DENSITY THEOREMS FOR POWER SERIES
AND COMPLETE SETS

BY

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1. Introduction. The problems considered in this paper are the completeness of sets of the form \( \{ t^{\lambda_n e^{-\epsilon t}} \} \) \( (c > 0, \lambda_n > 0) \) in \((0, \infty)\) (where the \( \lambda_n \)'s are not necessarily integers) and the analytic continuation of lacunary power series \( \sum c_n x^{\lambda_n} \) (where the \( \lambda_n \)'s are integers).

Let \( n(t) = n_\lambda(t) \) denote the number of \( \lambda_n \) not exceeding \( t \). W. H. J. Fuchs [3] proved that \( \{ t^{\lambda_n e^{-\epsilon t}} \} \) is complete with respect to \( L^p \), \( 1 \leq p \leq \infty \), if there is a constant \( A \) such that \( n(t) \geq t/2 - A \). I shall prove by a different method it is sufficient to have

\[
(1.1) \quad n(t) - t/2 \geq - t \delta(t),
\]

where \( \int_0^\infty t^{-1} \delta(t) dt \) converges. Fuchs has pointed out to me that the same result is obtainable by his original method; in addition, since the present paper was written, a paper [3a] by Fuchs has appeared, establishing the result (stronger if the \( \lambda_n \) satisfy \( \lambda_{n+1} - \lambda_n \geq \epsilon > 0 \)) that if \( \lambda_{n+1} - \lambda_n \geq \epsilon > 0 \), \( \{ t^{\lambda_n e^{-\epsilon t}} \} \) is complete with respect to \( L^2 \) if and only if

\[
(1.2) \quad \int_1^\infty r^{-2} \psi(r) dr = \infty, \quad \psi(r) = \exp \left\{ 2 \sum_{\lambda_n < r} \frac{\lambda_n^{-1}}{2} \right\},
\]

and that (1.2) is sufficient for completeness with respect to \( L^p, p \neq 2, p \geq 1 \). However, the proof given in the present paper is somewhat simpler than either of Fuchs's proofs.

Fuchs also showed [3] that if the set \( \{ n \} \) of all positive integers is divided into two complementary subsequences \( \{ \lambda_n \} \) and \( \{ \mu_n \} \), then at least one of \( \{ t^{\lambda_n e^{-\epsilon t}} \} \) and \( \{ t^{\mu_n e^{-\epsilon t}} \} \) is complete. (This is a trivial consequence of his later result [3a].) I shall show that \( \{ n \} \) can be replaced by any sequence \( \{ a_n \} \) such that \( a_n(t) \geq t - t \delta(t) \) with \( \int_0^\infty t^{-1} \delta(t) dt \) convergent (see Theorem 4, where an extension to \( k \) subsequences is given). This is a corollary of Fuchs's results in [3a] if \( a_{n+1} - a_n \geq \epsilon > 0 \).

The completeness of sets \( \{ t^{\lambda_n e^{-\epsilon t}} \} \) is equivalent to the completeness of various other sets. For example, the Fourier cosine transform leads (as was pointed out to me by H. Pollard) to the set \( \{ (\cos x)^{\lambda_n} \cos \lambda_n x \} \) in \((0, \pi/2)\); the Mellin transform leads to the set \( \{ \Gamma(\lambda_n + it) \} \) on \((-\infty, \infty)\).

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(5) Numbers in brackets refer to the references at the end of the paper.

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Almost the same reasoning as is used for the completeness problem leads to new criteria for the uniqueness of the solution of the generalized moment problem

\[ \mu_n = \int_0^\infty t^n d\alpha(t), \]

\( \alpha(t) \) nondecreasing (see §5).

Let \( f(z) \) have a power series of the form

\[ \sum_{n=0}^{\infty} c_n z^n, \]

where the \( \lambda_n \) are integers and

\[ \limsup_{n \to \infty} n\lambda(r)/r = D. \]

If \( f(z) \) is defined by (1.3) for small \( |z| \) and by analytic continuation (if possible) for large \( |z| \), Mandelbrojt has shown that, in every angle with vertex at the origin and opening \( 2\alpha > 2\pi D \), \( f(z) \) (if not a constant) either has a singular point or is unbounded. Mandelbrojt’s result applies to Dirichlet series, and is more general in other ways; but it requires the strict inequality \( \alpha > \pi D \). I shall give some results in which, if (1.4) is somewhat strengthened, \( \alpha = \pi D \) is permissible. In the first place, (1.4) can be replaced by

\[ n(r) \leq r(\pi^{-1} \alpha - \epsilon(r)), \]

where \( \int_0^\infty r^{-1} \epsilon(r) dr \) diverges; (1.4) with \( \alpha > \pi D \) is the case where \( \epsilon(r) \geq \epsilon > 0 \). In the second place, when \( \alpha = \pi/2 \), results on complete sets can be applied to show that \( f(z) \) (if not constant) either has a singular point or is unbounded in every half-plane containing \( z = 0 \) in its interior, provided only that

\[ n(r) \leq r(1/2 + \delta(r)), \]

where \( \delta(r) \) has the same properties as in (1.1). By applying Fuchs’s later result [3a] instead of the results of the present paper, (1.6) can be replaced by (1.2); either (1.6) or (1.2) requires less of \( n(r) \) than (1.4), but implies a little less about \( f(z) \). These results for \( \alpha = \pi/2 \) imply a corresponding improvement of results of Mandelbrojt and Ulrich [6] on a generalization of quasi-analyticity.

The power series result for \( \alpha = \pi/2 \) is not only a consequence of the completeness of \( \{t^{\lambda_n} e^{-t}\} \), but also implies it. It is interesting to observe that, as Fuchs showed [3], the completeness of \( \{t^{\lambda_n} e^{-t}\} \) is also equivalent to the following statement about differences: if \( \{a_n\} \) is a sequence such that \( a_n = o(n) \)
and \( \Delta^\ast a_0 = 0 \), then \( \{ a_\alpha \} \) is constant.(\(^9\))

There are also decomposition theorems for power series analogous to those for complete sets. For example, let \( \{ \nu \} = \{ \lambda_n \} + \{ \mu_n \} \), \( \alpha_1 > 0 \), \( \alpha_2 > 0 \), \( \alpha_1 + \alpha_2 > \pi \). If \( f_1(z) = \sum c_n z^{\lambda_n} \) and \( f_2(z) = \sum \gamma_n z^{\mu_n} \) and if we take any two angles of openings \( 2\alpha_1 \), \( 2\alpha_2 \), with vertices at the origin, then one of \( f_1(z) \), \( f_2(z) \) has a singular point, is unbounded, or is a constant in the corresponding angle. A stronger result can be obtained if we start from a sequence which already possesses gaps.

The theorems of this paper depend on some uniqueness results for functions analytic in a half-plane; these will be given first.

2. **Lemmas on entire functions.** We begin with some properties of special entire functions.

**Lemma 2.1.** Let \( \{ \lambda_n \} \) be an increasing sequence of positive numbers and let \( n(r) \) denote the number of \( \lambda_n \) not exceeding \( r \). Let \( \delta(t) \) be a positive function vanishing near \( t = 0 \). Let \( c > 0 \) and

\[
(2.2) \quad n(t) \geq ct - t\delta(t), \quad n(t) = O(t), \quad t \to \infty.
\]

Then the product

\[
(2.3) \quad \phi(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\lambda_n} \right)
\]

converges and there are constants \( A_1, A_2 \) such that for \( r > 0 \)

\[
(2.4) \quad \log |\phi(re^{i\theta})| \leq c\pi r |\sin \theta| - A_1 r \delta(r), \quad \pi/4 \leq |\theta| \leq \pi/2,
\]

where \( r \delta(r) \) is nondecreasing and \( \int_0^{\pi} \delta(r) dr \) converges or diverges with \( \int_0^{\pi} \delta(r) dr \), while for an unbounded sequence of \( r \)’s,

\[
(2.5) \quad \log |\phi(re^{i\theta})| \leq - A_2 r.
\]

The convergence of the product follows from (2.2), which also shows, since \( \phi(z) \) is even, that it is a function of order 1 and finite type; (2.5) is a well known result for such functions [10, p. 276].

We have, if \( z \) is not real,

\[
\log \phi(z) = \sum_{n=1}^{\infty} \log \left( 1 - \frac{z^2}{\lambda_n^2} \right) = \int_0^{\infty} \log \left( 1 - \frac{z^2}{t^2} \right) dn(t)
\]

\[
= \lim_{R \to \infty} \left\{ n(R) \log \left( 1 - \frac{z^2}{R^2} \right) - 2z^2 \int_0^R \frac{n(t) dt}{t(t^2 - z^2)} \right\}
\]

\[
= -2z^2 \int_0^{\infty} \frac{n(t) dt}{t(t^2 - z^2)};
\]

\(^9\) The case \( \{ \lambda_n \} = 2n \) was found independently by Agnew [1]; a simpler proof was given by Pollard [8]; the equivalence of "power series," "completeness," and "difference" theorems has been given a simplified proof by Boas and Pollard [2a].
the "integrated term" disappears by (2.2), since \( \log(1-z^2/R^3) = O(R^{-4}) \) as \( R \to \infty \). Hence

\[
\log | \phi(z) | = -2 \Re \left\{ \int_0^\infty \frac{n(t)}{t} \frac{z^2}{t^2 - z^2} \, dt \right\}.
\]

Let \( n(t) = ct + t \xi(t) \), where \( \xi(t) \geq -\delta(t) \). Then

\[
\log | \phi(z) | = -2 c R \left\{ \int_0^\infty \frac{z^2}{t^2 - z^2} \, dt \right\} - 2 \int_0^\infty \xi(t) \Re \left\{ \frac{z^2}{t^2 - z^2} \right\} \, dt
\]

\[= I_1 + I_2,\]
say. We have

\[(2.6)\]

\[I_1 = \pi c |y| \cdot\]

Also,

\[
\Re \left\{ \frac{z^2}{t^2 - z^2} \right\} = -r^2 \frac{r^2 - \rho^2 \cos 2\theta}{t^4 - 2t^2 r^2 \cos 2\theta + r^4}
\]

and so

\[I_2 = 2r^2 \int_0^\infty \xi(t) \frac{r^2 - t^2 \cos 2\theta}{t^4 - 2t^2 r^2 \cos 2\theta + r^4} \, dt.\]

If \( \pi/4 \leq |\theta| \leq \pi/2 \), \( \cos 2\theta \leq 0 \) and so we have,

\[I_2 \geq -2r^2 \int_0^\infty \delta(t) \frac{t^2 + r^2}{t^4 + r^4} \, dt = -r \delta_1(r),\]

where

\[
\delta_1(r) = 2r \int_0^\infty \delta(t) \frac{t^2 + r^2}{t^4 + r^4} \, dt,
\]

so that \( r \delta_1(r) \) is nondecreasing. We have

\[
\int_0^\infty r^{-1} \delta_1(r) \, dr = 2 \int_0^\infty \delta(t) \int_0^\infty \frac{t^2 + r^2}{t^4 + r^4} \, dt \, dr
\]

\[= 2 \int_0^\infty \delta(t) \, dt \int_0^\infty \frac{t^2 + r^2}{t^4 + r^4} \, dr
\]

\[= C \int_0^\infty t^{-1} \delta(t) \, dt,
\]

where \( C \) is a positive constant; consequently \( \int_0^\infty r^{-1} \delta_1(r) \, dr \) and \( \int_0^\infty t^{-1} \delta(t) \, dt \) converge or diverge together.
Lemma 2.7. With the hypotheses of Lemma 2.1, except that (2.2) is replaced by

\[ n(t) \geq ct + t\delta(t), \]

we have for \( r > 0 \)

\[ \log |\phi(re^{i\theta})| \geq c\pi r \sin \theta + A_3 r^2 \delta_2(r), \quad \pi/4 \leq |\theta| \leq \pi/2, \]

where \( \delta_2(r) \) has the same properties as \( \delta_1(r) \), and we have (2.5) for an unbounded sequence of \( r \)'s.

The proof is the same as for Lemma 2.1, through (2.6). We then have

\[
I_2 \geq 2r^2 \int_0^\infty \delta(t) \frac{r^2 - t^2 \cos 2\theta}{t^4 - 2r^2 \cos 2\theta + r^4} \, dt \\
\geq 2r^4 \int_0^\infty \frac{\delta(t)}{(t^2 + r^2)^2} \, dt = r\delta_2(r),
\]

\[
\delta_2(r) = 2r^3 \int_0^\infty \frac{\delta(t) \, dt}{(t^2 + r^2)^2}.
\]

Just as before, it follows that \( \int_0^\infty r^{-1} \delta_2(r) \, dr \) and \( \int_0^\infty r^{-1} \delta(r) \, dr \) converge or diverge together.

Lemma 2.10. If \( \eta(r) \) is nonincreasing, \( r\eta(r) \uparrow \infty \) as \( r \uparrow \infty \), and \( \int_0^\infty t^{-1} \eta(t) \, dt \) converges, there is an entire function \( \psi(z) \) such that, for \( |\theta| \leq \pi/2 \),

\[
\log |\psi(re^{i\theta})| \geq r\eta(r)/3, \quad r \geq r_0,
\]

where \( r_0 \) is a positive constant.

Let \( \{\lambda_n\} \) be an increasing sequence of positive numbers such that

\[
A r\eta(r) \geq n\lambda(r) \geq r\eta(r), \quad r \geq b > 0,
\]

where \( A \) is a constant greater than 1. We shall show that

\[
\psi(z) = \prod_{n=1}^\infty (1 + z/\lambda_n)
\]

has the required properties.

We have, as \( R \to \infty \),

\[
\sum_{\lambda_n \leq R} 1/\lambda_n = \int_0^R t^{-1}dn(t) = R^{-1}n(R) + \int_0^R t^{-2}n(t) \, dt \\
\leq A\eta(R) + \int_{\lambda_1}^R t^{-1}\eta(t) \, dt = O(1);
\]

hence \( \sum 1/\lambda_n \) converges, and so \( \psi(z) \) is entire.
We then have
\[ \log \psi(z) = \sum_{n=1}^{\infty} \log (1 + z/\lambda_n) = \int_0^\infty \log (1 + z/t) \, dn(t) = \int_0^\infty \frac{zn(t)}{t(t + z)} \, dt, \]
and consequently, if \( |\theta| \leq \pi/2, \)
\[ \log |\psi(z)| = \Re \{\log \psi(z)\} = \int_0^\infty \frac{n(t)}{t} \frac{r^2 + rt \cos \theta}{r^2 + 2rt \cos \theta + t^2} \, dt \]
\[ \geq r^2 \int_0^\infty \frac{n(t) \, dt}{t(r + t)^2} \geq r^2 \int_b^r \frac{\eta(t) \, dt}{(r + t)^2} \]
\[ \geq (r - b)\eta(r)/2 > r\eta(r)/3, \quad r \geq 2b. \]

3. **Analytic functions in a half-plane.** We can now prove the uniqueness theorems which we need.

**THEOREM 1.** Let \( F(z) \) be analytic in \( x > 0 \) and continuous in \( x \geq 0 \). For \( x \geq 0 \), let
\[ (3.1) \quad \log |F(z)| \leq mx \log x + Ax + \sigma(r), \quad m > 0, \]
where \( A \) is a constant, \( \sigma(r) \) is nondecreasing and \( \int^\infty t^{-2} \sigma(t) \, dt \) converges. Let \( F(\lambda_n) = 0, \) where \( \lambda_n \) is nonincreasing, \( \int t^{-1} t \sigma(t) \, dt \) converges. Then \( F(z) = 0. \)

Let \( \phi(z) \) be the function of Lemma 2.1, with the given sequence \( \{\lambda_n\} \) and \( c = m/2. \) (If \( n(t) \neq O(t) \), we can discard \( \lambda_n \)'s until \( n(t) = O(t) \) without affecting (3.2).) Let \( \eta(t) \) be nonincreasing, \( t\eta(t) \to \infty, \) with \( \int^\infty t^{-1} \eta(t) \, dt \) convergent and \( \eta(t) > 3A_1 \delta_1(t) + 3t^{-1} \sigma(t), \) where \( A_1 \) and \( \delta_1 \) are the quantities appearing in (2.4).

Let \( \psi(z) \) be the function of Lemma 2.10, and consider the function
\[ H(z) = \frac{F(z)}{B \Gamma(z)^{\eta(z)}}, \]
where \( B \) is a constant, to be chosen in a moment. Since the zeros of \( \phi(z) \) in \( x > 0 \) are cancelled by the zeros of \( F(z), \) \( H(z) \) is analytic in \( x > 0 \) and continuous in \( x \geq 0. \) We then have, for \( r > r_1, \) where \( r_1 > r_0 \) is a suitable constant,
\[ \log |H(z)| \leq mr \cos \theta \log r + mr \cos \theta \log \cos \theta + Ar \cos \theta + \sigma(r) \]
\[ - mr \cos \theta \log r + mr \theta \sin \theta - \log |\phi(re^{i\theta})| - r\eta(r)/2 - \log B \]
\[ \leq mr \cos \theta (A + \log \cos \theta) + mr \sin \theta + \sigma(r) \]
\[ - r\eta(r)/2 - \log |\phi(re^{i\theta})| - \log B. \]
Hence, for $\pi/4 \leq |\theta| \leq \pi/2$ and $r > r_1$,
\[
\log |H(z)| \leq mr \cos (A + \log \cos \theta) + mr \sin |\theta| (|\theta| - \pi/2)
+ \sigma(r) + Ar_0 \delta_1(r) - r_0/r - \log B
\leq mr \cos (A + \log \cos \theta) - \log B.
\]
If $B$ is chosen larger than $\max |BH(z)|$ for $|z| \leq r_1$, $x \geq 0$, we have $\log |H(z)| < 0$ for $|z| < r_1$, and so
\[
(3.3) \quad \log |H(z)| \leq mr \cos (A + \log \cos \theta), \quad r \geq 0, \pi/4 \leq |\theta| \leq \pi/2.
\]
For $0 \leq |\theta| \leq \pi/2$ and for an unbounded sequence of values of $r$ we also have
\[
\log |H(z)| \leq C,
\]
where $C$ is a constant.

Now let $K_\omega(z) = H(z)e^{\omega z}$, $\omega > 0$. Then by (3.3), for $\pi/4 \leq |\theta| \leq \pi/2$,
\[
\log |K_\omega(z)| \leq mr \cos (\log \cos \theta + A + \omega)
\]
while
\[
\log |K_\omega(z)| \leq (C + \omega)r, \quad 0 \leq |\theta| \leq \pi/2,
\]
for an unbounded sequence of values of $r$.

If $\theta_0$ is chosen so that $\cos \theta_0 < \varepsilon^{-\omega}$, we then have $\log |K_\omega(z)| \leq 0$ for $\theta = \pm \theta_0$, $\theta_0 < \pi/2$, and so, by a well known
Phragmén-Lindelöf theorem [10, p. 177],
\[
\log |K_\omega(z)| \leq 0 \text{ for } 0 \leq |\theta| \leq \theta_0. \text{ In particular, then,}
\]
\[
|H(x)e^{\omega x}| \leq 1
\]
for $x > 0$. Letting $\omega \to \infty$, we obtain $H(z) \equiv 0$. Hence $F(z) \equiv 0$.

**Theorem 2.** Let $F(z)$ be analytic in $x > 0$. For $x \geq 1$, let
\[
(3.4) \quad \log |F(x + iy)| \leq k |y| + \sigma(r),
\]
where $\int_0^x t^{-2} \sigma(t) dt$ converges. Let $F(\mu_n) = 0$, where $\mu_n > 0$ and
\[
n_\mu(t) \geq (k/\pi)t + \delta(t),
\]
$\delta(t)$ is nonincreasing, and $\int_0^x t^{-1} \delta(t) dt$ diverges. Then $F(z) \equiv 0$.

Let $\phi(z)$ be the function of Lemma 2.7, with $\mu_n$ replacing $\lambda_n$. In the part
of $x \geq 1$ where $\pi/4 \leq |\theta| \leq \pi/2$, we have
\[
\log |F(z)/\phi(z)| \leq \sigma(r) - Ar_0 \delta_2(r),
\]
and
\[
\log |F(z)/\phi(z)| = O(r)
\]
on a sequence of semicircles of unbounded radii. Applying Carleman's theorem
[10, p. 130] to $F(z)/\phi(z)$ in this half-plane, we obtain
0 \leq O(1) + \pi^{-1} \int_1^R \{ y^{-2} \sigma(y) - A_3 y^{-1} \delta_2(y) \} \, dy

= O(1) - A_3 \int_1^R y^{-1} \delta_2(y) \, dy.

Since \int_0^\infty y^{-1} \delta_2(y) \, dy diverges, this leads to a contradiction unless \( F(z) = 0 \).

4. Completeness of sets \( \{ t^{\lambda_n} e^{-t} \} \). We can always suppose that \( c = 1 \), by making a change of variable if necessary. We say that a set of functions \( \{ f_n(t) \}_{n=1}^\infty \) is complete with respect to a class \( C \) if

\[
\int_0^\infty f_n(t) g(t) \, dt = 0, \quad n = 1, 2, \ldots; g \in C,
\]

implies \( g(t) = 0 \) in \( C \).

**Theorem 3(\textsuperscript{\dagger}).** If \( \{ \lambda_n \} \) is an increasing positive sequence with

\begin{equation}
(4.1) \quad n(t) \geq t/2 - t \delta(t), \quad \int_0^\infty t^{-1} \delta(t) \, dt < \infty,
\end{equation}

\( \delta(t) \) nonincreasing, then the set \( \{ t^{\lambda_n} e^{-t} \} \) is complete with respect to every \( L^p(0, \infty) \), \( 1 \leq p \leq \infty \).

We have to show that if

\[
\int_0^\infty e^{-t} f_n(t) \, dt = 0, \quad n = 1, 2, \ldots; f(t) \in L^p(0, \infty),
\]

then \( f(t) = 0 \) almost everywhere. We consider the function

\[
F(z) = \int_0^\infty e^{-t^z} f(t) \, dt
\]

and show that, if \( F(\lambda_n) = 0 \) \( (n = 1, 2, \ldots) \), then \( F(z) = 0 \). That \( f(t) = 0 \) almost everywhere then follows from the uniqueness theorem for Mellin transforms. \( F(z) \) is analytic in \( x > 0 \) and continuous in \( x \geq 0 \). If \( p = \infty \), we have \( |f(t)| \leq M \) and so

\[
|F(z)| \leq \int_0^\infty e^{-t^z} dt = \Gamma(x + 1).
\]

If \( p = 1 \), let \( \int_0^\infty |f(t)| \, dt = M \); then

\[
|F(z)| \leq M \sup_{0 \leq t < \infty} e^{-t^z} = e^{-2x^2}.
\]

If \( 1 < p < \infty \), let \( p' = p/(p-1) \); then, by Hölder's inequality,

\[\text{(4) A result of Fuchs [3a], stronger when } \lambda_{n+1} - \lambda_n \geq \varepsilon > 0, \text{ is quoted in } \S 1.\]
Using Stirling's formula when \( p > 1 \), we thus have in all cases

\[
\log |F(z)| \leq x \log x - x + B \log x,
\]

where \( B \) depends only on \( p \).

\( F(z) \) thus satisfies the hypotheses of Theorem 1, with \( m = 1, \sigma(r) = B \log r \). Consequently \( F(z) = 0 \).

Theorem 4. Let \( \{a_n\} \) be a real positive sequence with

\[
n_o(t) > m(t/2) - \delta(t),
\]

where \( m \) is an integer greater than 1 and \( \delta(t) \) satisfies the conditions of Theorem 3. Then if \( \{a_n\} \) is divided into \( m \) exhaustive and nonoverlapping sequences \( \{a_{n_1}\}, \ldots, \{a_{n_m}\} \), at least one of the sets \( \{a_{n_k}e^{-t}\} \) is complete with respect to \( L^p(0, \infty) \) \( (1 \leq p \leq \infty) \).

Suppose that the first \( m - 1 \) of the sets are not complete in a specified \( L^p \); then there exist \( m - 1 \) functions \( \phi_k(t) \) of \( L^p \) such that

\[
F_k(z) = \int_0^\infty t^e^{-t}\phi_k(t)dt \neq 0,
\]

\[
F_k(a_{n_k}) = 0, \quad n = 1, 2, \ldots; \quad k = 1, 2, \ldots, m - 1.
\]

If \( \phi_m(t) \) is orthogonal to all the functions \( t^{\lambda_n}e^{-t} \), let

\[
F_m(z) = \int_0^\infty t^e^{-t}\phi_m(t)dt,
\]

and let

\[
F(z) = F_1(z)F_2(z) \cdots F_m(z).
\]

Then \( F(a_n) = 0 \) \( (n = 1, 2, \ldots) \). By the reasoning of Theorem 3, we have

\[
\log |F(z)| \leq m(x \log x - x + B \log x), \quad x \geq 0.
\]

\( F(z) \) now satisfies the hypotheses of Theorem 1, and consequently \( F(z) = 0 \). Hence \( F_m(z) = 0 \), and so the set \( \{t^{\lambda_n}e^{-t}\} \) is complete.

5. A generalized moment problem. Let \( \{\mu_n\} \) be a sequence of positive numbers such that

\[
\mu_n = \int_0^\infty t^\lambda d\alpha(t)
\]
for at least one nondecreasing \( \alpha(t) \). The problem is to find conditions on \( \{\mu_n\} \) which imply that \( \alpha(t) \) is unique if normalized \( (\alpha(0+) = 0, \alpha(t) = \{\alpha(t+) + \alpha(t-)\}/2) \). The following theorem gives a sufficient condition for uniqueness which is better than conditions given previously [2] for non-integral \( \lambda_n \), though weaker than Carleman's criterion [9, p. 20] for \( \lambda_n = n \).

**Theorem 5.** Let \( \{\lambda_n\} \) be an increasing sequence such that

\[
\lambda_n \leq \lambda_{n-1}(1 + 1/\log \lambda_{n-1})
\]

and \( n(t) \geq t(1 - \delta(t)) \), where \( \delta(t) \) satisfies the conditions of Theorem 3. Then two normalized nondecreasing functions \( \alpha_1(t), \alpha_2(t) \) satisfying

\[
\mu_n = \int_0^\infty t^n d\alpha_j(t), \quad j = 1, 2,
\]

are identical if there is a constant \( \sigma \) such that

\[
\mu_n^{1/(2\lambda_n)} \leq \sigma \lambda_n, \quad n = 1, 2, \ldots.
\]

Suppose that \( \alpha_1(t) \) and \( \alpha_2(t) \) satisfy (5.3), and let

\[
F(z) = \int_0^\infty t^z d\{\alpha_1(t) - \alpha_2(t)\},
\]

so that \( F(\lambda_n) = 0 \). Then

\[
|F(z)| \leq \int_0^\infty t^z d\{\alpha_1(t) + \alpha_2(t)\};
\]

and so \( |F(\lambda_n + iy)| \leq 2\mu_n \) and

\[
\log |F(\lambda_n + iy)| \leq \log 2 + \log \mu_n
\]

\[
\leq \log 2 + 2\lambda_n \log (\sigma \lambda_n)
\]

\[
\leq 2\lambda_{n-1} \log \lambda_{n-1} + A\lambda_{n-1},
\]

where \( A \) is independent of \( n \). For, we may assume without loss of generality that \( \lambda_{n-1} \geq e \), and then, since

\[
\lambda_n \leq \lambda_{n-1} + \lambda_{n-1}/\log \lambda_{n-1},
\]

we have

\[
\lambda_n \log (\sigma \lambda_n) \leq (\lambda_{n-1} + \lambda_{n-1}/\log \lambda_{n-1})(\log \lambda_n + \log \sigma)
\]

\[
\leq (\lambda_{n-1} + \lambda_{n-1}/\log \lambda_{n-1})(\log \lambda_{n-1} + \log (1 + 1/\log \lambda_{n-1}) + \log \sigma)
\]

\[
\leq (\lambda_{n-1} + \lambda_{n-1}/\log \lambda_{n-1})(\log \lambda_{n-1} + \log \sigma + 1/\log \lambda_{n-1})
\]

\[
\leq \lambda_{n-1} \log \lambda_{n-1} + \lambda_{n-1}(3 + 2 \log \sigma),
\]
and (5.5) follows. Hence, if $\lambda_{n-1} \leq x < \lambda_n$,

$$
\log |F(x + iy)| \leq \log |F(\lambda_n + iy)| \\
\leq 2\lambda_{n-1} \log \lambda_n \gamma + A\lambda_{n-1} \\
\leq 2x \log x + Ax.
$$

Theorem 1 now shows that $F(z) \equiv 0$, and consequently $\alpha_1(t) = \alpha_3(t)$.

6. Gap theorems for power series in half-planes. We next apply the results of §4 to power series.

**Theorem 6(6).** Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad |z| < 1,$$

where $\{\lambda_n\}$ is an increasing sequence of positive integers such that

$$(6.1) \quad n_\lambda(r) \leq r/2 + r\delta(r);$$

here $\delta(t)$ satisfies the conditions of Theorem 3; namely, $\delta(t)$ nonincreasing and $\int_{0}^{\infty} \delta(t) dt$ convergent. Then if $f(z)$ is not a constant it cannot be analytic and bounded in any half-plane having the origin as an interior point.

Theorem 6 follows, by Theorem 3, from the following result, which establishes the equivalence of gap theorems with conclusions like that of Theorem 6 and completeness theorems for sets $\{t^{\lambda}e^{-ct}\}$.

**Theorem 7.** Let $\{\lambda_n\}$ be a sequence of positive integers, $\{\mu_n\}$ the complementary sequence. The following two statements are equivalent.

(A) The set $\{t^{\mu}e^{-ct}\}$ is complete in $L^2(0, \infty)$.

(B) Every function $f(z) = \sum_{n=0}^{\infty} c_n z^{\lambda_n}$, not a constant, either has a singular point or is unbounded in every half-plane containing the origin in its interior.

For the application to Theorem 6, we need only note that (6.1) implies

$$n_\lambda(r) \geq r/2 - r\delta_1(r),$$

where $\delta_1(r)$ satisfies the same conditions as $\delta(r)$.

Theorem 7 is proved by Boas and Pollard [2a].

**Theorem 8.** Let $\{c_n\}$ be a sequence of complex numbers and let $\{\lambda_n\}, \{\mu_n\}$ be two complementary sequences of the positive integers. Let

$$F_1(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}, \quad F_2(z) = \sum_{n=1}^{\infty} c_{\mu_n} z^{\mu_n}.$$

Then, in every half-plane having the origin as an interior point, one at least of

(6) A stronger result follows by the same reasoning from the result of Fuchs [3a] quoted in §1.
$F_1(z)$ and $F_2(z)$ has a singular point, is unbounded, or is identically a constant.

This follows from Theorem 4, with $m = 2$, together with Theorem 7.

7. Gap theorems for power series in angles. We now establish the remainder of the theorems outlined in §1.

**Theorem 9.** Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| \leq 1,$$

where $c_n = 0$ except for $n = \lambda_n$. Let

$$n_\alpha(t) \leq \pi^{-1} \alpha t - t \delta(t), \quad \alpha < \pi,$$

where $\delta(t)$ satisfies the conditions of Theorem 2, namely, $\delta(t)$ nonincreasing, and $\int_0^\infty \delta(t) \, dt$ divergent. If $f(z)$ is not a constant, it is unbounded or has a singular point in every closed angle of opening $2\alpha$, with vertex at 0.

The special case in which the upper density of $\{\lambda_n\}$ is less than $\alpha/\pi$ is not quite included in the result of Mandelbrojt quoted in §1. The case $\alpha = \pi/2$ is included in Theorem 6.

We may suppose that the angle is $|\theta| \leq \alpha$. Let $f(z)$ be analytic and bounded in this angle; we shall show that $f(z)$ is a constant.

We have

$$2\pi c_n = -i \int_C z^{-n-1} f(z) \, dz,$$

where $C$ consists of the arc $\alpha \leq \theta \leq 2\pi - \alpha$ of the circle $|z| = 1$, the line segments $\theta = \pm \alpha$, $0 \leq \theta \leq R$ ($R > 1$), and the arc $-\alpha \leq \theta \leq \alpha$ of $|z| = R$. The integral along the latter arc is $R^{-n} \int_{-\alpha}^{\alpha} e^{i n \theta} f(R e^{i \theta}) \, d\theta$, which approaches zero as $R \to \infty$ provided that $n > 0$. Hence, for $n \geq 1$,

$$2\pi c_n = i e^{-i\alpha} \int_1^{\infty} f(i e^{i \alpha}) t^{-n-1} \, dt - i e^{-i\alpha(2\pi - \alpha)} \int_1^{\infty} f(i e^{-i \alpha}) t^{-n-1} \, dt$$

$$+ \int_{\alpha}^{2\pi - \alpha} e^{-i \delta(t)} e^{i \theta} \, d\theta.$$

Thus for $n \geq 1$,

$$(-1)^n 2\pi c_n = i e^{i\alpha(n-\alpha)} \int_1^{\infty} t^{-n-1} f(i e^{i \alpha}) \, dt - i e^{-i(n\pi - \alpha)} \int_1^{\infty} t^{-n-1} f(i e^{-i \alpha}) \, dt$$

$$- \int_{(\pi - \alpha)}^{\pi - \alpha} f(-e^{-i \theta}) e^{i \theta} \, d\theta.$$

We now write

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\[ F(z) = i e^{i \pi (\pi - \alpha)} \left( \int_{1}^{\infty} t^{-(\pi - \alpha)} f(te^{i\alpha}) dt - i e^{-i \pi (\pi - \alpha)} \int_{1}^{\infty} t^{-(\pi - \alpha)} f(te^{-i\alpha}) dt \right) - \int_{-(\pi - \alpha)}^{\pi - \alpha} f(-e^{-i\theta}) e^{i\theta} d\theta. \]

Then \( F(z) \) is analytic for \( x > 0, \)
\[ |F(x + iy)| \leq A e^{(\pi - \alpha)\nu}, \quad x \geq 1, \]
where \( A \) is a constant, and \( F(\mu_n) = 0, \) where \( \{\mu_n\} \) is the set of positive integers which are not \( \lambda_n \)'s. We have
\[ n_\nu(r) \geq ((\pi - \alpha)/\pi) r + r \delta(r) + \text{constant}. \]

By Theorem 2, \( F(z) \equiv 0, \) and so, in particular, \( c_n = 0 \) for \( n \geq 1. \) Thus \( f(z) \equiv c_0. \)

As an application of Theorem 9, we can improve a result of Mandelbrojt and Ulrich \([6]\) on quasi-analytic functions. In their Theorem 1, the condition \( \lim \sup (\nu_m/m) < 2, \) which can also be written \( \lim \sup n_\nu(r)/r^{1/2}, \) can be replaced by \( n_\nu(r) > r/2 + r \delta(r), \) where \( \delta(r) \) satisfies the conditions of Theorem 9.

It is natural to seek a decomposition theorem analogous to Theorem 8. The most obvious one would state that, if \( \{n\} = \{\lambda_n\} + \{\mu_n\}, \) then, in every angle of opening exceeding \( \pi, \) one of the series \( \sum c_{\lambda_n} z^{\lambda_n}, \sum c_{\mu_n} z^{\mu_n} \) has a singularity, is unbounded, or is constant; this is a weaker result than Theorem 8. However, the following two theorems are not contained in Theorem 8.

**Theorem 10.** Let \( \{c_n\} \) be any sequence. Let the positive integers be divided into disjoint sets \( \{\lambda_n\} (j = 1, 2, \ldots, m). \) Let \( \alpha_1 + \cdots + \alpha_m > \pi. \) Then, if functions \( f_j(z) \) are defined by the series
\[ f_j(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}, \]
and if we associate with \( f_j(z) \) an angle of opening \( 2\alpha_j, \) with vertex at the origin, at least one \( f_j(z) \) has a singular point, is unbounded, or is constant in the corresponding angle.

Suppose that every \( f_j(z) \) is analytic and bounded in the corresponding angle, and let \( c_0 = 0. \) By replacing \( f_j(z) \) by \( f_j(\exp(i\beta)) \) with a suitable \( \beta, \) we can make the \( j \)th angle be \( |\theta| \leq \alpha_j. \) We then form the functions \( F_j(z) \) as in the proof of Theorem 9; \( F_j(z) \) satisfies
\[ |F_j(z)| \leq A_\nu e^{(\pi - \alpha_j)\nu}. \]

We have \( F_j(\mu_n) = 0, \) where the \( \mu_n \) are the \( n \)'s which are not \( \lambda_n \)'s. Let
\[ F(z) = F_1(z)F_2(z) \cdots F_m(z). \]
Then $F(z)$ has an $(m - 1)$-fold zero at every $n$, and satisfies

$$|F(z)| \leq A e^{m \pi - (\alpha_1 + \cdots + \alpha_m) |\nu|}.$$ 

If $\pi(m - 1) > m \pi - (\alpha_1 + \cdots + \alpha_m)$, that is, $\alpha_1 + \cdots + \alpha_m > \pi$, $F(z) \equiv 0$ by Theorem 2 with $k = m \pi - (\alpha_1 + \cdots + \alpha_m)$; we do not use the full force of Theorem 2. Then one $F_j(z)$ at least must vanish identically.

We can obtain a somewhat different result if we start from a sequence which already has gaps.

**Theorem 11.** Let $\{c_n\}$ be a sequence in which $c_n = 0$ except for $n = \lambda_n$, where

$$n_\lambda(t) \leq ct - \delta(t),$$

with the same conditions on $\delta(t)$ as in Theorem 9. If $\{\lambda_n\}$ is divided into $m$ disjoint sequences $\{\lambda_n\}$ $(k = 1, \cdots, m; m > 1)$ and $f_k(z)$ are the corresponding functions, then in every angle of opening $2\pi c(1 - 1/m)$, with vertex at $z = 0$, at least one of $f_k(z)$ has a singular point, is unbounded, or is a constant.

We suppose that all the $f_k(z)$ are analytic and bounded in the angle $|\theta| \leq \alpha$ and form functions $F_k(z)$ as in the proof of Theorem 9. Let $F(z) = F_1(z) \cdots F_m(z)$. Let $\{\mu_n\}$ be the sequence of integers which are not $\lambda_n$'s. Then $F(\lambda_n) = 0$ for every $n$, and in addition $F(z)$ has an $m$-fold zero at each $\mu_n$. Hence the number of zeros of $F(z)$ in $(0, t)$ is

$$t + (m - 1) \left\{ (1 - c)t + \delta(t) \right\} + O(1)$$

$$= \left\{ m(1 - c) + c \right\} t + (m - 1) \delta(t) + O(1).$$

On the other hand, $\log |F_k(z)| = O(e^{(c - \alpha) |\nu|})$, and so

$$\log |F(z)| = O(e^{m(c - \alpha) |\nu|}).$$

By Theorem 2, $F(z) \equiv 0$, and so some $F_k(z) \equiv 0$, provided that

$$\pi \left\{ m(1 - c) + c \right\} \geq m(\pi - \alpha),$$

or $\alpha \geq \pi c(1 - 1/m)$.

**References**


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