ON ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER SERIES

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Introduction. The results of this paper are extensions of corresponding results for simple Fourier series, given by one of the authors (cf. [5])(1). The main problem was to study the relationship between the mean modulus of a function \( f(x) \) and series of the type \( \sum |c_n|^{\beta} \), \( \beta > 0 \), where the \( c_n \) are the Fourier coefficients of \( f(x) \). We obtain here analogous results, employing spherical means of a function of several variables. These means were first used by Bochner [1] in the study of summation of multiple Fourier series.

A particular result is: if \( a_{n_1} \ldots a_{n_k} \) are the Fourier coefficients of \( f(x_1, \ldots, x_k) \), and \( f \) satisfies a Lipschitz condition of degree \( \alpha \), then \( \sum |a_{n_1} \ldots a_{n_k}|^{\beta} < \infty \) for \( \beta > 2\kappa/(\kappa + 2\alpha) \), while the series may be divergent for \( \beta = 2\kappa/(\kappa + 2\alpha) \). For some previous results concerning the absolute convergence of double Fourier series cf. [3].

1. Notations. We denote by capital letters vectors in the \( k \)-dimensional space, so that \( X = (x_1, x_2, \ldots, x_k) \), \( N = (n_1, n_2, \ldots, n_k) \); \( |N| = (\sum n_i^2)^{1/2} \) is the norm of \( N \); \( NX = \sum n_i x_i \) is the scalar product of \( N \) and \( X \). The \( x_i, \ldots, x_k \) are real variables, the \( n_1, \ldots, n_k \) are integers. \( f(x_1, \ldots, x_k) = f(X) \) is a real-valued integrable function of period \( 2\pi \) in each variable. The formal Fourier series of \( f(X) \) is

\[
 f(X) \sim \sum_{n_1}^{+\infty} \cdots \sum_{n_k}^{+\infty} c_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \cdots + n_k x_k)} = \sum c_N e^{iNX},
\]

where

\[
 c_N = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(X) e^{-iNX} dX.
\]

\( J_\mu(x) \) is the Bessel function of order \( \mu \geq 0 \):

\[
 J_\mu(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{\mu+2r}}{r! \Gamma(\mu + r + 1)};
\]

we put

\[
 \alpha_\mu(x) = \frac{2^\mu \Gamma(\mu + 1) J_\mu(x)}{x^\mu} = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r} \Gamma(\mu + 1)}{4^r r! \Gamma(\mu + r + 1)},
\]

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(1) Numbers in brackets refer to the bibliography at the end of the paper.
\[ A_n(X) = \sum_{|N| \leq n} c_N \exp (iNX), \text{ so that } f(X) \sim \sum_{n=0}^{\infty} A_n(X). \]

We shall denote by \( \omega(t) \) a positive function of \( t \), decreasing to zero as \( t \to 0 \)

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**Lemma 1.** If \( R_x(n) \) is the number of lattice points in the sphere \( \sum_1^n x_i^2 \leq n \), then

\[ R_x(n) = O(n^{s/2}) \equiv O(n^{s/2}), \quad \text{as } n \to \infty. \]

Actually the sharper estimate is known \[\text{cf. 2, p. 825}\]:

\[ R_x(n) = \frac{\pi^{s/2} n^{s/2}}{\Gamma(1 + \kappa/2)} + O(n^{(s-1)/2(s+1)}). \]

**Lemma 2.** For \( \mu \geq 0 \), \( x \) real or complex,

\[ \alpha_\mu(x) = \frac{2\Gamma(\mu + 1)}{\Gamma(\mu + 1/2) \Gamma(1/2)} \int_0^{\pi/2} \cos (x \cos t) \sin^{2\mu} t dt \]

\[ = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1/2) \Gamma(1/2)} \int_0^{\pi} e^{ix \cos t} \sin^{2\mu} t dt. \]

The proof follows on using the cosine series or exponential series and integrating termwise \[6, \text{pp. 47-48}\].

**Corollary.** For real \( x \)

\[ |\alpha_\mu(x)| \leq \frac{2\Gamma(\mu + 1)}{\Gamma(\mu + 1/2) \Gamma(1/2)} \int_0^{\pi/2} \sin^{2\mu} t dt = \alpha_\mu(0) = 1. \]

For \( \mu = 0 \), (2.3) reduces to \( |J_0(x)| \leq 1 \), an inequality given by Hansen \[6, \text{p. 31}\].

**Lemma 3.** For any \( u > 0 \) and a corresponding constant \( b(u) > 0 \), \( b(u) < 1 \)

\[ -\alpha_\mu(x) < 2 \text{ for } x > u; \text{ moreover} \]

\[ 1 - \alpha_\mu(x) > \frac{x^2}{\pi^2(\mu + 1)} \quad \text{for } 0 < x < \pi, \]

and

\[ 1 - \alpha_\mu(x) < (x/2)^2 \frac{1}{\mu + 1} \quad \text{for } x > 0. \]

**Proof.** From (2.2) and (2.3), putting \( 2\Gamma(\mu + 1)/\Gamma(\mu + 1/2) \Gamma(1/2) = \gamma(\mu) \), we have

\[ 1 - \alpha_\mu(x) = \gamma(\mu) \int_0^{\pi/2} \left\{ 1 - \cos (x \cos t) \right\} \sin^{2\mu} t dt > 0 \quad \text{for } x > 0. \]
It is known that $J_\mu(x) \to 0$ as $x \to \infty$, hence $\alpha_\mu(x) \to 0$; thus for some $b(u) > 0$

$$1 - \alpha_\mu(x) > b(u)$$

for $x > u$.

Furthermore from (2.4) and (2.3)

$$1 - \alpha_\mu(x) < 2\gamma(\mu) \int_0^{\pi/2} \sin^{2\mu} \theta d\theta = 2,$$

for $x > 0$.

Finally, for $0 < x < \pi$,

$$1 - \cos (x \cos t) = 2 \sin^2 \left( \frac{x}{2} \cos t \right),$$

hence

$$1 - \alpha_\mu(x) > 2\gamma(\mu) \int_0^{\pi/2} \cos^2 \theta \sin^{2\mu} \theta d\theta = \frac{2\gamma(\mu)}{\pi^2} \frac{x^2}{\pi^2(\mu + 1)}$$

and

$$1 - \alpha_\mu(x) < \frac{\gamma(\mu)}{2} \int_0^{\pi/2} \cos^2 \theta \sin^{2\mu} \theta d\theta = \frac{x^2}{4(\mu + 1)};$$

this proves the lemma.

**Lemma 4.** Let $h$ be real, $r > 0$, $\delta > 0$, then the following statements are equivalent:

(2.5) \[ \sum_{n=1}^{\infty} n^{\rho h-1} \omega(\delta n^{-r}) < \infty, \]

(2.5') \[ \sum_{\lambda=1}^{\infty} 2^{\lambda h} \omega(\delta \cdot 2^{-\lambda r}) < \infty, \]

(2.6) \[ \int_1^{\infty} t^{h-1} \omega \left( \frac{1}{t} \right) dt < \infty. \]

**Proof.** We have for $rh \geq 1$

$$2^{(\rho-1)h} \omega(\delta \cdot 2^{-\lambda r}) < \sum_{\rho=rh-1}^{2\lambda-1} n^{\rho h-1} \omega(\delta \cdot n^{-r}) < 2^{\lambda h} \omega(\delta \cdot 2^{-(\rho-1)r}),$$

hence

$$2^{-rh} \sum_{\lambda=1}^{\infty} 2^{\lambda h} \omega(\delta \cdot 2^{-\lambda r}) < \sum_{n=1}^{\infty} n^{\rho h-1} \omega(\delta \cdot n^{-r}) < 2^h \sum_{\rho=0}^{\infty} 2^{\lambda h} \omega(\delta \cdot 2^{-\lambda r}),$$
with similar inequalities for $rh < 1$; hence (2.5) and (2.5') are equivalent. We also have for $rh \leq 1$

$$
\int_{n}^{n+1} x^{rh-1} \omega(\delta x^{-r}) \, dx < \sum_{n=1}^{n} x^{rh-1} \omega(\delta n^{-r}) < \int_{n-1}^{n} x^{rh-1} \omega(\delta x^{-r}) \, dx,
$$

hence

$$
\int_{1}^{\infty} x^{rh-1} \omega(\delta x^{-r}) \, dx < \sum_{1}^{\infty} n^{rh-1} \omega(\delta n^{-r}) < \int_{0}^{\infty} x^{rh-1} \omega(\delta x^{-r}) \, dx,
$$

with similar inequalities for $rh > 1$; the substitution $x^{r} = \delta t$ yields the equivalence of (2.5) and (2.6). This proves the lemma.

In view of (2.6), $r$ and $\delta$ are not necessarily the same in the different statements.

**Corollary.** The following statements are equivalent:

$$
\sum_{n=1}^{\infty} n^{(\kappa/r)-1} \omega(\delta n^{-1/2}) < \infty
$$

and

$$
\sum_{1}^{\infty} 2^{\lambda/2} \omega(\delta \cdot 2^{-\lambda/2}) < \infty.
$$

This follows on putting $r = 1/\kappa$ in (2.5), and $r = 1/2$ in (2.5').

**Lemma 5.** If $a_{r} \geq 0$, and $r > 0$, then the two statements are equivalent:

(2.7)

$$
\sum_{i=1}^{\infty} a_{r} \left| 1 - \alpha_{\mu}(tv^{1/2}) \right|^{r} = O_{\omega}(t) \quad \text{as } t \to 0,
$$

and

(2.8)

$$
n^{r} \sum_{1}^{n} v^{r} a_{r} + \sum_{n+1}^{\infty} a_{r} = O_{\omega}(\delta n^{-1/2}) \quad \text{as } n \to \infty,
$$

$\delta$ being an arbitrary positive number.

Assume first that (2.8) holds; given $t > 0$ choose

$$
n = \left[ \delta^{2} t^{-2} \right] \leq \delta^{2} t^{-2} < n + 1;
$$

then from Lemma 3

$$
\sum_{1}^{n+1} a_{r} \left| 1 - \alpha_{\mu}(tv^{1/2}) \right|^{r} < \frac{t^{2r}}{4r(\mu + 1)^{r}} \sum_{1}^{n+1} v^{r} a_{r} = O(t^{2} n^{r} \omega(\delta(n + 1)^{-1/2})) = O_{\omega}(t),
$$

and

$$
\sum_{n+2}^{\infty} a_{r} \left| 1 - \alpha_{\mu}(tv^{1/2}) \right|^{r} < 2 \sum_{n+2}^{\infty} a_{r} = O(\delta(n + 1)^{-1/2}) = O_{\omega}(t).
$$

Conversely, if (2.7) holds, choose for a given $n$ and $\delta > 0$
\[ t = \min (\pi n^{-1/2}, \delta n^{-1/2}), \]

then
\[ \sum_{1}^{n} a_{r} |1 - \alpha_{r}(t^{1/2})|^{r} > \frac{f^{2r}}{\pi^{2r}(\mu + 1)^{r}} \sum_{1}^{n} r^{r} a_{r}, \]

hence
\[ n^{-r} \sum_{1}^{n} r^{r} a_{r} = O(\omega(t)) = O(\delta n^{-1/2}). \]

Furthermore, using again Lemma 3, we have
\[ \sum_{n+1}^{\infty} a_{r} |1 - \alpha_{r}(t^{1/2})|^{r} b \sum_{n+1}^{\infty} a_{r} \quad \text{(}b \text{ a constant)}, \]

hence
\[ \sum_{n+1}^{\infty} a_{r} = O(\omega(t)) = O(\delta n^{-1/2}). \]

This proves the lemma. It follows that if (2.8) holds for some \( \delta > 0 \), it holds for any \( \delta > 0 \).

**Lemma 6.** Assume that for some \( \delta > 0 \)
\[ (2.9) \sum_{1}^{\infty} \omega(\delta 2^{-\lambda} n^{-1/2}) = O(\omega(\delta n^{-1/2})), \quad \text{as} \quad n \to \infty, \]

and let \( r > 0, a_{r} \geq 0 \); then the following statements are equivalent:
\[ (2.10) n^{-r} \sum_{1}^{n} r^{r} a_{r} + \sum_{n+1}^{\infty} a_{r} = O(\delta n^{-1/2}), \quad n \to \infty, \]
\[ (2.11) n^{-r} \sum_{1}^{n} r^{r} a_{r} = O(\delta n^{-1/2}), \]
\[ (2.12) \sum_{1}^{\infty} a_{r} |1 - \alpha_{r}(t^{1/2})|^{r} = O(\omega(t)), \quad t \to 0. \]

The equivalence of (2.10) and (2.11) follows from Lemma (2.5) in [5]; the equivalence of (2.11) and (2.12) follows from Lemma 5. This proves Lemma 6.

**Lemma 7.** Young-Hausdorff inequality. If \( 1 < p \leq 2 \), and
\[ f(X) \sim \sum c_{N} \exp (iNX), \]

then
\[ (2.13) \{ \sum |c_{N}|^{p'} \}^{1/p'} \leq M_{p}(f) = M_{p}f, \]

and
where \( \frac{1}{p} + \frac{1}{p'} = 1 \), and

\[
M_p^p f = \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} |f(x)|^p \, dx
\]

(cf. [4]).

Denote by \( f(x; t) \) the spherical mean of \( f(x) \) over the surface of the sphere of radius \( t \) and center \( x \); then [1, p. 177]

\[
f(x; t) = (2\pi)^{-k/2} \Gamma\left(\frac{k}{2}\right) \int_{\sigma} f(x_1 + t\xi_1, \ldots, x_k + t\xi_k) \, d\sigma_t
\]

(2.14)

\[
= \sum_{n=0}^{\infty} c_N \mu_n(t | N |) \exp (iNX)
\]

\[
\sim \sum_{n=0}^{\infty} \alpha_n(t n^{1/2}) A_n(x),
\]

\( \sigma \) denotes the unit sphere \( \xi_1^2 + \cdots + \xi_k^2 = 1 \), \( d\sigma_t \) its \( (k-1) \)-dimensional volume element. Thus, putting \( f(x; t) - f(x) = \phi(x; t) \), we have

\[
\phi(x; t) \sim \sum_{n=0}^{\infty} c_N \{ \alpha_n(t | N |) - 1 \} \exp (iNX)
\]

\[
\sim \sum_{n=0}^{\infty} \{ \alpha_n(t n^{1/2}) - 1 \} A_n(x).
\]

**Lemma 8.** If \( M_1 \phi(X; t) = O(\omega(t)) \) as \( t \to 0 \), then for any \( \delta > 0 \)

\[
c_N = O\omega\left(\frac{\delta}{|N|}\right)
\]

It follows from (1.1), (1.2) and (2.14) that

\[
c_N \{ \alpha_n(t | N |) - 1 \} = (2\pi)^{-k} \Gamma\left(\frac{k}{2}\right) \int_{\mathbb{R}^k} \phi(x; t) \exp (-iNX) \, dx,
\]

hence

\[
|c_N| |1 - \alpha_n(t | N |)| \leq M_1 \phi(X; t) = O(\omega(t)).
\]

Lemma 8 now follows from Lemma 3, on putting \( t | N | = \delta \).

**Lemma 9.** Let \( P_n(z) = \sum c_n z^n, 1 \leq p \leq \infty \); if

\[
M_p P_n(z) \leq 1
\]

for \( |z| \leq 1 \),

then

\[
M_p P_n'(z) \leq n
\]

(cf. [5, p. 385]).

**Note.** For \( p = \infty \), \( M_p P(z) = \max |P(z)| \) for \( |z| \leq 1 \).
We shall frequently use Hölder’s and Minkowski’s well known inequalities for multiple series and integrals (cf. Hardy, Littlewood, and Pólya, Inequalities, Cambridge, 1934).

3. A theorem on absolute convergence. We now present our main criterion for absolute convergence.

**Theorem 1.** If, with the notations of §2, 1 ≤ p ≤ 2, f(X) ∈ L_p,

\[(3.1)\quad M \phi(X; t) = O(\omega(t)) \quad \text{as} \ t \to 0,\]

and

\[(3.2)\quad \sum_{1}^{\infty} n^{-\beta/p'} \omega^n(\delta n^{-1/2}) < \infty \quad \text{for some } \beta > 0,\]

then

\[(3.3)\quad \sum |c_n|^\beta < \infty.\]

By (3.1) and Lemma 7 for 1 < p ≤ 2, \(\sum |c_n|^{p'}(1 - \alpha_n(t|N|)|^{p'} = O(\omega^{p'}(t)), or

\[(3.4)\quad \sum_{1}^{\infty} \rho_n^{p'}(1 - \alpha_n(tn^{1/2}))^{p'} = O(\omega^{p'}(t)),\]

where \(\rho_n = \rho_n(p)\) is defined by

\[\rho_n^p = \sum_{|N| = n} |c_N|^{p'} = \sum_{|N| = n} |c_{n_1} \cdots c_{n_s}|^{p'} \quad (n_1 + \cdots + n_s = n).\]

By Lemma 5, (3.4) is equivalent to

\[n^{p'} \sum_{1}^{n} \rho_n^{p'} + \sum_{n+1}^{\infty} \rho_n^{p'} = O(\omega^{p'}(\delta n^{-1/2})),\]

hence

\[(3.5)\quad \sum_{n+1}^{2n} \rho_n^{p'} = O(\omega^{p'}(\delta n^{-1/2})).\]

By the Hölder inequality for \(q > 1, 1/q + 1/q' = 1,\)

\[\sum_{n}^{2n} \rho_n^\beta = \sum_{n \leq 1, |N| \leq 2n} |c_N|^\beta \leq \left(\sum_{N} |c_N|^{\beta q} \left(\sum 1\right)^{1/q'}\right)^{1/q};\]

let first \(\beta < p';\) choose

\[\beta q = p', \quad \text{hence} \quad q' = \frac{q}{q-1} = \frac{p'}{p' - \beta}.\]

Now, from (3.5) and (2.1)
\[
\sum_{n} \rho_{\beta} = O_{\omega}(\delta n^{-1/2})(R_{\beta}(2n))^{1-\beta/p'} = O_{n^{(1-\beta/p')/2}} \omega(\delta n^{-1/2}) \].

Putting \(n = 2^\lambda, \lambda = 0, 1, \ldots\), and summing over \(\lambda\) yields
\[
\sum_{\lambda=0}^{\infty} \rho_{\beta} = O \sum_{\lambda=0}^{\infty} 2^{\lambda \alpha(1-\beta/p')} \omega(\delta 2^{-\lambda/2});
\]
the right side is convergent by the corollary to Lemma 4 (with \(h = \kappa(1-\beta/p')\)) and by (3.2). Hence (3.3) holds.

Next if \(\beta = p' > 1\), then (3.3) follows from (2.13), if we assume only, instead of (3.1), that \(f(X) \in L_p\); a fortiori
\[
\sum |c_{N}|^\beta < \infty
\]
for \(\beta \geq p'\).

Finally let \(p = 1\); (3.2) becomes
\[
\sum \omega(\delta n^{-1/2}) < \infty.
\]

Denote by \(r_\epsilon(n)\) the number of lattice points on the circle \(\sum x_i^2 = n\); thus
\[
\sum r_\epsilon(n) = R_\epsilon(n).
\]
From Lemma 8, for any \(\delta > 0\)
\[
\sum_{|N| \leq n} |c_{N}|^\beta = O \sum \omega(\delta |N|^{-1}) = Or_\epsilon(n) \omega(\delta n^{-1/2});
\]
furthermore from (3.6) and Lemma 4 (with \(h = \kappa\))
\[
\sum_{1}^\infty n^{\xi/2-1} \omega(\delta n^{-1/2}) < \infty.
\]

Now, using (2.1), we have
\[
\sum_{1}^n r_\epsilon(\nu) \omega(\delta \nu^{-1/2}) = \sum_{1}^n R_\epsilon(\nu) \omega(\delta \nu^{-1/2}) - \sum_{0}^{n-1} R_\epsilon(\nu) \omega(\delta (\nu + 1)^{-1/2})
\]
\[
\leq R_\epsilon(n) \omega(\delta n^{-1/2}) + \sum_{1}^{n-1} R_\epsilon(\nu) \{ \omega(\delta \nu^{-1/2}) - \omega(\delta (\nu + 1)^{-1/2}) \}
\]
\[
= O_{n^{\xi/2}} \omega(\delta n^{-1/2}) + O \sum_{1}^{n-1} \nu^{\xi/2} \{ \omega(\delta \nu^{-1/2}) - \omega(\delta (\nu + 1)^{-1/2}) \}
\]
\[
= O \sum_{1}^{n} \nu^{\xi/2} (\nu - 1)^{\xi/2} \omega(\delta \nu^{-1/2})
\]
\[
= O(1), \quad \text{as } n \to \infty.
\]

This completes the proof of Theorem 1.
Actually we can prove for $\beta = p'$ that
\[ \sum \rho_n^{p'} \log n < \infty. \]

4. Converse theorems. We give here two theorems to be employed in subsequent sections.

**Theorem 2.** Let $1 \leq p \leq 2$; assume that
\[ (4.1) \quad \sum_{\lambda=1}^{\infty} \omega^p(\delta^n n^{-1/2}) = O\omega^p(\delta n^{-1/2}), \quad \text{as } n \to \infty, \]
and that
\[ (4.2) \quad \sum_{1}^{n} p^{p} \rho_n = O n p \omega^p(\delta n^{-1/2}), \quad \text{as } n \to \infty; \]
then
\[ M_p \phi(X; t) = O\omega(t), \quad \text{as } t \to 0. \]

*Note. If $p = 1$, $p' = \infty$, then $M_p$ means the effective upper bound of $|\phi(X; t)|$ in the region of $X$.\]

Proof. By Lemma 6, (4.2) is equivalent to
\[ \sum_{1}^{n} \rho_n p |1 - \alpha_n(t n^{1/2})|^p = O\omega^p(t), \]
that is,
\[ \sum |c_N|^p |1 - \alpha_n(t |N|)|^p = O\omega^p(t). \]
Now from (2.14) and Lemma 7 (which holds also for $p = 1$)
\[ M_p \phi(X; t) = O\omega(t) \quad \text{as } t \to 0; \]
this proves the theorem.

Note that (4.2) means:
\[ \sum_{|N|^{1/2} \leq n} |N|^p |c_N|^p = O n^p \omega^p(\delta n^{-1/2}). \]

**Theorem 3.** Assume that $\omega(t) \downarrow 0$ as $t \downarrow 0$, and that
\[ (4.3) \quad \sum_{\lambda=1}^{\infty} \omega^p(2^{-\lambda}\delta n^{-1/2}) = O\omega^p(\delta n^{-1/2}) \quad \text{as } n \to \infty. \]
Then a necessary and sufficient condition that
\[ M_p \phi(X; t) = O\omega(t) \quad \text{as } t \to 0, \]
is that
First if (4.4) holds then (4.3) follows by Theorem 2 (for \( p = 2 \)). Conversely if (4.3) holds, then from (2.14) and Lemma 7
\[
\sum |c_n|^2 - \alpha_p(t | N |)^2 = O\omega^2(t),
\]
which by Lemma 6 is equivalent to (4.4).

5. Counter examples. For \( \beta = 1 \), Theorem 1 becomes:

**Theorem 1'.** If \( M_{p\phi}(t) = O\omega(t) \) as \( t \to 0 \), and
\[
\sum n^{-1/p'}\omega(\delta n^{-1}) < \infty, \quad 1 \leq p \leq 2,
\]
then
\[
\sum |c_n| < \infty.
\]

To show that this result is the best possible we shall prove:

**Theorem 4.** Let \( \omega(t) \), in addition to having the property \( \omega(t) \downarrow 0 \) as \( t \downarrow 0 \), be such that
\[
\int \omega(t^{-1})dt = O\omega(u^{-1}) \quad \text{as } u \to \infty,
\]
while
\[
\sum n^{-1/p'}\omega(\delta n^{-1}) = \infty, \quad \text{where } 1 \leq p \leq 2.
\]
Then there exists a function \( f(X) \in L_p, \) such that
\[
M_{p\phi}(X; t) = O\omega(t),
\]
while
\[
\sum |c_n| = \infty.
\]
By Lemma 4 and its corollary (with \( h = \kappa/p \)) (5.1) is equivalent to
\[
\sum_{\lambda=1}^{\infty} 2^{\lambda/p}\omega(2^{-\lambda}) < \infty,
\]
while (5.3) is equivalent to
\[
\sum 2^{\lambda/p}\omega(2^{-\lambda}) = \infty.
\]
We define \( \epsilon_n = \omega(2^{-n}) \), \( \lambda_n = 2^{n+1} + n - 2 \),
\[
g_n(x) = 2^{-n(1+1/p')} \left( \sum_0^{2^n} x^2 \right)^2, \quad n = 0, 1, 2, \ldots ;
\]
so that
\[ \lambda_{n+1} - \lambda_n = 2^{n+1} + 1. \]

Construct the power series
\[ G(Z) = \sum_{n=0}^{\infty} (\varepsilon_n - \varepsilon_{n+1})(z_1 \cdots z_n)^{\lambda_n} \prod_{r=1}^{x} g_n(z_r); \]
then \( G(Z) \) has the formal power series
\[ G(Z) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \gamma_{n_1} \cdots \gamma_{n_k} z_1^{n_1} \cdots z_k^{n_k}. \]

It is clear from the construction that \( \gamma_N \geq 0; \) putting \( Z = 1 \) we find
\[ \sum \gamma_N > \sum_{n=0}^{m} (\varepsilon_n - \varepsilon_{n+1})2^{-\varepsilon_n(1+1/p')(2^{n+1} - 1)}^{2^{n+1}/p} > \sum_{n=0}^{m} (\varepsilon_n - \varepsilon_{n+1})2^{\varepsilon_n/p} \]
\[ > \sum_{n=1}^{m} (\varepsilon_n - \varepsilon_{m+1})2^{\varepsilon_n/p}(1 - 2^{-\varepsilon/n/p}). \]

For a given integer \( l \) choose \( m \) so large that \( \varepsilon_l > 2\varepsilon_{m+1}, \) then
\[ \sum \gamma_N > \frac{1}{2} (1 - 2^{-\varepsilon/l/p}) \sum_{n=1}^{n} \varepsilon_{n}2^{\varepsilon_n/p} \to \infty \quad \text{as} \quad l \to \infty, \]
by (5.3'). Hence
\[ \sum \gamma_N = \infty. \]

We next show that for \( z_r = e^{i\pi r}, \) \( r = 1, 2, \ldots, k, \) \( G(Z) \) becomes the Fourier power series of a function \( F(X) \in L_p. \) Write
\[ (5.6) \quad u_n(Z) = (\varepsilon_n - \varepsilon_{n+1})(z_1 \cdots z_n)^{\lambda_n} \prod_{r=1}^{x} g_n(z_r), \]
then for \( z_r = e^{i\pi r}, \)
\[ M_p u_n = (\varepsilon_n - \varepsilon_{n+1}) \frac{1}{(2\pi)^x} 2^{-\varepsilon_n(1+1/p')} \left( \int_{-\pi}^{\pi} e^{i\pi x} \left| \frac{\sin (2^x + 1)x}{x} \right|^{2p} \sin (2^x + 1)x/2 \right)^{x/p} \]
\[ = O(\varepsilon_n - \varepsilon_{n+1})2^{-\varepsilon_n(1+1/p')} 2^{n(2p-1)x/p} = O(\varepsilon_n - \varepsilon_{n+1}); \]

hence, by Minkowski's inequality,
\[ M_p G \leq \sum_{n=0}^{\infty} M_p u_n = O(1). \]
We shall finally prove (5.4); we have

\[ F(X; t) - F(X) = \sum_{n=0}^{\infty} \{ u_r(X; t) - u_r(X) \}, \]

hence \( M_p \phi \leq \sum_{n=0}^{\infty} M_p \{ u_r(X; t) - u_r(X) \} = \sum_{n=0}^{\infty} + \sum_{n+1}^{\infty} = S_1 + S_2 \), say. Now, by Minkowski's inequality and (2.14),

\[ M_p u_r(X; t) = \Gamma \left( \frac{k}{2} \right) 2^{-r-1} \pi^{3k/2} \left( \int_{-\pi}^{\pi} | \int \{ u_r(x_1 + t\xi_1, \ldots, x_k + t\xi_k) \sigma \} |^p dX \right)^{1/p} \]

\[ \leq \Gamma \left( \frac{k}{2} \right) 2^{-r-1} \pi^{3k/2} \left( \int_{-\pi}^{\pi} | u_r(x_1 + t\xi_1, \ldots, x_k + t\xi_k) \sigma \} |^p dX \right)^{1/p} d\sigma_\xi \]

\[ = 2^{-r} \Gamma \left( \frac{k}{2} \right) \pi^{-k/2} \int_{-\pi}^{\pi} M_p(u_r) d\sigma_\xi; \]

hence, if we use (5.7), \( S_2 = O\epsilon_n \). Furthermore

\[ M_p \{ u_r(X; t) - u_r(X) \} \leq 2^{-r} \Gamma \left( \frac{k}{2} \right) \pi^{-k/2} \int_{-\pi}^{\pi} M_p \{ u_r(X; t) - u_r(X) \} d\sigma_\xi; \]

from the mean value theorem

\[ u_r(X; t) - u_r(X) = t \sum_{\lambda=1}^{\infty} \xi_\lambda \frac{\partial u_r}{\partial x_\lambda} (X; \theta t), \quad 0 < \theta < 1; \]

hence from Minkowski's inequality

\[ M_p \{ u_r(X; t) - u_r(X) \} \leq t \sum_{\lambda=1}^{\infty} | \xi_\lambda | M_p \frac{\partial u_r}{\partial x_\lambda} (X; \theta t) \]

\[ \leq (2\pi)^{-k/2} t \int_{-\pi}^{\pi} M_p \frac{\partial u_r}{\partial x_\lambda} (X) d\sigma_\xi. \]

We now employ Lemma 9; thus from (5.5) and (5.6)

\[ M_p \{ u_r(X; t) - u_r(X) \} = tO(\epsilon_\theta - \epsilon_{r+1}) (2^{r+1} + \lambda_r) = tO2^r(\epsilon_\theta - \epsilon_{r+1}). \]

It follows that

\[ S_1 = tO \sum_{r=0}^{n} 2^r(\epsilon_r - \epsilon_{r+1}) = Ot \sum_{r=0}^{n} (2^{r+1} - 2^r) \epsilon_r \]

\[ = Ot \sum_{r=0}^{n} (2^{r+1} - 2^r) \omega(\delta 2^{-r}) = Ot \int_{0}^{2^n} \omega(\delta x^{-1}) dx \]

\[ = Ot2^n \omega(\delta 2^{-n}), \]
by (5.2). We now choose \( n \) so that for a given positive \( t < \delta \)

\[
2^{n-1} < \delta t^{-1} \leq 2^n, \quad n \geq 1;
\]
then

\[
S_1 = O\omega(t), \quad \text{and} \quad S_2 = O\omega(t),
\]
and the proof of Theorem 4 is complete.

A simpler example, but of a special type, is

\[
G(Z) = \sum_{n=0}^{\infty} (\epsilon_n - \epsilon_{n+1}) \sum_{r=1}^{n} \lambda_r g_n(z_r).
\]

6. **The case \( p = 2 \) and arbitrary \( \beta > 0 \).** For the case \( p = 2 \), Theorem 1 becomes:

**Theorem 1'.** If \( M_{2\phi}(t) = O\omega(t) \), and for some \( \beta > 0 \)

\[
\sum n^{-\beta/2} \omega\left(\delta n^{-1/\beta}\right) < \infty,
\]
then

\[
\sum |c_N|^{\beta} < \infty.
\]

We now prove:

**Theorem 5.** Let \( \omega(t) \), in addition to having the property \( \omega(t) \downarrow 0 \) as \( t \downarrow 0 \), be such that

\[
\int_1^u x^2(x^{-1})dx = O\omega^2(u^{-1}) \quad \text{as} \quad u \to \infty,
\]
while for a given positive \( \beta < 2 \)

\[
\sum n^{-\beta/2} \omega\left(\delta n^{-1/\beta}\right) = \infty.
\]

Then there exists a function \( f(X) \in L_2 \), such that

\[
M_{2\phi}(X; t) = O\omega(t), \quad t \to 0,
\]
but

\[
\sum |c_N|^{\beta} = \infty.
\]

We employ again the polynomial (5.5), where now \( p' = 2 \), and the polynomial (5.6), replacing the factor \( \epsilon_n - \epsilon_{n+1} \) by

\[
(\epsilon_n - \epsilon_{n+1})^{1/\beta} = \alpha_n,
\]
say.

As before \( \epsilon_n = \omega(\delta 2^{-n}) \). On writing

\[
G(Z) = \sum_{n=0}^{\infty} u_n(Z) = \sum \gamma_N z_1^{n_1} \cdots z_\epsilon^{n_\epsilon},
\]
we have again $\gamma_N \geq 0$. Now

$$\sum \gamma_N^\beta > \sum \alpha_n 2^{-3n\beta/2} \left( \sum_{n=1}^{m} \beta_n^\beta \right)$$

$$> \frac{1}{(\beta + 1)^{\beta}} \sum_{n=0}^{m} \alpha_n 2^{n \varepsilon (1-\beta/2)} = \frac{1}{(\beta + 1)^{\beta}} \sum_{n=0}^{m} (\varepsilon_n - \varepsilon_{n+1}) 2^{n \varepsilon (1-\beta/2)}$$

$$> (2^{\varepsilon (1-\beta/2)} - 1) \sum_{n=1}^{m} \beta_n \sum_{1}^{n-1} (\varepsilon_n - \varepsilon_{n+1}) 2^{(n-1) (1-\beta/2)}.$$ 

Hence for $\epsilon^\beta > 2\epsilon_{m+1}$

$$\sum \gamma_N^\beta > \frac{1}{2} (2^{\varepsilon (1-\beta/2)} - 1) \sum_{1}^{n} \epsilon_n 2^{\varepsilon (n-1) (1-\beta/2)}.$$

By the corollary to Lemma 4, (6.2) is equivalent to

$$(6.5) \sum_{\lambda=1}^{\infty} 2^{(1-\beta/2)\lambda} \sum_{\omega} (\delta 2^{-\lambda}) = \infty, \quad \text{or} \quad \sum_{\lambda} 2^{(1-\beta/2)\lambda} \epsilon_n \epsilon_n = \infty,$$

hence $\sum \gamma_N^\beta = \infty$.

Next, in the same manner as in §5, one can prove that

$$(6.6) M^2_2(u_n) = O\alpha_n,$$

it is easily seen that [5, formula (6.14)]

$$(6.7) \alpha_n = (\varepsilon_n - \varepsilon_{n+1})^{2/\beta} = O(\varepsilon_n - \varepsilon_{n+1}),$$

hence

$$M^2_2(G) = \sum_{n=0}^{\infty} M^2_2(u_n) = O \sum_{n} (\varepsilon_n - \varepsilon_{n+1}) = O(1).$$

Finally, to prove (6.3), write

$$M^2_{2\phi} = \sum_{n=0}^{\infty} M^2_2 \{u_r(X; t) - u_r(X)\} = \sum_{n=0}^{\infty} + \sum_{n+1}^{\infty} = T_1 + T_2,$$

say. From (6.6) and (6.7), $T_2 = O\epsilon_n$, while, if we employ Lemma 9 (as in §5)

$$T_1 = i 0 \sum_{0}^{n} 2^{2\nu} \alpha_n = i 0 \sum_{0}^{n} 2^{2\nu} (\varepsilon_n - \varepsilon_{n+1})$$

$$= i 0 \sum_{0}^{n} \alpha_n (2^{2\nu} - 2^{2\nu-1}) = i 0 \sum_{0}^{n} (2^{2\nu} - 2^{2\nu-1}) 2^{\nu} \varepsilon_n$$

$$= i 0 \int_{1}^{x^n} (2^{2\nu} - 2^{2\nu-1}) d\nu = i 0 \int_{1}^{x^n} \gamma \omega^\nu (\gamma^{-1}) d\nu.$$
Employing (6.1), we now get
\[ T_1 = t^n \omega^2(\delta^2 - n). \]

Given a positive \( t \), choose \( n \) so that
\[ 2^n < \delta/t \leq 2^{n+1}; \]
then
\[ T_1 = t^n \omega^2(t) = O\omega^2(t), \quad \text{and} \quad T_2 = O\omega^2(t), \]
hence
\[ M_{\omega}^\phi(t) = O\omega(t) \quad \text{as} \quad t \to 0. \]

This proves Theorem 5.

**Remark.** The conditions (5.2) and (6.1) are equivalent (cf. [5, Remark 6.1]).

7. A continuous function as counter example. In [5, §6] we have employed polynomials

\[ g(z) = \sum_{r=0}^{2(q-1)} a_r^{(q)} z^r, \quad q \text{ a prime} \equiv 1 \pmod{4}, \]

with the following properties
\[ |g(z)| \leq 1 \quad \text{for} \quad |z| \leq 1, \]
\[ |a_r^{(q)}| = q^{-3/2}(\nu + 1), \quad \nu = 0, 1, \ldots, q - 2. \]

On putting \( g(z_1) \cdots g(z_e) = \sum b_N z_1^{a_1} \cdots z_e^{a_e} \), it follows that
\[ \sum |b_N|^\beta > (\sum |a_r^{(q)}|^\beta)^e > q^{-3\beta/2} (1^n + 2^n + \cdots + (q - 1)^n)^e \]
\[ > \frac{1}{e + 1} q^{-3\beta/2} (q - 1)^{\epsilon (\beta + 1)}. \]

Let \( 1 < q_1 < q_2 < \cdots \) be a sequence of primes congruent to 1 (mod 4), and such that for all large \( n \)
\[ 2^{n-1} < q_n < 2^n; \]
denote by \( g_n(z) \) the polynomial (7.1) with \( q = q_n \), and let
\[ \lambda_1 = 0, \quad \lambda_{n+1} = 2(q_1 + \cdots + q_n) - n, \quad n \geq 1; \]
\( \epsilon_n, \alpha_n, \) and \( \nu_n \) are defined as in §6. We assume that \( \omega(t) \) satisfies the conditions
of Theorem 5 and, in case \( 1 < \beta < 2 \), the additional conditions
\[ \int_1^\infty x^{-1} \omega(x^{-1})dx = \int_0^1 r^{-1} \omega(r)dr < \infty, \]
\[(7.6) \quad \int_{t-1}^{\infty} x^{-1} \omega(x^{-1}) \, dx = \int_{0}^{t} \tau^{-1} \omega(\tau) \, d\tau = O(\omega(t)) \quad \text{as } t \to 0.\]

Now, as shown in [5, §6],
\[(7.7) \quad \sum_{1}^{n} 2^n \alpha_n < 2 \int_{1}^{2^n} \omega(x^{-1}) \, dx,\]
and
\[(7.8) \quad \sum_{n+1}^{\infty} \alpha_n < \left\{ \begin{array}{ll}
\epsilon_{n+1} & \text{for } 0 < \beta \leq 1, \\
2 \int_{2^n}^{\infty} x^{-1} \omega(x^{-1}) \, dx & \text{for } 1 < \beta < 2.
\end{array} \right.\]

We define as before
\[(7.9) \quad G(z) = \sum_{1}^{\infty} u_n(z) = \sum \gamma_n z_1^{n_1} \cdots z_k^{n_k}.\]

By (7.1)
\[(7.10) \quad |u_n(z)| \leq \alpha_n \quad \text{for } |z_1| \leq 1, \ldots, |z_k| \leq 1,\]
hence the simple series in (7.9) converges uniformly and defines a continuous function in $|z_1| \leq 1, \ldots, |z_k| \leq 1$. Putting $z_\nu = \exp{(i\xi_\nu)}$, $\nu = 1, \ldots, \kappa$, (7.9) becomes the Fourier power series of a continuous function $F(x_1, \ldots, x_\kappa)$. Furthermore, using (7.2) and (7.3), we have
\[\sum |\gamma_n|^{\beta} > \frac{1}{\kappa + 1} \sum \alpha_n q_n - q_n^{\beta/2} (q_n - 1)^{\beta/(\beta + 1)} \]
\[> b \sum \epsilon_n^{\beta} - \epsilon_{n+1}^{\beta} 2^m (1^{\beta/2}), \quad b \text{ a constant},\]
and the divergence of this series follows from (6.2) as in §6.

We shall finally show that the modulus of continuity of $F(X)$ is majorized by $\omega(t)$. We define the modulus of continuity of $F(X)$ by
\[\max_{|H| \leq t} \max_{(x)} |F(X + H) - F(X)| = \xi(t),\]
where $|H| = (h_1^2 + \cdots + h_k^2)^{1/2}$, and each $x_\nu$ varies in $(-\pi, \pi)$. Now, in view of (7.9),
\[|F(X + H) - F(X)| \leq \sum_{1}^{\infty} |u_\nu(e^{i(x_1 + h_1)}, \ldots, e^{i(x_k + h_k)}) - u_\nu(e^{i\xi}, \ldots)|\]
\[= \sum_{1}^{n} + \sum_{n+1}^{\infty} = V_1 + V_2,\]
say. From (7.10) and (7.8)

$$V_2 < 2 \sum_{n+1}^{\infty} \alpha_n < \begin{cases} 2 \epsilon_n & \text{for } 0 < \beta \leq 1, \\ 2 \int_{\mathbb{R}^n} x^{-1} \omega(x^{-1}) dx & \text{for } 1 < \beta < 2; \end{cases}$$

in view of (7.6) we have in either case

$$V_2 = O(\delta 2^{-n}).$$

To estimate $V_1$, we employ as in §5 the mean value theorem, and Lemma 9 for $p = \infty$. We then get

$$V_1 < \left( \sum_{1}^{r} |h_r| \right) \left( \sum_{1}^{r} \alpha_{r+1} \right)$$

and, using (7.7) and (5.2) (which is equivalent to (6.1)),

$$V_1 = O(\delta 2^{-n}).$$

For $|H| \leq \delta$ choose $n$ so that $2^{n-1} < \delta^{-1} \leq 2^n$, then

$$V_1 = O(\delta) \quad \text{and} \quad V_2 = O(\delta)$$

to get

$$\xi(t) = O(\delta).$$

We have thus proved the theorem:

**Theorem 6.** If the assumptions of Theorem 5 are satisfied and if $0 < \beta \leq 1$, then there exists a continuous function $F(X)$ with modulus of continuity $\xi(t) < \omega(t)$, while $\sum |c_N| = \infty$. The same result holds for $1 < \beta < 2$ under the additional assumptions (7.5) and (7.6).

As an example choose $\omega(t) = t^\alpha$, $0 < \alpha < 1$; it is seen easily that now (6.1), (7.5), and (7.6) hold. Theorem 1'' yields the convergence of $\sum |c_N| = \infty$ whenever $M_{\delta \phi} = Ot^\alpha$, and if $\beta > 2\kappa/(\kappa + 2\alpha)$. For $\beta = 2\kappa/(\kappa + 2\alpha)$, however, there exists a continuous function whose modulus of continuity is less than $t^\alpha$, while $\sum |c_N| = \infty$.

**Closing remark.** In a similar manner the convergence of the series $\sum |N|^\alpha |c_N|^\beta$ can be discussed. The mode of procedure applies as well to Fourier integrals. We may also consider instead of the spherical mean (2.14) the more general average

$$f_\rho(X; t) = \frac{c_\rho}{\rho} \int_{0}^{t} \left( 1 - \frac{r^2}{\rho^2} \right)^{p-1} f(X; r) r^{p-1} dr.$$ 

Finally, if we denote the linear operator which transforms $f(X)$ into $f(X; t) - f(X)$ by $\Delta f(X; t)$, iteration yields
\[ \Delta_m f(X; t) \sim \sum c_N(\alpha \| N \|) - 1)^m \exp (iNX), \quad m = 1, 2, 3, \ldots, \]

and in Theorem 1 the assumption \( M_{\nu} \phi(X; t) \equiv M_\nu \Delta_1 f(X; t) = O\omega(t) \) can be replaced by \( M_\nu \Delta_m f(X; t) = O\omega(t) \).

**References**


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