi^i$$

such that $F - G$ dominates and $E$-dominates $L$ near $C_0$ on $\mathcal{D}$. In fact if along an extension of $C_0$ in $\mathcal{D}$ the functions $F_{\phi^i}(a, y, p)$ have continuous derivatives with respect to arc length then $G$ can be chosen to be the integrand of an invariant integral.

The hypothesis in the last statement of the theorem is satisfied, for example, when there is an extension of $C_0$ that is an extremal. It also holds when $F$ is of class $C^{''}$. 

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This theorem will be established in the next section. The principal theorem in this paper is the following:

**Theorem 4.3.** An admissible function $F$ is nonsingular relative to $\phi^1, \cdots, \phi^n$ and $E$-dominates $\phi^1, \cdots, \phi^n$ near $C_0$ on $\mathcal{D}$ if and only if $L = |p|$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{D}$.

The proof of this result will be given in §6 below. For the case in which there are no side conditions $\phi_0 = 0$ this result can be stated in the form given in the following corollary.

**Corollary 1.** The function $L$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{R}$ if and only if the determinant $|F_p^i\psi^j|$ has rank $n$ on $C_0$ and there is a neighborhood $\mathcal{R}_0$ of $C_0$ such that the inequality

$$(4.4) \quad E_F(a, y, p, q) \geq 0$$

holds whenever $(a, y, p)$ is in $\mathcal{R}_0$ and $(a, y, q)$ is in $\mathcal{R}$.

**Theorem 4.4.** Suppose that $F$ is of the form

$$F(a, y, p) = |g' \Sigma_{ij}(a, y) p^i p^j|^{1/2},$$

where $g''(a, y) \pi^i \pi^j$ is a positive definite quadratic form. Then $F$ dominates and $E$-dominates $L$ near $C_0$ on $\mathcal{R}$. Moreover an admissible function $H$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{D}$ if and only if it is dominated by $F$ near $C_0$ on $\mathcal{D}$.

For in this case the inequality (4.4) holds for all $(a, y, p)$ and $(a, y, q)$ on $\mathcal{R}$. Moreover the determinant $|F_p^i \psi^j|$ has rank $n$. It follows from the last corollary that $L$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{R}$. The last statement follows readily from the fact that each of the two functions $F$ and $L$ dominates and $E$-dominates the other.

5. **Proofs of Theorems 4.1 and 4.2.** As a first step observe that since the derivatives $F_{p^i}, H_{p^i}$ are positively homogeneous of order zero in $p$, there is a neighborhood $\mathcal{R}_0$ of $C_0$ and a constant $c > 0$ such that the inequalities

$$|F_p(a, y, p)q^i| \leq cL(q), \quad |H_p(a, y, p)q^i| \leq cL(q)$$

hold whenever $(a, y, p)$ is in $\mathcal{R}_0$. Using the relations

$$|H(a, y, q)| \leq |E_H(a, y, p, q)| + |q^i H_p(a, y, p)|,$$

$$|E_H(a, y, p, q)| \leq |H(a, y, q)| + |q^i H_p(a, y, p)|$$

one obtains the first two of the inequalities

$$(5.1) \quad |H(a, y, q)| \leq |E_H(a, y, p, q)| + cL(q),$$

$$(5.2) \quad |E_H(a, y, p, q)| \leq |H(a, y, q)| + cL(q),$$

$$F(a, y, q) \leq E_p(a, y, p, q) + cL(q),$$

$$(5.4) \quad E_F(a, y, p, q) \leq F(a, y, q) + cL(q),$$

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which hold whenever \((a, y, p)\) is in \(\mathcal{R}_0\) and \((a, y, q)\) is in \(\mathcal{R}\). The last two can be established in a similar manner.

Suppose now that \(F\) dominates \(L\) and \(E\)-dominates \(H\) near \(C_0\) on \(\mathcal{D}\). Then there is a neighborhood \(\mathcal{D}_0\) of \(C_0\) relative to \(\mathcal{D}\) and a constant \(b\) such that

\[
L(q) \leq bF(a, y, q), \\
|E_H(a, y, p, q)| \leq bE_F(a, y, p, q)
\]

hold whenever \((a, y, p)\) is in \(\mathcal{D}_0\) and \((a, y, q)\) is in \(\mathcal{D}\). We can suppose that \(\mathcal{D}_0\) is interior to \(\mathcal{R}_0\). By the use of Lemma 2.1 it is seen that we can select a neighborhood \(\mathcal{F}\) of \(C_0\) in \(ay\)-space such that if \((a, y)\) is in \(\mathcal{F}\) there is a value \(p\) such that \((a, y, p)\) is in \(\mathcal{D}_0\). Consider therefore an element \((a, y, p)\) in \(\mathcal{D}_0\) with \((a, y)\) in \(\mathcal{F}\) and select any value \(q\) such that \((a, y, q)\) is in \(\mathcal{D}\). Using (5.1), (5.6) and (5.4) we see that

\[
|H(a, y, q)| \leq bE_F(a, y, p, q) + cL(q) \leq bF(a, y, q) + (b + 1)cL(q).
\]

It follows from (5.5) that

\[
|H(a, y, q)| = bF(a, y, q)
\]

where \(b' = b + bc + bc\). Consequently \(F\) dominates \(H\) near \(C_0\) on \(\mathcal{D}\), as was to be proved.

Suppose conversely that \(F\) dominates \(H\) and \(E\)-dominates \(L\) near \(C_0\) on \(\mathcal{D}\). Let \(\mathcal{F}\) be a neighborhood of \(C_0\) in \((a, y)\)-space and \(b'\) be a constant chosen so that the inequality (5.7) holds for every set \((a, y, q)\) in \(\mathcal{D}\) with \((a, y)\) in \(\mathcal{F}\). By Theorem 3.1, parts IV and V, the function \(F\) weakly \(E\)-dominates \(H\) near \(C_0\) on \(\mathcal{D}\). We can accordingly select a constant \(b > 0\) so that (5.6) holds whenever \((a, y, p)\) and \((a, y, q)\) are in a neighborhood \(\mathcal{D}_1\) of \(C_0\) relative to \(\mathcal{D}\). Choose a neighborhood \(\mathcal{D}_0\) of \(C_0\) whose closure is in \(\mathcal{D}_1\) and whose components \((a, y)\) are in \(\mathcal{F}\). According to our hypothesis that \(L\) is \(E\)-dominated by \(F\) we can diminish \(\mathcal{D}_0\) if necessary so that there is a constant \(b''\) for which the inequality \(E_L(p, q) \leq b''E_F(a, y, p, q)\) holds whenever \((a, y, p)\) is in \(\mathcal{D}_0\) and \((a, y, q)\) is in \(\mathcal{D}\). Since the closure of \(\mathcal{D}_0\) is interior to \(\mathcal{D}_1\) we can select another constant \(c' > 0\) effective as in the relation

\[
L(q) \leq c'E_L(p, q) \leq c''E_F(a, y, p, q) \quad (c'' = c'b'')
\]

when \((a, y, p)\) is in \(\mathcal{D}_0\) and \((a, y, q)\) is in \(\mathcal{D}\) but not in \(\mathcal{D}_1\). Consider now a set \((a, y, p)\) in \(\mathcal{D}_0\) and select \(q\) so that \((a, y, q)\) is in \(\mathcal{D}\). If \((a, y, q)\) is in \(\mathcal{D}_1\) then (5.6) holds. Suppose therefore that \((a, y, q)\) is not in \(\mathcal{D}_1\). Combining the relations (5.2), (5.7), (5.3) and (5.8) we obtain

\[
|E_H(a, y, p, q)| \leq b'F(a, y, q) + cL(q) \\
\leq b'E_F(a, y, p, q) + c_1L(q) \quad (c_1 = b'c + c) \\
\leq c_2E_F(a, y, p, q) \quad (c_2 = b' + c_1c'').
\]
Consequently (5.6) holds in this case also provided \( b \geq c_2 \). It follows that \( H \)

is \( E \)-dominated by \( F \) near \( C_0 \) on \( \mathcal{D} \) and Theorem 4.1 is proved.

In order to prove Theorem 4.2 we shall make use of the following lemma.

**Lemma 5.1.** Let \( B_i(a, y) \) be a set of continuous functions having continuous
derivatives with respect to arc length along an extension of \( C_0 \) in \( \mathcal{D} \). There exist functions \( D_i(a, y) \) of class \( C' \) which coincide with \( B_i(a, y) \) along \( C_0 \) and which

satisfy the relations

\[
\frac{\partial D_i}{\partial y^j} = \frac{\partial D_j}{\partial y^i} \quad (i, j = 0, 1, \ldots, n).
\]

In the proof we can suppose that \( C_0 \) is part of the \( y^0 \)-axis since this can

be brought about by a nonsingular transformation of class \( C'' \). Under this

transformation the vector \( B_i \) is to be transformed covariantly. Moreover this

transformation can be carried out so as to preserve arc length along \( C_0 \). Setting \( x = y^0 \) we then have \( C_0 \) given by the set

\[
a_h = 0, \quad 0 \leq x \leq l, \quad y^i = 0 \quad (j = 1, \ldots, n).
\]

The given functions \( B_i(a, x, y) \) are such that \( B_i(0, x, 0) \) have continuous
derivatives. Set

\[
D_j(a, x, y) = B_j(0, x, 0) \quad (j = 1, \ldots, n)
\]

\[
D_0(a, x, y) = B_0(0, x, 0) + y_i \frac{\partial D_i}{\partial x}.
\]

It is clear that \( D_i = B_i \) along \( C_0 \) and that

\[
\frac{\partial D_0}{\partial y^i} = \frac{\partial D_i}{\partial x}, \quad \frac{\partial D_j}{\partial y^k} = \frac{\partial D_k}{\partial y^i} = 0 \quad (j, k = 1, \ldots, n).
\]

This proves the lemma.

We are now in position to prove Theorem 4.2. To this end let \( p^i(a, y) \)

be functions of class \( C' \) such that \( (a, y, p^i(a, y)) \) is in \( \mathcal{D} \) when \( (a, y) \) is in a

neighborhood \( \mathcal{G} \) of \( C_0 \) and is on \( C_0 \) whenever \( (a, y) \) is on \( C_0 \). If \( \mathcal{G} \) is taken sufficiently small, the fact that \( L \) is \( E \)-dominated by \( F \) implies the existence of a

positive constant \( c \) such that

\[
E_F(a, y, p(a, y), q) \leq cE_L(p(a, y), q)
\]

holds when \( (a, y) \) is in \( \mathcal{G} \) and \( (a, y, q) \) is in \( \mathcal{D} \). Setting

\[
B_i(a, y) = F_{p^i}(a, y, p(a, y)) - cL_{p^i}(p(a, y))
\]

it is seen that this inequality takes the form

\[
(5.10) \quad F(a, y, q) - B_i(a, y)q^i \geq cL(q).
\]

Since \( E_F = E_{F_{p^i}} \) this proves the first statement in Theorem 4.2.

In order to prove the last statement in the theorem observe that the functions \( B_i(a, y) \) have continuous derivatives with respect to arc length along an

extension of \( C_0 \). Select functions \( D_i \) related to \( B_i \) as described in Lemma 5.1.
Diminish $\xi$ so that

$$|D - B| < c/2$$

on $\xi$. Hence if $(a, y)$ is in $\xi$ and $(a, y, q)$ in $\mathbb{D}$ we have, by (5.10),

$$F(a, y, q) - Dq^i \geq cL(q) - (D_i - B_i)q^i \geq (c/2)L(q),$$

as was to be proved.

6. **Proof of Theorem 4.3.** If $L$ is $E$-dominated by $F$ near $C_0$ in $\mathbb{D}$, then $\phi^1, \ldots, \phi^m$ are $E$-dominated by $F$ near $C_0$ on $\mathbb{D}$, by virtue of Corollary 1 to Theorem 4.1. From Theorem 3.1 it follows that $F$ is nonsingular on $C_0$ relative to $\phi^1, \ldots, \phi^m$. Theorem 4.3 will be established if we show conversely that $L$ is $E$-dominated by $F$ near $C_0$ on $\mathbb{D}$ and $F$ is nonsingular on $C_0$ relative to $\phi^1, \ldots, \phi^m$.

In the remainder of this section we shall assume therefore that $F$ is nonsingular on $C_0$ relative to $\phi^1, \ldots, \phi^m$ and that there is a constant $b > 0$ and a neighborhood $\mathbb{D}_1$ of $C_0$ relative to $\mathbb{D}$ such that the inequality

$$(6.1) \quad E_F(a, y, p, q) \geq b \left| E_\phi(a, y, p, q) \right| = b \left| q \phi^\phi(a, y, p) \right|$$

holds whenever $(a, y, p)$ is in $\mathbb{D}_1$ and $(a, y, q)$ is in $\mathbb{D}$. By virtue of Theorem 3.1 the function $F$ weakly $E$-dominates $L$ near $C_0$ on $\mathbb{D}$. Hence we can suppose that $\mathbb{D}_1$ and $b$ have been chosen so that the relation

$$(6.2) \quad E_F(a, y, p, q) \geq bE_L(p, q)$$

holds whenever $(a, y, p)$ and $(a, y, q)$ are in $\mathbb{D}_1$.

Theorem 4.3 will be established if we show that after suitably diminishing $b$ (keeping $b > 0$) we can select a neighborhood $\mathbb{D}_0$ of $C_0$ relative to $\mathbb{D}$ and interior to $\mathbb{D}_1$ such that (6.2) holds whenever $(a, y, p)$ is in $\mathbb{D}_0$ and $(a, y, q)$ is in $\mathbb{D}$. Suppose this choice cannot be made. Then given a constant $b' > 0$ and a neighborhood $\mathbb{D}_0$ of $C_0$ in $\mathbb{D}_1$ the inequality

$$E_F(a, y, p, q) < b'E_L(p, q)$$

holds for a suitably chosen set $(a, y, p, q)$ with $(a, y, p)$ in $\mathbb{D}_0$ and $(a, y, q)$ in $\mathbb{D} - \mathbb{D}_1$. Because of the homogeneity properties of $E_F$ and $E_L$ this set can be chosen so that $|p| = |q| = 1$ and hence such that $E_L(p, q) \leq 2$. There exists therefore a sequence

$$(a_k, y_k, p_k, q_k) \quad (k = 1, 2, \ldots)$$

converging to a set $(a_0, y_0, p_0, q_0)$ having the following properties:

$$(6.3) \quad (a_k, y_k, p_k) \text{ in } \mathbb{D}_1, (a_k, y_k, q_k) \text{ in } \mathbb{D} - \mathbb{D}_1, (a_0, y_0, p_0) \text{ on } C_0,$$

$$(6.4) \quad |p_k| = |q_k| = |p_0| = |q_0| = 1,$$

$$(6.5) \quad \lim_{k \to \infty} E_F(a_k, y_k, p_k, q_k) = 0.$$
The set \((a_0, y_0, q_0)\) need not be in \(\mathcal{D}\). By the use of (6.1) and (6.5) it is seen that
\[
q_0 \phi_{\mu}(a_0, y_0, p_0) = 0.
\]

Consider now the functions \(r^i(a, y, p, v, c)\) described in Lemma 2.1. If \(\varepsilon\) is sufficiently small the functions
\[
r_k(v, c) = r^i(a_k, y_k, p_k, v, c) \quad |v| < \varepsilon, \quad |c - 1| < \varepsilon
\]
will be well defined for large values of \(k\) and will converge uniformly to
\[
r_0(v, c) = r^i(a_0, y_0, p_0, v, c).
\]
Moreover the elements
\[
(a_k, y_k, r_k(v, c)), \quad (a_0, y_0, r_0(v, c))
\]
will be in \(\mathcal{D}\). By (6.1) and (6.5) we have
\[
\lim_{k \to \infty} \left\{ E_F [a_k, y_k, r_k(v, c), q_k] - E_F(a_k, y_k, p_k, q_k) \right\} \geq 0
\]
and hence, by the definition of \(E_F\),
\[
\lim_{k \to \infty} q_0^i [F_{\mu}(a_k, y_k, p_k) - F_{\mu}(a_0, y_0, r_0(v, c))] \geq 0.
\]
It follows that
\[
q_0^i [F_{\mu}(a_0, y_0, p_0) - F_{\mu}(a_0, y_0, r_0(v, c))] \geq 0.
\]
If \(q_0 = -p_0\) this relation can be written in the form
\[
E_F(a_0, y_0, r_0(v, c), p_0) \leq 0.
\]
For \(c=1, v \neq 0\) we have \(|r_0| = 1, r_0 \neq p_0\) and (6.2) holds with \((a, y, p, q) = (a_0, y_0, r_0(v, 0), p_0)\), contrary to (6.8). Hence \(q_0 \neq -p_0\). We now select
\[
c = |p_0 + e q_0|, \quad \phi^\gamma(a_0, y_0, p_0 + e q_0) = \phi^\gamma(a_0, y_0, p_0)
\]
where \(\phi^\gamma\) are the functions defined in Lemma 2.1. Let \(r(e)\) be the corresponding values of \(r^i(v, c)\). Then \(r(0) = p_0\) and by Lemma 2.1
\[
|r(e)| = |p_0 + e q_0|, \quad \phi^\beta(a_0, y_0, r(e)) = 0, \quad \phi^\gamma(a_0, y_0, r(e)) = \phi^\gamma(a_0, y_0, p_0 + e q_0).
\]
Differentiating these relations with respect to \(e\) and setting \(e = 0\) we find that the derivative \(r^i_\varepsilon\) of \(r^i\) satisfies the relation
\[
r^i_\varepsilon \phi_{\mu}(a_0, y_0, p_0) = 0, \quad (r^i_\varepsilon - q_0^i) \phi^\gamma(a_0, y_0, p_0) = 0.
\]
Hence by (6.6) we have
Consider now the function

\[ Q(e) = q_0^{i} [F_{pl}(a_0, y_0, p_0) - F_{pl}(a_0, y_0, r(e))] \]

We have \( Q(0) = 0 \) since \( r(0) = p_0 \) and \( Q(e) \geq 0 \) by (6.7). It follows that \( Q'(0) = 0 \). Since \( r_{*}'(0) = q_0^{i} \), this gives

\[ 0 = Q'(0) = - q_0^{i} \int_{0}^{r} F_{pl}(a_0, y_0, p_0). \]

Since \( F \) weakly \( E \)-dominates \( L \) near \( C_0 \) on \( \mathcal{D} \) and \( q_0 \neq \pm p_0 \) we have by Theorem 3.1 and (6.6)

\[ q_0^{i} \int_{0}^{r} F_{pl}(a_0, y_0, p_0) > 0. \]

This contradiction completes the proof of Theorem 4.3.

7. Further theorems on \( E \)-dominance. The following theorem is of interest.

Theorem 7.1. Let \( r^{i}(a, y) \) be a set of \( n+1 \) continuous functions satisfying the equations

\[ \phi^{i}[a, y, r(a, y)] = 0 \]

and having the set \([a, y, r(a, y)]\) on \( C_0 \) whenever \((a, y)\) is on \( C_0 \). The function \( L = |p| \) is \( E \)-dominated by \( F \) near \( C_0 \) on \( \mathcal{D} \) if and only if there is a neighborhood \( \mathcal{F} \) of the points \((a, y)\) on \( C_0 \) and a constant \( b > 0 \) such that the inequality

\[ E_L(r(a, y), q) \leq b E_F(a, y, r(a, y), q) \]

holds for every set \((a, y, q)\) in \( \mathcal{D} \) with \((a, y)\) in \( \mathcal{F} \).

In order to establish this result suppose first that \( \mathcal{F} \) and \( b \) can be chosen so that the inequality (7.2) holds whenever \((a, y)\) is in \( \mathcal{F} \) and \((a, y, q)\) is in \( \mathcal{D} \). By an argument like that used in the proof of Theorem 3.1 it can be seen that the inequality (7.2) implies condition I of Theorem 3.1 and hence that \( F \) weakly \( E \)-dominates \( L \) near \( C_0 \) on \( \mathcal{D} \). Hence there is a neighborhood \( \mathcal{D}_1 \) of \( C_0 \) relative to \( \mathcal{D} \) and a constant \( b_1 > 0 \) such that the inequality

\[ E_L(p, q) \leq b_1 E_F(a, y, p, q) \]

holds whenever \((a, y, p)\) and \((a, y, q)\) are in \( \mathcal{D}_1 \).

As a next step in the proof of Theorem 7.1 we write \( E_F(a, y, p, q) \) in the form

\[ E_F(a, y, p, q) = E_F(a, y, r(a, y), q) + Q(a, y, p, q) \]

where

\[ Q = q^{i} [F_{pl}(a, y, r(a, y)) - F_{pl}(a, y, p)]. \]
Suppose now that $L$ is not $\mathcal{F}$-dominated by $F$ near $C_0$ on $\mathcal{D}$. Then there exist a sequence of pairs of elements $(a_k, y_k, p_k), (a_k, y_k, q_k)$ in $\mathcal{D}$ converging to a pair $(a_0, y_0, p_0), (a_0, y_0, q_0)$ such that $(a_0, y_0, p_0)$ is on $C_0$ and

$$
|p_k| = |q_k| = |r(a_k, y_k)|,
$$

$$
E_F(a_k, y_k, p_k, q_k) < (1/k)E_L(p_k, q_k).
$$

By (7.6) it follows that $p_0 = r(a_0, y_0)$ and hence by (7.5) that

$$
\lim_{k \to \infty} Q(a_k, y_k, p_k, q_k) = 0.
$$

Combining this result with (7.4) and (7.7) we obtain the inequality

$$
\limsup_{k \to \infty} E_F(a_k, y_k, r(a_k, y_k), q_k) \leq 0.
$$

As a consequence of this relation we have, by (7.2), $E_L(r(a_0, y_0), q_0) = 0$ and hence also $q_0 = r(a_0, y_0) = p_0$. For large values of $k$ the elements $(a_k, y_k, p_k)$ and $(a_k, y_k, q_k)$ accordingly will be in $\mathcal{D}_1$ and will satisfy the relations (7.3) and (7.7). This is impossible. The criterion stated in the theorem implies that $L$ is $\mathcal{E}$-dominated by $F$ near $C_0$ on $\mathcal{D}$. The converse is immediate and the theorem is established.

**Theorem 7.2.** Suppose that $F$ weakly $\mathcal{E}$-dominates $L$ near $C_0$ on $\mathcal{R}$. Then $L$ is $\mathcal{E}$-dominated by $F$ near $C_0$ on $\mathcal{D}$ if and only if there is a neighborhood $\mathcal{R}_1$ of $C_0$ and a constant $b > 0$ such that the inequality

$$
E_L(p, q) \leq bE_F(a, y, p, q)
$$

holds whenever $(a, y, p)$ is in $\mathcal{R}_1$ and $(a, y, q)$ is in $\mathcal{D}$.

In order to prove this result let $r^i(a, y, p)$ be a set of continuous functions defined on a neighborhood $\mathcal{R}_0$ of $C_0$ such that the set $(a, y, r(a, y, p))$ is in $\mathcal{D}$ and coincides with $(a, y, p)$ when $(a, y, p)$ is on $C_0$. The existence of such functions is established by the use of Lemma 2.1. As before we write

$$
E_F(a, y, p, q) = E_F(a, y, r(a, y, p), q) + Q(a, y, p, q)
$$

where $Q(a, y, p, q) = q^i [F_p^i(a, y, r(a, y, p)) - F_p^i(a, y, p)]$.

Suppose now that $L$ is $\mathcal{E}$-dominated by $F$ near $C_0$ on $\mathcal{D}$ and that the inequality (7.8) failed to hold as stated. Then there would exist a sequence of pairs of elements $(a_k, y_k, p_k), (a_k, y_k, q_k)$ converging to elements $(a_0, y_0, p_0), (a_0, y_0, q_0)$ such that $(a_k, y_k, q_k)$ is in $\mathcal{D},$

$$
|p_k| = |q_k| = 1, \quad \lim_{k \to \infty} r(a_k, y_k, p_k) = p_0,
$$

$$
E_F(a_k, y_k, p_k, q_k) < (1/k)E_L(p_k, q_k) \leq 2/k.
$$

Using (7.9) it follows that...
Since $L$ is $E$-dominated by $F$ near $C_0$ on $\mathcal{D}$ and the element $(a_k, y_k, r(a_k, y_k, p_k))$ is in $\mathcal{D}$ this relation can hold only in case
\[
\lim_{k \to \infty} E_L[r(a_k, y_k, p_k), q_k] = E_L(p_0, q_0) = 0,
\]
that is only in case $q_0 = p_0$. For large values of $k$ the elements $(a_k, y_k, p_k)$ and $(a_k, y_k, q_k)$ are in an arbitrarily small neighborhood $\mathcal{R}_1$ of $C_0$ and satisfy the relations (7.10). But this is impossible when $F$ weakly $E$-dominates $L$ near $C_0$ on $\mathcal{R}$. The inequality (7.8) therefore holds as stated. The converse is immediate and the theorem is established.

8. A consequence of $E$-dominance. It was seen in Theorem 3.1 that an admissible function $F$ weakly $E$-dominates $L$ near $C_0$ on $\mathcal{D}$ if and only if it can be modified on the set $\mathcal{R} - \mathcal{D}$ so that it weakly $E$-dominates $L$ near $C_0$ on $\mathcal{R}$. This result is also valid for $E$-dominance, as can be seen by the use of the following theorem.

**Theorem 8.1.** Let $F$ be an admissible function that $E$-dominates $L$ near $C_0$ on $\mathcal{D}$. There exists an admissible function $F^*$ of the form
\[
F^* = F + \theta(a, y, p)\phi^k\phi^k
\]
which $E$-dominates $L$ near $C_0$ on $\mathcal{R}$. The function $\theta(a, y, p)$ satisfies the homogeneity condition
\[
\theta(a, y, kp) = k^{-1}\theta(a, y, p) \quad (k > 0)
\]
on $\mathcal{R}$ and can be chosen to be of class $C^\infty$.

In order to establish this result we can suppose, by Theorem 3.1, that $F$ has been modified so that $F$ weakly $E$-dominates $L$ near $C_0$ on $\mathcal{R}$. By virtue of Theorem 7.2 we can select a neighborhood $\mathcal{R}_1$ of $C_0$ and a constant $c > 0$ such that the inequality
\[
E_F(a, y, p, q) \geq cE_L(p, q)
\]
holds whenever $(a, y, p)$ is in $\mathcal{R}_1$ and $(a, y, q)$ is in $\mathcal{D}$ or in $\mathcal{R}_1$. Select admissible regions $\mathcal{R}_2, \mathcal{R}_3, \ldots$ such that the closure of $\mathcal{R}_j$ is in $\mathcal{R}_{j+1}$ and such that
\[
\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \cdots.
\]

Let $\mathcal{R}_0$ be a neighborhood of $C_0$ whose closure is in $\mathcal{R}_1$. By virtue of (8.3) there is a constant $c' > 0$ such that the inequality
\[
E_F(a, y, p, q) > 2c'
\]
holds for normed sets $(a, y, p)$ and $(a, y, q)$ having $(a, y, p)$ in $\mathcal{R}_0$ and $(a, y, q)$ in $\mathcal{D}$ but not in $\mathcal{R}_1$. By continuity there is for each integer $j \geq 1$ a constant
\( \delta_i > 0 \) such that (8.4) holds subject to the conditions
\[
(8.5) \quad (a, y, p) \text{ in } \mathcal{R}_0, \quad (a, y, q) \text{ in } \mathcal{R}_{j+1} - \mathcal{R}_j, \quad |p| = |q| = 1,
\]
\[
(8.6) \quad \phi^\alpha(a, y, q)\phi^\beta(a, y, q) \leq \delta_j.
\]
Choose a constant \( b_j \) so that the inequality
\[
(8.7) \quad \mathcal{E}_\mathcal{R}(a, y, p, q) \geq b_j
\]
holds whenever (8.5) is satisfied and let \( c_i \) be a positive constant such that
\[
(8.8) \quad b_j + c_j \delta_j > 2c' \quad (j \text{ not summed}).
\]
Let \( \theta_i(a, y, p) \) be a function of normed sets \( (a, y, p) \) of class \( c^\infty \) such that \( \theta_i = 0 \) on \( \mathcal{R}_{j-1} \), \( \theta_i \geq 0 \) on \( \mathcal{R}_j \), \( \theta_i = 1 \) exterior to \( \mathcal{R}_j \). For an arbitrary set \( (a, y, p) \) on \( \mathcal{R} \) we define \( \theta_i \) by the formula
\[
\theta_i(a, y, p) = L^{-1}\theta_i(a, y, p/L) \quad (L = |p|).
\]
The function \( \theta \) defined by the sum
\[
(8.9) \quad \theta(a, y, p) = \sum_{i} \theta_i(a, y, p)
\]
can be shown to have the properties described in Theorem 8.1. It is well defined on \( \mathcal{R} \) since at most \( j+1 \) of the terms in (8.9) are different from zero on \( \mathcal{R}_j \). The relation (8.2) holds. Moreover
\[
(8.10) \quad \theta \geq c_j \text{ on } \mathcal{R}_{j+1} - \mathcal{R}_j \quad (j \geq 1).
\]
Since \( \theta = 0 \) on \( \mathcal{R}_0 \) the \( E \)-function for the function \( F^* \) given by (8.1) is expressible in the form
\[
(8.11) \quad \mathcal{E}_\mathcal{R}^*(a, y, p, q) = \mathcal{E}_\mathcal{R}(a, y, p, q) + \theta(a, y, q)\phi^\alpha(a, y, q)\phi^\beta(a, y, q)
\]
whenever \( (a, y, p) \) is in \( \mathcal{R}_0 \). Consider now normed sets \( (a, y, p) \) and \( (a, y, q) \) in \( \mathcal{R} \) with \( (a, y, p) \) in \( \mathcal{R}_0 \). If \( (a, y, q) \) is in \( \mathcal{R}_1 \), then (8.3) holds with \( \mathcal{F} \) replaced by \( \mathcal{F}^* \). Suppose therefore that \( (a, y, q) \) is in \( \mathcal{R}_{j+1} - \mathcal{R}_j \) \( (j \geq 1) \). If the inequality (8.6) holds, then by (8.4) and (8.11) we have
\[
(8.12) \quad \mathcal{E}_\mathcal{R}^*(a, y, p, q) > 2c' \geq c'\mathcal{E}_\mathcal{R}(p, q).
\]
If (8.6) fails to hold, then by (8.7), (8.8), (8.10), (8.11) it is seen that (8.12) still holds. Hence \( F \) \( E \)-dominates \( L \) near \( C_0 \) on \( \mathcal{R} \), as was to be proved.

Combining this result with Theorem 4.3 we have the following corollary.

**COROLLARY.** If \( F \) is nonsingular on \( C_0 \) relative to \( \phi^1, \cdots, \phi^m \) and \( E \)-dominates \( \phi^1, \cdots, \phi^m \) near \( C_0 \) on \( \mathcal{D} \), then there exists a function \( F^* \) of the form (8.1) that \( E \)-dominates \( L \) near \( C_0 \) on \( \mathcal{R} \).

If in the proof just given we set \( c' = 0 \), one obtains the following:

**THEOREM 8.2.** Let \( F \) be an admissible function such that at each element
(a, y, p) in a neighborhood $R_1$ of those on $C_0$ the inequality

$$E_F(a, y, p, q) > 0$$

holds whenever $(a, y, q)$ is in $D$ or in $R_1$ and $(q) \not=(kp)$ $(k > 0)$. There exists a function $F^*$ of the form (8.1) such that at each element $(a, y, p)$ in a neighborhood $R_0$ of those on $C_0$ one has

$$E_{F^*}(a, y, p, q) > 0$$

whenever $(a, y, q)$ is in $R$ and $(q) \not=(kp)$ $(k > 0)$.

9. The strengthened condition of Weierstrass. We now return to the study of the problem described in the introduction. The functions $f(a, y, p)$, $f^*(a, y, p)$, $\phi^\beta(a, y, p)$ are admissible functions of class $C''$. We form the function

$$(9.1) \quad F(a, y, p, \lambda, \mu) = \lambda_0 f + \lambda^s f^s + \mu^\beta \phi^\beta.$$ 

The arc $C_0$

$$a_0, y_0(t) \quad (t_1 \leq t \leq t_2)$$

will be said to satisfy the strengthened condition $II_N$ of Weierstrass with a set of multipliers

$$(9.2) \quad \lambda_0, \lambda^s, \mu^\beta(t) \quad (\sigma = 1, \cdots, s; \beta = 1, \cdots, m)$$

if the following conditions hold: The multipliers $\lambda_0$, $\lambda^s$ are constants and $\lambda_0 \geq 0$; the multipliers $\mu^\beta(t)$ are continuous functions of $t$ on $t_1 t_2$; at each element $(a, y, p, \lambda, \mu)$ in a neighborhood $N$ of those on $C_0$ ($\lambda_0$ being held fast) having $(a, y, p)$ in $D$ the inequality

$$(9.3) \quad E(a, y, p, \lambda, \mu, q) \geq 0$$

holds whenever $(a, y, q)$ is in $D$. Here

$$E = F(a, y, p, \lambda, \mu, q) - q^i F_{p^i}(a, y, p, \lambda, \mu).$$

The arc $C_0$ together with the set of multipliers (9.2) will be said to be nonsingular if the determinant

$$\begin{vmatrix} F_{p^i} & \phi^\beta \\ \phi^\gamma & 0 \end{vmatrix} \quad (i, j = 0, 1, \cdots, n) \quad (\beta, \gamma = 1, \cdots, m)$$

has rank $m+n$ on $C_0$.

Theorem 9.1. Let $l^0$, $l^s$, $m^\beta(a, y)$ be a set of continuous functions such that on $C_0$ we have

$$(9.4) \quad l^0 = \lambda_0, \quad l^s = \lambda^s, \quad m^\beta[a_0, y_0(t)] = \mu^\beta(t)$$
where the multipliers on the right are given by the set (9.2). The arc $C_0$ together with the multipliers (9.2) is nonsingular and satisfies the condition $\Pi_N$ of Weierstrass if and only if the admissible function
\begin{equation}
F[a, y, \dot{p}, l, m(a, y)]
\end{equation}
$E$-dominates the functions $L = |\dot{p}|$ and $f^\sigma$ ($\sigma = 1, \cdots, s$) near $C_0$ on $\mathcal{D}$.

In order to prove this result let $F^*(a, y, \dot{p})$ be the function (9.5). We have the identity
\begin{equation}
E(a, y, \dot{p}, \lambda, \mu, q) = E_{f^*}(a, y, \dot{p}, q) + (\lambda^\sigma - l^\sigma)E_{r^0}(a, y, \dot{p}, q) + (\mu^\beta - m^\beta)E_{\phi^\beta}(a, y, \dot{p}, q).
\end{equation}
Suppose now that $C_0$ together with the multipliers (9.2) is nonsingular and satisfies the condition $\Pi_N$ of Weierstrass. Then $F^*$ is obviously nonsingular on $C_0$ relative to $\phi^1, \cdots, \phi^m$. Moreover by virtue of condition $\Pi_N$ there is a neighborhood $\mathcal{D}_0$ of $C_0$ relative to $\mathcal{D}$ and a constant $b > 0$ such that the left member of (9.6) is positive whenever $(a, y, \dot{p})$ is in $\mathcal{D}_0$, $(a, y, q)$ is in $\mathcal{D}$ and
\begin{align*}
|\lambda^\sigma - l^\sigma(a, y)| &\leq b, \\
|\mu^\beta - m^\beta(a, y)| &\leq b.
\end{align*}
It follows from (9.6) that $E_{r^*} \geq b|E_{r^0}|$, $E_{r^*} \geq b|E_{\phi^\beta}|$ in this event. Consequently $f^*$ and $\phi^\beta$ are $E$-dominated by $F^*$ near $C_0$ on $\mathcal{D}$. By Theorem 4.3 the function $L = |\dot{p}|$ is also $E$-dominated by $F^*$ near $C_0$ on $\mathcal{D}$.

Suppose conversely that $L$ and $f^*$ are $E$-dominated by $F^*$ near $C_0$ on $\mathcal{D}$. Then, by Theorem 4.3, $F^*$ is nonsingular on $C_0$ relative to $\phi^1, \cdots, \phi^m$ and consequently $C_0$ and the set (9.2) are nonsingular. Moreover by the same theorem the functions $\phi^\beta$ are $E$-dominated by $F^*$ near $C_0$ on $\mathcal{D}$. By the use of the identity (9.6) it is seen that the condition $\Pi_N$ holds for $C_0$ and the multipliers (9.2). This completes the proof of the theorem.

**Theorem 9.2.** Let $\phi^\beta(a, y)$ be a set of continuous functions such that the set $[a, y, \phi(a, y)]$ is on $C_0$ when $(a, y)$ is on $C_0$ and such that
$$\phi^\beta[a, y, \phi(a, y)] = 0 \quad (\beta = 1, \cdots, m)$$
on a neighborhood of $C_0$. Let $l^\sigma, l, m^\beta(a, y)$ be a set of continuous multipliers satisfying the conditions (9.4) on $C_0$. Then $C_0$ and the set of multipliers (9.2) are nonsingular and satisfy the condition $\Pi_N$ if and only if there is a neighborhood $\mathcal{S}$ of $C_0$ in ay-space and a constant $b > 0$ such that the inequality
$$E_L[\phi(a, y), q] \leq bE[a, y, \phi(a, y), l, m(a, y), q]$$holds for every element $(a, y, q)$ in $\mathcal{D}$ with $(a, y)$ in $\mathcal{S}$.

This result is obtained by combining Theorems 9.1 and 7.1.

**Theorem 9.3.** Let $F(a, y, \dot{p})$ be a function of the form

\[ F(a, y, p) = l^0 f + l^* f^* + m^\theta(a, y) \phi^\theta \]

where \( l^0 \geq 0 \), \( l^* \) are constants, and \( m^\theta(a, y) \) are continuous near \( C_0 \). Let \( H(a, y, p) \) be a second admissible function of the form

\[ (a, y, p) = l^0 f + \lambda^\varepsilon(a, y) f^* + \mu^\theta(a, y) \phi^\theta \]

where \( \lambda^\varepsilon(a, y) \) and \( \mu^\theta(a, y) \) are continuous near \( C_0 \). If \( L = |p| \) and \( f^* \) are E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \), then the function \( H \) is E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \).

This result follows readily if we write \( H \) in the form

\[ H = F + F^* \]

where

\[ F^* = (\lambda^\varepsilon - l^*) f^* + (\mu^\theta - m^\theta) \phi^\theta. \]

Suppose now that \( L, f^* \) are E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \). Then by Theorem 4.3 the functions \( \phi^\theta \) are E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \). Consequently \( F^* \) is also E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \). The same is true for \( F \) and hence also for \( H \), as was to be proved.

**Corollary 1.** If a function \( F \) of the form (9.7) with \( l^0 > 0 \) E-dominates \( L \) and \( f^* \) near \( C_0 \) on \( \mathbb{D} \), then \( f \) is E-dominated by \( F \) near \( C_0 \) on \( \mathbb{D} \).

**Corollary 2.** Let \( l^0 \geq 0, l^*, m^\theta(a, y) \) be a set of continuous multipliers such that the function \( F(a, y, p) \) defined by (9.7) E-dominates \( L = |p| \) and \( f^* \) near \( C_0 \) on \( \mathbb{D} \). If \( l^0, l^*, \bar{m}^\theta(a, y) \) is a second set of continuous multipliers such that

\[ l^0 = l^0, \quad l^* = l^*, \quad \bar{m}^\theta(a, y) = m^\theta(a, y) \]

on \( C_0 \), then

\[ \bar{F}(a, y, p) = l^0 f + \bar{f} f^* + \bar{m}^\theta(a, y) \phi^\theta \]

also E-dominates \( L \) near \( C_0 \) on \( \mathbb{D} \). Moreover an admissible function \( H(a, y, p) \) is E-dominated by \( \bar{F}(a, y, p) \) near \( C_0 \) on \( \mathbb{D} \) if and only if it is E-dominated by \( F(a, y, p) \) near \( C_0 \) on \( \mathbb{D} \).

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