ON THE EXTENSION OF INTERVAL FUNCTIONS

BY

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Introduction. The problem of extending the range of definition of a function defined on a class of elementary figures—intervals, rectangles—has been treated in various ways in the literature. In the theory of Lebesgue measure a particular function—length of interval (area of rectangle)—is extended in a completely additive way to an additive class of sets. In the general extension problem we start, say, with a function (real, single-valued, and finite) of intervals \( \phi(I) \) and extend the range of definition to an additive class of sets obtaining a function \( \Phi(E) \) which is completely additive and which has the property that \( \Phi(E) = \phi(I) \) whenever "E is the set I." But what is the interval I? A priori \( \phi(I) \) is defined on a class of intervals \( I \), where \( I \) is considered neither open nor closed but merely as an interval. From the viewpoint of \( \Phi(E) \) an interval \( I \) must be considered as a definite point set—a closed interval, an open interval, a semi-open interval, and so on. Corresponding to open intervals and to closed intervals, \( \Phi(E) \) gives rise to two interval functions: \( \phi_1(I) = \Phi(I') \), \( \phi_2(I) = \Phi(I^o) \) where \( I' \) is understood to be closed and \( I^o \) open. If \( \phi(I) = \phi_2(I) \) identically, then \( \Phi(E) \) is an extension of \( \phi(I) \) considered as a function of closed intervals; if \( \phi(I) = \phi_2(I) \) identically, then \( \Phi(E) \) is an extension of \( \phi(I) \) considered as a function of open intervals. As a starting point in the general extension problem, the function \( \phi(I) \) has been considered, somewhat artificially and arbitrarily perhaps, a function either of closed intervals or of open intervals (see, for example, [10])(\(^1\)). Extensions \( \Phi(E) \) which have the property that \( \Phi(I') = \Phi(I^o) \) identically are of particular interest since then \( \Phi(E) \) is an extension of \( \phi(I) \) whether \( I \) be considered open or closed.

The main results of the paper concern the existence of \( B \)-extensions, a precise definition of which is given in \( \S \)1.6. Suffice it to say here that if \( \Phi(E) \) is a \( B \)-extension of \( \phi(I) \) then \( \Phi(I') = \Phi(I^o) = \phi(I) \). The idea of a \( B \)-extension was suggested by a result of Burkill [2] which we shall review in \( \S \)1.5. Burkill's theorem on extension is stated in terms of a sufficient condition while our results on \( B \)-extensions are stated in terms of necessary and sufficient conditions.

In Part 1 we explain notation, define terms, and summarize results. In Part 2 we present a proof of a theorem (Theorem 1) which states a necessary and sufficient condition that a non-negative function of closed intervals

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\(^1\) Numbers in brackets indicate references in the bibliography at the end of the paper.
admit a non-negative completely additive extension. This theorem was proved in [10] using the results of Radon [9]. The present proof makes use of the theory of outer measure in the sense of Carathéodory. Theorem 2 extends the result of Theorem 1 to the case of a function of arbitrary sign. Theorem 3 concerns the uniqueness and characterization of an extension of a function of intervals. Part 3 contains proofs of our results (Theorems 4, 5, and 6) on the $B$-extension. We present necessary and sufficient conditions that a non-negative function of intervals, a function of intervals of arbitrary sign, and an indefinite integral of a function of intervals, respectively, admit $B$-extensions.

1. Preliminaries and summary.

1.1. In modern literature intervals are considered as $k$-dimensional where $k$ is a positive integer. We shall consider functions of intervals in the $xy$-plane, that is, our intervals are two-dimensional. The word interval is used in the sequel only in the accepted point set sense: Given two points $(x_1, y_1)$, $(x_2, y_2)$ an interval is the set of points $(x, y)$ such that $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$; an open interval is the set of points $(x, y)$ such that $x_1 < x < x_2$, $y_1 < y < y_2$ (Saks [12, p. 57]). Let $R_0$ be the interval $0 \leq x \leq 1$, $0 \leq y \leq 1$. All sets considered in this paper are subsets of $R_0$ unless otherwise stated. We use the letters $I$, $J$, $R$ to denote intervals. The symbol $I^0$ denotes the open interval which corresponds to $I$, that is, $I^0$ is the interior of the set $I$. The letter $C$ is used to denote a class of intervals. A capital script letter, as $\mathcal{E}$, is used to denote an elementary system of intervals, that is, a finite set of intervals $I_1, I_2, \ldots, I_k$ such that $I_i^0 \cap I_j^0 = \emptyset$ whenever $i \neq j$. A capital script letter is also used as an operator in the sense that $\mathcal{E}I$ denotes an elementary system of intervals $I_1, I_2, \ldots, I_k$ which constitutes a subdivision of $I$, that is, $\sum_{i=1}^k I_i = I$. The parameter of regularity of an interval $I$, denoted by $p(I)$, is the ratio of the length of the shorter side of $I$ to the length of the longer side of $I$. The norm of an interval $I$, denoted by $||I||$, is the length of the diameter of $I$; the symbol $||\mathcal{E}||$ is defined as the largest of the numbers $||I||$ where $I \in \mathcal{E}$. The measure of an interval $I$ is denoted by $|I|$; the symbol $|\mathcal{E}|$ is defined as the number $\sum |I|$ where the sum is taken over $I \in \mathcal{E}$. The boundary of an interval $I$, denoted by $b(I)$, is the set $I - I^0$.

1.2. A class of sets in $R_0$ is said to be closed (relative to $R_0$)—and is generically denoted by $K$—if the following conditions are satisfied:

(i) Every set open relative to $R_0$ (denoted generically by $O$) is in $K$.

(ii) If a set $E$ is in $K$, then the complement of $E$ relative to $R_0$ (denoted by $CE$) is also in $K$.

(iii) If $\{E_n\}$ is a sequence of sets in $K$, then $\sum_{n=1}^\infty E_n$ is also a set in $K$.

Clearly every closed class $K$ contains all Borel sets $E \subset R_0$. In fact, the class of all Borel sets in $R_0$, which we denote by $B$, is identical with the product of all closed classes in $R_0$.

Let $\lambda$ be a fixed number satisfying the relation $0 \leq \lambda < 1$, and let $C_\lambda$ denote the class of all intervals $I$ such that $p(I) \geq \lambda$. A subscript $\lambda$ as in $I_\lambda$ and $\mathcal{E}_\lambda$
indicates that \( I_\lambda \subseteq C_\lambda \) and that \( I \subseteq C_\lambda \) for every \( I \subseteq E_\lambda \). The symbol \( C_0 \) will denote the class of all intervals \( I \) such that \( I \subseteq R_0 \). Let \( \phi(I) \) denote a real, finite, single-valued function which is defined for every \( I \subseteq C_\lambda \). This function is denoted briefly by the symbol \([\phi, C_\lambda]\) in which the first letter denotes the function and the second letter denotes the range of definition of the function.

A function \( \Phi(E) \) which is defined on a closed class \( K \), that is, the function \([\Phi, K]\), is a completely additive extension of the function \([\phi, C_\lambda]\) if the following conditions are satisfied:

(i) \([\Phi, K]\) is a completely additive set function.

(ii) \( \Phi(I) = \phi(I) \) for every \( I \subseteq C_\lambda \).

In Part 2 we shall prove the following theorems:

**Theorem 1.** (See [10, Theorem 3].) A necessary and sufficient condition that a non-negative function of intervals \([\phi, C_\lambda]\) have a non-negative completely additive extension is that it satisfy the following condition \( \mathcal{C} \): If \( \{ I_n \} \) is any sequence of intervals in \( C_\lambda \) such that \( I_i \cdot I_j = 0 \) when \( i \neq j \) and if \( \{ J_m \} \) is any sequence of intervals in \( C_\lambda \) such that \( \sum_m J_m \supset \sum_n I_n \) then \( \sum_m \phi(J_m) \geq \sum_n \phi(I_n) \).

**Theorem 2.** A necessary and sufficient condition that a function of intervals \([\phi, C_\lambda]\), of arbitrary sign, have a completely additive extension is that it be the difference of two non-negative functions each of which satisfies condition \( \mathcal{C} \).

**Theorem 3.** If \([\Phi, K]\) is a completely additive extension of a function of intervals \([\phi, C_\lambda]\), then the value of the number \( \Phi(E) \), where \( E \) is any Borel set, is uniquely determined by the function \([\phi, C_\lambda]\). If \([\phi, C_\lambda]\) is non-negative, then \([\Phi, K]\) is also non-negative and for every Borel set \( E \) we have the characterization: \( \Phi(E) = \text{g.l.b.} \sum_n \phi(I_n) \) for all sequences \( \{ I_n \} \) such that \( \sum_n I_n \supset E \) and \( I_n \subseteq C_\lambda \), \( n = 1, 2, \ldots \).

1.3. Given a function of intervals \([\phi, C_\lambda]\), we extend the range of definition of \([\phi, C_\lambda]\) to include all elementary systems of intervals \( E_\lambda \) as follows: \( \phi(E_\lambda) = \sum \phi(I) \) where the sum is taken over \( I \subseteq E_\lambda \). The function \([\phi, C_\lambda]\) is additive if for every \( I_\lambda \) and every \( E_\lambda \subseteq I_\lambda \) it is true that \( \phi(E_\lambda I_\lambda) = \phi(I_\lambda) \). If we replace \( \phi(E_\lambda I_\lambda) = \phi(I_\lambda) \) by \( \phi(E_\lambda I_\lambda) \geq \phi(I_\lambda) \) and then by \( \phi(E_\lambda I_\lambda) \leq \phi(I_\lambda) \) we obtain the definitions of a function which increases by subdivision and decreases by subdivision respectively. The function \([\phi, C_\lambda]\) is continuous if for every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that (i) \( |I_\lambda| < \delta \) implies \( |\phi(I_\lambda)| < \epsilon \) and (ii) \( I_{\lambda_1} \subseteq I_{\lambda_2}, \ |I_{\lambda_2} - I_{\lambda_1}| < \delta \) imply \( |\phi(I_{\lambda_1}) - \phi(I_{\lambda_2})| < \epsilon \). It is observed that condition (ii) in this definition is a consequence of condition (i) if the function is additive and \( \lambda = 0 \). The function \([\phi, C_\lambda]\) is absolutely continuous if for every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that \( |E_\lambda| < \delta \) implies \( |\phi(E_\lambda)| < \epsilon \).

1.4. Given a function of intervals \([\phi, C_\lambda]\), we define for every interval \( I \) the following two numbers:
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(i) \( L(\phi, I) = \lim \inf \phi(E_\lambda I) \) as \( \|E_\lambda I\| \to 0 \),
(ii) \( U(\phi, I) = \lim \sup \phi(E_\lambda I) \) as \( \|E_\lambda I\| \to 0 \),

and call these numbers, which are finite or infinite, the lower and upper integrals of \([\phi, C_\lambda]\) over the interval \( I \) respectively. Given any interval \( I \), any number \( \lambda \) such that \( 0 \leq \lambda < 1 \), and any number \( \delta > 0 \), it is easily shown that there exists an \( E_\lambda I \) such that \( \|E_\lambda I\| < \delta \). Consequently the lower and upper integrals are defined for every \( I \in C_0 \). In our bracket notation these functions (not necessarily finite) may be denoted by \([L(\phi), C_0]\) and \([U(\phi), C_0]\) respectively. In case \( L(\phi, I) = U(\phi, I) \) is a finite number, we denote the common value by \( F(\phi, I) \) and call it the integral of \( \phi \) over \( I \). Defining the integral in this manner, that is, for a function of intervals defined on a class \( C_\lambda \), makes it sufficiently flexible to include the integral in the extended sense of Burkill, the strong integral of Saks, and the regular integral of Kempisty, by suitably choosing \( \lambda(\cdot) \). If \([\phi, C_\lambda]\) is integrable over \( R_0 \) then it is integrable over every \( I \in C_0 \). The indefinite integral, which we may denote by \([F(\phi), C_0]\), is an additive function of intervals.

A function of intervals \([\phi, C_0]\) is said to be absolutely continuous in the restricted sense, briefly RAC, if the function \([U(\phi), C_0]\) is continuous.

1.5. The following theorem was stated and proved by Burkill [2, p. 289] for an integral which, under the stated assumptions, reduces to the integral as defined in 1.4 if \( \lambda = 0 \).

**Theorem.** If a function of intervals \([\phi, C_\lambda]\) is absolutely continuous and integrable, if \( E \) is a measurable set, and if \( \epsilon_n, n = 1, 2, \ldots \), is any decreasing sequence of positive numbers approaching 0, and corresponding to each \( n \), \( E \) is decomposed into \( E_n + e_n' - e_n'' \) where \( e_n' \) and \( e_n'' \) are measurable sets such that \( |e_n'| \) and \( |e_n''| \) are each less than \( \epsilon_n \), and \( E_n \) is an elementary system of intervals, then as \( n \to \infty \), \( F(\phi, E_n) \) approaches a limit which we call \( F(E) \) and which is independent of the particular decomposition of \( E \) for any \( n \).

Burkill showed that the function \( F(E) \), which is defined for all measurable subsets of \( R_0 \), is an absolutely continuous, completely additive function of measurable sets, which for intervals reduces to the integral. This is a strong type of extension in the sense that if any interval \( I \) is given, and if \( E \) is any set satisfying the relation \( I^0 \subset E \subset I \), then \( F(E) = F(\phi, I) \). This property is a direct implication of the absolute continuity and additivity of the function \([F(\phi), C_0]\).

1.6. Burkill’s result suggested the following type of extension. A completely additive set function \([\Phi, B]\) defined for all Borel sets in \( R_0 \) (briefly, an additive function of Borel sets) is a \( B \)-extension of a function \([\phi, C_\lambda]\) if \( \Phi(E) = \phi(I) \) for every \( I \in C_\lambda \) and for every Borel set \( E \) such that \( I^0 \subset E \subset I \). In Part 3 we shall establish the following theorems:

\( (*) \) For Burkill's definition, see [2, p. 279]; for Saks's definition, see [11, p. 212]; for Kempisty's definition, see [6, p. 212].
Theorem 4. A necessary and sufficient condition that a non-negative function of intervals \([\phi, C_\lambda]\) admit a non-negative B-extension is that it be an additive, continuous function.

Theorem 5. A necessary and sufficient condition that a function of intervals \([\phi, C_\lambda]\) admit a B-extension is that it be additive and RAC.

Theorem 6. A necessary and sufficient condition that the indefinite integral of an integrable function of intervals \([\phi, C_\lambda]\) admit a B-extension is that \([\phi, C_\lambda]\) be RAC.

2. Completely additive extensions of functions of intervals.

2.1. The necessity of condition C in Theorem 1 is an immediate consequence of the following property of a completely additive set function: If \([\Phi, K]\) is any non-negative completely additive set function, if \([e_n]\) is any sequence of mutually exclusive sets in \(K\), and if \([E_m]\) is any sequence of sets in \(K\) such that \(\sum_m E_m \supset \sum_n e_n\) then \(\sum_m \Phi(E_m) \geq \sum_n \Phi(e_n)\).

2.2. Let \([\phi, C_\lambda]\) be a non-negative function of intervals which satisfies condition (5). For every set \(E \subset R_0\) we define

\[
\phi(E) = \text{g.l.b.} \sum_n \phi(I_{\lambda n})
\]

for all sequences \([I_{\lambda n}]\) such that \(\sum_n I_{\lambda n} \supset E\). Obviously \(\phi(E)\) is a non-negative function. We shall show that it is an outer measure in the sense of Carathéodory (see [12, p. 43]), that is, we shall show that it satisfies the following conditions.

(i) \(\phi(E_1) \leq \phi(E_2)\) whenever \(E_1 \subset E_2\).

(ii) \(\phi(\sum_n E_n) \leq \sum_n \phi(E_n)\) for every sequence \([E_n]\) of sets.

(iii) \(\phi(E_1 + E_2) = \phi(E_1) + \phi(E_2)\) whenever the distance from \(E_1\) to \(E_2\), which we denote by \(d(E_1, E_2)\), is greater than 0. Conditions (i) and (ii) follow directly from the definition of \(\phi(E)\) and from condition C. We proceed to establish condition (iii).

A transversal of \(R_0\), denoted generically by \(t\), is a closed line segment which is parallel to either the \(x\)-axis (a horizontal transversal) or to the \(y\)-axis (a vertical transversal), and which satisfies the following two conditions: (i) the end points of \(t\) lie on the boundary of \(R_0\), (ii) the set which consists of \(t\) less its end points—denoted by \(t^0\)—is contained in \(R_0^0\).

Given any interval \(I \subset C_0\) we say that a subdivision \(E_\lambda I\) is a \(\phi'\)-subdivision of \(I\) if the set of intervals in \(E_\lambda I\) can be segregated into two sets, say \(I_1, I_2, \ldots, I_n\) and \(J_1, J_2, \ldots, J_m\), such that \(I_i \cdot I_j = 0\) for \(i \neq j\) and \(J_i \cdot J_j = 0\) for \(i \neq j\). Given a transversal \(t\), it may be verified that there exist intervals \(I\) such that \(I \supset t, I^0 \supset t^0\), and such that \(I\) has a \(\phi'\)-subdivision. A \(\phi'\)-subdivision of an interval \(I\), where \(I \supset t, I^0 \supset t^0\), is called a \(\phi'\)-covering of \(t\). For every transversal \(t\) we define \(\phi'(t) = \text{g.l.b.} \phi(E_\lambda I)\) for all elementary systems \(E_\lambda I\).
which are \( \phi' \)-coverings of \( t \). It follows from condition \( \mathfrak{C} \) that there are at most a denumerable number of transversals \( t \) for which \( \phi'(t) > 0 \).

Let \( I_0 \) be any interval in \( C_\lambda \); let \( t_1, t_2, \ldots, t_n \) be a finite number of horizontal transversals and \( t_{n+1}, \ldots, t_m \) a finite number of vertical transversals such that \( t_i \cap I_0 \neq \emptyset \) for \( i = 1, \ldots, m \). These transversals determine a subdivision of \( I_0 \) into \((n+1)(m+1)\) intervals, say \( I_1, I_2, \ldots, I_k \). We shall say that this elementary system is a regular subdivision of \( I_0 \) if \( \phi'(t_i) = 0 \) for \( i = 1, 2, \ldots, m \), and \( I_i \subseteq C_\lambda \) for \( i = 1, 2, \ldots, k \). Given a number \( \delta > 0 \) it may be shown that there exists a regular subdivision of \( I_0 \)—say \( I_0^* \)—such that \( \| E_{I_0} \| < \delta \).

Let \( I \) be one of the intervals in a regular subdivision of an interval \( I_0 \subseteq C_\lambda \). Let \( I^* \) denote the set \( I^* = b(I_0) \setminus I \). The set \( I - I^* \) is the sum of \( k \), \( 1 \leq k \leq 4 \), line segments. Let \( t_1, t_2, \ldots, t_k \) be the set of transversals such that \( t_i \cap I^* \neq \emptyset \) for \( i = 1, \ldots, k \), contains one of the line segments in \( I - I^* \). Given \( \varepsilon > 0 \) let \( E_{I, I_i} \) be a \( \phi' \)-covering of \( t_i \) such that \( \phi(E_{I, I_i}) < \varepsilon/4 \). Let \( J \subseteq C_\lambda \) be an interval such that \( I^* \supset J \) and \( I \cap J + \cdots + J_k \). It follows from condition \( \mathfrak{C} \) that \( \phi(I) \leq \phi(J) + \phi(E_{I, I_1}) + \cdots + \phi(E_{I, I_k}) < \phi(J) + \varepsilon \).

Let \( I_1, I_2, \ldots, I_k \) be the intervals in a regular subdivision of \( I_0 \subseteq C_\lambda \). Given \( \varepsilon > 0 \), let \( J_i, i = 1, \ldots, k \), be an interval in \( C_\lambda \) such that \( I_i^* \supset J_i \) and \( \phi(I_i) < \phi(J_i) + \varepsilon/k \). From condition \( \mathfrak{C} \) it follows that

\[
\phi(I_0) \leq \sum_{i=1}^{k} \phi(I_i) < \sum_{i=1}^{k} \phi(J_i) + \varepsilon \leq \phi(I_0) + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary it follows that \( \phi(I_0) = \phi(I_1) + \cdots + \phi(I_k) \). Thus \( [\phi, C_\lambda] \) is additive over regular subdivisions.

Let \( E \) be any set in \( R_\lambda \) and, given \( \varepsilon > 0 \), let \( \{ I_n \} \) be a sequence of intervals in \( C_\lambda \) such that \( \sum I_n \supseteq E \) and \( \phi(E) > \sum \phi(I_n) - \varepsilon \). Given \( \delta > 0 \) let \( E_{I_n} I_n \), \( n = 1, 2, \ldots \), be a regular subdivision of \( I_n \) such that \( \| E_{I_n} I_n \| < \delta \). Arrange the set of intervals \( J \) such that \( J \subseteq E_{I_n} I_n \) for some integer \( n \) into a sequence \( \{ J_m \} \). Then \( \sum J_m \supseteq E; \phi(E) > \sum \phi(J) - \varepsilon \); and \( \| J_m \| < \delta \) for \( m = 1, 2, \ldots \).

Let \( E_1 \) and \( E_2 \) be any two sets such that \( d(E_1, E_2) = \delta > 0 \), and let there be given a number \( \varepsilon > 0 \). Let \( \{ I_n \} \) be a sequence of intervals in \( C_\lambda \) such that \( \sum I_n \supseteq E_1 + E_2 \), \( \| I_n \| < \delta \), \( n = 1, 2, \ldots \), and \( \phi(E_1 + E_2) > \sum \phi(I_n) - \varepsilon \). Let \( I_{1j}, j = 1, 2, \ldots \), be the intervals in \( \{ I_n \} \) such that \( I_{1j} \cap E_1 \neq \emptyset \). Let \( I_{2j}, j = 1, 2, \ldots \), be the remainder of the intervals in \( \{ I_n \} \). Then \( \sum I_{1j} \supseteq E_1 \); \( \sum I_{2j} \supseteq E_2 \) and \( \phi(E_1 + E_2) \leq \sum \phi(I_{1j}) + \sum \phi(I_{2j}) = \sum \phi(I_n) < \phi(E_1 + E_2) + \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary it follows that \( \phi(E_1 + E_2) \leq \phi(E_1 + E_2) + \varepsilon \). This result together with condition (ii) establishes condition (iii) in the definition of outer Carathéodory measure.

Applying the theory of outer Carathéodory measure (see [12, chap. 2]) we may now complete our proof of Theorem 1. A set \( E \) is \( \phi \)-measurable if \( \phi(P + Q) = \phi(P) + \phi(Q) \) for every pair of sets \( P \) and \( Q \) contained, respectively, in the set \( E \) and in its complement \( CE \). The class of all sets which are \( \phi \)-meas-
urable—we denote it by \( K_\phi \)—is an additive class in the sense of Saks, that is, (i) it contains the empty set, (ii) if \( E \in K_\phi \) then \( C \subseteq K_\phi \), and (iii) if \( \{ E_n \} \) is a sequence of sets in \( K_\phi \), then \( \sum_n E_n \) is a set in \( K_\phi \). Furthermore, the class \( K_\phi \) contains all sets which are open relative to \( R_\phi \). Thus \( K_\phi \) is a closed class of sets (1.2). The function \([\phi, K_\phi]\) is a completely additive set function; it is non-negative; and it follows from condition \( C \) that \( \phi(I) = \Phi(I) \) for every \( I \in C_\lambda \). Thus \([\phi, K_\phi]\) is a non-negative completely additive extension of the function \([\phi, C_\lambda]\). This completes a proof of Theorem 1; we proceed to outline a proof of Theorem 2.

2.3. Let \([\phi, C_\lambda]\) be a function of intervals and let \([\Phi, K]\) be a completely additive extension of \([\phi, C_\lambda]\). Then \([\Phi, K]\) is the difference of two non-negative completely additive set functions (see [13, p. 90]), call them \([\Phi_1, K]\) and \([\Phi_2, K]\). We suppose that \( \Phi(E) = \Phi_1(E) - \Phi_2(E) \) for every \( E \in K_\phi \). Each of the functions \([\Phi_i, K]\), \( i = 1, 2 \), satisfies the condition stated in 2.1; in particular, each of the functions \([\Phi_i, C_\lambda]\) satisfies condition \( C \). But \( \phi(I_\lambda) = \Phi(I_\lambda) = \Phi_1(I_\lambda) - \Phi_2(I_\lambda) \); thus the condition in Theorem 2 is necessary.

Let \([\phi, C_\lambda], [\phi_1, C_\lambda], [\phi_2, C_\lambda]\) be finite, single-valued functions of intervals. We assume that \([\phi_1, C_\lambda]\) and \([\phi_2, C_\lambda]\) are non-negative functions each of which satisfies condition \( C \) and that \( \phi(I) = \phi_1(I) - \phi_2(I) \) for every \( I \in C_\lambda \). Let \([\Phi_1, K_1]\) and \([\Phi_2, K_2]\) be non-negative completely additive extensions of \([\phi_1, C_\lambda]\) and \([\phi_2, C_\lambda]\) respectively. Let \( K \) denote the closed class \( K_1 \cap K_2 \). For every \( E \in K \) we define \( \Phi(E) = \Phi_1(E) - \Phi_2(E) \). The function \([\Phi, K]\) is completely additive, and furthermore, the relation

\[
\phi(I) = \phi_1(I) - \phi_2(I) = \Phi_1(I) - \Phi_2(I) = \Phi(I)
\]

holds for every \( I \in C_\lambda \). Thus \([\Phi, K]\) is a completely additive extension of \([\phi, C_\lambda]\). This establishes the sufficiency of the condition in Theorem 2.

2.4(3). Let \([\Phi, B]\) be a non-negative additive function of Borel sets (1.6). For every set \( E \in B \) define \( \Phi(E) = \inf \sum_n \Phi(I_n) \) for all sequences \( \{ I_n \} \) of intervals in \( C_\lambda \) such that \( \sum_n I_n \supseteq B \). The function \([\Phi, B]\) is a non-negative completely additive extension of the function \([\phi, C_\lambda]\). Therefore \([\phi, C_\lambda]\) satisfies condition \( C \) and the function \( \Phi(E) \) is completely additive on the class of all Borel sets. It follows from the definition of \( \Phi(E) \) and from condition \( C \) that \( \Phi(R_\phi) = \Phi(R_\phi) \) and that \( \Phi(E) \geq \Phi(E) \) for every \( E \in B \). Thus, for \( E \subseteq B \), we have \( \Phi(E) \geq \Phi(E) \), \( \Phi(CE) \geq \Phi(CE) \), and \( \Phi(E) + \Phi(CE) = \Phi(R_\phi) \) \( \Phi(E) + \Phi(CE) \). Obviously then, \( \Phi(E) = \Phi(E) \).

2.5. Let \([\Phi_1, B]\) and \([\Phi_2, B]\) be any two non-negative additive functions of Borel sets and suppose that \( \Phi_1(I) \geq \Phi_2(I) \) for every \( I \in C_\lambda \). For \( E \subseteq B \) and for \( i = 1, 2 \), define \( \Phi_i(E) = \inf \sum_n \Phi_i(I_n) \) for all sequences \( \{ I_n \} \) such that \( \sum_n I_n \supseteq E \). From 2.4 it follows that \( \Phi_i(E) = \Phi_i(E) \) for every \( E \subseteq B \). But \( \Phi_1(E) \geq \Phi_2(E) \), and hence \( \Phi_1(E) \geq \Phi_2(E) \).

(3) The proof of Theorem 3 as presented here in §§2.4–2.7 was suggested by Professor Earl Mickle. For another treatment of the uniqueness of a completely additive extension, see [9].
2.6. Let \([\Phi_1, B] \) and \([\Phi_2, B] \) be any two additive functions of Borel sets such that \(\Phi_1(I_\lambda) \geq \Phi_2(I_\lambda) \) for every \(I_\lambda \subseteq C_\lambda \). We express each of the functions \(\Phi_1, \Phi_2 \) as the difference of two non-negative additive functions of Borel sets:

\[
\Phi_1(E) = \Phi_{11}(E) - \Phi_{12}(E); \quad \Phi_2(E) = \Phi_{21}(E) - \Phi_{22}(E).
\]

Then for \(I \subseteq C_\lambda \) we have \(\Phi_{11}(I) - \Phi_{12}(I) = \Phi_{21}(I) - \Phi_{22}(I) \) and \(\Phi_{11}(I) + \Phi_{22}(I) \geq \Phi_{12}(I) + \Phi_{21}(I) \). From 2.5 it follows that if \(E \subseteq B \) then \(\Phi_{11}(E) + \Phi_{22}(E) \geq \Phi_{12}(E) + \Phi_{21}(E) \). From this it follows immediately that \(\Phi_1(E) \geq \Phi_2(E) \).

2.7. Let \([\Phi_1, K] \) and \([\Phi_2, K] \) be two completely additive extensions of a function \([\phi, C_\lambda] \). Then \(\Phi_1(I) = \Phi_2(I) = \phi(I) \) for every \(I \subseteq C_\lambda \). It follows from 2.6 that \(\Phi_1(E) = \Phi_2(E) \) for every \(E \subseteq B \). In other words, if \([\Phi, K] \) is a completely additive extension of a function \([\phi, C_\lambda] \), and if \(E \subseteq B \), then the value of \(\Phi(E) \) is uniquely determined by the function \([\phi, C_\lambda] \). If \([\phi, C_\lambda] \) is non-negative, the number \(\Phi(E) \) has the characterization as defined in 2.2. Thus we have established Theorem 3.

3. \(B\)-extensions. This part contains our results on the \(B\)-extension as stated in \S 1.6. In \S\S 3.1 and 3.2 we state and prove two lemmas which are used in the proofs of the main theorems.

3.1. Let \([\Phi, B] \) be a completely additive extension of a function of intervals \([\phi, C_\lambda] \). A necessary and sufficient condition that \([\Phi, B] \) be a \(B\)-extension of \([\phi, C_\lambda] \) is that \(\Phi(I) = \Phi(I^0) \) for every \(I \subseteq C_\lambda \).

Proof. Let \([\Phi, B] \) be a \(B\)-extension of \([\phi, C_\lambda] \). Then \(\Phi(I) = \Phi(I^0) \) for every \(I \subseteq C_\lambda \). Let \(I \subseteq C_\lambda \) and let \(E_1, \cdots, E_n \) be a subdivision of \(I \) into the intervals \(I_1, \cdots, I_n \). Express the set \(I - I^0 \) as the sum of mutually exclusive Borel sets \(E_1 + E_2 + \cdots + E_n \) where \(E_i \subseteq I_i - I^0_i \), \(i = 1, \cdots, n \). Since \(\Phi(E_i) = 0 \) it follows that \(\Phi(I - I^0) = 0 \) and that \(\Phi(I) = \Phi(I^0) \).

Let \([\Phi, K] \) be a completely additive extension of \([\phi, C_\lambda] \) and assume that \(\Phi(I) = \Phi(I^0) \) for every \(I \subseteq C_\lambda \). Let \(i, i^0 \subseteq R_0 \), be any closed linear interval which is parallel to either the \(x\)- or the \(y\)-axis. Let \(I_1 \supset I_2 \supset \cdots \) be a descending sequence of intervals in \(C_\lambda \) such that \(\prod_n I_n = i \) and \(\prod_n I_n^0 \) is the empty set. It follows from the additivity of \([\Phi, B] \) that

\[
0 = \lim_{n \to \infty} \Phi(I_n^0) = \lim_{n \to \infty} \Phi(I_n) = \Phi(i).
\]

Let \(i \) be a transversal or a boundary segment of \(R_0 \). The function \(\Phi(E) \) where \(E \subseteq B, \ E \subseteq t \) is a completely additive extension of the function of linear intervals \(\Phi(i) \) where \(i \subseteq t \). But such an extension is unique on Borel sets. Therefore, since \(\Phi(i) = 0 \), it follows that \(\Phi(E) = 0 \) for \(E \subseteq B, \ E \subseteq t \). Let \(I \) be any interval in \(C_\lambda \) and let \(E \subseteq B \) be any set such that \(I^0 \subseteq E \subseteq I \). Then \(\Phi(I - E) = 0 \) and it follows that

\[
\phi(I) = \Phi(I) = \Phi(I - E) + \Phi(E) = \Phi(E).
\]
Thus \([\Phi, B]\) is a \(B\)-extension of \([\phi, C_\lambda]\).

3.2. If a function of intervals \([\phi, C_\lambda]\) admits a \(B\)-extension, then \([\phi, C_\lambda]\)
is additive.

**Proof.** Let \([\Phi, B]\) be a \(B\)-extension of \([\phi, C_\lambda]\). Let \(I_\lambda\) and \(E_\lambda I_\lambda\) be given. Denote the intervals in \(E_\lambda I_\lambda\) by \(I_1, I_2, \ldots, I_n\). Then

\[
\phi(I_\lambda) = \Phi(I_\lambda) \geq \sum_{i=1}^{n} \Phi(I_i) = \sum_{i=1}^{n} \phi(I_i).
\]

But \(\sum_{i=1}^{n} I_i \supseteq I_\lambda\). It follows from condition \(C\) that \(\phi(I_\lambda) \leq \sum_{i=1}^{n} \phi(I_i)\). Thus \([\phi, C_\lambda]\) is additive. If we extend the range of definition of \([\phi, C_\lambda]\) from \(C_\lambda\) to \(C_0\) by defining \(\phi(I) = \Phi(I)\), then the function \([\phi, C_0]\) is also additive.

3.3. We proceed to a proof of Theorem 4. Let \([\phi, C_\lambda]\) be a non-negative function of intervals and let \([\Phi, B]\) be a non-negative \(B\)-extension of \([\phi, C_\lambda]\). Extend the range of definition of \([\phi, C_\lambda]\) from \(C_\lambda\) to \(C_0\) by defining \(\phi(I) = \Phi(I)\) for all \(I \in C_0\). Let \(R_\bullet\) be a fixed interval such that \(R_\bullet \supseteq R_0\). Let \(C_\bullet\) denote the class of all intervals \(I \subseteq R_\bullet\). Define the function \([\phi_\bullet, C_\bullet]\) by the relation \(\phi_\bullet(I) = \phi(I \cap R_0)\) if \(I \cap R_0\) is an interval in \(C_0\); by the relation \(\phi_\bullet(I) = 0\) for all other \(I \subseteq C_\bullet\). Let \(B_\bullet\) denote the class of all Borel sets \(E \subseteq R_\bullet\) and define the function \([\Phi_\bullet, B_\bullet]\) by the relation \(\Phi_\bullet(E) = \Phi(E \cap R_0)\) for every \(E \subseteq B_\bullet\). Then \([\Phi_\bullet, B_\bullet]\) is a non-negative \(B\)-extension of \([\phi_\bullet, C_\bullet]\). Let \(t\) be any transversal of \(R_0\) or a closed boundary segment of \(R_0\). Let \(\{I_n\}\) be a sequence of intervals in \(C_\bullet\) such that \(\prod_{n=1}^{\infty} I_n = t\). Then

\[
\lim_{n \to \infty} \phi_\bullet(I_n) = \lim_{n \to \infty} \Phi_\bullet(I_n) = \lim_{n \to \infty} \Phi_\bullet(I_n^0) = \Phi_\bullet(t) = \Phi(t) = 0.
\]

Given \(\varepsilon > 0\) let each vertical transversal of \(R_0\) and each of the two vertical sides of \(R_0\) be covered by an open interval \(I_0^\bullet\) such that \(I \subseteq C_\bullet\) and \(\phi_\bullet(I) < \varepsilon\). Then the closed set \(R_0\) is covered by this class of open intervals. A finite number of these open intervals, say \(I_{10}^\bullet, I_{20}^\bullet, \ldots, I_{n0}^\bullet\), suffice to cover \(R_0\). Let \(t_1, t_2\) denote the vertical boundary segments of \(R_0\) and let \(t_3, t_4, \ldots, t_m\) be the set of vertical transversals of \(R_0\) which lie on the boundaries of the intervals \(I_i, i = 1, \ldots, n\). Let \(\delta_1\) be the minimum of the numbers \(d(t_i, t_j)\) where \(i \neq j\) and \(i, j = 1, 2, \ldots, m\). Let \(I \subseteq C_0\) be any interval whose horizontal dimension is less than \(\delta_1\). Then \(I \subseteq I_i\) for some integer \(i, i = 1, \ldots, n\), and it follows that \(\phi(I) = \phi_\bullet(I) \leq \phi_\bullet(I_i) < \varepsilon\). Similarly we may show that there exists a number \(\delta_2 > 0\) such that if \(I \subseteq C_0\) is an interval whose vertical dimension is less than \(\delta_2\) then \(\phi(I) < \varepsilon\). Let \(\delta_3 = \min(\delta_1, \delta_2)\), let \(\delta\) be a number such that \(0 < \delta < \delta_3\), and let \(I \subseteq C_0\) be any interval such that \(|I| < \delta\). Then at least one dimension of \(I\) is less than \(\delta\) and it follows that \(\phi(I) < \varepsilon\). Condition (ii) in the definition of continuity follows from the additivity of \([\phi, C_\lambda]\) which was established in 3.2. Thus the condition in Theorem 4 is necessary.

3.4. Let \([\phi, C_\lambda]\) be a non-negative, additive, continuous function of intervals. Let \(I\) be any interval in \(C_0\) and let \(E_\lambda I\) and \(\gamma_\lambda I\) be any two subdivisions
of \( I \). Complete the elementary system \( \mathcal{E}_\lambda I \) with an elementary system of intervals \( \mathcal{G}_\lambda \) to a subdivision of \( R_0 \). The system \( \mathcal{Y}_\lambda + \mathcal{G}_\lambda \) also forms a subdivision of \( R_0 \) and we have that \( \phi(\mathcal{E}_\lambda I) + \phi(\mathcal{G}_\lambda I) = \phi(R_0) = \phi(\mathcal{Y}_\lambda I) + \phi(\mathcal{G}_\lambda I) \). Thus \( \phi(\mathcal{E}_\lambda I) = \phi(\mathcal{Y}_\lambda I) \). We extend the range of definition of \( [\phi, C_\lambda] \) from \( C_\lambda \) to \( C_0 \) by defining \( \phi(I) = \phi(\mathcal{E}_\lambda I) \), where \( \mathcal{E}_\lambda I \) is any subdivision of \( I \subseteq C_0 \). The function \( [\phi, C_0] \) is non-negative and additive. Using the definitional properties of continuity on the class \( C_\lambda \), the additivity on the class \( C_0 \), and a technique similar to that employed in 3.3, it may be proved that \( [\phi, C_0] \) is a continuous function.

From the non-negative and additive properties of \( [\phi, C_0] \) it follows that \( [\phi, C_0] \) satisfies the following condition \( \mathcal{C}' \): If \( I_1, I_2, \ldots, I_n \) is a finite set of intervals in \( C_0 \) such that \( I_i \cap I_j = \emptyset \) when \( i \neq j \), and if \( J_1, J_2, \ldots, J_m \) is a finite set of intervals in \( C_0 \) such that \( \sum_{i=1}^{n} I_i \supseteq \sum_{j=1}^{m} J_i \), then \( \sum_{i=1}^{n} \phi(I_i) \geq \sum_{j=1}^{m} \phi(J_j) \).

We shall show that \( [\phi, C_0] \) satisfies condition \( \mathcal{C} \) as stated in Theorem 1. Let \( R_\epsilon \) be a fixed interval such that \( R_\epsilon \supseteq R_0 \). Let the class \( C_\epsilon \) and the function \( [\phi_\epsilon, C_\epsilon] \) be defined as in 3.3. Then \( [\phi_\epsilon, C_\epsilon] \) is a continuous, additive function. Let \( I_1, I_2, \ldots, I_n \) be a finite set of mutually exclusive intervals in \( C_\epsilon \) and let \( J_1, J_2, \ldots \) be an infinite sequence of intervals in \( C_\epsilon \) such that \( \sum_{i=1}^{n} I_i \supseteq \sum_{j=1}^{m} J_j \). Given \( \epsilon > 0 \), let \( R_i, i = 1, 2, \ldots \), be an interval in \( C_\epsilon \) such that \( \sum_{i=1}^{n} R_i \supseteq \sum_{j=1}^{m} J_j \) and \( \phi_\epsilon(R_i) < \phi_\epsilon(J_j) + \epsilon/2^i \). Then \( \sum_{i=1}^{n} R_i \supseteq \sum_{j=1}^{m} J_j \). Since the set \( \sum_{i=1}^{n} I_i \) is closed it follows that there is an integer \( m^* \) such that \( \sum_{i=1}^{n} R_i \supseteq \sum_{j=1}^{m^*} J_j \).

Thus we have

\[
\sum_{i=1}^{n} \phi(I_i) = \sum_{i=1}^{n} \phi_\epsilon(I_i) \leq \sum_{i=1}^{m^*} \phi_\epsilon(R_i) < \sum_{i=1}^{m^*} \phi_\epsilon(J_i) + \epsilon = \sum_{i=1}^{m^*} \phi(J_i) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that \( \sum_{i=1}^{n} \phi(I_i) \leq \sum_{i=1}^{m^*} \phi(J_i) \).

Thus condition \( \mathcal{C}' \) is fulfilled when the finite set of \( J \)'s is replaced by a sequence of \( J \)'s. That it may be further extended by replacing the finite set of \( I \)'s by a sequence of \( I \)'s is obvious. Thus the function \( [\phi, C_0] \) satisfies condition \( \mathcal{C} \). Let \( [\Phi, B] \) be the non-negative completely additive extension of \( [\phi, C_0] \) to the class \( B \). We shall show that \( [\Phi, B] \) is a \( B \)-extension of \( [\phi, C_0] \). Let \( I \) be any interval in \( C_0 \). Given a number \( \epsilon > 0 \), let \( J \subseteq C_0 \) be an interval such that \( J \subseteq I \) and \( \phi(J) > \phi(I) - \epsilon \). Then

\[
\Phi(I) = \phi(I) < \phi(J) + \epsilon = \Phi(J) + \epsilon = \Phi(I^0) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that \( \Phi(I) = \Phi(I^0) \) and from 3.1 it follows that \( [\Phi, B] \) is a \( B \)-extension of \( [\phi, C_0] \); obviously it is also a \( B \)-extension of the function \( [\phi, C_\lambda] \). Thus the condition in Theorem 4 is sufficient.

3.5. Given a function of intervals \( [\phi, C_\lambda] \) and a transversal \( t \) of \( R_0 \) we define\(^4\):

\[
\lambda(t) = \lim \sup [\phi(\mathcal{E}_\lambda I) - \phi(\mathcal{Y}_\lambda I)] \text{ for all } \mathcal{E}_\lambda \text{ and } \mathcal{Y}_\lambda \text{ such that } \| \mathcal{E}_\lambda \| \to 0,
\]

\(^4\) This definition is an extension of the concept of ecart as employed by Saks for functions of linear intervals. See [11, p. 211].
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\[ |\mathcal{F}_0| \to 0, \text{ and } I \cdot t \neq 0 \text{ for all } I \in \mathcal{E}_\lambda \text{ and } I \in \mathcal{F}_\lambda. \]

\[ a(t) = \liminf \left[ \phi(\mathcal{E}_\lambda) - \phi(\mathcal{F}_\lambda) \right] \text{ for all } \mathcal{E}_\lambda \text{ and } \mathcal{F}_\lambda \text{ as in the definition of } A(t). \]

The numbers \( \Omega(t) = 2^{-1} \left[ A(t) + |A(t)| \right] \geq 0 \) and \( \omega(t) = 2^{-1} \left[ a(t) - |a(t)| \right] \leq 0 \) are called the non-negative and non-positive \textit{ecarts} of \( [\phi, C_\lambda] \) on \( t \). If both \( \Omega(t) = 0 \) and \( \omega(t) = 0 \), then \( t \) is a transversal of zero \textit{ecart}; otherwise \( t \) is a transversal of nonzero \textit{ecart}.

Let \( [\phi, C_\lambda] \) be a function of intervals which is RAC, that is, the function

\[ [U(|\phi|), C_\lambda] \text{ is continuous.} \]

Let \( t \) be a transversal of \( R_0 \). Given \( \varepsilon > 0 \), let \( I \) be an interval in \( C_0 \) such that \( I^0 \supseteq t^0, I \supseteq t \), and \( U(|\phi|, I) < \varepsilon /4 \). Let \( \eta > 0 \) be a number such that \( \|\mathcal{E}_\lambda I\| < \eta \) implies \( |\phi(\mathcal{E}_\lambda I) - U(|\phi|, I)| < \varepsilon /4 \). Then if \( \mathcal{F}_\lambda \) and \( \mathcal{F}_\lambda^\varepsilon \) are any two elementary systems such that \( \|\mathcal{F}_\lambda^\varepsilon\| < \eta, i = 1, 2, \) and \( J \subseteq I \) for every interval \( J \in \mathcal{F}_\lambda^\varepsilon \), we have

\[ |\phi(\mathcal{F}_\lambda^\varepsilon) - \phi(\mathcal{F}_\lambda)| \leq |\phi(\mathcal{F}_\lambda) + |\phi(\mathcal{F}_\lambda)| < \varepsilon. \]

Thus \( \phi(\mathcal{F}_\lambda^\varepsilon) < U(|\phi|, I) + \varepsilon/4 < \varepsilon/2. \) Since \( \varepsilon > 0 \) is arbitrary, it follows that \( A(t) = a(t) = 0 \) and \( R_0 \) has no transversals of nonzero \textit{ecart}.

3.6. If \( [\phi, C_\lambda] \) increases (decreases) by subdivision and is RAC, then \( U(\phi, I) (L(\phi, I)) \) is finite and additive.

\textbf{Proof.} Since \( [\phi, C_\lambda] \) increases by subdivision, it follows from a theorem of Kempisty that \( U(\phi, I) > -\infty \) for all \( I \in C_0 \) and that \( U(\phi, I) \) is additive. It follows from the continuity of \( U(|\phi|, I) \) that \( U(\phi, I) \) is also a continuous function. Let \( \delta > 0 \) be a number such that \( |I| < \delta \) implies \( U(\phi, I) < 1 \). Let \( \mathcal{E}I_0 \) be a subdivision of a fixed interval \( I_0 \) into intervals \( I \) such that \( |I| < \delta \). It follows that \( U(\phi, I_0) \) is less than the number of intervals in \( \mathcal{E}I_0 \).

If \( [\phi, C_\lambda] \) decreases by subdivision, the result follows as a corollary if we consider the function \( -\phi, C_\lambda \).

3.7. If \( [\phi, C_\lambda] \) is RAC and increases (decreases) by subdivision, then \( [\phi, C_\lambda] \) is integrable.

\textbf{Proof.} Let \( [\phi, C_\lambda] \) be an RAC function which increases by subdivision. Then by 3.6, \( U(\phi, R_0) \) is a finite number. Given \( \varepsilon > 0 \), let \( \mathcal{E}_\lambda R_0 \) be a subdivision of \( R_0 \) such that \( \phi(\mathcal{E}_\lambda R_0) > U(\phi, R_0) - \varepsilon \). Let \( \{ \mathcal{E}_{\lambda m} R_0 \} \) be a sequence of subdivisions of \( R_0 \) such that \( \|\mathcal{E}_{\lambda m} R_0\| \to m_0 \) and \( \phi(\mathcal{E}_{\lambda m} R_0) \to L(\phi, R_0) \). Let \( t_1, t_2, \ldots, t_k \) be the set of all transversals of \( R_0 \) such that \( t_i, i = 1, \ldots, k \), contains at least one boundary segment of an interval \( I \in \mathcal{E}_\lambda R_0 \). Let \( \mathcal{J}_m, m = 1, 2, \ldots \), be the subdivision of \( R_0 \) which is formed by the transversals \( t_i, i = 1, 2, \ldots, k \), and all of the boundary segments of intervals \( I \in \mathcal{E}_{\lambda m} \). Let \( T = \sum_{i=1}^k t_i \) and let \( \mathcal{E}_{\lambda m}^T \) be the elementary system consisting of all the intervals \( I \) such that \( I \in \mathcal{E}_{\lambda m} \) and \( I \cdot T \neq 0 \). Each interval \( I \) in \( \mathcal{F}_m \) which is in the class \( C_0 - C_\lambda \) is contained in an interval \( I \in \mathcal{E}_{\lambda m}^T \). We replace each such interval \( I \in \mathcal{F}_m \) by a subdivision \( \mathcal{E}_{\lambda m} I \) and denote the resulting elementary system by \( \mathcal{F}_{\lambda m} \). Let \( \mathcal{E}_{\lambda m}^C \) denote the elementary system of intervals \( I \) such that \( I \in \mathcal{F}_{\lambda m} \) and \( I \) is contained in some interval \( J \in \mathcal{E}_{\lambda m}^T \). The elementary systems \( \mathcal{E}_{\lambda m} - \mathcal{E}_{\lambda m}^T \) and \( \mathcal{F}_{\lambda m} - \mathcal{F}_{\lambda m}^T \) are identical. Therefore \( \phi(\mathcal{E}_{\lambda m}) - \phi(\mathcal{F}_{\lambda m}) \)
= \phi(E_{\lambda m}) - \phi(J_{\lambda m})$. It follows from 3.5 that \( \lim_m [\phi(E_{\lambda m}) - \phi(J_{\lambda m})] = 0 \). Thus \( \lim_m \phi(J_{\lambda m}) = \lim_m \phi(E_{\lambda m}) = L(\phi, R_0) \). Since \( \phi, C_\lambda \) increases by subdivision we have
\[
\phi(J_{\lambda m}) \geq \phi(E_{\lambda}) > U(R_0) - \epsilon, \quad m = 1, 2, \ldots.
\]
Therefore \( L(\phi, R_0) = \lim_m \phi(J_{\lambda m}) > U(\phi, R_0) - \epsilon \) and it follows that \( \phi, C_\lambda \) is integrable. The case in which \( \phi, C_\lambda \) decreases by subdivision follows as an immediate corollary since the function \( -\phi, C_\lambda \) increases by subdivision.

3.8. We proceed to the proof of Theorem 5. Let \( \Phi, B \) be a \( B \)-extension of the function of intervals \( \phi, C_\lambda \). Express the function \( \Phi, B \) as the difference of two non-negative additive functions of Borel sets, say \( \phi_1, B \) and \( \phi_2, B \), where \( \Phi(E) = \Phi_1(E) - \Phi_2(E), E \in B \). For \( \delta = 1, 2 \) and \( I \in C_0 \) define \( \phi_\delta(I) = \Phi_\delta(I) \). Then \( \phi_1, C_0 \) and \( \phi_2, C_0 \) are non-negative additive functions of intervals, and \( \Phi_1, B \), \( \Phi_2, B \) are non-negative \( B \)-extensions of \( \phi_1, C_0 \), \( \phi_2, C_0 \) respectively. It follows from Theorem 4 that \( \phi_1, C_0 \), \( \phi_2, C_0 \) are continuous functions. Since \( |\phi(I_\lambda)| \leq |\phi_1(I_\lambda)| + |\phi_2(I_\lambda)| \) for every \( I \in C_\lambda \) it follows that \( U(\phi, I) \leq U(\phi_1, I) + U(\phi_2, I) = \phi_1(I) + \phi_2(I) \) for every \( I \in C_0 \). Thus \( U(\phi, I) \) is a continuous function, that is, the function \( \phi, C_\lambda \) is RAC. The necessity of the condition in Theorem 5 follows from this result and 3.2.

Let \( \phi, C_\lambda \) be an additive, RAC function of intervals. For every \( I \in C_\lambda \) define \( \phi_1(I) = 2^{-1}[\phi(I) + |\phi(I)|], \phi_2(I) = 2^{-1}[|\phi(I)| - \phi(I)]. \) Then \( \phi_1, C_\lambda \), \( \phi_2, C_\lambda \) are non-negative functions and \( \phi(I_\lambda) = \phi_1(I_\lambda) - \phi_2(I_\lambda) \) for every \( I \in C_\lambda \). It is readily proved that \( \phi_1, C_\lambda \), \( \phi_2, C_\lambda \) are RAC and increase by subdivision. It follows from 3.7 that they are integrable. Their indefinite integrals, denoted by \( [F_1, C_0] \) and \( [F_2, C_0] \) respectively, are continuous, additive functions. Let \( \Phi_1, B \) and \( \Phi_2, B \) denote the \( B \)-extensions of \( [F_1, C_0] \) and \( [F_2, C_0] \) respectively. For every \( E \in B \) define \( \Phi(E) = \Phi_1(E) - \Phi_2(E) \). For \( I \in C_\lambda \) we have \( \Phi(I) = \Phi_1(I) - \Phi_2(I) = \Phi_1(I_0) - \Phi_2(I_0) = \Phi(I_0) \). It follows from 3.1 that \( \Phi, B \) is a \( B \)-extension of \( \phi, C_\lambda \).

3.9. If \( \phi, C_\lambda \) is integrable, then \( U(|\phi|, I) = U(|F(\phi)|, I) \) for every \( I \in C_0 \).

Proof. Given a number \( \epsilon > 0 \) let \( \delta > 0 \) be a number such that \( \|E_\lambda\| < \delta \) implies \( |\phi(E_\lambda) - F(\phi, E_\lambda)| < \epsilon/2 \) (see [6, Theorem 3]). Given an interval \( I_0 \in C_0 \) let \( E_\lambda I_0 \) be a subdivision of \( I_0 \) which consists of the intervals \( I_1, \ldots, I_n \), and which is such that \( \|E_\lambda I_0\| < \delta \). Let \( E_1 \) consist of the intervals \( I \in E_\lambda I_0 \) which satisfy the relation \( \phi(I) - F(\phi, I) \geq 0 \) and let \( E_2 \) consist of the intervals \( I \in E_\lambda I_0 \) which satisfy the relation \( \phi(I) - F(\phi, I) < 0 \). Then \( |\phi(E_i) - F(\phi, E_i)| < \epsilon/2, i = 1, 2, \) and \( \sum_{i=1}^{n} |\phi(I_i) - F(\phi, I_i)| < \epsilon \). But \( |F(\phi, I_i)| \leq |\phi(I_i)| + |F(\phi, I_i) - \phi(I_i)| \);
\[
|\phi(I_i)| \leq |F(\phi, I_i)| + |F(\phi, I_i) - \phi(I_i)|.
\]
Therefore \( |F(\phi, I_i) - \phi(I_i)| \leq |F(\phi, I_i) - \phi(I_i)| \), and \( \sum_{i=1}^{n} |F(\phi, I_i)| \).
$-|\phi(I_i)| < \epsilon$. It follows that $U(|\phi|, I_0) = U(|F(\phi)|, I_0)$.

3.10. Since the integral of a function of intervals is additive, it follows from Theorem 5 that: *A necessary and sufficient condition that the integral of a function of intervals admit a $B$-extension is that the integral be $RAC$.* Theorem 6 follows immediately from 3.9.

**Bibliography**