RINGS OF REAL-VALUED CONTINUOUS FUNCTIONS. I

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Research in the theory of topological spaces has brought to light a great deal of information about these spaces, and with it a large number of ingenious special methods for the solution of special problems. Very few general methods are known, however, which may be applied to topological problems in the way that general techniques are utilized in classical analysis. For nearly every new problem in set-theoretic topology, the student is forced to devise totally novel methods of attack.

With every topological space satisfying certain restrictions there may be associated two non-trivial algebraic structures, namely, the ring of real-valued continuous bounded functions and the ring of all real-valued continuous functions defined on that space. These rings enjoy a great number of interesting algebraic properties, which are connected with corresponding topological properties of the space on which they are defined. It may be hoped that by utilizing appropriate algebraic techniques in the study of these rings, some progress may be made toward establishing general methods for the solution of topological problems.

The present paper, which is the first of a projected series, is concerned with the study of these function rings, with a view toward establishing their fundamental properties and laying the foundation for applications to purely topological questions. It is intended first to give a brief account of the theory of rings of bounded real-valued continuous functions, with the adduction of proofs wherever appropriate. The theory of rings of bounded real-valued continuous functions has been extensively developed by mathematicians of the American, Russian, and Japanese schools, so that our account of this theory will in part be devoted to the rehearsal and organization of known facts. Second, it is proposed to extend the theory to include rings of *all* real-valued continuous functions. These rings have in the past received but scanty attention, although they exhibit a number of properties interesting *per se* and also possess various advantages over rings of bounded real-valued continuous functions for purposes of application. Analogues between the two theories, as well as differences, will be pointed out as they appear.

For a general topological background the reader may be referred to the treatise of Alexandroff and Hopf [2](2), which is designated throughout the

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sequel as AH. A similar algebraic background is provided by the treatise of van der Waerden [30], referred to in the sequel as VDW.

Throughout the present paper, the following terminological conventions are employed: the terms *bicompact* and *compact* are used in the same senses as in AH (see AH, Kap. II, *passim*); the term *ideal* is taken to mean, unless the contrary is specified, an ideal distinct from the entire ring in which it lies; the term *algebra* is used without any restriction on the cardinal number of a basis and with the assumption of the associative law.

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Part I. General properties of function rings

1. Preliminary definitions and remarks. We commence with precise definitions of the rings which form the objects of our present attention.

Definition 1. Let X be any topological space and let R be the space of real numbers with its usual topology. $\mathfrak{C}^*(X, R)$ is the set of all real-valued continuous bounded functions with domain X, and $\mathfrak{C}(X, R)$ is the set of all continuous real-valued functions with domain X. If K denotes the space of complex numbers with its usual topology, then $\mathfrak{C}^*(X, K)$ and $\mathfrak{C}(X, K)$ are defined similarly.

Theorem 1. *If X is any topological space, then $\mathfrak{C}^*(X, R)$ is a Banach*

algebra over R[3], *where*

$$(f+g)(p) = f(p) + g(p), \tag{1}$$

$$(fg)(p) = f(p)g(p), \tag{2}$$

$$(\alpha f)(p) = \alpha(f(p)), \tag{3}$$

for all $p \in X$ *and* $\alpha \in R$,

(4) *the ring unit is* 1, *where* $1(p) = 1$ *for all* $p \in X$;

$$\|f\| = \sup_{p \in X} |f(p)|. \tag{5}$$

The operations of addition and multiplication are continuous in the topology defined by the norm (5), *and the operation of inversion is continuous throughout the open subset of* $\mathfrak{C}^*(X, R)$ *in which it is defined. The set of functions* $\mathfrak{C}(X, R)$ *forms an algebra over* R, *with addition, scalar multiplication, multiplication, and ring unit defined by* (1)–(4) *above; the operation of multiplication is commutative in this ring.* $\mathfrak{C}(X, R)$ *is a complete metric space, where the distance function* ρ *is defined by*

$$\rho(f, g) = \sup_{p \in X} \frac{|f(p) - g(p)|}{1 + |f(p) - g(p)|}. \tag{6}$$

The operations of addition and scalar multiplication are continuous in the metric (6), *but multiplication is not in general continuous. The set of elements of* $\mathfrak{C}(X, R)$ *possessing inverses is not in general an open set under the metric* (6).

Verification that $\mathfrak{C}^*(X, R)$ is a linear space complete under the norm (5) is quite simple and may be left to the reader. To show that $\mathfrak{C}^*(X, R)$ is a Banach algebra, we need only show that the equality $fg = gf$ obtains, that $\|1\| = 1$, both of which are obvious, and that $\|fg\| \leqq \|f\| \cdot \|g\|$. This inequality is easily established, as follows. For every positive real number ϵ, there is a point $p \in X$ such that $\|fg\| - \epsilon < |f(p)g(p)| = |f(p)| \cdot |g(p)| \leqq \|f\| \cdot \|g\|$. These inequalities being valid for all $\epsilon > 0$, it follows that $\|fg\| \leqq \|f\| \cdot \|g\|$.

The operations of addition and multiplication are continuous here, as in any Banach algebra. It is easy to verify that the elements in $\mathfrak{C}^*(X, R)$ with inverse form an open set. If f is any element in $\mathfrak{C}^*(X, R)$ with inverse, and if $\|f-g\| < 1/\|f^{-1}\|$, then the series $1 + \sum_{k=1}^{\infty} ((f-g)f^{-1})^k$ converges to the element fg^{-1}. Continuity of the inverse in the open set where it is defined is also easy to establish. Let ϵ be any positive real number, and let f be any element in $\mathfrak{C}^*(X, R)$ with inverse. The number $a = \inf_{p \in X} |f(p)|$ is clearly positive, and the equality $a^{-1} = \|f^{-1}\|$ is obviously correct. Let $\eta = \min(\epsilon a^2/2, a/2)$. Then if $\|f-g\| < \eta$, we have $\inf_{p \in X} |g(p)| \geqq a/2$, and $\|f^{-1} - g^{-1}\| \leqq \|f-g\| \cdot \|f^{-1}\| \cdot \|g^{-1}\|$

(3) The term *Banach algebra* is here used as synonymous with the term *normed ring* [11].

$\leqq \|f-g\| \cdot 1/a \cdot 2/a < \epsilon$. This proves the continuity of the operation of inversion.

We now turn to the space $\mathfrak{C}(X, R)$, which is clearly an algebra over R satisfying all the conditions set forth in the statement of the present theorem. The function $\rho(f, g)$ defined by (6) is a true metric, as may be easily proved from the inequality

$$\frac{|\alpha-\beta|}{1+|\alpha-\beta|} \leqq \frac{|\alpha-\gamma|}{1+|\alpha-\gamma|} + \frac{|\gamma-\beta|}{1+|\gamma-\beta|},$$

where α, β, and γ are arbitrary real numbers. Completeness of $\mathfrak{C}(X, R)$ under the metric ρ is proved by arguments familiar from elementary analysis.

The remaining statements of the present theorem may be established by simple examples. Let $X=R$, and, as is customary, let $\mathfrak{S}_t(f)$ be the set of all $g \in \mathfrak{C}(R, R)$ such that $\rho(f, g) < t$, where t is an arbitrary positive real number. Let η be any positive real number. The functions $e^x+\eta/2$ and $e^{-x}+\eta/2$ are in the neighborhoods $\mathfrak{S}_\eta(e^x)$ and $\mathfrak{S}_\eta(e^{-x})$, respectively, but the product $(e^x+\eta/2) \cdot (e^{-x}+\eta/2)$ lies in no neighborhood $\mathfrak{S}_\zeta(e^x \cdot e^{-x}) = \mathfrak{S}_\zeta(1)$, for $\zeta > 0$. Multiplication is therefore not continuous in $\mathfrak{C}(R, R)$. The function $f(x) = (1+x^2)^{-1}$ is a function in $\mathfrak{C}(R, R)$ which clearly has an inverse, but in every neighborhood $\mathfrak{S}_\eta(f)$ there are functions which do not have inverses.

2. Various topologies in $\mathfrak{C}(x, R)$ and $\mathfrak{C}^*(X, R)$. The introduction of metrics into $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$, as described in Theorem 1, naturally leads to the consideration of other topologies in function rings and function spaces.

DEFINITION 2. Let X be any topological space and let M be any metric space. Let the set of all continuous mappings of X into M be denoted by $\mathfrak{C}(X, M)$. The set $\mathfrak{C}(X, M)$ may be made a topological space, where neighborhoods $\mathfrak{U}_\eta(f)$ are defined as $E[g; g \in \mathfrak{C}(X, M), |g(p)-f(p)| < \eta$ for all $p \in X]$, η being an arbitrary positive real number and f being an arbitrary element of $\mathfrak{C}(X, M)$. The resulting topology is called the u-topology. $\mathfrak{C}(X, M)$ may also be made a topological space by considering an arbitrary bicompact subset K of X and any positive real number η. The neighborhood $\mathfrak{U}_{K,\eta}(f)$ is then defined as $E[g; g \in \mathfrak{C}(X, M), |g(p)-f(p)| < \eta$ for all $p \in K]$. The topology in $\mathfrak{C}(X, M)$ which results by taking arbitrary sets K and arbitrary positive real numbers η is called the k-topology. Finally, $\mathfrak{C}(X, M)$ may be made a topological space by substituting arbitrary finite subsets of X for arbitrary bicompact subsets of X in the definition of the k-topology. The resulting topology is called the p-topology. Bounded functions in $\mathfrak{C}(X, M)$ being defined in the usual way, and the set of all such functions being denoted by $\mathfrak{C}^*(X, M)$, the u-, k-, and p-topologies in $\mathfrak{C}^*(X, M)$ are defined as the relative topologies for $\mathfrak{C}^*(X, M)$ as a subspace of $\mathfrak{C}(X, M)$.

DEFINITION 3. If X is an arbitrary topological space, the set $\mathfrak{C}(X, R)$ may be made into a topological space by considering an arbitrary function

$\pi \in \mathfrak{C}(X, R)$ having the property that $\pi(p)$ is positive for all $p \in X$. For any such function π, and any function f in $\mathfrak{C}(X, R)$, the neighborhood $\mathfrak{U}_\pi(f)$ is defined as $E[g; g \in \mathfrak{C}(X, R), |g(p) - f(p)| < \pi(p)$ for all $p \in X]$. The resulting topology(4) is called the m-topology.

For many purposes, the u-topology appears to be the most natural in considerations involving rings $\mathfrak{C}^*(X, R)$, while the m-topology enjoys great advantages for the study of rings $\mathfrak{C}(X, R)$. The k- and p-topologies, which have been well known for a number of years, have been extensively studied in other connections, and will be considered briefly in the sequel. It will be observed that the u-topology and the norm topology described in Theorem 1 are equivalent, and that for topological spaces on which every continuous real-valued function is bounded, the u-topology and the m-topology coincide. In the presence of unbounded real-valued continuous functions, the two topologies are not equivalent. In the case of bicompact topological spaces X, the k-, u-, and m-topologies coincide.

We now consider the structures of $\mathfrak{C}(X, R)$ under the u- and m-topologies.

THEOREM 2. *The topology introduced into $\mathfrak{C}(X, R)$ by the metric* (6) *of Theorem* 1 *is equivalent to the topology introduced into $\mathfrak{C}(X, R)$ by the u-topology.*

Let η be an arbitrary positive real number; let $\mathfrak{U}_\eta(f)$ be the η-neighborhood of $f \in \mathfrak{C}(X, R)$ in the u-topology; and let $\mathfrak{S}_\eta(f)$ be the open sphere of radius η about the element f in the metric ρ of Theorem 1. Suppose that g is in the open sphere $\mathfrak{S}_{\eta/1+\eta}(f)$. Then the inequality

$$\frac{|g(p) - f(p)|}{1 + |g(p) - f(p)|} < \frac{\eta}{1 + \eta}$$

obtains for all $p \in X$, and, the real function $x(1+x)^{-1}$ being strictly monotone increasing for $x > 0$, it follows that $|g(p) - f(p)| < \eta$. Therefore we have $\mathfrak{S}_{\eta/1+\eta}(f) \subset \mathfrak{U}_\eta(f)$. Likewise, for $0 < \eta < 1$, it is true that $\mathfrak{U}_{\eta/1-\eta}(f) \subset \mathfrak{S}_\eta(f)$, and the equivalence of the metric topology defined by ρ and the u-topology is established.

The natural character of the m-topology for rings $\mathfrak{C}(X, R)$ is made clear by the following observations.

THEOREM 3. *Let X be an arbitrary topological space. Under the m-topology, and with the definitions of addition, scalar multiplication, and multiplication set forth in Theorem* 1, *the set $\mathfrak{C}(X, R)$ is a commutative algebra with unit over R, in which the operations of addition and multiplication are continuous, the set of elements with inverses is an open set, and the inverse is a continuous operation wherever defined. If there exists an unbounded function in $\mathfrak{C}(X, R)$, then $\mathfrak{C}(X, R)$ is not metrizable under the m-topology, and, indeed, fails at every point*

(4) This m-topology for $\mathfrak{C}(R, R)$ was introduced by E. H. Moore [21]. The writer is indebted to Professor Anna Pell Wheeler for the reference to Moore's work.

to satisfy the first axiom of countability. If every function in $\mathfrak{C}(X, R)$ *is bounded, then the m-topology and the u-topology coincide in* $\mathfrak{C}(X, R)$.

Let f and g be arbitrary elements of $\mathfrak{C}(X, R)$, and let π be an arbitrary function in $\mathfrak{C}(X, R)$ which is positive everywhere. If $\bar{f}\in\mathfrak{U}_{\pi/2}(f)$ and $\bar{g}\in\mathfrak{U}_{\pi/2}(g)$, then it is obvious that $\bar{f}+\bar{g}\in\mathfrak{U}_{\pi}(f+g)$. Hence addition is continuous in the m-topology for $\mathfrak{C}(X, R)$. To prove that multiplication is continuous, let f, g, and π be as above, and let $\psi=[\{(|f|+|g|)^2+4\pi^2\}^{1/2}/2-(|f|+|g|)]$. It is plain that ψ is an element of $\mathfrak{C}(X, R)$ positive everywhere; and it is further clear that if $\bar{f}\in\mathfrak{U}_{\psi}(f)$ and $\bar{g}\in\mathfrak{U}_{\psi}(g)$, then $\bar{f}\cdot\bar{g}\in\mathfrak{U}_{\pi}(f\cdot g)$. To prove that the set of elements in $\mathfrak{C}(X, R)$ with inverses is an open set, we select an arbitrary element f such that f^{-1} exists; we then observe that $|f|$ is a function in $\mathfrak{C}(X, R)$ positive everywhere and that every function in the open set $\mathfrak{U}_{|f|}(f)$ possesses an inverse.

We now prove that the operation of inversion is continuous throughout the open subset of $\mathfrak{C}(X, R)$ in which it is defined. Let f be any function in $\mathfrak{C}(X, R)$ having an inverse, and let π be an arbitrary element of $\mathfrak{C}(X, R)$ which is positive everywhere and has the further property that the function $|f|-\pi/2$ is positive everywhere. Let $\psi=\min\{\pi\cdot|f|\cdot(|f|-\pi/2), \pi/2\}$. Then if $g\in\mathfrak{U}_{\psi}(f)$, it follows that $g^{-1}\in\mathfrak{U}_{\pi}(f^{-1})$. This fact is demonstrated easily, as follows. Let p be an arbitrary point of X, and let g be an arbitrary function in $\mathfrak{U}_{\psi}(f)$. Then we have $|f^{-1}(p)-g^{-1}(p)|=|f(p)-g(p)|\cdot|f^{-1}(p)|\cdot|g^{-1}(p)|<\pi(p)\cdot|f(p)|\cdot(|f(p)|-\pi(p)/2)\cdot|f^{-1}(p)|\cdot|g^{-1}(p)|=\pi(p)\cdot(|f(p)|-\pi(p)/2)\cdot|g^{-1}(p)|$. However, if $g\in\mathfrak{U}_{\psi}(f)$, then we have $g(p)>f(p)-\pi(p)/2$, $g(p)+\pi(p)/2>f(p)$, $|g(p)|+\pi(p)/2>|f(p)|$, $|g(p)|>|f(p)|-\pi(p)/2$, and hence $|g^{-1}(p)|<(|f(p)|-\pi(p)/2)^{-1}$. It follows that $|f^{-1}(p)-g^{-1}(p)|<\pi(p)$; hence $g^{-1}\in\mathfrak{U}_{\pi}(f^{-1})$, and the inverse is continuous wherever defined.

To verify the final assertions of the present theorem, we suppose that there exists an unbounded function f in $\mathfrak{C}(X, R)$. Then $g=f^2+1$ is an unbounded function with inverse which is positive everywhere. There must accordingly exist a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers which is strictly increasing and has limit $+\infty$, and a countable subset $\{p_n\}_{n=1}^{\infty}$ of X such that $g(p_n)=a_n$ $(n=1, 2, 3, \cdots)$. Now consider any arbitrary countable set $\{\pi_n\}_{n=1}^{\infty}$ of functions in $\mathfrak{C}(X, R)$ which are positive everywhere. We shall show that there exists a neighborhood $\mathfrak{U}_{\psi}(0)$ in $\mathfrak{C}(X, R)$ which contains no neighborhood $\mathfrak{U}_{\pi_n}(0)$. Let $b_n=2^{-1}\min[\pi_1(p_n), \pi_2(p_n), \cdots, \pi_n(p_n)]$, for $n=1, 2, 3, \cdots$. There obviously exists a function σ in $\mathfrak{C}(R, R)$ such that $\sigma(x)$ is positive for all positive real numbers x and such that $\sigma(a_n)=b_n^{-1}$ $(n=1, 2, 3, \cdots)$. Let $\psi=(\sigma(g))^{-1}$; at the point p_n, we have $\psi(p_n)\leqq 2^{-1}\pi_n(p_n)$. Hence the neighborhood $\mathfrak{U}_{\psi}(0)$ contains no neighborhood $\mathfrak{U}_{\pi_n}(0)$, and the first axiom of countability fails to obtain at the point 0. Since $\mathfrak{C}(X, R)$ is a topological group under addition, it follows that $\mathfrak{C}(X, R)$ satisfies the first axiom of countablity at no point, and that $\mathfrak{C}(X, R)$ is not metrizable under the m-topology.

Finally, if every function in $\mathfrak{C}(X, R)$ is bounded, then every function in

$\mathfrak{C}(X, R)$ which is positive everywhere has a positive lower bound. (See Theorem 27 below for a proof of this fact.) The functions $(1/n)\cdot 1$, where 1 is the function identically equal to 1, may be used to define a countable complete neighborhood system at an arbitrary point in $\mathfrak{C}(X, R)$, in the m-topology, and it is patent that in this case the m-topology and the u-topology coincide.

3. **Useful categories of spaces.** In order to ensure that the rings $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}(X, R)$ may reflect in some measure distinguishing topological properties of the space X, it is necessary severely to limit the class of spaces considered.

DEFINITION 4. Two subsets A and B of a topological space X are said to be completely separated if there exists a function ϕ in $\mathfrak{C}(X, R)$ such that $\phi(p) \leqq 0$ for all $p \in A$ and $\phi(p) \geqq 1$ for all $p \in B$.

DEFINITION 5. A T_0-space X (see AH, pp. 58–60) is said to be completely regular if every closed subset F of X is completely separated from every point $p \in X$ such that p non$\in F$.

DEFINITION 6. A topological space X is said to be a Stone space if every pair of distinct points of X are completely separated.

Completely regular spaces were first discussed by Tychonoff [28], who showed that they are identifiable as the class of subspaces of Cartesian products of closed intervals in R, or, equivalently, as the class of dense subspaces of bicompact Hausdorff spaces. Completely regular spaces have been extensively discussed by a number of other authors. (See, for example, [26, Chapter III], [6, Chapter II], [10], [14], and [15].) Stone spaces have been mentioned by Urysohn [29] and Chittenden [9], and have been described in detail by Stone (in a letter to the writer), who proved that they are the class of spaces which can be continuously mapped onto bicompact Hausdorff spaces.

We shall limit ourselves for the most part throughout the sequel to consideration of completely regular spaces. If X is completely regular, the ring $\mathfrak{C}(X, R)$ necessarily contains a sufficient number of elements to describe the topology of X, and the algebraic properties of $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$ may reasonably be expected to enjoy close connections with the topological structure of X. If we widen our field of investigation to include all regular spaces X (see AH, p. 67), we cannot expect that the function rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$ will yield useful general information, since a large family of regular spaces X exist for which $\mathfrak{C}(X, R) = R$. (See [16].) If we generalize in another direction, on the other hand, and consider all Stone spaces, we obtain nothing novel, as the following theorem shows.

THEOREM 4. *Let X be a Stone space. Then there exists a completely regular space Y such that Y is a one-to-one continuous image of X, $\mathfrak{C}(X, R)$ can be mapped isomorphically and homeomorphically (in the m-topology or u-topology)*

onto $\mathfrak{C}(Y, R)$, *and* $\mathfrak{C}^*(X, R)$ *can be mapped isomorphically and isometrically* (*in the norm topology*) *onto* $\mathfrak{C}^*(Y, R)$.

The points of the space Y are to be taken as those of the space X, and the family of open subsets of Y is to be taken as a certain subfamily of the family of open subsets in X. If f is an element of $\mathfrak{C}(X, R)$, and α and β are real numbers such that $\alpha<\beta$, then the set $E[p;\ p\in Y,\ \alpha<f(p)<\beta]$ is to be open in Y, and the family of all open subsets of Y is formed by taking arbitrary unions and finite intersections of all such sets, with arbitrary α, β, and $f\in\mathfrak{C}(X, R)$. Since the functions f are all continuous, it is plain that the identity mapping carrying X onto Y is continuous, and since X is a Stone space, Y is a Hausdorff space. It is likewise obvious that every function $f\in\mathfrak{C}^*(X, R)$ and every function $f\in\mathfrak{C}(X, R)$ can be considered as a continuous function on Y. The mapping $f\rightarrow\bar{f}$, where $f(p)=\bar{f}(p)$ for all $p\in X$, is therefore a norm-preserving isomorphism carrying $\mathfrak{C}^*(X, R)$ into $\mathfrak{C}^*(Y, R)$. Since Y is a continuous image of X, it is clear that this isomorphism has the whole ring $\mathfrak{C}^*(Y, R)$ as its range. The rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ may be treated in precisely the same manner.

The foregoing discussion shows the natural domain of investigations involvings rings $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}(X, R)$ to be the class of completely regular spaces.

From our present point of view, normal spaces may be regarded as a specialized family of spaces whose consideration may occasionally permit simplified proofs. (See AH, p. 68, for a definition of normal spaces.) They also appear as a natural complement to Stone spaces and completely regular spaces, from an axiomatic point of view.

THEOREM 5. *A* T_1-*space* X (*see* AH, *p.* 58) *is normal if and only if every pair of subsets of* X *having disjoint closures are complete separated.*

The present theorem is obvious.

4. **Representation of rings** $\mathfrak{C}^*(X, R)$. In studying function rings, it is natural to inquire after intrinsic criteria for determining under what conditions a given Banach algebra over R is the ring $\mathfrak{C}^*(X, R)$ for some space X.

THEOREM 6. *Let* $\mathfrak{E}$ *be any Banach algebra over the field* R. *There exists a bicompact Hausdorff space* X *such that* $\mathfrak{E}$ *can be mapped onto* $\mathfrak{C}^*(X, R)$ *by a norm-preserving algebraic isomorphism if and only if the following conditions obtain in* $\mathfrak{E}$:

(1) $(x^2+e)^{-1}$ *exists for all* $x\in\mathfrak{E}$, e *being the unit in* E;

(2) $(e\|x\|^2-x^2)^{-1}$ *exists for no* $x\in\mathfrak{E}$.

The present theorem can be proved as a consequence of a theorem of Gelfand (see [11, p. 16, Satz 16]), and the proof is accordingly omitted. A direct proof may also be given.

5. **Definition of βX.** A cardinal property of rings $\mathfrak{C}^*(X, R)$ is the fact that for every completely regular space, there exists a unique bicompact Hausdorff space, commonly denoted as βX, having the properties that $X \subset \beta X$, $X^- = \beta X$, and $\mathfrak{C}^*(X, R)$ is algebraically isomorphic to $\mathfrak{C}^*(\beta X, R)$. The existence and uniqueness of βX were first proved by Stone (see [26, Theorems 78, 79, 88]), by methods dependent upon the theory of representation of topological spaces as maps in Boolean spaces. A second, simpler, proof was given by Čech [7]. A third construction of β, valid for normal spaces only, was obtained by Wallman [31], and A. Weil has presented a construction based on the theory of uniform structures [32]. A simplified version of Stone's original construction was given in 1941 by Gelfand and Shilov (see [13]). Kakutani has given a construction of β based on Banach lattices [18]. Finally, Alexandroff, using a modification of Wallman's construction, has produced a construction of β and of yet more general bicompact T_1-spaces in which arbitrary regular spaces can be imbedded as dense subsets. (See [1].) Spaces βX thus appear as truly protean entities, arising in the most diverse manner from apparently unrelated constructions. It is not our purpose at the present time to elaborate on the inner connections which obtain among the various constructions of β, or to present any essential variants thereof. We shall briefly describe the construction obtained by Gelfand and Shilov [13], with the aim of completing and simplifying their proof and of exhibiting the details of their construction for use in certain applications.

6. **Existence of βX.** We commence with certain useful definitions.

DEFINITION 7. Let X be any topological space. An ideal $\mathfrak{J}$ in $\mathfrak{C}^*(X, R)$ or $\mathfrak{C}(X, R)$ is said to be a free ideal if, for every point $p \in X$, there exists an element f of $\mathfrak{J}$ such that $f(p) \neq 0$. An ideal which is not free is said to be fixed.

DEFINITION 8. Let X be any topological space and let f be a function in $\mathfrak{C}(X, R)$. The set of points in X for which f vanishes is said to be the zero set of f and is denoted by $Z(f)$. A subset A of X which is the set $Z(f)$ for some f in $\mathfrak{C}(X, R)$ is said to be a Z-set.

THEOREM 7. *Let X be any completely regular space. Then the ring $\mathfrak{C}^*(X, R)$ contains a free ideal if and only if X is non-bicompact.*

Suppose first that X is a bicompact, and assume that a free ideal exists in $\mathfrak{C}^*(X, R)$. Then, for every $p \in X$, there is a function f in I such that $f(p) \neq 0$ and, consequently, $f^2(p) > 0$. Since f^2 is continuous, there is an open neighborhood of p, which may be denoted by $U(p)$, such that $f^2(q) > 0$ for all $q \in U(p)$. The space X being bicompact, a finite number of these neighborhoods, say $U_1, U_2, \cdots, U_n$, suffice to cover the space X. It follows that the function $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2$, where the function f_i^2 is associated as above with the neighborhood U_i, is an element of $\mathfrak{J}$ which vanishes nowhere and is positive. Such functions on bicompact Hausdorff spaces have positive lower bounds and accordingly possess inverses. This contradicts the hy-

pothesis that $\mathfrak{J}$ is a proper ideal, and proves that $\mathfrak{J}$ must be a fixed ideal.

On the other hand, let us suppose that X is non-bicompact; and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an open covering of X for which no finite subcovering exists. For each $p\in X$, there is an index $\lambda\in\Lambda$ such that $p\in G_\lambda$. Selecting such a λ for each $p\in X$, we define a function $f_{p,\lambda}$ in $\mathfrak{C}^*(X, R)$ such that $f_{p,\lambda}(p)=1$ and $f_{p,\lambda}(q)=0$ for all $q\in G_\lambda'$. Such functions $f_{p,\lambda}$ exist in virtue of the complete regularity of X. The ideal $\mathfrak{J}$ generated by the set of functions $f_{p,\lambda}$, where p runs through all the elements of X and λ runs through all appropriate indices in Λ, is certainly a free ideal. It must be a proper ideal as well. If $\mathfrak{J}$ were not a proper ideal, then for an appropriate choice of $p_1, \lambda_1, p_2, \lambda_2, \cdots, p_n, \lambda_n$ and functions $g_1, g_2, \cdots, g_n$ in $\mathfrak{C}^*(X, R)$, we should have the equality

$$1 = \sum_{i=1}^{n} f_{p_i,\lambda_i} g_i,$$

where 1 is the function identically equal to unity. This equality implies that the open sets $G_{\lambda_1}, G_{\lambda_2}, \cdots, G_{\lambda_n}$ cover X, in contradiction to our hypothesis regarding the family $\{G_\lambda\}_{\lambda\in\Lambda}$.

It may be remarked that if $\mathfrak{C}(X, R)$ contains an unbounded function f, then a free ideal in $\mathfrak{C}^*(X, R)$ may be obtained at once. The function $(f^2+1)^{-1}$ is positive everywhere, lies in $\mathfrak{C}^*(X, R)$, but clearly has no inverse in $\mathfrak{C}^*(X, R)$. The function $(f^2+1)^{-1}$ therefore generates a proper free ideal.

The fixed maximal ideals in $\mathfrak{C}^*(X, R)$ may be characterized very simply, as the following theorem shows. We remark that $\mathfrak{C}^*(X, R)/\mathfrak{M}$ is the field R, for any maximal ideal $\mathfrak{M}$ in $\mathfrak{C}^*(X, R)$. (See [26, Theorem 76].)

THEOREM 8. *Let X be a completely regular space. The fixed maximal ideals of $\mathfrak{C}^*(X, R)$ are precisely those ideals of the form $E[f; f\in\mathfrak{C}^*(X, R), f(p)=0]$, where p is some fixed point in X; we denote such an ideal by the symbol $\mathfrak{M}_p$. In the homomorphism onto R defined by $\mathfrak{M}_p$, $\mathfrak{M}_p(f)=f(p)$, for all $f\in\mathfrak{C}^*(X, R)$.*

First, it is obvious that every set $\mathfrak{M}_p$ is an ideal in $\mathfrak{C}^*(X, R)$. If g non$\in\mathfrak{M}_p$, then $g(p)\neq 0$, and the function $g-g(p)$ is in $\mathfrak{M}_p$. The ideal generated by $\mathfrak{M}_p$ and g contains the function $(g-g(p))-g=g(p)$, and hence this ideal is the whole ring $\mathfrak{C}^*(X, R)$, so that $\mathfrak{M}_p$ is a maximal ideal. Conversely, suppose that $\mathfrak{J}$ is a fixed maximal ideal. Since $\mathfrak{J}$ is fixed, the set $A=\coprod_{f\in\mathfrak{J}} Z(f)$ is non-void, and can obviously contain but one point if $\mathfrak{J}$ is to be maximal. By the same token, $\mathfrak{J}$ must contain all of the functions vanishing at the single point in A. Finally, in the quotient field $\mathfrak{C}^*(X, R)/\mathfrak{M}_p$, the equalities $\mathfrak{M}_p(f-f(p))=0$, $\mathfrak{M}_p(f)=\mathfrak{M}_p(f(p))=f(p)$ obtain.

We now introduce a topology into the set of maximal ideals in $\mathfrak{C}^*(X, R)$ and show that this space, so topologized, is βX. This topology was introduced by Stone [26], and has been exploited by Jacobson [17] and Gelfand and Shilov [13].

THEOREM 9. *Let X be a completely regular space, and let $\mathfrak{M}^*$ be the set of all maximal ideals in the ring $\mathfrak{C}^*(X, R)$. Let $\mathcal{U}_f$ be the set $E[\mathfrak{M}; \mathfrak{M}\in\mathfrak{M}^*, f \text{ non}\in\mathfrak{M}]$ and let every $\mathcal{U}_f$ be designated as a neighborhood of every maximal ideal $\mathfrak{M}$ which it contains. As f assumes all possible values, a complete neighborhood system is defined for every point of $\mathfrak{M}^*$. The family $\{\mathcal{U}_f\}_{f\in\mathfrak{C}^*(X, R)}$ is closed under the formation of finite intersections. Under the topology so defined, the space $\mathfrak{M}^*$ is a bicompact Hausdorff space containing a homeomorphic image $\mathcal{X}$ of X. The space $\mathcal{X}$ is dense in $\mathfrak{M}^*$, and every function in $\mathfrak{C}^*(\mathcal{X}, R)$ can be continuously extended over the whole space $\mathfrak{M}^*$. The rings $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(\mathfrak{M}^*, R)$ are therefore isomorphic, and $\mathfrak{M}^*$ is the space βX. The points of $\mathfrak{M}^*$ corresponding to points of X are the fixed maximal ideals in $\mathfrak{C}^*(X, R)$, and the points of $\mathfrak{M}^*\cap\mathcal{X}'$ are precisely the free maximal ideals of $\mathfrak{C}^*(X, R)$.*

We note first that $\mathcal{U}_{f_1}\cap\mathcal{U}_{f_2}\cap\cdots\cap\mathcal{U}_{f_n}$ is equal to $\mathcal{U}_{f_1\cdot f_2\cdots f_n}$; hence the family $\{\mathcal{U}_f\}$ is closed under the formation of finite intersections. Suppose next that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two distinct elements of $\mathfrak{M}^*$, and let f be a function such that $f\in\mathfrak{M}_1\cap\mathfrak{M}_2'$. Then we have $\mathfrak{M}_2\in\mathcal{U}_f$, while $\mathfrak{M}_1\in\mathcal{U}_f'$. $\mathfrak{M}^*$ is hence a T_1-space. We postpone the proof that $\mathfrak{M}^*$ is a Hausdorff space. $\mathfrak{M}^*$ is bicompact, as proved by Jacobson [17].

We next map the space X into the space $\mathfrak{M}^*$ by a function Ψ, where $\Psi(p)$ is the ideal $\mathfrak{M}_p$, and denote the image space $\Psi(X)$ by $\mathcal{X}$. Since it is quite elementary to show that Ψ is a homeomorphism, we omit the details of the verification. Every function f in $\mathfrak{C}^*(X, R)$ may be represented faithfully on $\mathcal{X}$ by the homomorphisms which the ideals $\mathfrak{M}_p$ define: $f(\mathfrak{M}_p)=\mathfrak{M}_p(f)=f(p)$. Each such function f (we make no notational distinction between functions defined on X and functions defined on $\mathcal{X}$) may be extended over all of $\mathfrak{M}^*$ by defining $f(\mathfrak{M})$ to be $\mathfrak{M}(f)$, for each free maximal ideal $\mathfrak{M}$ in $\mathfrak{M}^*$. This plainly defines an extension of f over the bicompact space $\mathfrak{M}^*$. We now prove that the extension is continuous. Let $\mathfrak{M}$ be any element of $\mathfrak{M}^*$, f any function in $\mathfrak{C}^*(\mathcal{X}, R)$, and ϵ any positive real number. There are two cases.

Case I: $f\in\mathfrak{M}$. In this case, $f(\mathfrak{M})=0$. We observe first that $\mathfrak{M}(|f|)=|\mathfrak{M}(f)|$ and that in consequence $\mathfrak{M}(\min(f, g))=\min(\mathfrak{M}(f), \mathfrak{M}(g))$, since $\min(f, g)=2^{-1}(-|f-g|+(f+g))$. We now set g equal to the function $\min(|f|, \epsilon)-\epsilon$ and consider the neighborhood $\mathcal{U}_g$, which clearly contains $\mathfrak{M}$. Let $\mathfrak{N}$ be an arbitrary maximal ideal not containing g; we assert that $|\mathfrak{N}(f)|<\epsilon$. Making the contrary assumption, that $|\mathfrak{N}(f)|\geqq\epsilon$, we have $\mathfrak{N}(g)=\mathfrak{N}(\min(|f|, \epsilon)-\epsilon)=\min(|\mathfrak{N}(f)|, \epsilon)-\epsilon=0$. This implies that $g\in\mathfrak{N}$, contrary to our hypothesis on $\mathfrak{N}$. It follows at once that the extended function f is continuous at the point $\mathfrak{M}$.

Case II: f non$\in\mathfrak{M}$. In this case, the function $f-\mathfrak{M}(f)$ is in $\mathfrak{M}$, and is continuous at $\mathfrak{M}$ by Case I. Since the constant function $\mathfrak{M}(f)$ is continuous everywhere, it follows that f is continuous.

We now infer that $\mathfrak{M}^*$ is a Hausdorff space from the fact that for $\mathfrak{M}_1$ and

$\mathfrak{M}_2$ distinct maximal ideals in $\mathcal{M}^*$, there exists a function $f \in \mathfrak{C}^*(X, R)$ such that $f(\mathfrak{M}_1) = 0$ and $f(\mathfrak{M}_2) \neq 0$. Finally, it is easy to see that the rings $\mathfrak{C}^*(\mathcal{X}, R)$ and $\mathfrak{C}^*(\mathcal{M}^*, R)$ are algebraically isomorphic. Every function ϕ in $\mathfrak{C}^*(\mathcal{M}^*, R)$ is continuous and bounded on $\mathcal{X}$ and may hence be considered as a function defined on $\mathcal{X}$ alone and in $\mathfrak{C}^*(\mathcal{X}, R)$. The ring $\mathfrak{C}^*(\mathcal{M}^*, R)$ can thus be mapped onto a subring of the ring $\mathfrak{C}^*(\mathcal{X}, R)$. Since $\mathcal{X}$ is dense in $\mathcal{M}^*$, a function in $\mathfrak{C}^*(\mathcal{X}, R)$ can be extended continuously over $\mathcal{M}^*$ in at most one way, so that the mapping of $\mathfrak{C}^*(\mathcal{M}^*, R)$ into $\mathfrak{C}^*(\mathcal{X}, R)$ is one-to-one; since this mapping is obviously a homomorphism, it follows that $\mathfrak{C}^*(\mathcal{M}^*, R)$ is isomorphic to an analytic subring (i.e., a subring containing all constant functions and closed in the u-topology) of $\mathfrak{C}^*(\mathcal{X}, R)$. By preceding remarks in the present proof, however, it is clear that every function in $\mathfrak{C}^*(\mathcal{X}, R)$ is the image of a function in $\mathfrak{C}^*(\mathcal{M}^*, R)$, so that the two rings in question are indeed isomorphic. Theorems enunciated by Stone and Čech ([26, Theorem 88], and [7, p. 831]) state that βX is completely determined by the properties of being a bicompact Hausdorff space containing X as a dense subset and of having its ring of all real-valued continuous functions isomorphic to the ring $\mathfrak{C}^*(X, R)$. With this observation, the present proof is complete.

THEOREM 10. *If X_1 and X_2 are bicompact Hausdorff spaces such that $\mathfrak{C}^*(X_1, R)$ is algebraically isomorphic to $\mathfrak{C}^*(X_2, R)$, then X_1 is homeomorphic to X_2.*

This theorem, which is an immediate consequence of Theorem 10, has been well known for some time. (See [26, Theorem 86] and [12]).

7. General properties of βX. We turn now to the consideration of certain useful properties of βX.

THEOREM 11. *Let X be a non-bicompact completely regular space and let Y be a bicompact Hausdorff space. If $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(Y, R)$ are algebraically isomorphic, then X can be imbedded as a dense subset of Y; and if the space Z satisfies the inclusion relations $X \subset Z \subset Y$, then the rings $\mathfrak{C}^*(Z, R)$ and $\mathfrak{C}^*(X, R)$ are algebraically isomorphic.*

The first statement is obvious. Next suppose that Z is a space satisfying the inclusion relations $X \subset Z \subset Y = \beta X$. Then $\mathfrak{C}^*(Z, R)$ is isomorphic to an analytic subring of $\mathfrak{C}^*(X, R)$, under the mapping which carries a function f defined on Z onto the same function defined only on X. To show that this mapping is indeed an isomorphism carrying $\mathfrak{C}^*(Z, R)$ onto $\mathfrak{C}^*(X, R)$, let f be any function in $\mathfrak{C}^*(X, R)$. Then there exists a (necessarily unique) extension of f over all of the space $Y = \beta X$. This extension of f, restricted to the domain Z, is clearly a function in $\mathfrak{C}^*(Z, R)$, and maps onto the original function f under the mapping described above. Since X is dense in Z, the mapping of $\mathfrak{C}^*(Z, R)$ onto $\mathfrak{C}^*(X, R)$ is one-to-one and is therefore an isomorphism.

THEOREM 12. *Let X be a completely regular space and let Y be a topological*

space such that the inclusion relations $X \subset Y \subset \beta X$ obtain. Then Y is completely regular and has the properties that $\beta X = \beta Y$ and that $\mathfrak{C}^*(X, R)$ is algebraically isomorphic to $\mathfrak{C}^*(Y, R)$.

This remark is an immediate consequence of Theorems 10 and 11 above.

As a necessary preliminary to our next assertions concerning βX, we prove a theorem on Cartesian products of some interest *per se.*

THEOREM 13. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a nonvoid family of topological spaces such that the Cartesian product $\boldsymbol{P}_{\lambda \in \Lambda} X_\lambda$ is a dense subset of a bicompact Hausdorff space K. Let $\bar{x} = \{\bar{x}_\lambda\}$ be a fixed point of $\boldsymbol{P}_{\lambda \in \Lambda} X_\lambda$, and let $Y_{\lambda_0} = E[x; x \in \boldsymbol{P}_{\lambda \in \Lambda} X_\lambda, x_\lambda = \bar{x}_\lambda$ for all $\lambda \neq \lambda_0]$. The index λ_0 is to be an arbitrary element of the index class Λ. For every $\lambda_0 \in \Lambda$, let $B_{\lambda_0} = Y_{\lambda_0}^-$ (closure in K). Then K is homeomorphic to the Cartesian product $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$.*

We first remark that the spaces X_λ, for all $\lambda \in \Lambda$, are necessarily completely regular; and next that the spaces B_λ, for all $\lambda \in \Lambda$, are bicompact Hausdorff spaces since they are closed subsets of the bicompact Hausdorff space K. (See AH, p. 86, Satz IV.) Furthermore, the Cartesian product $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$, as the product of bicompact Hausdorff spaces, is itself a bicompact Hausdorff space (see [7, p. 830]). We now define a mapping Φ carrying K into $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$. Let k be an arbitrary point of K. If $k \in B_\lambda$ then we set $\Phi_\lambda(k) = k$. If k non$\in B_\lambda$, then we synthesize the mapping Φ_λ as follows. Consider $\{U_\gamma(k)\}_{\gamma \in \Gamma}$, the family of all open subsets of K which contain k (the set Γ is an appropriate index class). For each $\lambda \in \Lambda$, let $A_{\gamma,\lambda}$ be the set of all points $\{x_\lambda\}$ in Y_λ such that $x_\lambda \in U_\gamma(k)$. It is clear that the sets $A_{\gamma,\lambda}^-$ (γ running through all elements of Γ, λ fixed, closure in K) are a family of closed subsets of B_λ such that every finite subfamily has nonvoid intersection. Accordingly, there is at least one point S_λ in B_λ which is contained in the intersection $\prod_{\gamma \in \Gamma} A_{\gamma,\lambda}^-$. Select one such point s_λ, and define $\Phi_\lambda(k)$ as the point $s_\lambda \in B_\lambda$. Finally, let $\Phi(k) = \{\Phi_\lambda(k)\}_{\lambda \in \Lambda}$. The mapping Φ, so defined, is clearly a single-valued transformation carrying K into $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$, and it is obviously one-to-one as well. If k is in the subspace $\boldsymbol{P}_{\lambda \in \Lambda} X_\lambda$, so that $k = \{x_\lambda\}$, then each point s_λ must reduce to that point of Y_λ for which the λth co-ordinate is x_λ, so that Φ carries $\boldsymbol{P}_{\lambda \in \Lambda} X_\lambda$ onto $\boldsymbol{P}_{\lambda \in \Lambda} Y_\lambda$ in the natural homeomorphic manner. It follows of course that $\Phi(K)$ contains $\boldsymbol{P}_{\lambda \in \Lambda} Y_\lambda$. To prove that Φ is continuous, let $\Phi(k)$ be an arbitrary point in $\Phi(K)$, and let G be any open set in $\Phi(K)$ containing $\Phi(k)$. The set G is the intersection with $\Phi(K)$ of an open set in $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$, and there accordingly exists a neighborhood in $\boldsymbol{P}_{\lambda \in \Lambda} B_\lambda$ of the form $D_{\lambda_1} \otimes D_{\lambda_2} \otimes \cdots \otimes D_{\lambda_m}$ whose intersection with $\Phi(K)$ is contained in G. Let the point $\Phi(k)$ be $\{s_\lambda\}$, and let C_{λ_i} be a neighborhood of s_{λ_i} in B_{λ_i} such that $C_{\lambda_i}^- \subset D_{\lambda_i}$ ($i = 1, 2, \cdots, m$). Such neighborhoods C_{λ_i} can be found, since the spaces B_λ are bicompact Hausdorff spaces and are hence regular. (See AH, p. 89, Satz IX.) Next, let V_{λ_i} be $C_{\lambda_i} \cap Y_{\lambda_i}$ ($i = 1, 2, \cdots, m$), and let U_{λ_i} be the set in X_{λ_i} corresponding in the natural way to the set V_{λ_i}. The set $U_{\lambda_1} \otimes U_{\lambda_2}$

$\otimes \cdots \otimes U_{\lambda_m}$ is an open set in $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$, and as such is the intersection with $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$ of an open subset H of K. Since $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$ is dense in K, the set H is uniquely determined. Let p be any point in H. Then $\Phi(p)=\{q_\lambda\}$, which is in $\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda$, has the property that its λ_ith co-ordinate, q_{λ_i}, is in the set $V_{\lambda_i}^-$; however, the relations $q_{\lambda_i}\in V_{\lambda_i}^-\subset C_{\lambda_i}^-\subset D_{\lambda_i}$ all obtain $(i=1, 2, \cdots, m)$. These relations show that $\Phi(p)$ lies in G and hence that Φ is a continuous mapping. It is now a simple matter to complete the proof. Since Φ is a one-to-one continuous mapping of the bicompact Hausdorff space K onto the Hausdorff space $\Phi(K)$, it follows that $\Phi(K)$ is homeomorphic to K. (See AH, p. 95, Satz III.) Clearly $\Phi(K)$ contains the space $\boldsymbol{P}_{\lambda\in\Lambda}Y_\lambda$, so that we have $(\boldsymbol{P}_{\lambda\in\Lambda}Y_\lambda)^-\subset\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda\subset\Phi(K)^-=\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda$, and $\Phi(K)$ is dense in $\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda$. Since $\Phi(K)$ is bicompact, it must be closed in $\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda$ (see AH, p. 91, Satz XI), and we conclude that the equality $\Phi(K)=\boldsymbol{P}_{\lambda\in\Lambda}B_\lambda$ obtains.

We can now describe the simple relation obtaining among the spaces β of factor spaces and the space β of a Cartesian product.

THEOREM 14. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a nonvoid family of completely regular spaces. Then $\beta(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda)$ is homeomorphic to the space $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$.*

Using the notation of the preceding theorem, and writing K for $\beta(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda)$ to secure complete uniformity, we first show that B_{λ_0} is homeomorphic to βX_{λ_0}, for all $\lambda_0\in\Lambda$. To do this, let ϕ be any function in the ring $\mathfrak{C}^*(X_{\lambda_0}, R)$. We extend ϕ first to a function ϕ' continuous throughout $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$ by defining ϕ first on the space Y_{λ_0} (which may be considered identical with X_{λ_0}) and defining $\phi'(\{x_\lambda\})$ as $\phi(\{y_\lambda\})$, where $\{y_\lambda\}$ is in Y_{λ_0} and $y_{\lambda_0}=x_{\lambda_0}$. It is obvious that ϕ' is continuous on $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$. The function ϕ' can be continuously extended over all of K, by virtue of K's special properties, and hence can be extended to be continuous throughout $Y_{\lambda_0}^-=B_{\lambda_0}$. This proves that B_{λ_0} is homeomorphic to $B(X_{\lambda_0})$; and the present theorem follows immediately from this observation together with Theorem 13.

THEOREM 15. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of completely regular spaces; and let $\mathfrak{G}$ be the set of functions in $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ which are of the form $f_\lambda\pi_\lambda$, where f_λ is an arbitrary function in $\mathfrak{C}^*(X_\lambda, R)$ and π_λ is the projection of the space $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$ onto the co-ordinate space X_λ. Then the subring of $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ generated by the functions in $\mathfrak{G}$ is dense in the u-topology for $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}X, R)$.*

The present theorem is an easy consequence of Theorem 14. Since $\beta\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda=\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$, it is clear that every function f in $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ can be continuously extended over the space $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$ to be a function $\bar{f}$, and that the correspondence $f\rightarrow\bar{f}$ is a norm-preserving algebraic isomorphism of the rings $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ and $\mathfrak{C}^*(\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda, R)$. Let $\bar{\pi}_\lambda$ represent the projection of the space $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$ onto the λth co-ordinate space, and let f_λ represent a generic function of $\mathfrak{C}^*(X_\lambda, R)$. It is plain the functions of the form $\bar{f}_\lambda\bar{\pi}_\lambda$ serve to distinguish between any pair of distinct points of $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$, and hence the

ring generated by these functions is dense in the u-topology of $\mathfrak{C}^*(P_{\lambda\in\Lambda}\beta X_\lambda, R)$. (See [15] for a proof of this fact.) The inverse image of the set of functions $\bar{f}_\lambda\bar{\pi}_\lambda$ therefore generates a subring of the ring $\mathfrak{C}^*(P_{\lambda\in\Lambda}X_\lambda, R)$ which is dense in the u-topology for $\mathfrak{C}^*(P_{\lambda\in\Lambda}X_\lambda, R)$; but such inverse images are obviously in $\mathfrak{G}$, and the assertion of the present theorem is established.

8. **Extensions of continuous real-valued functions.** Since spaces which are β's for proper subsets of themselves enjoy the extremely strong property that all bounded continuous real-valued functions on certain dense subsets can be continuously extended throughout the whole space, we entertain a natural interest in the local properties of a topological space which allow or deny the possibility of such extensions.

THEOREM 16. *Let X be any topological space, p a point of X, and f a function in $\mathfrak{C}^*(X-p, R)$ which cannot be continuously extended over the point p. Then there exists a real number α and a positive real number η such that, for every neighborhood $U(p)$, there are points q_1 and q_2 in $U(p)$ such that $f(q_1)<\alpha-\eta$ and $f(q_2)>\alpha+\eta$.*

Let $\mathcal{U}=\{U_\lambda(p)\}_{\lambda\in\Lambda}$ be a complete family of neighborhoods for the point p in X. It is obvious that the point p is not isolated in X, since every function in $\mathfrak{C}^*(X-p, R)$ could be extended continuously over p if p were isolated. Let the number γ be defined as $\inf_{\lambda\in\Lambda}(\sup_{q\in U_\lambda}f(q))$. The supremum and infimum in question certainly exist, since f is bounded. Since f cannot be defined at p so as to be continuous there, a positive real number ζ exists such that for every $U_\lambda(p)(\lambda\in\Lambda)$ there are points q and $\bar{q}$ in $U_\lambda(p)$ with the property that $|f(q)-f(\bar{q})|>\zeta$. We define the number η as $\zeta/4$ and the number α as $\gamma-\zeta/2$. We can show without difficulty that α and η enjoy the properties set forth in the statement of the theorem. There exists a neighborhood $U_{\lambda_0}(p)$ such that $f(q)<\gamma+\eta$ for all $q\in U_\lambda(p)$, as the definition of γ shows. It follows also from the definition of γ that in every neighborhood $U_\lambda(p)$ there are points r such that $|f(r)-\gamma|<\eta$. Furthermore, in every neighborhood $U_\lambda(p)$ there are points q_1 and q_2 such that $|f(q_1)-f(q_2)|>\zeta$. If such points be selected in $U_\lambda(p)\cap U_{\lambda_0}(p)$, where λ is an arbitrary element of Λ, it is clear that $f(q_1)$ or $f(q_2)$ must be less than $\gamma+\eta-\zeta=\gamma-3\zeta/4=\alpha-\eta$. Thus we have points q_1 (say) and r such that $f(q_1)<\alpha-\eta$ and $f(r)>\alpha+\eta$.

THEOREM 17. *Let X be a completely regular space and let p be a non-isolated point of X which satisfies Hausdorff's first axiom of countability. Then there exist functions in $\mathfrak{C}^*(X-p, R)$ which cannot be continuously extended over the point p, and there also exist unbounded continuous real-valued functions on $X-p$.*

Let $\{U_n\}_{n=1}^\infty$ be a countable family of neighborhoods of the point p such that any open set containing p contains one of the sets U_n. Since p is non-isolated, all of the sets U_n are infinite. Let p_1 be any point in U_1 and V_1 a neighborhood of p_1 such that p non$\in V_1^-$, and such that also $V_1\subset U_1$. Let U_{n_1}

be any neighborhood of p (the neighborhood with least subscript, for example) disjoint from V_1^-, and let p_{n_1} be any point of U_{n_1} distinct from p. Setting n_0 equal to 1, it is plain that we can prove by induction the existence of a sequence $\{p_{n_k}\}_{k=0}^{\infty}$ of points of X and of neighborhoods $\{V_{n_k}\}_{k=0}^{\infty}$, where each V_{n_k} is a neighborhood of p_{n_k}, such that $V_{n_i} \cap V_{n_j} = 0$ for $i \neq j$, and such that for any neighborhood U_n, all but a finite number of the neighborhoods V_{n_k} are contained in U_n. We now make use of the fact that X is completely regular to define a set of continuous real-valued functions $\{\psi_{n_k}\}_{k=0}^{\infty}$, as follows: for k an even integer, $\psi_{n_k}(p_{n_k}) = 1$ and $\psi_{n_k}(q) = 0$ for $q \in V'_{n_k}$; for k an odd integer, ψ_{n_k} is identically 0.

We now define a function ψ at every point q in X distinct from p:

$$\psi(q) = \sum_{k=0}^{\infty} \psi_{n_k}(q).$$

It is easy to see that ψ is continuous on the space $X-p$. Let q be any point in $X-p$; then there are a neighborhood $W(q)$ and a neighborhood $U_n(p)$ such that $W(q) \cap U_n(p) = 0$. Only a finite number of the sets V_{n_k} can have nonvoid intersection with the set $W(q)$, since $U_n(p)$ contains all but a finite number of these sets, and only a finite number, therefore, of the functions ψ_{n_k} can be different from 0 within $W(q)$. It follows that ψ is continuous at q. Replacing ψ if necessary by the function min $(\psi, 1)$, we obtain a bounded function, which may be denoted by $\bar{\psi}$, continuous throughout the space $X-p$. The function $\bar{\psi}$ can in no wise be defined at p so as to be continuous at that point, since in every neighborhood $U_n(p)$, there are points $q_{n_{2k+1}}$ at which $\bar{\psi}$ vanishes and points $q_{n_{2k}}$ at which it is equal to unity. The existence of unbounded functions on $X-p$ follows in exactly the same way, if we merely employ functions ϕ_{n_k} such that: $\phi_{n_k}(p_{n_k}) = n_k$, $\phi_{n_k}(q) = 0$ for all $q \in V'_{n_k}$, and $0 \leqq \phi_{n_k}(q) \leqq n_k$ for all $q \in X$.

Since the validity of the first axiom of countability at a point p in a completely regular space X implies the existence of functions in $\mathfrak{C}^*(X-p, R)$ which cannot be continuously extended over the point p (except in the trivial case that p is an isolated point), it is natural to inquire after properties of a space at a point p which will ensure that every function in $\mathfrak{C}^*(X-p, R)$ can be continuously extended over p. Such properties cannot be described in terms of the directed set of neighborhoods of the point p, as the following theorem shows.

THEOREM 18. *Let Γ be any directed set with order relation denoted by $\prec$. (See* [27, p. 10] *for a definition of such sets.) Then there exists a completely regular space X and a point p in X such that the system of neighborhoods of the point p, ordered by set-inclusion, is order-isomorphic to the directed set Γ, and such that there exist functions in $\mathfrak{C}^*(X-p, R)$ which cannot be continuously extended over p.*

It is simple to construct a completely regular space Y containing a point p whose neighborhood system, ordered by set-inclusion, is order-isomorphic to the directed set Γ. Let the points of Y be the elements of the directed set Γ, with the addition of a single additional point, p. Let every point of Y be isolated, with the exception of the point p, and let the neighborhood $U_\gamma(p)$ of p in Y be defined as $p \cup E[\delta, \delta \in \Gamma, \delta \succ \gamma] \cup \delta$. It is obvious that Y is completely regular (it is, in fact, totally disconnected ([5])) and that the neighborhood system at p satisfies the conditions imposed above. If Γ possesses a last element, we replace p by that element.

We now let Γ^* be an exact replica of Γ, but disjoint from Γ, and let Y^* be the space defined with the point p and the directed set Γ^* just as Y was defined with the point p and the directed set Γ. (Note that we use the same point p for both Y and Y^*.) The space X is defined as the union of the two spaces Y and Y^*. Clearly the neighborhood system at p in X is order-isomorphic to the directed set Γ, and the space $X-p$ is obviously disconnected. The function ϕ which is equal to 0 on $Y-p$ and equal to 1 on Y^*-p is certainly continuous on $X-p$ but is not continuously extensible over the point p.

It thus appears that we must turn to more refined properties of neighborhoods of a point than merely their direction by set-inclusion in order to describe the extensibility of continuous real-valued functions. We can obtain a complete description of this phenomenon by considering how p is related to certain Z-sets in $X-p$. We first state a simple theorem on complete separation.

THEOREM 19. *Two nonvoid subsets A and B of a topological space X are completely separated if and only if they are contained in disjoint Z-sets.*

Suppose first that A and B are completely separated, and that f is a separating functions as specified in Definition 4 above. Then the functions $\phi_1 = \min(f, 2/3) - 2/3$ and $\phi_2 = \max(f, 1/3) - 1/3$ have the properties that $Z(\phi_2) \supset A$, $Z(\phi_1) \supset B$, and $Z(\phi_1 \cap Z(\phi_2) = 0$. Conversely, if A is contained in the set $Z(f_1)$ and B is contained in the set $Z(f_2)$, where f_1 and f_2 are in $\mathfrak{C}(X, R)$ and $Z(f_1) \cap Z(f_2) = 0$, then the function $\psi = f_1^2 + f_2^2$ vanishes nowhere in X and consequently has an inverse in $\mathfrak{C}(X, R)$. The function $f_1^2 \cdot \psi^{-1}$ clearly vanishes on $Z(f_1)$ and is equal to unity on $Z(f_2)$. Hence A and B are completely separated.

THEOREM 20. *Let X be any completely regular space, and let p be any point of X. Then every function in $\mathfrak{C}^*(X-p, R)$ can be continuously extended over p*

([5]) Of the various definitions of total disconnectedness in use at the present time, we choose the following: a topological space X is said to be totally disconnected if for every point $p \in X$ and every closed set F in X not containing p, there exists an open and closed set A such that $p \in A$ and $A \subset F'$. It is obvious that every totally disconnected space is completely regular.

if and only if, whenever A_1 and A_2 are subsets of $X-p$ such that $p \in A_1^- \cap A_2^-$ (closures in X), A_1 and A_2 fail to be completely separated in $X-p$.

If there is a function ϕ in $\mathfrak{C}^*(X-p, R)$ not continuously extensible over p, let α and $\eta > 0$ be the real numbers described in Theorem 16 as being associated with ϕ and p. Then, if $A_1 = E[q;\ q \in X-p,\ \phi(q) \geqq \alpha+\eta]$ and $A_2 = E[q;\ q \in X-p,\ \phi(q) \leqq \alpha-\eta]$, it follows that A_1 and A_2 are completely separated in $X-p$, and that $p \in A_1^- \cap A_2^-$. Conversely, if p is contained in the intersection $A_1^- \cap A_2^-$, where A_1 and A_2 are completely separated subsets of $X-p$, then there exists a function ψ in $\mathfrak{C}^*(X-p, R)$ which vanishes on A_1 and is equal to unity on A_2; since $p \in A_1^- \cap A_2^-$, every neighborhood of p contains points at which ψ is 0 and points at which ψ is 1; hence ψ is not continuously extensible over p.

The foregoing theorem yields useful information concerning certain special classes of spaces. Let X be a locally bicompact Hausdorff space which is not bicompact. There exists a unique bicompact Hausdorff space, $X \cup p$, obtained by adjoining a single point to X. Neighborhoods of the point p are simply the complements of bicompact subsets of X. It is a simple matter to prove that this extension of X, which we may denote by γX, is a bicompact Hausdorff space and that it is completely determined by X. (See AH, p. 93, Satz XIV.) If X is a non-bicompact completely regular space, and if q is any point of $\beta X - X$, then the space $\beta X - q$ has the property that $\beta(X-q) = \gamma(X-q) = \beta X$. We may ask what properties a locally bicompact Hausdorff space Y must have in order for the relation $\gamma Y = \beta Y$ to obtain.

THEOREM 21. *If Y is a locally bicompact Hausdorff space, then the relation $\gamma Y = \beta Y$ obtains if and only if, given two completely separated closed subsets of Y, at least one of them is bicompact in its relative topology.*

Suppose that the condition stated in the theorem holds and that A_1 and A_2 are completely separated subsets of Y. Then A_1^- and A_2^- are completely separated in Y, and by hypothesis at least one of them, say A_1^-, must be bicompact. In this case, the set $A_1^{-\prime}$ is a neighborhood of the adjoined point q in γY, so that q non$\in A_1^- \cap A_2^-$. It follows from Theorem 20 that every function in $\mathfrak{C}^*(Y, R)$ can be extended continuously over q, so that by a familiar argument, we infer that $\mathfrak{C}^*(Y, R)$ and $\mathfrak{C}^*(\gamma Y, R)$ are isomorphic, and hence that γY is homeomorphic to βY. On the other hand, if two non-bicompact completely separated subsets A_1 and A_2 exist in Y, then in every neighborhood of the adjoined point q of γY, there must be points of both A_1 and A_2. As before, it follows from Theorem 20 that some function in $\mathfrak{C}^*(Y, R)$ cannot be continuously extended over q, and hence that γY is not homeomorphic to βY.

From Theorem 9, we infer the following statement: if Y is a locally bicompact Hausdorff space, then γY is homeomorphic to βY if and only if the ring $\mathfrak{C}^*(Y, R)$ contains precisely one free maximal ideal.

THEOREM 22. *A normal locally bicompact space Y has the property that γY is homeomorphic to βY if and only if at least one of every pair of disjoint closed subsets of X is bicompact in its relative topology.*

The present theorem is a special case of Theorem 21, obvious in the light of the fact that every pair of disjoint closed subsets of a normal space are completely separated.

9. **Examples of spaces βX.** Every known construction of βX relies heavily on the axiom of choice. The construction described in Theorem 9 above uses this axiom in two places: first, in proving the existence of free maximal ideals in $\mathfrak{C}^*(X, R)$; and second, in making use of the family of all maximal ideals in $\mathfrak{C}^*(X, R)$. In view of the non-constructive character of βX, therefore, it is not astonishing that concrete examples of βX for non-bicompact completely regular spaces X are difficult to exhibit. A few such have been found, however, which we now proceed to discuss.

It may be remarked at the outset of this discussion that every bicompact Hausdorff space is its own β, so that we are provided at once with a large but trivial family of spaces β.

Let Δ be any ordinal number, and let T_Δ be the set of all ordinal numbers δ such that $\delta < \Delta$. For an arbitrary $\delta_0 \in T_\Delta$, we define $U_\alpha(\delta_0)$ as the set $E[\delta;\ \delta \in T_\Delta,\ \alpha < \delta \leqq \delta_0]$. As the ordinal number α assumes all possible values less than δ_0, the sets $U_\alpha(\delta_0)$ define a complete set of neighborhoods for the point δ_0. Under the topology defined by these neighborhoods, T_Δ is obviously a totally disconnected (and therefore completely regular) space.

THEOREM 23. *Let $\aleph$ be any infinite cardinal number which is not expressible as the sum of $\aleph_0$ cardinal numbers each smaller than $\aleph$. Let Σ be the least ordinal number with cardinal number $\aleph$. Then, for the space T_Σ, we have $\beta T_\Sigma = T_{\Sigma+1}$.*

A closely related family of spaces for which β may be explicitly described is discussed in the following theorem.

THEOREM 24. *Let Σ be an ordinal number of the type described in Theorem 23, and let Σ^* be the smallest ordinal number with corresponding cardinal number $\aleph^*$, where $\aleph^*$ is an arbitrary infinite cardinal number. Then we have $\beta(T_{\Sigma+1} \otimes T_{\Sigma^*+1} - (\Sigma, \Sigma^*)) = T_{\Sigma+1} \otimes T_{\Sigma^*+1}$.*

Theorems 23 and 24 repose upon the facts that $T_{\mu+1}$ is bicompact for every ordinal number μ and that every continuous real-valued function on the spaces considered is constant beyond some point in the spaces considered. Details of the proofs are omitted. These spaces have β's which are easily obtained from the spaces themselves on account of the fact that there are comparatively few continuous real-valued functions defined on them. These spaces need not all be compact (for example, the space $T_{\Omega+1} \otimes T_{\omega+1} - (\Omega, \omega)$

is not compact ([6])), but they come very close to being bicompact, and it is not astonishing that their spaces β should be simple to find.

10. **β for a discrete space.** If we turn our attention to discrete spaces, we find, as might be expected, a situation differing widely from the one considered above. If N_α is an infinite discrete space of cardinal number $\aleph_\alpha$, then $\mathfrak{C}(N_\alpha, R)$ is the direct sum of $\aleph_\alpha$ replicas of R, and $\mathfrak{C}^*(N_\alpha, R)$ is the subring of this direct sum consisting of all bounded elements in it. The spaces βN_α are very large. As Pospíšil has proved [23], the cardinal number of βN_α is given by the equation([7]).

$$|\beta N_\alpha| = 2^{2^{\aleph_\alpha}}.$$

Interpreted in the light of Theorem 9, this equality states that the ring $\mathfrak{C}^*(N_\alpha, R)$ contains $2^{2^{\aleph_\alpha}}$ free maximal ideals. The space βN_0 enjoys a number of curious properties. Some of these are discussed by Čech [7], more have been set forth by Nakamura and Kakutani [22], and others will be described below.

We first give an explicit construction for βN_0, where N_0 may be taken as the set of all positive integers under their discrete topology. We shall exhibit N_0 as a certain closed subset of the set D of all characteristic functions of subsets of the closed interval $[0, 1]$, which is denoted by I. If A is any subset of I and ϕ_A is the characteristic function of the set A, we define neighborhoods for ϕ_A in the usual fashion. If $x_1, x_2, \cdots, x_n$ are in A and $y_1, y_2, \cdots, y_m$ are in A', then the neighborhood $U_{x_1,x_2,\cdots,x_n;\ y_1,y_2,\cdots,y_m}(\phi_A)$ is defined as the set of all characteristic functions ϕ_X such that $\phi_X(x_i)=1$ $(i=1, 2, \cdots, n)$ and $\phi_X(y_j)=0$ $(j=1, 2, \cdots, m)$. This is obviously the same as the usual Cartesian product topology in D when D is considered as the product of $2^{\aleph_0}$ T_1-spaces each containing exactly two points. As the finite sets $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_m\}$ assume all possible values, the corresponding neighborhoods define a complete neighborhood system for the function ϕ_A. It is easy to show that, under this topology, D is a bicompact Hausdorff space which is of dimension 0 and which has point-character $2^{\aleph_0}$ at every point ([8]).

We now consider the set B of numbers t in $[0, 1]$ such that the triadic development of t contains only 0's and 1's. The selection of B establishes a one-to-one correspondence between the set of all countably infinite sequences consisting of 0's and 1's and the set B contained in $[0, 1]$([9]). Next, let sets X_n $(n=1, 2, 3, \cdots)$ be defined as follows. The set X_n is the set of all t in

([6]) Ω denotes the least ordinal number with cardinal number $\aleph_1$, and ω denotes the least infinite ordinal number.

([7]) We denote the cardinal number of a set A by the symbol $|A|$.

([8]) The point-character of a point p in a topological space is the least cardinal number of a complete family of neighborhoods of the point p.

([9]) The writer is indebted to Dr. Kurt Gödel for the suggestion of making this correspondence by means of triadic developments.

B such that $t(n)$, the nth element in the triadic development of t, is equal to 0. It is obvious that every intersection $X_1^{\delta(1)} \cap X_2^{\delta(2)} \cap X_3^{\delta(3)} \cap \cdots \cap X_n^{\delta(n)} \cdots$, where $X_k^{\delta(k)}$ may be either X_k or $X_k' \cap B$, is nonvoid. In fact, this intersection is simply the number t such that $t(k)=0$ if and only if $X^{\delta(k)}=X_k$.

The set of characteristic functions $\{\phi_{X_n}\}_{n=1}^{\infty}$, considered in its relative topology as a subspace of D, is a discrete space. This is obvious from the fact that the number t_n such that $t_n(k)=0$ if and only if $k=n$ is in X_n but is in no set X_m for $m \neq n$. By identifying ϕ_{X_k} with the kth element of the space N_0, we establish a homeomorphism between N_0 and the space $\{\phi_{X_n}\}_{n=1}^{\infty}$. We now assert that the closure of N_0, considered as being identical with the space $\{\phi_{X_n}\}_{n=1}^{\infty}$, is indeed βN_0. We first establish two necessary conditions for a function ϕ_Z to be an element of N_0^-. Let T_0 denote the set of all numbers t in B such that $t(n)$ is 0 for all but a finite number of values of n, and let T_1 denote the set of all numbers t in B such that $t(n)$ is 1 for all but a finite number of values of n. If ϕ_Z is an element of N_0^-, then we have $T_0 \subset Z \subset B \cap T_1'$. These inclusion relations are verified by contradiction. Assume that for some $t \in T_0$, we have $t \in Z'$. Then the neighborhood $U_{;t}(\phi_Z) \cap N_0$ consists of all functions ϕ_{X_n} such that t non$\in X_n$; but there are only a finite number of such sets X_n, and ϕ_Z accordingly is not an element of N_0^-. To establish the second inclusion relation, assume that a number t exists such that $t \in Z$ and $t \in B' \cup T_1$. If $t \in B'$, then the neighborhood $U_{t;}(\phi_Z)$ has void intersection with N_0, and if $t \in T_1$, then the neighborhood $U_{t;}(\phi_Z)$ contains only a finite number of functions ϕ_{X_n}.

Finally, to prove that N_0^- is βN_0, we make use of the following theorem, due to Čech [7, p. 833]: let S be a normal space, and X a bicompact Hausdorff space containing S as a dense subset, with the property that if F_1 and F_2 are disjoint closed subsets of S, then $F_1^- \cap F_2^- = 0$ (closures in X); under these conditions, X is βS. Since the space N_0 is normal, this theorem may be applied to it, and since every subset of N_0 is closed, we must show that disjoint subsets of N_0 have disjoint closures in N_0^-. If $\{n_1, n_2, n_3, \cdots, n_k, \cdots\}$ and $\{m_1, m_2, m_3, \cdots, m_k, \cdots\}$ are disjoint subsets of the set of all positive integers, we must prove that

$$\phi_{S_n} \in \Big(\sum_{k=1}^{\infty} \phi_{X_{nk}}\Big)^- \text{ and } \phi_{S_m} \in \Big(\sum_{k-1}^{\infty} \phi_{X_{mk}}\Big)^-$$

imply that $S_n \neq S_m$. This non-equality is easily verified by examination of the properties of S_n and S_m. Let the number t be in $B \cap T_1'$, and suppose that $t(n_k)=0$ for $k=1, 2, 3, \cdots$. Then, plainly, t is in S_n or in S_n', and, if $t \in S_n'$, we find that $U_{;t}(\phi_{S_n})$ contains no functions $\phi_{X_{nk}}$. Hence we have $t \in S_n$, since ϕ_{S_n} is in $(\sum_{k=1}^{\infty} \phi_{X_{nk}})^-$. Similarly, if $t(n_k)=1$ for $k=1, 2, 3, \cdots$, then t is in S_n'. Now consider any number $\bar{t}$ in B such that $\bar{t}(n_k)=0$ and $\bar{t}(m_k)=1$ for all $k=1, 2, 3, \cdots$. Since the sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ are disjoint, such a number $\bar{t}$ exists, and from the preceding remarks, it is obvious

that $\bar{t}$ is in the set $S_n \cap S_m'$. Therefore S_n is distinct from S_m, and our construction is complete.

By means of similar arguments (not using effective constructions, to be sure), it is plain that a construction for βN_α can be given, $\aleph_\alpha$ being an arbitrary infinite cardinal number.

We next show that every space βN_α enjoys a curious topological property, that of extremal disconnectivity([10]). We have shown elsewhere [14, p. 326] that a Hausdorff space is totally disconnected if it is extremally disconnected.

THEOREM 25. *If N_α is a discrete space of (infinite) cardinal number $\aleph_\alpha$, then βN_α is extremally disconnected.*

Let G_1 and G_2 be any pair of disjoint open sets in βN_α. Then the sets A_1 and A_2, where $A_i = N_\alpha \cap G_i$ $(i=1, 2)$ are disjoint open and closed subsets of N_α. Hence the characteristic function ϕ_{A_1} is continuous on N_α and is zero throughout the set A_2. It can be continuously extended over all of βN_α; when so extended we denote it by $\bar{\phi}_{A_1}$. In view of the relations $G_i \subset A_i^-$ $(i=1, 2)$, it follows that $\bar{\phi}_{A_1}$ is equal to unity on G_1 and is zero on G_2. Hence G_1 and G_2 have disjoint closures.

We conclude the present section with a simplified proof of the theorem of Pospíšil referred to above.

THEOREM 26. *The cardinal number of the space βN_α is $2^{2^{\aleph_\alpha}}$.*

Let D_α be the Cartesian product of $\aleph_\alpha$ replicas of the T_1-space whose elements are the numbers 0 and 1. We may describe D_α as the set of all functions $p(\lambda)$, for λ in a domain Λ having cardinal number $\aleph_\alpha$ and assuming the values 0 and 1. Let 0 designate the point of D_α such that $0(\lambda)=0$ for all $\lambda \in \Lambda$, and let p_{λ_0} be the element of D_α such that $p_{\lambda_0}(\lambda)=1$ if and only if $\lambda=\lambda_0$. Let Q denote the set of all such p_{λ_0} in D_α. Let $\mathcal{U}=\{V(\lambda_1, \lambda_2, \cdots, \lambda_n)\}$ be a basis for open sets in D_α, where $|\mathcal{U}|$ is equal to $\aleph_\alpha$, and where a one-to-one correspondence has been set up between $\mathcal{U}$ and the family of all finite subsets $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of Λ. Let $T(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be defined as the set $(V(\lambda_1, \lambda_2, \cdots, \lambda_n) \cap Q') \cup (p_{\lambda_1} \cup p_{\lambda_2} \cup \cdots \cup p_{\lambda_n})$. Now consider the space C of all characteristic functions on the set D_α, with the topology described in the construction of βN_0 above. We write, throughout the present proof, $\phi(A)$ for the characteristic function of the subset A of D_α. It is plain that the set $\{\phi(T(\lambda_1, \lambda_2, \cdots, \lambda_n))\}$, where $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ runs through the family of all finite subsets of Λ, is a discrete space in its relative topology in C. Since the set Q has 0 as its only limit point in D_α, we can certainly find a non-void open subset H of D_α such that $H \cap Q$ is void; if X is any subset whatever of H, it is further obvious that the sets $\overline{V}(\lambda_1, \cdots, \lambda_n)$ can be chosen so

([10]) A Hausdorff space X is said to be extremally disconnected if every pair of disjoint open subsets of X have disjoint closures.

that $\phi(X \cup Q)$ is in the closure of the set $\{\phi(T(\lambda_1, \lambda_2, \cdots, \lambda_n))\}$. Since H has cardinal number $2^{\aleph_\alpha}$, we infer that the equality

$$|\{\phi(T(\lambda_1, \lambda_2, \cdots, \lambda_n))\}^-| = 2^{2^{\aleph_\alpha}}$$

obtains. By a theorem of Stone and Čech (see [26, p. 476, Theorem 88]), there is a continuous mapping of βN_α onto $\{\phi(T(\lambda_1, \lambda_2, \cdots, \lambda_n))\}^-$. This fact, together with well known estimates for the cardinal numbers of Hausdorff spaces containing dense subsets of given cardinal number, implies that βN_α has cardinal number $2^{2^{\aleph_\alpha}}$.

Part II. Rings $\mathfrak{C}(X, R)$ and spaces vX

1. **Introduction.** In turning our attention to rings $\mathfrak{C}(X, R)$, we encounter problems differing widely from those met with in the study of bounded functions alone. First, in order to render our considerations non-vacuous, we must confine our investigations to spaces on which unbounded real-valued continuous functions exist; and this necessary restriction immediately excludes all compact spaces. We can therefore hope for no reduction to bicompact spaces, like that described for rings $\mathfrak{C}^*(X, R)$ in the preceding chapter, and must define a new category of spaces which will serve to exemplify all rings $\mathfrak{C}(X, R)$.

In one direction at least, we find a simplification: the structure of ideals in $\mathfrak{C}(X, R)$ is connected with the structure of the space X in a highly perspicuous manner. We shall use this connection as an essential adjunct to our investigation and also as a means of illuminating further the relation between ideals in $\mathfrak{C}^*(X, R)$ and the structure of X.

In other directions, we find enormous complications. The quotient fields $\mathfrak{C}(X, R)/\mathfrak{M}$, where $\mathfrak{M}$ is a maximal ideal in $\mathfrak{C}(X, R)$, need not be isomorphic to R, but may be very large non-Archimedean ordered, formally real extensions of R. We shall describe these fields in as much detail as possible.

2. **Pseudo-compact spaces.** Our first problem is to determine the completely regular spaces on which unbounded real-valued continuous functions exist.

Definition 9. A completely regular space X is said to be pseudo-compact if $\mathfrak{C}(X, R)$ is identical with $\mathfrak{C}^*(X, R)$.

Theorem 27. *The following three properties of a completely regular space X are equivalent:*

(1) *X is pseudo-compact.*

(2) *Every function in $\mathfrak{C}^*(X, R)$ assumes its greatest lower bound and least upper bound for some point or points in X.*

(3) *If $f \in \mathfrak{C}^*(X, R)$, then $f(X)$ is a compact subset of R.*

We establish this result by proving the implications (1)→(2)→(3)→(1). Suppose that (2) does not hold. Then there is a function f in $\mathfrak{C}^*(X, R)$ which

does not assume both of its bounds; we may suppose that f fails to assume its greatest lower bound, which number we denote by t. Then $f-t$ is a positive function taking on arbitrarily small values, and the function $(f-t)^{-1}$ is an unbounded function in $\mathfrak{C}(X, R)$. This establishes the implication (1)$\rightarrow$(2). Next, suppose that (3) does not hold. If $f(X)$ is a bounded, non-compact subset of R, then $f(X)$ must be non-closed. Let u be a limit point of the set $f(X)$ which is not in $f(X)$. If u is the greatest lower bound or least upper bound of $f(X)$, then obviously (2) fails. On the other hand, if there are numbers t_1 and t_2 in $f(X)$ such that $t_1<u<t_2$, then the set $f(X)$ is disconnected, and the sets $A_1=f^{-1}(E[t;\ t\in f(X),\ t>u])$ and $A_2=f^{-1}(E[t;\ t\in f(X),\ t<u])$ are complementary, nonvoid, open sets in X. Suppose that for every positive real number ϵ, the interval $[u, u+\epsilon)$ contains points of $f(X)$. Then let a function ϕ be defined on X, as follows: $\phi(p)=f(p)$ for all $p\in A_1$; $\phi(p)=u+1$ for $p\in A_2$. It is patent that ϕ is a function in $\mathfrak{C}^*(X, R)$, that ϕ has greatest lower bound u, and that $\phi(p)$ is greater than u for all $p\in X$. A very similar construction may be applied if every interval $[u, u-\epsilon)$ contains points of $f(X)$. Hence condition (2) fails if condition (3) fails, and the implication (2)$\rightarrow$(3) is established. Finally, suppose that (1) fails and that the function f is continuous, real-valued, and unbounded. Then $(f^2+1)^{-1}$ is a function in $\mathfrak{C}^*(X, R)$ which is positive everywhere but which assumes values smaller than any preassigned positive real number. Condition (3) thus fails, and the implication (3)$\rightarrow$(1) is verified.

A complete characterization of pseudo-compact spaces is given by the following theorem.

THEOREM 28. *A completely regular space Y is pseudo-compact if and only if Y is equal to $\beta X-A$, where X is a completely regular space, A is a subset of $\beta X-X$, and A contains no set which is a closed G_δ in βX. Equivalently, Y is pseudo-compact if and only if $\beta Y-Y$ contains no closed G_δ except the void set.*

Let Y be a pseudo-compact space, and consider the space $\beta Y-Y$. If $\beta Y-Y$ contains a nonvoid closed G_δ, which we may denote by A, then there is a function ϕ in $\mathfrak{C}^*(\beta Y, R)$ such that $Z(\phi)=A$. (This observation follows from the normality of βY and the fact that every closed G_δ in a normal space is a Z-set. The second statement is proved in [7, p. 829].) Since we have $A\subset\beta Y-Y$, the function ϕ^2 cannot vanish on Y, but since Y is dense in βY, ϕ^2 must have 0 as its greatest lower bound on Y. It follows from Theorem 27 that Y is not pseudo-compact. Conversely, if Y is not pseudo-compact, then there is a function ϕ in $\mathfrak{C}^*(Y, R)$ which is positive everywhere and has greatest lower bound 0. The function ϕ is continuously extensible over all of βY; since βY is bicompact, the set $Z(\bar{\phi})$ is non-void, where $\bar{\phi}$ represents the extension over βY of ϕ, and it is a closed G_δ lying in $\beta Y-Y$.

It will be proved below (Theorem 49) that every closed G_δ in $\beta X-X$ has cardinal number not less than $2^{2^{\aleph_0}}$ (X being any completely regular space)

We thus have the following simple consequence of Theorem 28.

THEOREM 29. *A completely regular space X with the property that $|\beta X - X|$ is less than $2^{2^{\aleph_0}}$ is pseudo-compact.*

The space $T_{\Omega+1} \otimes T_{\omega+1} - (\Omega, \omega)$ is pseudo-compact, as Theorems 24 and 29 show. It is not compact, however, the countable set $A = E[(\Omega, n), n = 1, 2, 3, \cdots]$ having no limit point. This space is, furthermore, non-normal, since A and the set $B = E[(\alpha, \omega); \alpha < \Omega]$ are not completely separable. For normal spaces, indeed, the notions of compactness and pseudo-compactness coincide.

THEOREM 30. *A normal space X is pseudo-compact if and only if it is compact.*

It is patent that every compact normal space is pseudo-compact. To prove the converse, let X be a non-compact normal space, and let $P = \{p_1, p_2, p_3, \cdots, p_n, \cdots\}$ be any countably infinite subset of X having no limit point. The set P is closed, and the function ψ defined on P such that $\psi(p_n) = n$ $(n = 1, 2, 3, \cdots)$ is continuous on P. It is known that every real-valued function defined and continuous on a closed subset of a normal space can be continuously extended over the whole space. (See AH, p. 76, Bemerkung II.) Hence, the function ψ may be continuously extended over X, and X accordingly fails to be pseudo-compact.

We may also note the following immediate consequence of Theorems 28 and 30.

THEOREM 31. *If X is a non-bicompact completely regular space, and if A is a non-void subset of $\beta X - X$ which contains no closed G_δ, then the space $\beta X - A$ is either non-normal or compact.*

3. **The family** $\mathfrak{Z}(X)$. We next consider an important subfamily of the family of closed subsets of a topological space.

DEFINITION 10. Let X be any completely regular space. $\mathfrak{Z}(X)$ is defined as the family of all subsets of X of the form $Z(f)$, where f is an element of $\mathfrak{C}(X, R)$. If $\mathfrak{A}$ is any subset of $\mathfrak{C}(X, R)$, then $\mathfrak{Z}(\mathfrak{A})$ is defined as the family of all $Z(f)$ for $f \in \mathfrak{A}$.

It is clear that every set A in $\mathfrak{Z}(X)$ is a closed G_δ in X, and, as noted above, if the completely regular space X is normal, $\mathfrak{Z}(X)$ coincides with the family of all closed G_δ's in X. This property is not equivalent to normality, however, as the space $T_{\Omega+1} \otimes T_{\omega+1} - (\Omega, \omega)$ shows. It is easy to verify that every closed G_δ in this space is a Z-set, although the space itself is non-normal. An example is given in Appendix A of a non-normal completely regular space in which not every closed G_δ is a Z-set.

For various special categories of spaces, the family $\mathfrak{Z}$ assumes familiar forms. If M is a metric space, for example, $\mathfrak{Z}(M)$ coincides with the family of

all closed subsets of M. More generally, one may consider spaces in which every closed subset is a G_δ and which enjoy the property that every subspace is normal in its relative topology. These are the "perfectly normal" spaces of Čech (see [8]).

We now state certain elementary facts about the family $\mathfrak{Z}(X)$.

THEOREM 32. *The expression "$\mathfrak{C}(X, R)$" may be replaced by "$\mathfrak{C}^*(X, R)$" in Definition* 12 *above without altering the family* $\mathfrak{Z}(X)$.

This theorem follows from the fact that $Z(f)=Z(\min(f^2, 1))$, for all functions f in $\mathfrak{C}(X, R)$, and that $\min(f^2, 1)$ is a function in $\mathfrak{C}^*(X, R)$.

THEOREM 33. *The family* $\mathfrak{Z}(X)$ *is closed under the formation of finite unions and all countable intersections. It is not in general closed under the formation of countably infinite unions.*

The present theorem follows immediately from the two following relations: $Z(f_1)\cup Z(f_2)\cup \cdots \cup Z(f_n)=Z(\min(f_1^2, f_2^2, \cdots, f_n^2))$; and $\prod_{n=1}^{\infty} Z(f_n) = Z(\sum_{n=1}^{\infty} 2^{-n}\cdot g_n)$, where $g_n=\min(f_n^2, 1)$ $(n=1, 2, 3, \cdots)$.

Clearly the series $\sum_{n=1}^{\infty} 2^{-n}\cdot g_n$ converges uniformly and hence represents a continuous function on X. The last statement of the theorem is easily verified by an example. If we take X to be the space R, then every rational point $t\in R$ is a Z-set; but the union of all these Z-sets is not even closed.

We next obtain a second characterization of pseudo-compactness in terms of $\mathfrak{Z}(X)$.

THEOREM 34. *A completely regular space X is pseudo-compact if and only if, whenever $\{Z_n\}_{n=1}^{\infty}$ is a countable subfamily of $\mathfrak{Z}(X)$ enjoying the finite intersection property* (11), *the intersection $\prod_{n=1}^{\infty} Z_n$ is non-void.*

Suppose that the stated condition does not obtain, and that $\{Z_n\}_{n=1}^{\infty}$ is a subfamily of $\mathfrak{Z}(X)$ such that $Z_1\cap Z_2\cap \cdots \cap Z_m\neq 0$ for each natural number m, but such that $\prod_{n=1}^{\infty} Z_n$ is void. Let $A_n=Z_1\cap Z_2\cap \cdots \cap Z_n$ $(n=1, 2, 3, \cdots)$. Then we have $A_1\supset A_2\supset \cdots \supset A_n\supset \cdots$, with $\prod_{n=1}^{\infty} A_n$ void. Since A_n is in $\mathfrak{Z}(X)$, there exists a function f_n in $\mathfrak{C}(X, R)$ such that $Z(f_n)$ is equal to A_n. The function g_n, defined as $\min(f_n^2, 1)$, likewise has the property that $Z(g_n)=A_n$. Consider the function $\phi=\sum_{n=1}^{\infty} 2^{-n}\cdot g_n$. Since every $p\in X$ lies in A_m' for some natural number m, we have $\phi(p)\geqq 2^{-m}\cdot g_m(p)>0$, so that ϕ is positive everywhere. On the other hand, if $p\in A_n$, then $g_1(p)=g_2(p)=\cdots =g_n(p)=0$, so that we have $\phi(p)=\sum_{k=n+1}^{\infty} 2^{-k}\cdot g_k(p)\leqq 2^{-n}$. Hence, the relation $\inf_{p\in X}\phi(p)=0$ is valid, and the function violates condition (2) of Theorem 27, and X is not pseudo-compact. Conversely, if X is not pseudo-compact, there is an unbounded function f in $\mathfrak{C}(X, R)$. Let the set K_n be defined as $E[p; p\in X, f^2(p)\geqq n]$, for all natural numbers n. It is clear that $K_1\cap K_2$

(11) A family $\mathcal{A}$ of sets is said to have the finite intersection property if every set in $\mathcal{A}$ is non-void and $\mathcal{A}$ is closed under the formation of finite intersections.

$\cap \cdots \cap K_n$ is a non-void set, for every n, and that $\prod_{n=1}^{\infty} K_n$ is void. Since $K_n = Z(\max(f^2, n) - n)$, we find that the condition stated in the present theorem fails in the presence of an unbounded real-valued continuous function on X.

4. Ideals in $\mathfrak{C}(X, R)$. As in the case of rings $\mathfrak{C}^*(X\ R)$, we find that maximal ideals in $\mathfrak{C}(X, R)$ are indispensable aids to our study. As in Theorem 9, the fixed maximal ideals are easily described.

THEOREM 35. *The fixed maximal ideals in the ring $\mathfrak{C}(X, R)$ are the sets of all functions in $\mathfrak{C}(X, R)$ vanishing at a given point $p \in X$; for a fixed $p \in X$, this ideal may be denoted by $\mathfrak{M}_p$.*

The proof of the present theorem is identical with that of Theorem 8, and hence does not require repetition.

The free maximal ideals of $\mathfrak{C}(X, R)$, on the other hand, may exhibit the most extreme pathology, their peculiarities being closely connected with those of the space X. In particular, the family $\mathfrak{Z}(\mathfrak{M})$, where $\mathfrak{M}$ is an arbitrary free maximal ideal, can be identified by purely set-theoretic properties, which makes it, within the family $\mathfrak{Z}(\mathfrak{X})$, an exact analogue of the ultra-filters of Cartan (see [6, chap. I, pp. 20–31]). The corresponding property for free maximal ideals in $\mathfrak{C}^*(X, R)$, which establishes a complete equivalence between maximal ideals in the two rings, is somewhat less perspicuous. We now consider the properties of $\mathfrak{Z}(\mathfrak{M})$.

THEOREM 36. *Let X be any completely regular space. Consider the following properties of a non-void subfamily $\mathcal{A}$ of $\mathfrak{Z}(X)$:*

(1) *$\mathcal{A}$ enjoys the finite intersection property;*

(2) *If $A \in \mathcal{A}$, $B \in \mathfrak{Z}(X)$, and $B \supset A$, then $B \in \mathcal{A}$.*

(3) *If $W \in \mathfrak{Z}(X)$ and W non$\in \mathcal{A}$, then $W \cap A = 0$ for some $A \in \mathcal{A}$.*

(4) *The intersection of all sets in $\mathcal{A}$ is void.*

(5) *$\mathcal{A}$ contains no sets bicompact in their relative topologies.*

The family $\mathcal{A}$ is a family $\mathfrak{Z}(\mathfrak{J})$ for some ideal $\mathfrak{J}$ in $\mathfrak{C}(X, R)$ if and only if (1) *and* (2) *obtain; $\mathcal{A}$ is the family $\mathfrak{Z}(\mathfrak{M})$ for some maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ if and only if* (1), (2), *and* (3) *obtain; $\mathcal{A}$ is the family $\mathfrak{Z}(\mathfrak{M})$ for a free maximal ideal in $\mathfrak{C}(X, R)$ if and only if* (1), (2), (3), *and* (4) *obtain. Finally, if $\mathcal{A}$ is the family $\mathfrak{Z}(\mathfrak{M})$ for a free maximal ideal $\mathfrak{M}$, then* (5) *obtains; but the converse is not in general true.*

The present theorem is simply a translation of ideal properties into properties of sets. Let $\mathcal{A}$ be a subfamily of $\mathfrak{Z}(X)$ enjoying properties (1) and (2). Let $\mathfrak{A}$ be the set of all functions f in $\mathfrak{C}(X, R)$ such that $Z(f) \in \mathcal{A}$. If f and g are in $\mathfrak{A}$, then we have $Z(f+g) \supset Z(f^2+g^2) = Z(f) \cap Z(g)$. In view of (1) and (2), we find that $f+g$ is in $\mathfrak{A}$. Since the void set is not in $\mathcal{A}$, every function in $\mathfrak{A}$ vanishes at some point in X, and $\mathfrak{A}$ must be a proper subset of $\mathfrak{C}(X, R)$. Next, if $f \in \mathfrak{A}$ and ψ is any function in $\mathfrak{C}(X, R)$, we have, obviously, $Z(f\psi)$

$=Z(f)\cup Z(\psi)\supset Z(f)$; from (2), it follows that $Z(f\psi)$ is in $\mathcal{A}$, and hence that $f\psi$ is in $\mathfrak{A}$. $\mathfrak{A}$ is therefore an ideal in $\mathfrak{C}(X, R)$. Conversely, given any ideal $\mathfrak{J}$ in $\mathfrak{C}(X, R)$, it is plain that the family $\mathfrak{Z}(\mathfrak{J})$ enjoys properties (1) and (2).

Next, suppose that the family $\mathcal{A}$ enjoys properties (1), (2), and (3), and consider the ideal $\mathfrak{A}$ of all functions f in $\mathfrak{C}(X, R)$ such that $Z(f)\in\mathcal{A}$. If ϕ is any function not in $\mathfrak{A}$, then $Z(\phi)$ is not in $\mathcal{A}$, and from (3) we infer that $Z(\phi)\cap Z(f)=0$ for some $f\in\mathfrak{A}$. The function $f^2+\phi^2$ is positive throughout X and accordingly possesses an inverse in $\mathfrak{C}(X, R)$. It follows that $\mathfrak{A}$ has no proper superideals and is therefore by definition maximal. On the other hand, if $\mathfrak{M}$ is any maximal ideal in $\mathfrak{C}(X, R)$ and g is any function in $\mathfrak{C}(X, R)\cap\mathfrak{M}'$, there are elements $f\in\mathfrak{M}$ and $\psi\in\mathfrak{C}(X, R)$ such that $f+g\psi=1$; hence we find $0=Z(f+g\psi)\supset Z(f)\cap Z(g\psi)=Z(f)\cap(Z(g)\cup Z(\psi))\supset Z(f)\cap Z(g)$. Thus, if $Z(g)$ non$\in\mathfrak{Z}(\mathfrak{M})$, we have $Z(f)\cap Z(g)=0$ for some $Z(f)\in\mathfrak{Z}(\mathfrak{M})$, and it follows that the family $\mathfrak{Z}(\mathfrak{M})$ enjoys properties (1), (2), and (3). Property (4) is merely a restatement of the definition of freeness for an ideal, and need not detain us. Consider next the necessity of condition (5) for a free maximal ideal. If $\mathfrak{M}$ is a free maximal ideal and B is a bicompact set such that $B\in\mathfrak{Z}(\mathfrak{M})$, then, for all $A\in\mathfrak{Z}(\mathfrak{M})$, $B\cap A$ is a nonvoid closed subset of B and is therefore bicompact also. Since $\mathfrak{Z}(\mathfrak{M})$ enjoys the finite intersection property, the family $\{B\cap A\}$, A running through all elements of $\mathfrak{Z}(\mathfrak{M})$, likewise enjoys this property; since B is bicompact, there is a point common to all of the sets $B\cap A$. This circumstance contradicts the original hypothesis that $\mathfrak{M}$ is free. To show that properties (1), (2), (3), and (5) do not suffice to characterize families $\mathfrak{Z}$ for free maximal ideals, consider the Cartesian product $\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$, where Λ is uncountably infinite, and each R_λ is a replica of R. It is easy to see that any function in $\mathfrak{C}(\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda, R)$ which vanishes at a point $\{t_\lambda\}\in\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$ must vanish also on a closed set F containing $\{t_\lambda\}$ such that F is homeomorphic to the whole space $\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$, which is obviously not compact. Hence the fixed maximal ideal $\mathfrak{M}$ of all functions in $\mathfrak{C}(\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda, R)$ vanishing at the point $\{t_\lambda\}$ has the property that $\mathfrak{Z}(\mathfrak{M})$ contains only non-compact sets.

The foregoing theorem completely elucidates the analogy between ultrafilters and families $\mathfrak{Z}(\mathfrak{M})$, where $\mathfrak{M}$ is a maximal ideal in $\mathfrak{C}(X, R)$. Every ultrafilter enjoys property (1), is closed under the operation of adjoining arbitrary supersets, and is maximal with regard to this property and property (1). Our families $\mathfrak{Z}(\mathfrak{M})$ are thus ultrafilters restricted to the family $\mathfrak{Z}(X)$.

Incidence of free maximal ideals may be described as follows.

THEOREM 37. *Let X be any completely regular space. An element f of $\mathfrak{C}(X, R)$ lies in some free maximal ideal if and only if $Z(f)$ is non-bicompact.*

The necessity of the condition stated has been established in Theorem 36 above. To prove that it is sufficient, let $Z(f)$ be a non-bicompact subset of X, and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an open covering of $Z(f)$ admitting no finite subcovering. For every point $p\in Z(f)$ and every λ such that $p\in G_\lambda$, let $\phi_{p,\lambda}$ be a func-

tion in $\mathfrak{C}(X, R)$ such that $\phi_{p,\lambda}(p)=1$, and $\phi_{p,\lambda}(q)=0$ for all q in G_λ'. Such functions exist, by virtue of the complete regularity of X. The set of functions $f\cup\{\phi_{p,\lambda}\}$, where p runs through all points of $Z(f)$ and λ through all indices such that $p\in G_\lambda$, generate a free ideal $\mathfrak{J}$. $\mathfrak{J}$ must be a proper ideal, as well. If $\mathfrak{J}$ is improper, then the relation $1=gf+g_1\phi_{p_1,\lambda_1}+g_2\phi_{p_2,\lambda_2}+\cdots+g_k\phi_{p_k,\lambda_k}$ must obtain for appropriate $(\lambda_1, p_1), \cdots, (\lambda_k, p_k)$ and functions $g_1, \cdots, g_k$, and g in $\mathfrak{C}(X, R)$. However, there is a point in $Z(f)$ which lies in $(G_{\lambda_1}\cup G_{\lambda_2}\cup\cdots\cup G_{\lambda_k})'$, and at this point the function $gf+g_1\phi_{p_1,\lambda_1}+g_2\phi_{p_2,\lambda_2}+\cdots+g_k\phi_{p_k,\lambda_k}$ must vanish, contradicting the relation set forth above. This proves that $\mathfrak{J}$ is not the whole ring $\mathfrak{C}(X, R)$; by Zorn's lemma, it can be proved that $\mathfrak{J}$ is contained in a (free) maximal ideal, which must of course contain f.

It is to be noted, finally, that maximal ideals in $\mathfrak{C}(X, R)$ are not necessarily closed in the u-, k-, or p-topologies for $\mathfrak{C}(X, R)$. It is obvious that every fixed maximal ideal $\mathfrak{M}_p$ is closed in all three topologies, while the following simple example may be adduced to show that a free maximal ideal need be closed in none of them. Let X be the space R, and let f_n be defined as follows:

$$\begin{aligned} f_n(t) &= 1 && \text{for } -\infty < t \leqq n; \\ f_n(t) &= (n+1)-t && \text{for } n < t \leqq n+1; \\ f_n(t) &= 0 && \text{for } n+1 < t < +\infty; \end{aligned}$$

n is to be any natural number. The family of functions $\{f_n\}_{n=1}^{\infty}$ generates a free proper ideal in $\mathfrak{C}(R, R)$, which may be imbedded in a free maximal ideal $\mathfrak{M}$. Every u-, k-, and p-neighborhood of the function e^{-t} clearly contains elements of $\mathfrak{M}$, but e^{-t} is not in $\mathfrak{M}$, since e^{-t} possesses the inverse function e^t. The m-topology, on the other hand, enjoys close connections with the maximal ideals in $\mathfrak{C}(X, R)$, which we now examine.

THEOREM 38. *Let X be any completely regular space. The m-closure of any ideal $\mathfrak{J}$ in $\mathfrak{C}(X, R)$ is again an ideal in $\mathfrak{C}(X, R)$.*

Let f and g be any elements of $\mathfrak{J}^-$ (throughout the present discussion the closure operator in $\mathfrak{C}(X, R)$ refers to the m-topology), and let $\mathfrak{U}_\pi(f+g)$ be an arbitrary m-neighborhood of $f+g$. In view of Theorem 3, there exist neighborhoods $\mathfrak{U}_\sigma(f)$ and $\mathfrak{U}_\rho(g)$ such that $\bar{f}\in\mathfrak{U}_\sigma(f)$ and $\bar{g}\in\mathfrak{U}_\rho(g)$ imply that $\bar{f}+\bar{g}\in\mathfrak{U}_\pi(f+g)$. Since f and g are in $\mathfrak{J}^-$, it follows that $\mathfrak{U}_\pi(f+g)$ contains an element f_0+g_0, where f_0 and g_0, and hence f_0+g_0, are in $\mathfrak{J}$. It follows that $f+g\in\mathfrak{J}^-$. If f is in $\mathfrak{J}^-$, ψ is any function in $\mathfrak{C}(X, R)$, and $\mathfrak{U}_\pi(\psi f)$ is an arbitrary neighborhood of ψf, then as above there exists a neighborhood $\mathfrak{U}_\sigma(f)$ such that $\bar{f}\in\mathfrak{U}_\sigma(f)$ implies that $\psi f\in\mathfrak{U}_\pi(\psi f)$. We infer from this relation that $\psi f\in\mathfrak{J}^-$, and hence that $\mathfrak{J}^-$ is an ideal in $\mathfrak{C}(X, R)$[12].

THEOREM 39. *Every maximal ideal in $\mathfrak{C}(X, R)$ is m-closed.*

[12] We note that Theorem 38 holds for any topological ring.

If $\mathfrak{M}$ is a maximal ideal in $\mathfrak{C}(X, R)$, the closure $\mathfrak{M}^-$ of $\mathfrak{M}$ must be either $\mathfrak{M}$ itself or the ring $\mathfrak{C}(X, R)$, as Theorem 38 shows. If the latter contingency should arise, then there would be elements of $\mathfrak{M}$ in every neighborhood of 1; since 1 has an inverse, and the set of elements with inverse is open in $\mathfrak{C}(X, R)$, it would follow that $\mathfrak{M}$ contains an element with inverse. This is impossible if $\mathfrak{M}$ is to be a proper ideal, and $\mathfrak{M}^- = \mathfrak{M}$ remains as the only possibility.

THEOREM 40. *Let A be any nonvoid subset of the completely regular space X. Then the set of all functions vanishing on A is an m-closed ideal in $\mathfrak{C}(X, R)$.*

This theorem follows immediately from the preceding result and the fact that the set of functions vanishing on A is the intersection $\prod_{p \in A^-} \mathfrak{M}_p$.

It is an open question to identify all of the m-closed ideals in $\mathfrak{C}(X, R)$. It is conjectured that an ideal in $\mathfrak{C}(X, R)$ is m-closed if and only if it is the intersection of maximal ideals in $\mathfrak{C}(X, R)$; but we have not succeeded in finding either a proof of or a counter-example to this conjecture.

5. **Quotient fields of rings** $\mathfrak{C}(X, R)$. We turn now to a discussion of those fields which are homomorphic images of rings $\mathfrak{C}(X, R)$.

DEFINITION 11. Let X be a completely regular space and let $\mathfrak{M}$ be a maximal ideal in $\mathfrak{C}(X, R)$. If the quotient field $\mathfrak{C}(X, R)/\mathfrak{M}$ is isomorphic to R, the ideal $\mathfrak{M}$ is said to be a real ideal. If $\mathfrak{C}(X, R)/\mathfrak{M}$ fails to be isomorphic to R and contains R as a proper subfield, then $\mathfrak{M}$ is said to be a hyper-real ideal.

THEOREM 41. *If $\mathfrak{M}$ is a maximal ideal in $\mathfrak{C}(X, R)$, then the quotient field $\mathfrak{C}(X, R)/\mathfrak{M}$ is an ordered field containing R as a subfield, and every maximal ideal is therefore either real or hyper-real. Every fixed maximal ideal $\mathfrak{M}_p$ is real, satisfying the relation $\mathfrak{M}_p(f) = f(p)$, for every $f \in \mathfrak{C}(X, R)$*[13]. *If X is pseudo-compact, then every maximal ideal in $\mathfrak{C}(X, R)$ is real; if X is not pseudo-compact, there exist hyper-real ideals in $\mathfrak{C}(X, R)$.*

We commence the present proof by displaying an order relation in $\mathfrak{C}(X, R)/\mathfrak{M}$. In the sequel, we shall designate the quotient field $\mathfrak{C}(X, R)/\mathfrak{M}$ by the symbol $\mathfrak{C}_{\mathfrak{M}}$, in cases where no ambiguity arises. Let f be an arbitrary element of $\mathfrak{C}(X, R)$; let $P(f)$ denote the set $E[p;\ p \in X,\ f(p) \geqq 0]$; and let $N(f)$ denote the set $E[p;\ p \in X, f(p) \leqq 0]$. Clearly, $P(f) \cap N(f)$ is the set $Z(f)$.

We now define the order relation in $\mathfrak{C}_{\mathfrak{M}}$: $\mathfrak{M}(f)$ is positive ($\mathfrak{M}(f) > 0$) if and only if $P(f)$ is in $\mathfrak{Z}(\mathfrak{M})$ and $Z(f)$ is not in $\mathfrak{Z}(\mathfrak{M})$; $\mathfrak{M}(f)$ is negative ($\mathfrak{M}(f) < 0$) if and only if $N(f)$ is in $\mathfrak{Z}(\mathfrak{M})$ and $Z(f)$ is not in $\mathfrak{Z}(\mathfrak{M})$. Under these definitions, the following statements are true:

(1) precisely one of the relations $\mathfrak{M}(f) > 0$, $\mathfrak{M}(f) = 0$, $\mathfrak{M}(f) < 0$, obtains;

(2) $\mathfrak{M}(f) > 0$ and $\mathfrak{M}(g) > 0$ imply that $\mathfrak{M}(f) + \mathfrak{M}(g) > 0$ and $\mathfrak{M}(f) \cdot \mathfrak{M}(g) > 0$.

It is clear that $\mathfrak{M}(f) = 0$ if and only if $Z(f) \in \mathfrak{Z}(\mathfrak{M})$. If $P(f)$ and $N(f)$ are both in $\mathfrak{Z}(\mathfrak{M})$, then $Z(f) = P(f) \cap N(f)$ is in $\mathfrak{Z}(\mathfrak{M})$, and $\mathfrak{M}(f)$ is 0. Hence, if

[13] The symbol $\mathfrak{M}(f)$ denotes, as above, the image of f in the homomorphism defined by the ideal $\mathfrak{M}$.

$\mathfrak{M}(f)$ is not 0, $\mathfrak{M}(f)$ cannot be both positive and negative. Since $P(f)$ is equal to $Z(|f|-f)$ and $N(f)$ is equal to $Z(|f|+f)$, and since $0=(|f|-f)(|f|+f)$ is in $\mathfrak{M}$, it follows that at least one of the functions $|f|-f$ and $|f|+f$ is in $\mathfrak{M}$, $\mathfrak{M}$ being a prime ideal since it is maximal. As we have remarked, if f non$\in\mathfrak{M}$, then at least one of the functions $|f|-f$, $|f|+f$ must fail to be in $\mathfrak{M}$; hence statement (1) is established. To prove (2), we observe that $\mathfrak{M}(f)$ is positive if and only if $f(p)$ is positive for all $p\in A$, where A is a set in $\mathfrak{Z}(\mathfrak{M})$. For, if $\mathfrak{M}(f)$ is positive, then we have $P(f)\in\mathfrak{Z}(\mathfrak{M})$ but $Z(f)$ non$\in\mathfrak{Z}(\mathfrak{M})$. Since $\mathfrak{M}$ is maximal, there exists a set $Z(g)\in\mathfrak{Z}(\mathfrak{M})$ such that $Z(f)\cap Z(g)=0$. On the other hand, we have $Z(g)\cap P(f)\in\mathfrak{Z}(\mathfrak{M})$, and it follows that $f(p)$ is positive for all $p\in Z(g)\cap P(f)$. Furthermore, if f is positive throughout a set A in $\mathfrak{Z}(\mathfrak{M})$, it is true that $P(f)$ contains A and hence $P(f)$ is in $\mathfrak{Z}(\mathfrak{M})$. Since $Z(f)\cap A$ is void, it cannot be true that $Z(f)\in\mathfrak{Z}(\mathfrak{M})$, and consequently $\mathfrak{M}(f)$ is positive. Similarly, $\mathfrak{M}(f)$ is negative if and only if f is negative throughout a set A which is in $\mathfrak{Z}(\mathfrak{M})$. Statement (2) is obvious after these observations: if $f(p)$ is positive for all $p\in A$ and $g(p)$ is positive for all $p\in B$, where A and B are elements of $\mathfrak{Z}(\mathfrak{M})$, then $f(p)+g(p)$ and $f(p)\cdot g(p)$ are positive for all $p\in A\cap B$, which is again an element of $\mathfrak{Z}(\mathfrak{M})$. Let α be the constant function in $\mathfrak{C}(X, R)$ with value $\alpha\in R$. It is clear that $\mathfrak{M}(\alpha)$ is positive if and only if α is positive. Likewise, it is clear that $\mathfrak{M}(\alpha_1)\neq\mathfrak{M}(\alpha_2)$ if $\alpha_1\neq\alpha_2$. The homomorphism $\mathfrak{M}$ is therefore an isomorphism carrying the field of constant functions onto R; as such, it must be the identity, and we have $\mathfrak{M}(\alpha)=\alpha$, for all $\alpha\in R$. It follows that $R\subset\mathfrak{C}\mathfrak{M}$.

Using standard terminology from the literature on ordered fields (see [4] and [5]), we say that an element a of an ordered field F is infinitely large if the inequality $a>n\cdot 1$ obtains for all natural numbers n, and that a is infinitely small if the inequalities $0<a<n^{-1}\cdot 1$ obtain for all natural numbers n.

An element f of $\mathfrak{C}(X, R)$ has a real image α under $\mathfrak{M}$ if and only if $Z(f-\alpha)$ is in $\mathfrak{Z}(\mathfrak{M})$. Furthermore, R has no proper Archimedean ordered superfields. (See VDW, pp. 218–227.) It follows that $\mathfrak{C}_{\mathfrak{M}}$ is isomorphic to R if and only if $\mathfrak{C}_{\mathfrak{M}}$ contains no infinitely large elements. Now we have $\mathfrak{M}(f)>n\cdot 1$ if and only if $\mathfrak{M}(f-n)$ is positive, which circumstance is equivalent to the relation $f(p)>n$ for all p in some set A belonging to $\mathfrak{Z}(\mathfrak{M})$. Thus $\mathfrak{M}(f)$ can be infinitely large only if f is unbounded, so that for a pseudo-compact space T, we have $\mathfrak{C}(T, R)/\mathfrak{M}$ isomorphic to R for all $\mathfrak{M}$. On the other hand, if X is not pseudo-compact, and if ϕ is an unbounded function in $\mathfrak{C}(X, R)$, then there is a maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ with respect to which $|\phi|$ is infinitely large. Denote $P(|\phi|-(n+1))$, the set on which the relation $\phi(p)\geqq n+1$ obtains, by A_n. It is clear that the family $\{A_n\}_{n=1}^{\infty}$ enjoys the finite intersection property and also has total intersection void. A simple argument utilizing the well-ordering principle shows that the family $\{A_n\}_{n=1}^{\infty}$ may be imbedded in a superfamily $\mathcal{W}$ such that: (1) $\mathcal{W}\subset\mathfrak{Z}(X)$; (2) $\mathcal{W}$ enjoys the finite intersection property; (3) $B\in\mathcal{W}$, $C\in\mathfrak{Z}(X)$, and $C\supset B$ imply that $C\in\mathcal{W}$; (4) $Z\in\mathfrak{Z}(X)$

and Z non$\in\mathcal{W}$ imply that $Z\cap W=0$ for some $W\in\mathcal{W}$. The details of this construction may be left to the reader. In view of Theorem 36, it may be asserted that there is a maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ such that $\mathfrak{Z}(\mathfrak{M})=\mathcal{W}$. The element $\mathfrak{M}(|\phi|)$ is infinitely large in $\mathfrak{C}_{\mathfrak{M}}$, since $|\phi(p)|>n$ for all $p\in A_n$, and A_n is an element of $\mathfrak{Z}(\mathfrak{M})$. It follows from our previous remarks that $\mathfrak{C}_{\mathfrak{M}}$ is a proper superfield of R; and this completes the present proof.

We proceed to a closer examination of properties common to all fields $\mathfrak{C}_{\mathfrak{M}}$, along with examples of individual peculiarities.

THEOREM 42. *If X is any completely regular space and $\mathfrak{M}$ is any maximal ideal in $\mathfrak{C}(X, R)$, then $\mathfrak{C}_{\mathfrak{M}}$ is a real-closed field. If $\mathfrak{C}_{\mathfrak{M}}$ is hyper-real, then $\mathfrak{C}_{\mathfrak{M}}$ has degree of transcendency at least $2^{\aleph_0}$ over R. If i denotes an element satisfying the equation $x^2+1=0$, then $\mathfrak{C}_{\mathfrak{M}}(i)$ is algebraically closed. If X contains a countable dense subset, then $\mathfrak{C}_{\mathfrak{M}}(i)$ is isomorphic to K.*

For definitions of the terms *formally real* and *real-closed*, see VDW, p. 235. It is proved in [4] that a field is formally real if and only if it can be ordered. Consequently, the field $\mathfrak{C}_{\mathfrak{M}}$ is formally real, a fact easily demonstrable by direct arguments. For, assume that there are functions $f_1, f_2, \cdots, f_n$ in $\mathfrak{C}(X, R)$ such that $\mathfrak{M}(f_1)^2+\mathfrak{M}(f_2)^2+\cdots+\mathfrak{M}(f_n)^2=-1$. Then we have $\mathfrak{M}(f_1^2+f_2^2+\cdots+f_n^2+1)=0$, so that $f_1^2+f_2^2+\cdots+f_n^2+1$ is an element of $\mathfrak{M}$. However, $(f_1^2+f_2^2+\cdots+f_n^2+1)^{-1}$ clearly exists, so that we are presented with an immediate contradiction. To prove that $\mathfrak{C}_{\mathfrak{M}}$ is real-closed, it suffices to show that (1) every positive element in $\mathfrak{C}_{\mathfrak{M}}$ has a square root, and (2) every polynomial of odd degree in the polynomial ring $\mathfrak{C}_{\mathfrak{M}}[x]$ has a root. (For a proof that these conditions imply real closure, see [4, pp. 89–90, Satz 3a and Satz 4].) Condition (1) is easily verified for $\mathfrak{C}_{\mathfrak{M}}$. If $\mathfrak{M}(f)$ is a non-negative element of $\mathfrak{C}_{\mathfrak{M}}$, then we have $P(f)\in\mathfrak{Z}(\mathfrak{M})$, and since $Z(\max(f, 0)-f)$ is equal to $P(f)$, it follows that $\max(f, 0)\equiv f \pmod{\mathfrak{M}}$. Like every non-negative function in $\mathfrak{C}(X, R)$, the function $\max(f, 0)$ has a non-negative square root $(\max(f, 0))^{1/2}$; and it is obvious that $(\mathfrak{M}(\max(f, 0)^{1/2})^2)=\mathfrak{M}(f)$. Hence $\mathfrak{M}(f)$ has a non-negative square root.

To establish the validity of condition (2) in $\mathfrak{C}_{\mathfrak{M}}$, we first prove a "fundamental theorem of algebra" in $\mathfrak{C}(X, R)$. Let $\mathfrak{p}(x)=f_{2n+1}x^{2n+1}+f_{2n}x^{2n}+\cdots+f_1x+f_0$ be any polynomial with coefficients in $\mathfrak{C}(X, R)$ such that f_{2n+1} vanishes nowhere. Then there is a function ϕ_0 in $\mathfrak{C}(X, R)$ such that $\mathfrak{p}(\phi_0)=0$. Since f_{2n+1} vanishes nowhere, the function f_{2n+1}^{-1} exists, and we may, by considering the polynomial $f_{2n+1}^{-1}\cdot\mathfrak{p}(x)$, suppose that f_{2n+1} is 1. For each $p\in X$, the polynomial $\mathfrak{p}(x)$ becomes merely a polynomial with real coefficients and of odd degree: $x^{2n+1}+f_{2n}(p)x^{2n}+\cdots+f_1(p)x+f_0(p)$. This equation has a non-void set of real roots, $\phi_0(p), \phi_1(p), \cdots, \phi_m(p)$, where the natural number m depends upon p; we choose our notation in such a way that the inequalities $\phi_0(p)\leqq\phi_1(p)\leqq\cdots\leqq\phi_m(p)$ obtain. We now define the function ϕ_0 on X as assuming the value $\phi_0(p)$ for all $p\in X$. It is easy to see that the function

ϕ_0 is a continuous function of the coefficients $f_0, f_1, \cdots, f_{2n}$ (in the obvious sense); since these functions are continuous throughout X, it follows at once that ϕ_0 is in $\mathfrak{C}(X, R)$.

Returning to $\mathfrak{C}_{\mathfrak{M}}$, we consider any polynomial of odd degree with coefficients in $\mathfrak{C}_{\mathfrak{M}}$: $\mathfrak{p}(x) = \mathfrak{M}(f_{2n+1})x^{2n+1} + \mathfrak{M}(f_{2n})x^{2n} + \cdots + \mathfrak{M}(f_1)x + \mathfrak{M}(f_0)$, where $\mathfrak{M}(f_{2n+1})$ is not zero. As before, we may suppose that $\mathfrak{M}(f_{2n+1})$ is 1. The equation $x^{2n+1} + f_{2n}x^{2n} + \cdots + f_1x + f_0$ has a root ϕ_0 in $\mathfrak{C}(X, R)$, as we have seen, and it follows at once that $\mathfrak{M}(\phi_0)$ is a root of the equation $\mathfrak{p}(x) = 0$ in $\mathfrak{C}_{\mathfrak{M}}$. This observation completes the proof that $\mathfrak{C}_{\mathfrak{M}}$ is real-closed.

We now consider those fields $\mathfrak{C}_{\mathfrak{M}}$ which contain the field R as a proper subfield. It is plain from the developments of Theorem 41 that $\mathfrak{C}_{\mathfrak{M}}$ is hyperreal if and only if there is an element $\phi \in \mathfrak{C}(X, R)$ such that $\mathfrak{M}(\phi)$ is infinitely large. We may clearly suppose that ϕ is non-negative throughout X. Since $\mathfrak{M}(\phi)$ is infinitely large, it must be true that $P(\phi - n)$ is in $\mathfrak{Z}(\mathfrak{M})$ for all natural numbers n. We now consider the family of functions $\{e^{\alpha\phi^2}\}$, where α runs through all the elements of R; and shall show that the elements $\mathfrak{M}(e^{\alpha\phi^2})$ are all algebraically independent over R in $\mathfrak{C}_{\mathfrak{M}}$. If we assume the contrary, we find that some polynomial with real coefficients $\mathfrak{p}(u_1, u_2, \cdots, u_k)$, in the indeterminates $u_1, u_2, \cdots, u_k$, vanishes when $\mathfrak{M}(e^{\alpha_1\phi^2}), \mathfrak{M}(e^{\alpha_2\phi^2}), \cdots, \mathfrak{M}(e^{\alpha_k\phi^2})$ are substituted for $u_1, u_2, \cdots, u_k$. Any monomial part of this polynomial may be reduced in the following way:

$$\begin{aligned} t\cdot\mathfrak{M}(e^{\alpha_1\phi^2})^{m_1}\cdot\mathfrak{M}(e^{\alpha_2\phi^2})^{m_2}\cdots\mathfrak{M}(e^{\alpha_k\phi^2})^{m_k} &= t\cdot\mathfrak{M}(e^{m_1\alpha_1\phi^2})\cdot\mathfrak{M}(e^{m_2\alpha_2\phi^2})\cdots\mathfrak{M}(e^{m_k\alpha_k\phi^2}) \\ &= t\cdot\mathfrak{M}(e^{(m_1\alpha_1+m_2\alpha_2+\cdots+m_k\alpha_k)\phi^2}). \end{aligned}$$

It is thus clear that the equality $\mathfrak{p}(\mathfrak{M}(e^{\alpha_1\phi^2}), \cdots, \mathfrak{M}(e^{\alpha_k\phi^2})) = 0$ is equivalent to

$$\mathfrak{M}(t_1e^{\beta_1\phi^2} + t_2e^{\beta_2\phi^2} + \cdots + t_ne^{\beta_n\phi^2}) = 0,$$

for appropriately selected real numbers $t_1, \cdots, t_n$ and $\beta_1, \cdots, \beta_n$. If we set $f = \sum_{i=1}^n t_ie^{\beta_i\phi^2}$, we clearly have $f \in \mathfrak{M}$ and $Z(f) \in \mathfrak{Z}(\mathfrak{M})$. Suppose that the numbers $\beta_1, \beta_2, \cdots, \beta_n$ have been so arranged that $\beta_1 < \beta_2 < \cdots < \beta_n$. We next define a new function g, which depends upon β_n. If β_n is positive, then g is equal to t_nf. If β_n is non-positive, then g is equal to $t_ne^{(-\beta_n+1)}f$. It is clear that $g \in \mathfrak{M}$, and it is also clear that g is a function of the form $s_1e^{\gamma_1\phi^2} + s_2e^{\gamma_2\phi^2} + \cdots + s_ne^{\gamma_n\phi^2}$, where γ_n and s_n are positive real numbers and $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. It is a fact familiar from elementary analysis that the function (in $\mathfrak{C}(R, R)$) $h(t) = s_1e^{\gamma_1t^2} + s_2e^{\gamma_2t^2} + \cdots + s_ne^{\gamma_nt^2}$ approaches $+\infty$ as $t \to \infty$. Hence there is a natural number n_0 such that $h(t) > 0$ for all t such that $t^2 \geqq n_0$. If n' is any natural number greater than $(n_0)^{1/2}$, then $Z(g) \cap P(|\phi| - n') = 0$. Since $Z(g) \in \mathfrak{Z}(\mathfrak{M})$ and $P(|\phi| - n') \in \mathfrak{Z}(\mathfrak{M})$, however, the equality $Z(g) \cap P(|\phi| - n') = 0$ cannot obtain, it being at variance with known properties of $\mathfrak{Z}(\mathfrak{M})$. The assumption that $M(e^{\alpha_1\phi^2}), M(e^{\alpha_2\phi^2}), \cdots, M(e^{\alpha_k\phi^2})$ are algebraically dependent must accordingly be rejected; and we have shown

that $\mathfrak{C}_{\mathfrak{M}}$ must have degree of transcendency at least $2^{\aleph_0}$ over R.

To verify the last statement of the present theorem, let X be any completely regular space containing a countable dense subset. It is well known that in this case $|\mathfrak{C}(X, R)|$ is $2^{\aleph_0}$, and consequently $|\mathfrak{C}_{\mathfrak{M}}|$ is $2^{\aleph_0}$. Accordingly, $\mathfrak{C}_{\mathfrak{M}}$ is either R itself or a real-closed extension of R having degree of transcendency $2^{\aleph_0}$ over R and hence over the rational field as well. If $\mathfrak{C}_{\mathfrak{M}}$ is R, then $\mathfrak{C}_{\mathfrak{M}}(i)$ is K, by definition. If $\mathfrak{C}_{\mathfrak{M}}$ is an extension of R, then $\mathfrak{C}_{\mathfrak{M}}(i)$ is an algebraically closed field (see VDW, p. 237, Satz 3) having degree of transcendency $2^{\aleph_0}$ over its prime subfield. By a celebrated theorem of Steinitz [25], it follows that $\mathfrak{C}_{\mathfrak{M}}(i)$ is isomorphic to K.

We also note that $\mathfrak{C}(X, K)/\mathfrak{M}$ is algebraically closed and that $\mathfrak{C}(X, K)/\mathfrak{M}$ is isomorphic to K if X contains a countable dense subset. In certain special cases, we can establish further interesting properties for $\mathfrak{C}_{\mathfrak{M}}$.

THEOREM 43. *Let X be a Hausdorff space which is locally bicompact and which can be represented as a union $\sum_{n=1}^{\infty} B_n$, where each subspace B_n is bicompact in its relative topology*(14). *Let M be any hyper-real ideal in $\mathfrak{C}(X, R)$ and let $\{t_n\}_{n=1}^{\infty}$ be any countable subset of $\mathfrak{C}_{\mathfrak{M}}$. Then there exists an element u of $\mathfrak{C}_{\mathfrak{M}}$ such that $u > t_n$ for $n = 1, 2, 3, \cdots$.*

It is well known that X must be a completely regular space. We next observe that $X = \sum_{n=1}^{\infty} A_n$, where $A_n \subset A_{n+1}'^{-\prime}$ and A_n is bicompact ($n = 1, 2, 3, \cdots$). First, let $C_n = \sum_{i=1}^{n} B_i$. It is clear that C_n is bicompact. For every point $p \in C_1$, choose a neighborhood $V(p)$ (in X) such that $V^-(p)$ is bicompact; and select a finite covering $V(p_1), V(p_2), \cdots, V(p_m)$ of C_1. Let $A_1 = \sum_{i=1}^{m} V^-(p_i)$. Let the bicompact set $A_1 \cup C_2$ be imbedded similarly in a set $\sum_{i=1}^{n} V^-(q_i) = A_2$. Continuing this process by an obvious finite induction, we find that every A_n is bicompact, that $A_n \subset A_{n+1}'^{-\prime}$ ($n = 1, 2, 3, \cdots$), and that $X = \sum_{n=1}^{\infty} A_n$.

We observe next that the sets A_n and $A_{n+1}'^{-}$ are completely separated ($n = 1, 2, 3, \cdots$). Since we have $A_n \subset A_{n+1}'^{-\prime}$, there exists, for an arbitrary point $p \in A_n$, a function f_p in $\mathfrak{C}(X, R)$ such that $f_p(p) = 2$ and $f_p(q) = 0$ for all $q \in A_{n+1}'^{-}$. Since f_p is continuous, there exists a neighborhood U_p of p such that $f_p(q) > 1$ for all $q \in U_p$. Since A_n is bicompact, we find that there is a finite subset $p_1, p_2, \cdots, p_m$ of A_n such that $\sum_{i=1}^{m} U_{p_i}$ covers A_n. It follows that the function $\psi = \max(f_{p_1}, f_{p_2}, \cdots, f_{p_m})$ is greater than unity throughout A_n and is zero throughout $A_{n+1}'^{-}$, which fact shows that A_n and $A_{n+1}'^{-}$ are completely separated.

Returning to our main assertion, we may suppose without loss of generality that all of the elements $t_1, t_2, t_3, \cdots, t_n, \cdots$ are infinitely large in $\mathfrak{C}_{\mathfrak{M}}$ and the inequalities $t_1 < t_2 < t_3 < \cdots < t_n < \cdots$ obtain. We now consider functions $f_1, f_2, f_3, \cdots, f_n, \cdots$ in $\mathfrak{C}(X, R)$ such that $\mathfrak{M}(f_n) = t_n$; since $\mathfrak{M}(|f_n|)$

(14) Typical examples of spaces satisfying these conditions are N_0 and Euclidean m-space, m being any natural number.

$=|\mathfrak{M}(f_n)|=\mathfrak{M}(f_n)$, we may suppose that all of the functions f_n are non-negative. We seek a function ϕ in $\mathfrak{C}(X, R)$ such that $\mathfrak{M}(\phi)>\mathfrak{M}(f_n)$, for all n; such a function will certainly yield an upper bound for the set $\{t_n\}_{n=1}^{\infty}$. We construct the desired function ϕ by the following process. Let the function ϕ_1 be defined as f_1+1. Let ψ_{12} be a function in $\mathfrak{C}(X, R)$ such that $\psi_{12}(p)=0$ for all $p\in A_1$, $\psi_{12}(p)=1$ for all $p\in A_2'$, and $0\leqq\psi_{12}(p)\leqq 1$ for all $p\in X$. Defining the function ϕ_2 as $f_1+\psi_{12}f_2+1$, we find that ϕ_2 is equal to f_1+1 on A_1 and is equal to f_1+f_2+1 on A_2'. Let ψ_{34} be a function in $\mathfrak{C}(X, R)$ such that $\psi_{34}(p)=0$ for all $p\in A_3$, $\psi_{34}(p)=1$ for all $p\in A_4'$, and $0\leqq\psi_{34}(p)\leqq 1$ for all $p\in X$. Let the function ϕ_3 be defined as $\phi_2+\psi_{34}f_3$. It is plain that $\phi_3=f_1+1$ on A_1, that $\phi_3=f_1+f_2+1$ on $A_3\cap A_2'$, and that $\phi_3=f_1+f_2+f_3+1$ on A_4'. Continuing this construction by an inductive process, we find that a function ϕ_n can be defined for every natural number n such that:

$$\begin{array}{lll}
\phi_n = f_1 + 1 & \text{on } A_1; & \\
\phi_n = f_1 + f_2 + 1 & \text{on } A_3 \cap A_2'; & \\
\cdots\cdots\cdots\cdots & & \\
\phi_n = f_1 + f_2 + \cdots + f_i + 1 & \text{on } A_{2i-1} \cap A_{2i-2}' & (i = 2, 3, \cdots, n-2); \\
\phi_n = f_1 + f_2 + \cdots + f_n + 1 & \text{on } A_{2n-2}'. &
\end{array}$$

If we let $\phi=\lim_{n\to\infty}\phi_n$, it is clear that ϕ is an element of $\mathfrak{C}(X, R)$ and that ϕ is not less than $f_1+f_2+\cdots+f_n+1$ throughout A_{2n-2}'. Since A_n is bicompact, $\mathfrak{Z}(\mathfrak{M})$ must contain, for every natural number n, a set Z_n contained in A_n'. For, if every Z in $\mathfrak{Z}(\mathfrak{M})$ had nonvoid intersection with some A_{n_0}, then the intersection $\prod_{Z\in\mathfrak{Z}(\mathfrak{M})}(A_{n_0}\cap Z)$ would be nonvoid, in contradiction to our hypothesis that $\mathfrak{M}$ is a hyper-real and therefore free ideal. Thus, for an arbitrary natural number n, there is a set Z in $\mathfrak{Z}(\mathfrak{M})$ such that $Z\subset A_{2n-2}'$. On this set Z, the function $\phi-f_n$ is positive, and therefore we have $\mathfrak{M}(\phi)>\mathfrak{M}(f_n)$. Setting $u=\mathfrak{M}(\phi)$, we have a proof of the present theorem.

The questions naturally arise as to what functions in $\mathfrak{C}(X, R)$ always have non-real images in quotient fields $\mathfrak{C}_{(\mathfrak{M})}$, where $\mathfrak{M}$ is a hyper-real ideal in $\mathfrak{C}(X, R)$, and what functions always have real images in such quotient fields. We obtain answers to both of these queries, as follows.

THEOREM 44. *Let X be a non-pseudo-compact completely regular space. A function f in $\mathfrak{C}(X, R)$ has the property that $\mathfrak{M}(f)$ is non-real for every free ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ if and only if every set $E[p; p\in X, f(p)=\alpha]$ is bicompact, where α is an arbitrary real number. The function f has the property that $\mathfrak{M}(f)$ is non-real for some maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ if and only if there is a countable closed subset $\{p_n\}_{n=1}^{\infty}$ of X, discrete in its relative topology, such that $f(p_n)\neq f(p_m)$ for $m\neq n$.*

The first equivalence enunciated in the present theorem is evident in the light of Theorems 36 and 41. We accordingly proceed directly to the second.

If $\mathfrak{M}(f)$ is non-real for some maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$, then there are two possibilities: $\mathfrak{M}(f)$ is infinitely large or $\mathfrak{M}(f)$ is not infinitely large. In the former case, we merely need to select a strictly monotonic increasing sequence of real numbers, $\{t_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty} t_n = +\infty$, such that $f(p_n) = t_n$, for some $p_n \in X$ $(n=1, 2, 3, \cdots)$. The latter case can be reduced to the former very simply. If $\mathfrak{M}(f)$ is non-real and not infinitely large, then there exists a real number α such that $(\mathfrak{M}(f)-\alpha)$ is infinitely small. (See Theorem 46 below for a proof of this fact.) It is plain that $(\mathfrak{M}(f)-\alpha)^{-1}$ is infinitely large; and we may apply the argument used in the former case.

On the other hand, suppose that f satisfies the condition stated above for some set $\{p_n\}_{n=1}^{\infty}$. We may evidently exclude the possibility that $f(p_{n_0})=0$ for some n_0. Let F_n be defined as $E[p; p\in X, f(p)=f(p_n)]$, and let K_n be defined as $\sum_{i=n}^{\infty} F_i$ $(n=1, 2, 3, \cdots)$. It can be proved that each set K_n is a Z-set, the argument used being identical with that set forth in the proof of Theorem 49 below. It is obvious that each set K_n is non-compact. By a simple application of the well-ordering principle, one can show that there exists a maximal ideal $\mathfrak{M}$ such that $\mathfrak{Z}(\mathfrak{M})$ contains all of the sets K_n. It is evident that $\mathfrak{M}(f)$ is non-real in $\mathfrak{C}(X, R)/\mathfrak{M}$.

6. **Relations between $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$.** We now proceed to a comparison of maximal ideals in $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$ and to a new construction for βX.

THEOREM 45. *Let X be a completely regular space, and let $\mathfrak{M}^*$ be any maximal ideal in $\mathfrak{C}^*(X, R)$. Let $\mathcal{S}(\mathfrak{M}^*)$ be the family of all sets $N(|f|-\epsilon)$ (where f runs through all the elements of $\mathfrak{M}^*$ and ϵ runs through the set of all positive real numbers) together with all sets $Z(f)$, where $f\in\mathfrak{M}^*$ and $Z(f)$ is nonvoid. Then there exists a unique maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$ such that $\mathfrak{Z}(\mathfrak{M})=\mathcal{S}(\mathfrak{M}^*)$; for all $f\in\mathfrak{C}^*(X, R)$, f is an element of $\mathfrak{M}^*$ if and only if $\mathfrak{M}(f)$ is 0 or is infinitely small in $\mathfrak{C}_{\mathfrak{M}}$. Conversely, if $\mathfrak{M}$ is any maximal ideal in $\mathfrak{C}(X, R)$, then the set $\mathfrak{A}$ of elements $f\in\mathfrak{C}^*(X, R)$ such that $N(|f|-\epsilon)\in\mathfrak{Z}(\mathfrak{M})$ for all positive real numbers ϵ constitutes a maximal ideal in $\mathfrak{C}^*(X, R)$. If $\mathfrak{N}$ is an arbitrary maximal ideal in $\mathfrak{C}(X, R)$, then $\mathfrak{N}\cap\mathfrak{C}^*(X, R)$ is contained in precisely one maximal ideal of $\mathfrak{C}^*(X, R)$, and $\mathfrak{N}\cap\mathfrak{C}^*(X, R)$ is a maximal ideal in $\mathfrak{C}^*(X, R)$ if and only if $\mathfrak{N}$ is a real ideal.*

The first assertion will be established by showing that $\mathcal{S}(\mathfrak{M}^*)$ enjoys the properties (1), (2), and (3) listed in Theorem 36. First, $\mathcal{S}(\mathfrak{M}^*)$ does not contain the void set. If $f\in\mathfrak{M}^*$, then f^{-1} does not exist in $\mathfrak{C}^*(X, R)$, and, for every positive real number ϵ, there must be a point $p\in X$ such that $|f(p)|<\epsilon$. Next, if $f_1, f_2\in\mathfrak{M}^*$ and ϵ_1, ϵ_2 are positive real numbers, we find that $N(\max(|f_1|, |f_2|)-\min(\epsilon_1, \epsilon_2))\subset N(|f_1|-\epsilon_1)\cap N(|f_2|-\epsilon_2)$. Since $|f|\in\mathfrak{M}^*$ if $f\in\mathfrak{M}^*$ (see [26, p. 457, Theorem 76]), it follows that $\max(f_1, f_2)=(|f_1-f_2|+f_1+f_2)/2$ is in $\mathfrak{M}^*$ whenever $f_1, f_2\in\mathfrak{M}^*$. Therefore the intersection of any two elements of $\mathcal{S}(\mathfrak{M}^*)$ contains a third element of $\mathcal{S}(\mathfrak{M}^*)$. We next observe

that if $A \in \mathfrak{S}(\mathfrak{M}^*)$, $B \in \mathfrak{Z}(X)$, and $B \supset A$, then $B \in \mathfrak{S}(\mathfrak{M}^*)$. For, if $Z(g) = B$, and $A = N(|f| - \epsilon)$, for some $f \in \mathfrak{M}^*$, $g \in \mathfrak{C}^*(X, R)$, and positive real number ϵ, then $fg \in \mathfrak{M}^*$, and $Z(fg) = Z(f) \cup Z(g) = Z(f) \cup B = B$. It is therefore proved that $B \in \mathfrak{S}(\mathfrak{M}^*)$ and that $\mathfrak{S}(\mathfrak{M}^*)$ enjoys properties (1) and (2) of Theorem 36.

To show that $\mathfrak{S}(\mathfrak{M}^*)$ enjoys property (3) of Theorem 36, we make use of the maximality of $\mathfrak{M}^*$. Suppose that A non$\in \mathfrak{S}(\mathfrak{M}^*)$ and that $A = Z(g)$, for some $g \in \mathfrak{C}^*(X, R)$. Since A non$\in \mathfrak{S}(\mathfrak{M}^*)$, g non$\in \mathfrak{M}^*$, and consequently there are functions $\phi \in \mathfrak{M}^*$ and $\psi \in \mathfrak{C}^*(X, R)$ such that $\phi + \psi g = 1$. Since g vanishes on A, $\phi(p) = 1$ for all $p \in A$, and the set $N(|\phi| - 3/4)$, which is an element of $\mathfrak{S}(\mathfrak{M}^*)$, is disjoint from A. We have thus proved that $\mathfrak{S}(\mathfrak{M}^*)$ possesses properties (1), (2), and (3) of Theorem 36, and we infer from that theorem the existence of a (necessarily unique) maximal ideal $\mathfrak{M}$ in $C(X, R)$ such that $\mathfrak{Z}(\mathfrak{M}) = \mathfrak{S}(\mathfrak{M}^*)$.

Suppose next that $f \in \mathfrak{M}^*$. Then there are two possibilities: $Z(f) \in \mathfrak{S}(\mathfrak{M}^*)$, or $N(|f| - \epsilon) \in \mathfrak{S}(\mathfrak{M}^*)$ for every positive real number ϵ while $Z(f)$ non$\in \mathfrak{S}(\mathfrak{M}^*)$. If $Z(f) \in \mathfrak{S}(\mathfrak{M}^*)$, then $\mathfrak{M}(f) = 0$. If $N(|f| - \epsilon) \in \mathfrak{S}(\mathfrak{M}^*)$, for every positive real number ϵ, then $P(\epsilon - |f|) \in Z(\mathfrak{M})$, and $\mathfrak{M}(|f|) = |\mathfrak{M}(f)|$ must be infinitely small in $\mathfrak{C}_{\mathfrak{M}}$. Conversely, suppose that $|\mathfrak{M}(f)|$ is infinitely small or zero in $\mathfrak{C}_{\mathfrak{M}}$. If $|\mathfrak{M}(f)| = 0$, then $Z(f) \in \mathfrak{Z}(\mathfrak{M}) = \mathfrak{S}(\mathfrak{M}^*)$, and $f \in \mathfrak{M}^*$. If $|\mathfrak{M}(f)|$ is infinitely small, there must exist, for every positive real number ϵ, a set $A \in \mathfrak{Z}(\mathfrak{M})$ such that $P(\epsilon - |f|) \supset A$. This implies that $N(|f| - \epsilon) \in \mathfrak{Z}(\mathfrak{M})$; hence $N(|f| - \epsilon) \in \mathfrak{S}(\mathfrak{M}^*)$, and $f \in \mathfrak{M}^*$.

The next statement of the theorem is easily verified. If f and g are in the set $\mathfrak{A}$, and if ϵ is any positive real number, then $N(|f| - \epsilon/2)$ and $N(|g| - \epsilon/2)$ are in $\mathfrak{Z}(\mathfrak{M})$, and $N(|f+g| - \epsilon) \supset N(|f| - \epsilon/2) \cap N(|f| - \epsilon/2)$; hence $f+g$ is in $\mathfrak{A}$. If ψ is an arbitrary function different from 0 in $\mathfrak{C}^*(X, R)$, and if $N(|f| - \epsilon \cdot \|\psi\|^{-1}) \in \mathfrak{Z}(\mathfrak{M})$, then $N(|\psi f| - \epsilon)$ is in $\mathfrak{Z}(\mathfrak{M})$, since $N(|\psi f| - \epsilon) \supset N(|f| - \epsilon \|\psi\|^{-1})$. Thus we find that $\psi f \in \mathfrak{A}$. $\mathfrak{A}$ is hence an ideal in $\mathfrak{C}^*(X, R)$. To prove that $\mathfrak{A}$ is maximal, select any function ϕ in $\mathfrak{C}^*(X, R)$ which is not in $\mathfrak{A}$. There exist positive real numbers ϵ and ϵ' and a function ϕ' in $\mathfrak{A}$ such that $N(|\phi| - \epsilon) \cap N(|\phi'| - \epsilon') = 0$. The function $\phi^2 + \phi'^2$ thus has greatest lower bound not less than min $(\epsilon^2, \epsilon'^2)$, and the function

$$g = \phi\left(\frac{\phi}{\phi^2 + \phi'^2}\right) + \phi'\left(\frac{\phi'}{\phi^2 + \phi'^2}\right)$$

is identically 1; since g is in the ideal generated by ϕ and $\mathfrak{A}$, it is clear that $\mathfrak{A}$ is a maximal ideal in $\mathfrak{C}^*(X, R)$.

The final statements of the present theorem follow as direct consequences of our preceding remarks. If $\mathfrak{N}$ is any maximal ideal in $\mathfrak{C}(X, R)$, then $\mathfrak{N} \cap \mathfrak{C}^*(X, R)$ is the set of all functions f in $\mathfrak{C}^*(X, R)$ such that $Z(f) \in \mathfrak{Z}(\mathfrak{M})$. Clearly this set of functions forms a (proper) ideal in $\mathfrak{C}^*(X, R)$; by Zorn's lemma, it can be extended to be a maximal ideal in $\mathfrak{C}^*(X, R)$. Any such maximal extension $\mathfrak{H}^*$ must contain all the elements $g \in \mathfrak{C}^*(X, R)$ such that

$N(|g|-\epsilon)\in\mathfrak{Z}(\mathfrak{M})$ for all positive real numbers ϵ. If any such g were not in $\mathfrak{H}^*$, then the set of functions of the form $fg+h$, where f runs through all elements of $\mathfrak{C}^*(X, R)$ and h runs through all elements of $\mathfrak{H}^*$, would be a proper extension of $\mathfrak{H}^*$. On the other hand, we have already seen that $\mathfrak{H}^*$ is a maximal ideal in $\mathfrak{C}^*(X, R)$. The final remark of the present theorem is obvious: if $\mathfrak{N}$ is a real ideal in $\mathfrak{C}(X, R)$, then there are no infinitely small elements in $\mathfrak{C}_{\mathfrak{N}}$; consequently there are no elements in $\mathfrak{C}^*(X, R)$ with, $N(|f|-\epsilon)\in\mathfrak{Z}(\mathfrak{N})$ (for all positive real numbers ϵ) except those for which $\mathfrak{Z}(f)\in\mathfrak{Z}(\mathfrak{N})$. The converse is also obvious.

We now prove a theorem stated but not proved in detail by Gelfand and Kolmogoroff [12].

THEOREM 46. *Let X be any completely regular space, and let $\mathcal{M}$ be the set of all maximal ideals in $\mathfrak{C}(X, R)$. Then if $\mathcal{M}$ is topologized in accordance with the definition set forth in Theorem* 10, *$\mathcal{M}$ is homeomorphic to βX.*

As in Theorem 9, let $\mathcal{X}$ be the set of all fixed ideals in $\mathcal{M}$. The proofs that $\mathcal{M}$ is a bicompact T_1-space containing $\mathcal{X}$ as a dense subset and that $\mathcal{X}$ is homeomorphic to X in its relative topology may be carried over from Theorem 9 with only notational changes. We accordingly need only to prove that every function in $\mathfrak{C}^*(X, R)$ can be continuously extended over $\mathcal{M}$. To do this, we first observe that if $f\in\mathfrak{C}^*(X, R)$ and $\mathfrak{M}\in\mathcal{M}$, then there is a unique real number $\mathfrak{M}^*(f)$ such that $|\mathfrak{M}(f)-\mathfrak{M}^*(f)|$ is infinitely small or 0 in $\mathfrak{C}_{\mathfrak{M}}$. If $\mathfrak{M}(f)$ is real, then we take $\mathfrak{M}^*(f)=\mathfrak{M}(f)$; our assertion is obviously true in this case. If $\mathfrak{M}(f)$ is not real, we can still assert that $\mathfrak{M}(f)$ is not infinitely large, since there is a positive real number α such that $\alpha\geqq f(p)\geqq-\alpha$ for all $p\in X$. These inequalities imply that $\mathfrak{M}(\alpha)=\alpha\geqq\mathfrak{M}(f)\geqq\mathfrak{M}(-\alpha)=-\alpha$ (these inequalities in $\mathfrak{C}_{\mathfrak{M}}$). We define $\mathfrak{M}^*(f)$ as the real number sup $[\beta;\beta\leqq\mathfrak{M}(f)]$ (inequality in $\mathfrak{C}_{\mathfrak{M}}$). To show that $|\mathfrak{M}(f)-\mathfrak{M}^*(f)|$ is infinitely small, we remark that $\mathfrak{M}(f)-\mathfrak{M}^*(f)\geqq 0$; if $\mathfrak{M}(f)-\mathfrak{M}^*(f)$ were not infinitely small, there would be a positive real number δ such that $\mathfrak{M}(f)-\mathfrak{M}^*(f)-\delta>0$. Then we should have $\mathfrak{M}(f)>\mathfrak{M}^*(f)+\delta$, which contradicts our definition of $\mathfrak{M}^*(f)$. The number $\mathfrak{M}^*(f)$ is unique, since if α_1 and α_2 are real numbers such that $|\mathfrak{M}(f)-\alpha_1|$ and $|\mathfrak{M}(f)-\alpha_2|$ are both infinitely small, then $|\alpha_1-\alpha_2|=|(\mathfrak{M}(f)-\alpha_1)+(\alpha_2-\mathfrak{M}(f)|\leqq|\mathfrak{M}(f)-\alpha_1|+|\mathfrak{M}(f)-\alpha_2|$; and consequently $|\alpha_1-\alpha_2|$ is infinitely small; this of course implies that $\alpha_1=\alpha_2$.

If ϕ is any function in $\mathfrak{C}^*(\mathcal{X}, R)$ (we regard the spaces X and $\mathcal{X}$ as identical) we extend ϕ throughout all of $\mathcal{M}$ by the definition $\phi(\mathfrak{M})=\mathfrak{M}^*(\phi)$. To prove the continuity of ϕ at an arbitrary point $\mathfrak{M}_0\in\mathcal{M}$, we have, as in the proof of Theorem 9, two cases.

Case I: $\phi(\mathfrak{M}_0)=0$. Let ϵ be any positive real number, and let $f=\min(|\phi|,\epsilon)-\epsilon$; then $\mathfrak{M}_0(f)=\min(\mathfrak{M}_0(|\phi|),\epsilon)-\epsilon=|\mathfrak{M}_0(\phi)|-\epsilon$. This element of $\mathfrak{C}_{\mathfrak{M}}$ cannot be zero, since $|\mathfrak{M}_0(\phi)|$ is infinitely small. Hence $\mathcal{U}_f(\mathfrak{M}_0)$ is an open set in $\mathcal{M}$ containing $\mathfrak{M}_0$. If $\mathfrak{M}\in\mathcal{U}_f(M_0)$, then $\mathfrak{M}(f)=\min(|\mathfrak{M}(\phi)|,\epsilon)$

$-\epsilon \neq 0$, and hence we have $|\mathfrak{M}(\phi)| < \epsilon$. This means of course that $\phi(\mathfrak{M}) = \mathfrak{M}^*(\phi)$ must be less than ϵ in absolute value. Hence ϕ is continuous at $\mathfrak{M}_0$.

Case II: $\phi(\mathfrak{M}_0) \neq 0$. In this event, the function $\psi = \phi - \phi(\mathfrak{M}_0)$ has the property that $\psi(\mathfrak{M}_0) = 0$; we effect in this way a reduction to Case I. Since ψ is continuous at $\mathfrak{M}_0$, and since ϕ differs from ψ only by a constant, it follows that ϕ is continuous at $\mathfrak{M}_0$. This observation completes the proof.

If we consider the infinite discrete space N_α of cardinal number $\aleph_\alpha$, and apply Theorems 46, 36, and 26, we obtain the following interesting theorem.

THEOREM 47. *There are exactly* $2^{2^{\aleph_\alpha}}$ *ultra-filters on a set of cardinal number* $\aleph_\alpha$.

Our next result identifies the closed G_δ's in βX by algebraic means.

THEOREM 48. *Let X be an arbitrary completely regular space, and let βX be considered as the space $\mathfrak{N}$ of all maximal ideals in $\mathfrak{C}(X, R)$. The closed G_δ's in βX are precisely the sets $S_f = E[\mathfrak{M}; \mathfrak{M} \in \mathfrak{N}, |\mathfrak{M}(f)|$ is infinitely small or $0]$, f being an arbitrary element of $\mathfrak{C}(X, R)$.*

Since $\mathfrak{N}$ is bicompact, it is normal; and in a normal space, the closed G_δ's are simply the Z-sets. Thus the closed G_δ's in $\mathfrak{N} = \beta X$ are simply the subsets of $\mathfrak{N}$ on which continuous real-valued functions on $\mathfrak{N}$ vanish. But all of these functions are, by the preceding theorem, extensions of functions in $\mathfrak{C}(X, R)$; and these extensions vanish on just those maximal ideals $\mathfrak{M}$ in βX for which $|\mathfrak{M}(f)|$ is infinitely small or zero.

Čech has proved (see [7, p. 835]) that every closed G_δ in $\beta X - X$ has cardinal number not less than $2^{\aleph_0}$, X being an arbitrary completely regular space. We find it possible to strengthen this result, obtaining, indeed, the best possible estimate for the cardinal numbers of such sets.

THEOREM 49. *Let X be a completely regular space, and let A be a nonvoid closed G_δ in βX such that $A \subset \beta X - X$. Then the inequality $|A| \geqq 2^{2^{\aleph_0}}$ obtains; and for appropriate choice of X and A, the equality holds.*

Since X is dense in βX and since A is contained in $\beta X - X$, it is clear that A can contain no nonvoid open subset; in particular, the set A is not open. From Theorem 48, it is clear that A may be considered as the set of all maximal ideals $\mathfrak{M}$ in $\mathfrak{N}$ for which $|\mathfrak{M}(f)|$ is infinitely small or zero, f being some function in $\mathfrak{C}^*(X, R)$; equivalently, $A = Z(f)$, for some f in $\mathfrak{C}^*(\beta X, R)$. Replacing f by its absolute value if necessary, we suppose throughout the rest of the present proof that f is non-negative. Since A is not an open set, every set $E[p; p \in \beta X, f(p) < 1/n]$ must contain points other than those in A; and, since X is dense in βX, the sets $E[p; p \in \beta X, f(p) < 1/n]$ must contain points lying in X ($n = 1, 2, 3, \cdots$). Since $Z(f) = A$, and $A \subset \beta X - X$, it is clear that f vanishes nowhere on X. We may thus select points $p_0, p_1, p_2, \cdots \in X$ and find positive real numbers $\{t_n\}_{n=0}^{\infty}$ with these properties: $f(p_n) = t_n$, $t_n > t_{n+1}$

$(n=0, 1, 2, \cdots)$, and $\lim_{n\to\infty} t_n=0$. We next choose sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $t_{n-1}>a_n>t_n>b_n>a_{n+1}$, for $n=1, 2, 3, \cdots$, and define the set B_n as $E[p;\ p\in X,\ a_n\geqq f(p)\geqq b_n]$. We now consider an arbitrary sequence of strictly increasing natural numbers, $\{n_k\}_{k=1}^{\infty}$, and turn our attention to the set $\sum_{k=1}^{\infty} B_{n_k}$. It will be observed that this set is non-compact; for if it were compact, then the function f would vanish for some point in X. We assert that every set $\sum_{k=1}^{\infty} B_{n_k}$ is in the family $\mathfrak{Z}(X)$, proving this assertion for the case $n_k=k$. (The general case is proved by exactly the same method.) Let ψ_0 be defined as $\max(f, a_1)-a_1$: clearly $Z(\psi_0)=E[p;\ p\in X,\ f(p)\leqq a_1]$. For $n=1, 2, 3, \cdots$, let ϕ_n and ψ_n be the following functions:

$$\phi_n = b_n - \min(f, b_n);\quad \psi_n = \max(f, a_{n+1}) - a_{n+1}.$$

It is plain that $Z(\phi_n)=E[p;\ p\in X,\ f(p)\geqq b_n]$ and that $Z(\psi_n)=E[p;\ p\in X,\ f(p)\leqq a_{n+1}]$. The function $\phi=\psi_0+\sum_{k=1}^{\infty}\phi_k\psi_k$ clearly has the property that $Z(\phi)=\sum_{k=1}^{\infty}B_k$, and is obviously continuous. From Theorem 47, we see that there are $2^{2^{\aleph_0}}$ families $\mathfrak{F}_\lambda$ having total intersection void (where λ runs through an index class Λ of cardinal number $2^{2^{\aleph_0}}$) of subsets of any countably infinite set such that (1) each $\mathfrak{F}_\lambda$ consists only of infinite sets, (2) each $\mathfrak{F}_\lambda$ enjoys the finite intersection property, (3) each $\mathfrak{F}_\lambda$ contains all supersets of all of its members, (4) each $\mathfrak{F}_\lambda$ is maximal with respect to the properties (1), (2), and (3). From (1)–(4) inclusive we readily infer: (5) for $\lambda_1\neq\lambda_2$, there are sets $F_{\lambda_1}\in\mathfrak{F}_{\lambda_1}$ and $F_{\lambda_2}\in\mathfrak{F}_{\lambda_2}$ such that $F_{\lambda_1}\cap F_{\lambda_2}=0$. We may apply these considerations directly to the countable set whose elements are $B_1, B_2, B_3, \cdots$. There are $2^{2^{\aleph_0}}$ families of sets $\mathcal{G}_\lambda$, each $\mathcal{G}_\lambda$ being a family of sets each of the form $\sum_{k=1}^{\infty}B_{n_k}$, such that the families $\mathcal{G}$ all enjoy properties (1)–(5) inclusive. Each $\mathcal{G}_\lambda$ thus forms a family $\mathfrak{Z}(\mathfrak{J}_\lambda)$ for a free proper ideal $\mathfrak{J}_\lambda$ in $\mathfrak{C}(X, R)$. The ideal $\mathfrak{J}_\lambda$ can be imbedded in a maximal ideal, $\mathfrak{M}_\lambda$, in $\mathfrak{C}(X, R)$. On account of (5), the $\mathfrak{M}_\lambda$'s are all distinct. The function f is infinitely small with respect to all of the ideals $\mathfrak{M}_\lambda$. It follows that A, considered as a subset of $\mathfrak{M}$, must contain all of the maximal ideals $\mathfrak{M}_\lambda$. This observation completes the present proof, except for the presentation of an example. Consider the spaces N_0 and βN_0, where we take N_0 to be the positive integers. Let f be that function on N_0 such that $f(n)=1/n$. The function f is continuous on N_0 and hence can be continuously extended over βN_0. On the bicompact space βN_0, this extension must vanish; as it can vanish only for points in βN_0-N_0, we are presented with a closed G_δ in βN_0-N_0. Since the cardinal number of βN_0 is $2^{2^{\aleph_0}}$, it follows that f vanishes on a closed G_δ in βN_0-N_0 having cardinal number precisely $2^{2^{\aleph_0}}$.

We may remark that there is no upper limit on the cardinal number of closed G_δ's in $\beta X-X$, for appropriately chosen completely regular spaces X. If X is a normal space containing a closed subset B of cardinal number $\aleph_\alpha$ which is discrete in its relative topology, then $\beta X-X$ contains at least one closed G_δ having cardinal number $2^{2^{\aleph_\alpha}}$.

7. **Q-spaces.** It was proved in Part I that the study of rings $\mathfrak{C}^*(X, R)$ may be confined to the case in which X is a bicompact Hausdorff space. This arises from the fact that with every completely regular space, there is associated the unique bicompact Hausdorff space βX having the property that $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(\beta X, R)$ are algebraically isomorphic. In the present section, we shall demonstrate that an analogous category of spaces exists relative to rings $\mathfrak{C}(X, R)$. These spaces, which we have called Q-spaces, are characterized by no topological property so simple as bicompactness; indeed, their description may be considered somewhat recondite. Furthermore, we find that the class of Q-spaces is very extensive, including but not exhausted by the families of all separable metric spaces, all bicompact spaces, and all discrete spaces. The failure to obtain a severe restriction on the spaces to be considered is counterpoised by the fact that the very diversification of Q-spaces makes rings $\mathfrak{C}(X, R)$ much more powerful instruments than rings $\mathfrak{C}^*(X, R)$ for the study of topological properties of a completely regular space X.

We commence with a definition of our new family of spaces.

DEFINITION 12. A completely regular space X is said to be a Q-space if every free maximal ideal in $\mathfrak{C}(X, R)$ is hyper-real.

To identify the class of Q-spaces by purely topological properties, we introduce another definition.

DEFINITION 13. A subfamily $\mathcal{A}$ of the family $\mathfrak{Z}(X)$, where X is a completely regular space, is said to be $\mathfrak{Z}$-maximal if A enjoys the finite intersection property and no proper superfamily of $\mathcal{A}$ included in $\mathfrak{Z}(X)$ does so.

It is easy to characterize Q-spaces in terms of $\mathfrak{Z}$-maximality.

THEOREM 50. *A completely regular space X is a Q-space if and only if every $\mathfrak{Z}$-maximal family $\mathcal{A}$ in $\mathfrak{Z}(X)$ with total intersection void contains a countable subfamily with total intersection void.*

To establish this equivalence, we first suppose that X satisfies the stated condition. Let $\mathfrak{M}$ be any free maximal ideal in $\mathfrak{C}(X, R)$; $\mathfrak{Z}(\mathfrak{M})$ is then a $\mathfrak{Z}$-maximal family, as was proved in Theorem 36, and since $\mathfrak{M}$ is free, the intersection $\prod_{Z\in\mathfrak{Z}(\mathfrak{M})} Z$ is void. According to our hypothesis, there exists a countable subfamily $\{Z_n\}_{n=1}^{\infty}$ of $\mathfrak{Z}(\mathfrak{M})$ such that $\prod_{n=1}^{\infty} Z_n = 0$. Let the set A_n be defined as $Z_1 \cap Z_2 \cap \cdots \cap Z_n$, for $n = 1, 2, 3, \cdots$. It is evident that the inclusions $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ are valid; and that A_n is nonvoid and an element of $\mathfrak{Z}(\mathfrak{M})$, since $\mathfrak{Z}(\mathfrak{M})$ enjoys the finite intersection property. Also it is obvious that the intersection $\prod_{n=1}^{\infty} A_n$ is void. Since $A_n \in \mathfrak{Z}(\mathfrak{M})$, there is a function f_n in $\mathfrak{C}(X, R)$ such that $Z(f_n) = A_n$, and having the additional property that $0 \leqq f_n(p) \leqq 1$ for all $p \in X$. We define a function ϕ in the following way: $\phi = \sum_{n=1}^{\infty} 2^{-n} \cdot f_n$. It is obvious that $\phi \in \mathfrak{C}(X, R)$. If $p \in A_n$, we have $\phi(p) = f_{n+1}(p) \cdot 2^{-(n+1)} + f_{n+2}(p) 2^{-(n+2)} + \cdots \leqq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}$. On the other hand, since $\prod_{n=1}^{\infty} A_n = 0$, there is, for every $p \in X$, a natural num-

ber m such that p non$\in A_m$, whence we find that $\phi(p) \geqq f_m(p) \cdot 2^{-m} > 0$. Since ϕ is positive throughout X, ϕ^{-1} exists and ϕ non$\in \mathfrak{M}$. However, if $Z \in \mathfrak{Z}(\mathfrak{M})$, we have $Z \cap A_n \neq 0$ for all n, and if $q \in Z \cap A_n$, it follows that $\phi(q) \leqq 2^{-n}$. Thus, ϕ assumes arbitrarily small values on every Z in $\mathfrak{Z}(\mathfrak{M})$, and is consequently infinitely small in $\mathfrak{C}_{\mathfrak{M}}$. This proves that $\mathfrak{M}$ is a hyper-real ideal.

The converse is easily established. Let X be a completely regular space such that $\mathfrak{C}(X, R)/\mathfrak{M}$ is hyper-real for every free maximal ideal $\mathfrak{M}$. Let $\mathcal{A}$ be any $\mathfrak{Z}$-maximal family. It is evident from Theorem 36 that $\mathcal{A}$ is equal to $\mathfrak{Z}(\mathfrak{M})$ for some maximal ideal in $\mathfrak{C}(X, R)$, and it is patent that $\mathfrak{M}$ is free if and only if $\mathcal{A}$ has total intersection void. By hypothesis, the quotient field $\mathfrak{C}(X, R)/\mathfrak{M}$ is hyper-real, and there must be an element f of $\mathfrak{C}(X, R)$ such that $P(f-n)$ is in the family $\mathfrak{Z}(\mathfrak{M})$ for all natural numbers n. It is obvious that $\prod_{n=1}^{\infty} P(f-n)$ is void, and it follows that every $\mathfrak{Z}$-maximal family with total intersection void contains a countable subfamily with intersection void.

We now turn our attention to the classification of Q-spaces, finding that many commonly studied spaces lie in this category. Following Alexandroff and Urysohn, we define a weak type of compactness. (See [3, p. 17].)

DEFINITION 14. Let $\aleph$ be an arbitrary infinite cardinal number. A topological space X is said to be upper $\aleph$-compact if every open covering of X of cardinal number not less than $\aleph$ has a subcovering of cardinal number less than $\aleph$.

We shall show that every upper $\aleph_1$-compact completely regular space is a Q-space, and, by an example, that there exist Q-spaces which are not upper $\aleph_1$-compact. A preliminary theorem may first be enunciated.

THEOREM 51. *The following four properties of a completely regular space X are equivalent*:

(1) *X is upper $\aleph_1$-compact.*

(2) *If $\{F_\lambda\}_{\lambda \in \Lambda}$ is a family of closed subsets of X such that $\prod_{\lambda \in \Lambda^*} F_\lambda \neq 0$, where Λ^* is an arbitrary countable subclass of the index class Λ, then $\prod_{\lambda \in \Lambda} F_\lambda \neq 0$.*

(3) *If $\{A_\lambda\}_{\lambda \in \Lambda}$ is any covering of X, where the sets A_λ are of the form $P(f_\lambda) \cap Z'(f_\lambda)$, for $f_\lambda \in \mathfrak{C}(X, R)$, then there exists a countable subcovering $\{A_\lambda\}_{\lambda \in \Lambda^*}$ of the covering $\{A_\lambda\}_{\lambda \in \Lambda}$.*

(4) *If $\{Z(f_\lambda)\}_{\lambda \in \Lambda}$ is any subfamily of $\mathfrak{Z}(X)$ such that every countable subfamily of $\{Z(f_\lambda)\}_{\lambda \in \Lambda}$ has nonvoid intersection, then $\prod_{\lambda \in \Lambda} Z(f_\lambda) \neq 0$.*

Verification of the equivalence (1)$\leftrightarrow$(2) is quite elementary and is therefore passed over. The equivalence (3)$\leftrightarrow$(4) is likewise obvious when one observes that every set $P(f) \cap Z'(f)$ is equal to $Z'(\max(f, 0))$ and that $Z'(f)$ is equal to $P(f^2) \cap Z'(f^2)$. We may thus complete the present proof by demonstrating the equivalence (1)$\leftrightarrow$(3). The implication (1)$\rightarrow$(3) is of course obvious. To verify the converse (3)$\rightarrow$(1), let $\{G_\mu\}_{\mu \in M}$ be any open covering of X. Since X is completely regular, there exists, for every $p \in G_\mu$, a function

$f_{p,\mu}\in\mathfrak{C}(X, R)$ such that $f_{p,\mu}(p)=1, f_{p,\mu}(q)=0$ for all $q\in G_\mu'$, and $0\leqq f_{p,\mu}(q)\leqq 1$ for all $q\in X$. If we set $A_{p,\mu}=P(f_{p,\mu})\cap Z'(f_{p,\mu})$ we have $A_{p,\mu}\subset G_\mu$. The family $\{A_{p,\mu}\}$ for $\mu\in M+p\in G_\mu$ is a covering of X as described in (3). If (3) obtains, there exists a countable subcovering $A_{p_1,\mu_1}, A_{p_2,\mu_2}, \cdots, A_{p_n,\mu_n}, \cdots$, of the covering $\{A_{p,\mu}\}$. Since we have $A_{p,\mu}\subset G_\mu$, it follows that $\{G_{\mu_n}\}_{n=1}^{\infty}$ is a covering of X. Hence (1) obtains; and the present theorem is completely proved.

THEOREM 52. *Every upper $\aleph_1$-compact completely regular space is a Q-space; but the converse is false, as the discrete space N_α of arbitrary non-countable cardinal number $\aleph_\alpha$ is a Q-space but is not upper $\aleph_1$-compact.*

The first statement of the present theorem is an immediate consequence of Theorems 50 and 51. In proving the second, we first observe that the covering $\{p\}_{p\in N_\alpha}$ of N_α, each open set containing precisely one point, has no countable subcovering. Hence N_α is not upper $\aleph_1$-compact. Consider any maximal free ideal $\mathfrak{M}$ in $\mathfrak{C}(N_\alpha, R)$. To prove that N_α is a Q-space, it is sufficient, in virtue of Theorem 50, to show that $\mathfrak{Z}(\mathfrak{M})$ contains a countably infinite subfamily having void intersection. Since N_α is discrete, the family $\mathfrak{Z}(\mathfrak{M})$ is simply a free ultra-filter on N_α; that is, it is a family which enjoys the finite intersection property, contains all supersets of all of its elements, is maximal with respect to the first two properties, and has total intersection void. We may suppose that the family $\mathfrak{Z}(\mathfrak{M})$ is well-ordered: $\mathfrak{Z}(\mathfrak{M})=A_1, A_2, \cdots, A_n, \cdots, A_\gamma, \cdots$, where the ordinal numbers γ run through all ordinals less than the first ordinal of cardinal number $2^{\aleph_\alpha}$, which is the cardinal number of $\mathfrak{Z}(\mathfrak{M})$ (15). We now employ the axiom of choice to select certain points p_μ in N_α. Every ordinal γ may be written as $\lambda+n$, where λ is a limit ordinal and n is a finite ordinal. Since $\mathfrak{Z}(\mathfrak{M})$ enjoys the finite intersection property, the set $B_\gamma=A_\lambda\cap A_{\lambda+1}\cap\cdots\cap A_{\lambda+n}$ is nonvoid, and we choose p_γ as an arbitrary point in B_γ. Let $C_n=\sum p_\mu$ $(n=0, 1, 2, 3, \cdots)$, where μ runs through all ordinal numbers μ such that $\mu=\lambda+i$, where λ is a limit ordinal and i is a finite ordinal not less than n. It is obvious that $C_n\cap A_\gamma\neq 0$ for every γ, and hence $C_n\in\mathfrak{Z}(\mathfrak{M})$, since $\mathfrak{Z}(\mathfrak{M})$ is maximal. It is likewise obvious that $\prod_{n=0}^{\infty}C_n=0$. Hence $\mathfrak{Z}(\mathfrak{M})$ contains a countable subfamily with void intersection, and N_α is accordingly a Q-space.

It is well known that every topological space satisfying the second axiom of countability is upper $\aleph_1$-compact (see AH, p. 78, Satz I). From this fact and Theorem 52, we may infer the following result.

THEOREM 53. *Any completely regular space satisfying the second axiom of countability is a Q-space.*

Tychonoff has proved [28] that every regular (and hence every completely regular) space satisfying the second axiom of countability is normal, and by

(15) The cardinal number $|\mathfrak{Z}(\mathfrak{M})|$ is $2^{\aleph_\alpha}$ because, as may readily be seen, $\mathfrak{Z}(\mathfrak{M})$ contains precisely one of the sets A and A', where A is an arbitrary subset of N_α.

a celebrated theorem of Urysohn (AH, pp. 81–83) every normal space satisfying the second axiom of countability is metrizable. Theorem 53 therefore states that every separable metric space is a Q-space.

THEOREM 54. *A pseudo-compact completely regular space is a Q-space if and only if it is bicompact.*

If X is pseudo-compact and not bicompact, then the function-ring $\mathfrak{C}(X, R)=\mathfrak{C}^*(X, R)$ contains a free maximal ideal, $\mathfrak{M}$. The quotient field $\mathfrak{C}(X, R)/\mathfrak{M}$ must be R, however, so that X is not a Q-space.

Another special family of Q-spaces is described in the following theorem.

THEOREM 55. *Let X be any locally bicompact Hausdorff space such that $X=\sum_{n=1}^{\infty}B_n$, where each B_n is bicompact $(n=1, 2, \cdots)$. Then X is a Q-space.*

If X should itself be bicompact, then there is nothing to prove. If X is non-bicompact, then there is a free maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(X, R)$. As in the proof of Theorem 43, we may infer that $X=\sum_{n=1}^{\infty}A_n$, where A_n is bicompact and A_n and A'_{n+1} are completely separated $(n=1, 2, 3, \cdots)$. Since A_n and A'_{n+1} are completely separated, there exists a set $B_n\in\mathfrak{Z}(X)$ such that $A'_n\supset B_n\supset A'^{-}_{n+1}\supset A'_{n+1}$ $(n=1, 2, 3, \cdots)$. Let C be any set in $\mathfrak{Z}(\mathfrak{M})$. Since $B'_n\subset A_{n+1}$, the set B'^{-}_n is bicompact, and it is clear that C must have nonvoid intersection with $B'^{-\prime}_n$ and hence with B_n for every natural number n. For, if $C\subset B'^{-}_n$, then C would be bicompact, and $\mathfrak{M}$ could not be a free ideal. It follows at once from the maximal property enjoyed by the family $\mathfrak{Z}(\mathfrak{M})$ that every set B_n is in $\mathfrak{Z}(\mathfrak{M})$. Since $\prod_{n=1}^{\infty}B_n\subset\prod_{n=1}^{\infty}A'_{n+1}=0$, we apply Theorem 50 to infer that X is a Q-space.

As an example of a non-Q-space, we may mention the space T_Ω, which is compact but not bicompact. More generally, if X is a completely regular non-bicompact space such that the cardinal number $|\beta X-X|$ is less than $2^{2^{\aleph_0}}$, then X is not a Q-space. The connection (or lack of it) between the classes of normal spaces and Q-spaces is of some interest. The space T_Ω, easily proved to be normal, gives an example of a normal non-Q-space, while a space C_1 is constructed in Appendix A which is a Q-space but non-normal.

8. **The relation of Q-spaces to rings $\mathfrak{C}(X, R)$.** We now discuss the properties of Q-spaces, finding that they provide an exact analogue of bicompact spaces when applied to rings $\mathfrak{C}(X, R)$.

THEOREM 56. *Every completely regular space X can be imbedded as a dense subset of a Q-space vX such that $\mathfrak{C}(X, R)$ is algebraically isomorphic with $\mathfrak{C}(vX, R)$.*

We consider the family $\mathfrak{M}$ of all maximal ideals $\mathfrak{M}$ in $\mathfrak{C}(X, R)$, and single out for special attention the subfamily $\mathfrak{N}$ of all real ideals in $\mathfrak{M}$. As was pointed out in Theorem 46, the set $\mathfrak{M}$ may be made a bicompact Hausdorff space by the usual construction. In its relative topology, the space $\mathfrak{N}$ is a

completely regular space. We shall take $\mathcal{N}$ to be our space vX. It is clear that X can be mapped homeomorphically into $\mathcal{N}$: every fixed maximal ideal in $\mathfrak{C}(X, R)$ is real, and the set $\mathcal{X}$ of all fixed maximal ideals in $\mathfrak{C}(X, R)$ consitutes a subspace of $\mathcal{N}$ which is homeomorphic to X in its relative topology. We may therefore identify X and $\mathcal{X}$ in the subsequent discussion. It is also true (and obvious) that $\mathcal{X}$ is dense in $\mathcal{N}$. Since $\mathcal{X}$ is dense in $\mathcal{N}$, it is clear that $\mathfrak{C}(\mathcal{X}, R)$ contains $\mathfrak{C}(\mathcal{N}, R)$ as a subring, since every function in $\mathfrak{C}(\mathcal{N}, R)$ is continuous when its domain is restricted to $\mathcal{X}$, and every function in $\mathfrak{C}(\mathcal{X}, R)$ can be continuously extended over $\mathcal{N}$ in at most one way. Now every function in $\mathfrak{C}(\mathcal{X}, R)$ can be extended over $\mathcal{N}$. If $f \in \mathfrak{C}(\mathcal{X}, R)$, then $f(p) = \mathfrak{M}_p(f)$ for all $\mathfrak{M}_p \in \mathcal{X}$, and by defining $f(\mathfrak{M})$ to be $\mathfrak{M}(f)$ for all $\mathfrak{M} \in \mathcal{N}$, we obtain an extension of f from $\mathcal{X}$ to $\mathcal{N}$. It remains to show that f is continuous on $\mathcal{N}$. This verification may be carried out just as the corresponding fact was proved in Theorem 9. We therefore see that $\mathfrak{C}(X, R)$, which we consider as being identical with $\mathfrak{C}(\mathcal{X}, R)$, is isomorphic to a subring of $\mathfrak{C}(\mathcal{N}, R)$; as in the proof of Theorem 9, it follows that $\mathfrak{C}(\mathcal{N}, R)$ and $\mathfrak{C}(X, R)$ are algebraic isomorphs.

We must also show that $\mathcal{N}$ is a Q-space, that is, that every real ideal in $\mathfrak{C}(\mathcal{N}, R)$ is fixed. Let $\mathfrak{M}$ be any real ideal in $\mathfrak{C}(\mathcal{N}, R)$, and let $\mathcal{A}$ be any set in $\mathfrak{Z}(\mathfrak{M})$. Then $\mathcal{X} \cap \mathcal{A}$ is a nonvoid Z-set in $\mathcal{X}$, since if $\mathcal{X} \cap \mathcal{A} = 0$, there would be a function with inverse in $\mathfrak{C}(\mathcal{X}, R)$ corresponding under the natural isomorphism to a function in $\mathfrak{C}(\mathcal{N}, R)$ without inverse. The family of sets $\mathcal{A} \cap \mathcal{X}$, where $\mathcal{A}$ runs through all the elements of $\mathfrak{Z}(\mathfrak{M})$, is therefore a family $\mathfrak{Z}(\mathfrak{H})$ for some real ideal $\mathfrak{H}$ in $\mathfrak{C}(\mathcal{X}, R)$. Every such ideal being a point of $\mathcal{N}$, it follows that every function in $\mathfrak{M}$ must vanish at $\mathfrak{H} \in \mathcal{N}$, and hence we see that $\mathfrak{M}$ is a fixed ideal. This completes the present proof.

The foregoing theorem shows that in studying rings $\mathfrak{C}(X, R)$ we may restrict the domain of our investigations to Q-spaces, as all the possible characteristics of such rings are displayed on such spaces. We now prove an important result closely related to the foregoing.

THEOREM 57. *Two Q-spaces X and Y are homeomorphic if and only if the rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ are algebraically isomorphic.*

The necessity of the condition stated for the homeomorphism of X and Y being obvious, we confine ourselves to proving its sufficiency. If the ring $\mathfrak{C}(X, R)$ is given, we may reconstruct X as the family of all real ideals in $\mathfrak{C}(X, R)$, topologized as in Theorems 9, 46, and 56. Since X is a Q-space, every real ideal in $\mathfrak{C}(X, R)$ is fixed, and the space X is completely determined by the algebraic properties of $\mathfrak{C}(X, R)$.

From Theorems 56 and 57, we may show at once that vX is unique in the same sense that βX is unique.

THEOREM 58. *Let X be any completely regular space. Then the space vX is completely determined (to within homeomorphisms) by the following properties:*

(1) *vX is a Q-space*; (2) *vX contains X as a dense subspace*; (3) *every function in $\mathfrak{C}(X, R)$ can be continuously extended over vX.*

We now describe a second construction of the space vX.

THEOREM 59. *Let X be any completely regular space. Let X be mapped into the space $\boldsymbol{P}_{f\in\mathfrak{C}(X,R)}R_f$, where $R_f=R$, by the mapping Φ: $\Phi(x)=\{f(x)\}\in\mathfrak{P}_{f\in\mathfrak{C}(X,R)}R_f$. The mapping Φ is a homeomorphism and the closure of $\Phi(x)$ is the space vX.*

Verification that Φ is a homeomorphism is very simple and is hence passed over. It is furthermore clear that every function f in $\mathfrak{C}(\Phi(X), R)$ can be continuously extended over $(\Phi(X))^-$; for any such f, there is a co-ordinate space—namely, R_f— such that $f(p)=\pi_f(p)$ for every $p\in\Phi(X)$. If $q=\{q_f\}$ is in $(\Phi(X))^-\cap(\Phi(X))'$, set $\bar{f}(q)=q_f$. The function $\bar{f}$ is clearly a continuous extension of f over the space $(\Phi(X))^-$. Finally, let $\mathfrak{M}$ be any real ideal in $\mathfrak{C}((\Phi(X))^-, R)$. For every f in the ring $\mathfrak{C}((\Phi(X))^-, R)$, which we may identify with the ring $\mathfrak{C}(\Phi(X), R)$, $\mathfrak{M}(f)$ is a real number, and $\{\mathfrak{M}(f)\}_{f\in\mathfrak{C}(X,R)}$ is a point of $\boldsymbol{P}_{f\in\mathfrak{C}(X,R)}R_f$. For any finite set $f_1, f_2, \cdots, f_m$ of functions in $\mathfrak{C}(X, R)$, there is a point $x_0\in X$ such that $f_i(x_0)=\mathfrak{M}(f_i)$, for $i=1, 2, \cdots, m$, since $f_i(x)-\mathfrak{M}(f_i)=0$ for all x in some $A_i\in\mathfrak{Z}(\mathfrak{M})$ $(i=1, 2, 3, \cdots, m)$; and we may select any $x_0\in\prod_{i=1}^m A_i$. Hence an arbitrary neighborhood $U_{f_1,f_2\cdots,f_m;\epsilon}(\{M(f)\})$ contains points of $\Phi(X)$, and it follows that $\{M(f)\}$ is in $(\Phi(X))^-$. Now for $f\in\mathfrak{C}(\Phi(X), R)$, $f\in\mathfrak{M}$ if and only if $\mathfrak{M}(f)=0$; this equality means that the co-ordinate $\mathfrak{M}(f)=0$, that f vanishes at $\mathfrak{M}(f)$, and that $\mathfrak{M}$ is the fixed ideal of all functions in $\mathfrak{C}((\Phi(X))^-, R)$ which vanish at $\{\mathfrak{M}(f)\}$. This implies that $(\Phi(X))^-$ is a Q-space and, in virtue of Theorem 58, that $(\Phi(X))^-$ is vX.

THEOREM 60. *Every Q-space is homeomorphic to a closed subset of some product space $\boldsymbol{P}_{\lambda\in\Lambda}R$, where each $R_\lambda=R$.*

This remark follows immediately from Theorem 59. It is an open question to determine whether or not every closed subset of $\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$ is a Q-space.

We next prove the analogue of Theorem 14 for spaces vX.

THEOREM 61. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be any family of completely regular spaces. Then the following relation obtains: $v\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda=\boldsymbol{P}_{\lambda\in\Lambda}vX_\lambda$.*

We may ignore the trivial case in which Λ or some X_λ is void. We first prove that every function f in $\mathfrak{C}(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ can be continuously extended over $\boldsymbol{P}_{\lambda\in\Lambda}vX_\lambda$. It is obvious that $\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$ is a dense subset of $\boldsymbol{P}_{\lambda\in\Lambda}vX_\lambda$, and that the extension $\bar{f}$ of f is unique if it exists. As we have observed above (Theorem 56), the space vX_λ is a subspace of βX_λ, and $\boldsymbol{P}_{\lambda\in\Lambda}vX_\lambda$ is accordingly a subspace of $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda=\beta\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda$. Since every bounded function g in $\mathfrak{C}(\boldsymbol{P}_{\lambda\in\Lambda}X_\lambda, R)$ can be extended continuously over $\boldsymbol{P}_{\lambda\in\Lambda}\beta X_\lambda$, it follows that a unique continuous extension for such a function g exists over the space

$P_{\lambda\in\Lambda}\upsilon X_\lambda$. Now let f be an unbounded function in $\mathfrak{C}(P_{\lambda\in\Lambda}X_\lambda, R)$ and let $g=\tan^{-1} f$. Then g, being bounded, can be continuously extended over $P_{\lambda\in\Lambda}\upsilon X_\lambda$, and f will be likewise extensible provided that $g(\{\mathfrak{M}_\lambda\})\neq\pm\pi/2$ for any point $\{\mathfrak{M}_\lambda\}$ in $P_{\lambda\in\Lambda}\upsilon X_\lambda$. We use the notation $\mathfrak{M}_\lambda$ to designate a generic point in υX_λ, $\mathfrak{M}_\lambda$ being a real ideal in the ring $\mathfrak{C}(X_\lambda, R)$. Assuming that $g(\{\mathfrak{M}_\lambda^*\})=\pi/2$ for some $\{\mathfrak{M}_\lambda^*\}\in P_{\lambda\in\Lambda}\upsilon X_\lambda$, we see at once that some element $\mathfrak{M}_{\lambda_0}^*$ must lie in $\upsilon X_{\lambda_0}\cap X_{\lambda_0}'$, since $g(\{x_\lambda\})\neq\pi/2$ if $x_\lambda\in X_\lambda$ for all $\lambda\in\Lambda$. The set of points $A_n=E[\{\mathfrak{M}_\lambda\};\ \{\mathfrak{M}_\lambda\}\in P_{\lambda\in\Lambda}\upsilon X_\lambda;\ g(\{M_\lambda\})>\pi/2-1/n]$ is an open set in $P_{\lambda\in\Lambda}\upsilon X_\lambda$, and the set $A=\prod_{n=1}^{\infty} A_n$ contains the point $\{\mathfrak{M}_\lambda^*\}$. It is clear that A must contain a set of the form $D_{\lambda_1}\times D_{\lambda_2}\times\cdots\times D_{\lambda_n}\times P_{\lambda\neq\lambda_1,\lambda_2,\ldots\lambda_n} X_\lambda$, where the set D_{λ_n} is a closed G_δ in the space X_{λ_n} such that $\mathfrak{M}_{\lambda_n}^*\in D_{\lambda_n}$ $(n=1, 2, 3, \cdots)$. It is also clear that for some λ_n, $D_{\lambda_n}\cap X_{\lambda_n}=0$, since in the contrary case, we should have $g(\{x_\lambda\})=\pi/2$ for some $\{x_\lambda\}P_{\lambda\in\Lambda}X_\lambda$. Since D_{λ_n} is a G_δ, Theorem 17 implies that the space X_{λ_n}, in its relative topology, admits a continuous real-valued function which is not extensible so as to be continuous at $\{\mathfrak{M}_{\lambda_n}^*\}$, in the relative topology of the space $X_{\lambda_n}\upsilon\mathfrak{M}_{\lambda_n}^*$. This, however, contradicts the fact that $\mathfrak{M}_{\lambda_n}\in\upsilon X_{\lambda_n}$. It follows that $g(\{\mathfrak{M}_\lambda^*\})$ cannot assume the value $\pi/2$ or $-\pi/2$, and that the function f can therefore be continuously extended over $P_{\lambda\in\Lambda}\upsilon X_\lambda$.

We next show that $P_{\lambda\in\Lambda}\upsilon X_\lambda$ is a Q-space. Suppose that $\mathfrak{M}^*$ is a real maximal ideal in $\mathfrak{C}(P_{\lambda\in\Lambda}\upsilon X_\lambda, R)$. The ring $\mathfrak{C}(P_{\lambda\in\Lambda}\upsilon X_\lambda, R)$ contains an isomorph $\mathfrak{A}_\lambda$ of the ring $\mathfrak{C}(X_\lambda, R)$, for every $\lambda\in\Lambda$, the functions in $\mathfrak{A}_\lambda$ being those of the form $f_\lambda\pi_\lambda$, f_λ being a generic element of $\mathfrak{C}(X_\lambda, R)$ and π_λ being the projection of $P_{\lambda\in\Lambda}\upsilon X_\lambda$ onto the λth co-ordinate space υX_λ. The intersection $\mathfrak{A}_\lambda\cap\mathfrak{M}^*$ is clearly a real maximal ideal in the ring $\mathfrak{A}_\lambda$, and consequently is a point of the space υX_λ. Designate the ideal $\mathfrak{A}_\lambda\cap\mathfrak{M}^*$ by $\mathfrak{M}_\lambda^*$, and consider the point $\{\mathfrak{M}_\lambda^*\}$ in $P_{\lambda\in\Lambda}\upsilon X_\lambda$. For any function g in $\mathfrak{A}_\lambda$, it is clear that $g(\{\mathfrak{M}_\lambda^*\})=\mathfrak{M}_\lambda^*(g)=\mathfrak{M}^*(g)$. Let ϕ be an arbitrary bounded function in $\mathfrak{C}(P_{\lambda\in\Lambda}\upsilon X_\lambda, R)$. By virtue of Theorem 15, there exist functions $g_1, g_2, \cdots, g_n$, each of which is in some ring $\mathfrak{A}_\lambda$, such that $|\phi(\{\mathfrak{M}_\lambda\})-P(g_1(\{\mathfrak{M}_\lambda\}), \cdots, g_n(\{\mathfrak{M}_\lambda\}))|<\epsilon$ where $P(t_1, t_2, \cdots, t_n)$ is a suitable polynomial with real coefficients in the indeterminants $t_1, t_2, \cdots, t_n$, ϵ is an arbitrary positive real number, and $\{\mathfrak{M}_\lambda\}$ is any point in $P_{\lambda\in\Lambda}\upsilon X_\lambda$. Then we may infer: $|\phi(\{\mathfrak{M}_\lambda^*\})-\mathfrak{M}^*(P(g_1, g_2, \cdots, g_n))|<\epsilon$; and since $\mathfrak{M}^*$ is a real ideal, $|\mathfrak{M}^*(\phi)-\mathfrak{M}^*(P(g_1, g_2, \cdots, g_n))|<\epsilon$. It follows that $|\mathfrak{M}^*(\phi)-\phi(\{\mathfrak{M}_\lambda^*\})|<2\epsilon$, and hence $\phi\in\mathfrak{M}^*$ if and only if $\phi(\{\mathfrak{M}_\lambda\})=0$. From Theorem 36, we infer that an unbounded function $\psi\in\mathfrak{C}(P_{\lambda\in\Lambda}\upsilon X_\lambda, R)$ is in $\mathfrak{M}^*$ if and only if $\min(\psi^2, 1)$ is in $\mathfrak{M}^*$, and this implies that for any function $\psi\in\mathfrak{C}(P_{\lambda\in\Lambda}\upsilon X_\lambda, R)$, $\psi\in\mathfrak{M}^*$ if and only if $\psi(\{\mathfrak{M}_\lambda^*\})=0$.

The two facts now established concerning $P_{\lambda\in\Lambda}\upsilon X_\lambda$ enable us to refer to Theorem 58 to verify the present theorem.

We infer from Theorem 61 the following immediate results:

THEOREM 62. *The Cartesian product of Q-spaces is a Q-space.*

THEOREM 63. *Any Cartesian product of R with itself is a Q-space.*

In concluding the present discussion, we remark without proof that the space vX, for any completely regular space X, can be regarded as the completion of X in the coarsest uniform structure on X for which all functions in $\mathfrak{C}(X, R)$ are uniformly continuous. (See [6, pp. 85–121].)

From Theorem 60 and Theorem 53, we obtain the following result.

THEOREM 64. *Any separable metric space is homeomorphic to a closed subset of the Cartesian product $\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$, where the index class Λ has cardinal number $2^{\aleph_0}$.*

If M is a separable metric space, it is well known that the number $|\mathfrak{C}(M, R)|$ is equal to $2^{\aleph_0}$. The present theorem follows at once from this observation.

As a curious special case of Theorem 64, we remark that any subset of the Euclidean plane, however pathological it may be, is imbeddable as a closed subset of $\boldsymbol{P}_{\lambda\in\Lambda}R_\lambda$, where $|\Lambda|$ is $2^{\aleph_0}$.

We continue with another imbedding theorem.

THEOREM 65. *Let X be a Q-space, and let Y be an arbitrary completely regular space. Then the rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ are algebraically isomorphic if and only if Y can be imbedded as a dense subset of X in such a way that every function in $\mathfrak{C}(Y, R)$ can be continuously extended over X.*

The present proof is carried out by methods now familiar to the reader.

It is natural to ask how powerful the rings $\mathfrak{C}(X, R)$ are in distinguishing among non-homeomorphic spaces. That is, given two non-homeomorphic completely regular spaces X_1 and X_2, when will the rings $\mathfrak{C}(X_1, R)$ and $\mathfrak{C}(X_2, R)$ be non-isomorphic? A related question may also be posed: what spaces X are there whose rings $\mathfrak{C}(X, R)$ are the rings for no other spaces? Complete answers to these questions are not known to the writer. We can, however, obtain partial answers which are closely analogous to known results for rings $\mathfrak{C}^*(X, R)$, the results for rings $\mathfrak{C}^*(X, R)$ appearing as special cases of the theorems that we are about to state. We first show that rings $\mathfrak{C}(X, R)$ are no less effective than rings $\mathfrak{C}^*(X, R)$ in distinguishing among spaces.

THEOREM 66. *If X and Y are complete regular spaces with the property that the rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ are algebraically isomorphic, then the rings $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(Y, R)$ are algebraically isomorphic.*

Let τ be an algebraic isomorphism carrying $\mathfrak{C}(X, R)$ onto $\mathfrak{C}(Y, R)$; it is obvious that $\tau(\alpha)=\alpha$, for all real numbers α. Let f be any bounded function in $\mathfrak{C}(X, R)$. The assertion that f is bounded may be reformulated as the statement that there are real numbers α and β such that if γ is any real number not in the interval $[\alpha, \beta]$, then $(f-\gamma)^{-1}$ exists. It is clear that

$(\tau(f)-\gamma)^{-1}$ exists, since we have $(\tau(f)-\gamma)^{-1}=\tau((f-\gamma)^{-1})$, and that τ, consequently, carries $\mathfrak{C}^*(X, R)$ onto $\mathfrak{C}^*(Y, R)$.

The foregoing theorem shows that if $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(Y, R)$ are non-isomorphic, then $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ are non-isomorphic. That rings $\mathfrak{C}(X, R)$ actually have higher distinguishing power than rings $\mathfrak{C}^*(X, R)$ is shown by the following evident theorem.

THEOREM 67. *Let X be any non-pseudo-compact completely regular space; then X and βX have the property that $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(\beta X, R)$ are algebraically isomorphic, while $\mathfrak{C}(X, R)$ and $\mathfrak{C}(\beta X, R)=\mathfrak{C}^*(\beta X, R)$ are not algebraically isomorphic.*

We now take up the known cases in which the ring $\mathfrak{C}(X, R)$ completely determines X, that is, in which algebraic isomorphism between $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ implies the existence of a homeomorphism between X and Y.

THEOREM 68. *The ring $\mathfrak{C}(X, R)$ completely determines the topological space X if and only if X is a Q-space with the property that for every non-isolated point $p\in X$, there is a function in $\mathfrak{C}(X-p, R)$ which cannot be continuously extended over X.*

Suppose first that the space X enjoys the property described above, and suppose that Y is a space such that $\mathfrak{C}(Y, R)$ is isomorphic to $\mathfrak{C}(X, R)$. If Y is a Q-space, then we know from Theorem 57 that X and Y are homeomorphic. If Y is not a Q-space, Theorem 65 implies that Y can be imbedded as a (proper) subset of X which is dense in X and has the property that every function in $\mathfrak{C}(Y, R)$ can be continuously extended over X. As a dense subset of X, Y must contain all of the isolated points of X, and as a proper subset of X, it must omit some non-isolated point of X. The properties ascribed to X present us with an immediate contradiction. The converse is obviously true.

THEOREM 69. *If X is any separable metric space, then X is completely determined by $\mathfrak{C}(X, R)$. If X is a discrete space, then X is completely determined by $\mathfrak{C}(X, R)$.*

Any separable metric space is, in view of Theorem 53, a Q-space. Since the first axiom of countability is satisfied at every point p in X, Theorem 17 implies that every subspace $X-p$ (p being a non-isolated point of X) admits continuous real-valued bounded functions which are not continuously extensible over X. Consequently X falls within the purview of Theorem 68 and is completely determined by $\mathfrak{C}(X, R)$. If X is a discrete space, then the conditions of Theorem 68 are certainly satisfied. Theorems 52, 53, and 54 show that any discrete space is a Q-space, whatever its cardinal number; the extension property of Theorem 68 is vacuously satisfied because there are no non-isolated points in X.

As an application of the foregoing theorem to a familiar situation, we

remark that the ring $\mathfrak{C}(R, R)$ is isomorphic to the ring $\mathfrak{C}(X, R)$ for no completely regular space X other than R. Similar observations are valid for Euclidean n-space, Hilbert space, and so on. If X is a compact metric space, then $\mathfrak{C}(X, R)$ is identical to $\mathfrak{C}^*(X, R)$, and in this case $\mathfrak{C}^*(X, R)$ determines X. We also point out that Theorems 66 and the second theorem on page 835 of [7] show immediately that two completely regular spaces X and Y satisfying the first axiom of countability have isomorphic rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ if and only if they are homeomorphic.

Appendix A. The space C_1

We shall exhibit here a space C_1 which is (a) completely regular, (b) nonnormal, (c) a Q-space, and (d) contains a closed G_δ which is not a Z-set. The points of C_1 are those points (x, y) in the Euclidean plane such that $|x| \leqq 1$ and $|y| \leqq 1$. If $y \neq 0$, and n is any natural number, then the neighborhood $U_n(x, y)$ is the ordinary Euclidean neighborhood of (x, y): $U_n(x, y) = E[(z, w);\ (z, w) \in C_1,\ \{(z-x)^2+(w-y)^2\}^{1/2} < 1/n]$. The neighborhoods $U_n(x, y)$ comprise a complete neighborhood system for all points (x, y) with $y \neq 0$ $(n=1, 2, 3, \cdots)$. For points $(x, 0)$ in C_1, and arbitrary natural numbers n, neighborhoods $V_n(x, 0)$ are defined as follows:

$$V_n(x, 0) = E[(z, w);\ (z, w) \in C_1,\ |w| < 1/n,\ |w| \cdot |z - x|^{-1} > n] \cup E[(x, w);\ (x, w) \in C_1,\ |w| < 1/n].$$

As n runs through the set of all natural numbers, the neighborhoods $V_n(x, 0)$ define complete neighborhood systems for the points $(x, 0)$. Since every Euclidean neighborhood of every point lying in C_1 contains a C_1-neighborhood of that point, it is apparent that C_1 is a Hausdorff space. To prove that C_1 is completely regular, we first consider points (x, y) with $y \neq 0$. The function

$$\phi(z, w) = n\{(z - x)^2 + (w - y)^2\}^{1/2}$$

is evidently continuous on C_1 and satisfies the conditions

$$\phi(x, y) = 0, \qquad \phi(z, w) \geqq 1 \quad \text{for} \quad (z, w) \in U_n'(x, y).$$

Construction of appropriate functions is naturally somewhat more complicated in the case of neighborhoods $V_n(x, 0)$. For arbitrary $(x, 0) \in C_1$ such that $|x| < 1$ and an arbitrary natural number n, we define $\phi(z, w)$ as follows:

For $(z, w) \in V_n'(x, 0)$, $\phi(z, w) = 1$;

For points $(x, w) \in V_n(x, 0)$, $\phi(x, w) = n|w|$;

For points $(z, w) \in V_n(x, 0)$ with $z > x$, we make $\phi(z, w)$ a linear interpolation between the value $n|w|$ assumed at (x, w) and the value 1 assumed at $(x+n^{-1}|w|, w)$;

For points $(z, w) \in V_n(x, 0)$ with $z < x$, we make $\phi(z, w)$ a linear interpolation between the value 1 assumed at $(x-n^{-1}|w|, w)$ and the value $n|w|$ assumed at (x, w). Functions ϕ for the points $(\pm 1, 0)$ are defined in a similar

way. It is obvious that the ϕ are continuous throughout C_1, and by their definition the functions ϕ show that the space C_1 is completely regular.

We next show that C_1 enjoys properties (b) and (d) by exhibiting a closed G_δ in C_1 which is not a Z-set. As this set, we select $A=E[(x, 0); |x| \leqq 1, x$ is rational$]$. The set A is easily seen to be closed: if $0<|y| \leqq 1$, then there is a neighborhood $U_n(x, y)$ not intersecting A, for all $(x, y) \in C_1$; if t is irrational and $|t|<1$, then $V_n(t, 0) \cap A=0$ for all natural numbers n. To show that A is a G_δ, consider the set $G_n=\sum_r V_n(r, 0)$, where r runs through all rational numbers in $[-1, 1]$. Clearly we have $G_n \supset A$, and also $\prod_{n=1}^{\infty} G_n=A$. Now suppose that f is any function in $\mathfrak{C}(C_1, R)$ such that $Z(f) \supset A$. We shall show that $Z(f) \cap B \neq 0$, where $B=E[(x, 0); |x|<1, x$ is irrational$]$. The proof of this fact proceeds by contradiction. Assume that $\psi=|f|$ is positive for all $(t, 0) \in B$. Then, for every $(t, 0) \in B$, there is a neighborhood $V_n(t, 0)$ such that $(x, y) \in V_n(t, 0)$ implies that $\psi(x, y)>2^{-1}\psi(t, 0)>0$. Let $B_{m,n}$ be the set of all $(t, 0) \in B$ such that for all $(x, y) \in V_m(t, 0)$, the inequality $\psi(x, y)>1/n$ obtains $(m, n=1, 2, 3, \cdots)$. It is clear that $B=\sum_{m,n=1}^{\infty} B_{m,n}$. Furthermore, every set $B_{m,n}$ is nowhere dense in the Euclidean topology of the set $D=E[(x, 0), -1 \leqq x \leqq 1]$. For, assume that $B_{m,n}$ is dense in some open interval I of D. Select any rational point r in this interval, and a neighborhood $V_k(r, 0)$—in the topology of C_1—such that for $(x, y) \in V_k(r, 0)$, the inequality $\psi(x, y)<1/n$ obtains. Since $B_{m,n}$ is dense in the interval I, there must be neighborhoods $V_m(t, 0)$ for $(t, 0) \in I \cap B_{m,n}$ such that $F=V_m(t, 0) \cap V_k(r, 0) \neq 0$. For a point (x, y) in F, we have $\psi(x, y)<1/n$ and $\psi(x, y)>1/n$, a palpable absurdity. Hence $B_{m,n}$ is nowhere dense in the Euclidean topology of D. Now A, being countably infinite, is a set of the first category in D, and the set $\sum_{m,n=1}^{\infty} B_{m,n}$, by what we have just proved, is also a set of the first category. Since $D=A \cup \sum_{m,n=1}^{\infty} B_{m,n}$, and since D is a complete metric space in its Euclidean topology, a theorem of Baire (see AH, p. 108, Satz V) may be applied to show that D cannot be of the first category; we may thus infer that $Z(f) \cap B \neq 0$.

Finally, we wish to show that C_1 is a Q-space. In virtue of Theorem 50, it will suffice to show that for every free maximal ideal $\mathfrak{M}$ in $\mathfrak{C}(C_1, R)$, there exists a countable subfamily $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{Z}(\mathfrak{M})$ such that $\prod_{n=1}^{\infty} A_n=0$. We have two cases to consider.

Case I. D non$\in \mathfrak{Z}(\mathfrak{M})$. In this event, there is a set $H \in \mathfrak{Z}(\mathfrak{M})$ such that $H \cap D=0$. Since $\mathfrak{M}$ is free, the set H is non-bicompact in its relative topology in C_1. It is clear that in the Euclidean topology of C_1, the closure of H must have nonvoid intersection with D; otherwise this Euclidean closure of H would be a bicompact set in $\mathfrak{Z}(\mathfrak{M})$. Every element Z of $\mathfrak{Z}(\mathfrak{M})$ must intersect H in a non-bicompact set; as before, it follows that the Euclidean closure of Z has nonvoid intersection with D. If we now define the sets J_n as $E[(x, y); (x, y) \in C_1, |y| \leqq 1/n]$, it is clear that: (1) $J_n \in \mathfrak{Z}(C_1)$; (2) $J_n \cap Z \neq 0$, for all natural numbers n and all $Z \in \mathfrak{Z}(\mathfrak{M})$; (3) $J_n \cap H \in \mathfrak{Z}(\mathfrak{M})$ for all natural num-

bers n; (4) $\prod_{n=1}^{\infty} J_n \cap H = 0$. (4) shows that M is a non-real ideal.

Case II. $D \in \mathfrak{Z}(\mathfrak{M})$. We first show that if a is a real number such that $|a| < 1$, then there is a function $\psi \in \mathfrak{C}(C_1, R)$ such that $\psi(x, 0) = 0$ for $1 \geqq x > a$, $\psi(x, 0) = 1$ for $-1 \leqq x \leqq a$, and $0 \leqq \psi(x, y) \leqq 1$ for all $(x, y) \in C_1$. We commence with the function ψ defined on D alone, with values as stated, and show that ψ can be continuously extended over all of C_1. Select any neighborhood $V_n(a, 0)$, and let $\psi(x, y) = 1$ for all $(x, y) \in V_n^-(a, 0)$. Further, let $\psi(x, y) = 1$ for all (x, y) such that $|y| \geqq 1/n$ or $x \leqq a$. For points $(x, y) \in V_n'(a, 0)$ such that $0 < |y| < 1/n$ and $a < x < a + 1/n^2$, let $\psi(x, y)$ be a linear interpolation between the value 0 assumed at $(x, 0)$ and the value 1 assumed at $(x, n(x-a))$ (if y is positive) or $(x, -n(x-a))$ (if y is negative). For points (x, y) such that $0 < |y| < 1/n$ and $x \geqq a + 1/n^2$, let $\psi(x, y)$ be a linear interpolation between the value 0 assumed at $(x, 0)$ and the value 1 assumed at $(x, 1/n)$ (if y is positive) or at $(x, -1/n)$ (if y is negative). The function ψ is patently continuous throughout C_1 and enjoys the other properties stated above. It is to be noted that $Z(\psi) = E[(x, 0); (x, 0) \in C_1, a < x \leqq 1]$. Now let $\{Z_\lambda\}_{\lambda \in \Lambda}$ represent the family $\mathfrak{Z}(\mathfrak{M})$. Since $D \in \mathfrak{Z}(\mathfrak{M})$, it is clear that the family $\{D \cap Z_\lambda\}_{\lambda \in \Lambda}$ is a family of Z-sets in C_1 enjoying the finite intersection property. Denoting closure in the Euclidean topology of D by the symbol $\bigstar$, we observe that the family $\{(D \cap Z_\lambda)_\lambda^\bigstar\}_{\lambda \in \Lambda}$ is a family of bicompact subsets of D (in its Euclidean topology) enjoying the finite intersection property. There exists, therefore, a point $(a, 0)$ in D such that $(a, 0) \in \prod_{\lambda \in \Lambda} (D \cap Z_\lambda)^\bigstar$; hence, every open interval in D containing $(a, 0)$ contains points of every set $D \cap Z_\lambda$. Let $I_n = E[(x, 0); (x, 0) \in C_1, a - 1/n < x < a]$ and let $J_n = E[(x, 0); (x, 0) \in C_1, a < x < a + 1/n]$. From the construction of ψ set forth above, it is plain that the sets $K_n = J_n \cup I_n$ are all in $\mathfrak{Z}(C_1)$, for $n = 1, 2, 3, \cdots$. It is also plain that $(a, 0)$ non$\in \mathfrak{Z}(\mathfrak{M})$, since $\mathfrak{M}$ is free. Since the sets $K_n \cap D \cap Z_\lambda$ are nonvoid, for all natural numbers n and all indices $\lambda \in \Lambda$, and $\mathfrak{M}$ is a maximal ideal, we infer that $K_n \in \mathfrak{Z}(\mathfrak{M})$, $n = 1, 2, \cdots$. The intersection $\prod_{n=1}^{\infty} K_n$ is void, and it follows that $\mathfrak{M}$ is a hyper-real ideal. C_1 is accordingly a Q-space.

Appendix B. Other rings of continuous functions

It is natural to inquire why the rings $\mathfrak{C}(X, R)$ and $\mathfrak{C}^*(X, R)$ are chosen as instruments wherewith to study topological spaces X, to the exclusion of rings $\mathfrak{C}(X, K)$, $\mathfrak{C}^*(X, K)$, and more generally rings $\mathfrak{C}(X, T)$, where T may be any topological ring. First, the limitation to topological fields F offers considerable advantages in rings $\mathfrak{C}(X, F)$ by virtue of the simple characterization of elements with inverse in such rings. Second, the topological field used should be connected and locally bicompact, in order to admit reasonable numbers of continuous images of connected and bicompact spaces, respectively. By a celebrated theorem of Pontrjagin these properties limit us already to R, K, or Q, the field of quaternions, so that our only choice lies among the three fields mentioned. We now show that for distinguishing among topo-

logical spaces, choice of R, K, or Q is immaterial.

THEOREM. *Let X and Y be completely regular spaces. The following statements concerning X and Y are equivalent*:

(1) $\mathfrak{C}^*(X, R)$ *is algebraically isomorphic to* $\mathfrak{C}^*(Y, R)$;

(2) $\mathfrak{C}^*(X, K)$ *is algebraically isomorphic to* $\mathfrak{C}^*(Y, K)$;

(3) $\mathfrak{C}^*(X, Q)$ *is algebraically ismorphic to* $\mathfrak{C}^*(Y, Q)$ ([16]).

The implication (1)→(2) is obvious when the observation is made that $\mathfrak{C}^*(X, K)$ is obtained from $\mathfrak{C}^*(X, R)$ by adjoining i: $\mathfrak{C}^*(X, K)$ is the set of all functions $\phi+i\psi$, where $\phi, \psi \in \mathfrak{C}^*(X, R)$. We now consider the converse implication (2)→(1). Let $\mathfrak{M}_X$ be the family of all maximal ideals in the ring $\mathfrak{C}^*(X, K)$. By an argument identical with that used in the proof of Theorem 9, it may be shown that $\mathfrak{M}_X$ can be made a bicompact Hausdorff space containing a homeomorphic image $\mathfrak{X}$ of X as a dense subset and having the property that every function in $\mathfrak{C}^*(\mathfrak{X}, K)$ (and hence every function in the subring $\mathfrak{C}^*(\mathfrak{X}, R)$) can be continuously extended over $\mathfrak{M}_X$. This proves that $\mathfrak{M}_X$ is the space βX. By virtue of (2), $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ are homeomorphic, both being defined by purely algebraic means from algebraically isomorphic rings, from which (1) follows immediately.

A similar argument is applied to prove the equivalence (1)↔(3). We need only to replace the term "maximal ideal" by the term "maximal two-sided ideal" to prove (3)→(1) as above; and (1)→(3) is obvious.

THEOREM. *Let X and Y be completely regular spaces. The following statements concerning X and Y are equivalent*:

(1) $\mathfrak{C}(X, R)$ *is algebraically isomorphic to* $\mathfrak{C}(Y, R)$;

(2) $\mathfrak{C}(X, K)$ *is algebraically isomorphic to* $\mathfrak{C}(Y, K)$;

(3) $\mathfrak{C}(X, Q)$ *is algebraically isomorphic to* $\mathfrak{C}(Y, Q)$.

The proof of the present theorem is identical with that of the preceding, except that βX and βY are to be replaced by vX and vY.

Various writers have approached the study of topological spaces X through metrical and Banach space properties of the space $\mathfrak{C}^*(X, R)$, provided with the u-topology. (See, for example, [10], [26], and [19].) It can be shown that this approach cannot bring topological features of the space X into sharper focus than can the study of algebraic properties of $\mathfrak{C}^*(X, R)$. The following theorem renders this statement precise.

THEOREM. *Let X and Y be any completely regular spaces. Any algebraic isomorphism τ mapping $\mathfrak{C}^*(X, R)$ onto $\mathfrak{C}^*(Y, R)$ has the property that $\|f\| = \|\tau(f)\|$ and is hence a Banach space equivalence and a fortiori an isometry. On the other hand, if an isometry σ exists carrying $\mathfrak{C}^*(X, R)$ onto $\mathfrak{C}^*(Y, R)$ then*

([16]) A function ϕ in $\mathfrak{C}(X, Q)$ or $\mathfrak{C}(X, K)$ is said to be bounded and hence to be in $\mathfrak{C}^*(X, Q)$ or $\mathfrak{C}^*(X, K)$ if $\phi(X)$ is a bounded subset of Q (considered as R^4) or K (considered as R^2).

there also exists an algebraic isomorphism carrying $\mathfrak{C}^*(X, R)$ *onto* $\mathfrak{C}^*(Y, R)$.

The norm-preserving property enjoyed by every algebraic isomorphism τ is established by giving an algebraic definition of the norm in $\mathfrak{C}^*(X, R)$. If $\phi \in \mathfrak{C}^*(X, R)$, there exists a non-negative real number α such that for all real numbers $t > \alpha$, $(\phi - t)^{-1}$ exists. There also exists a non-positive real β number such that for all real numbers $t < \beta$, $(\phi - t)^{-1}$ exists. Let α_0 be the infimum of all α having the properties ascribed to α, and let β_0 be the supremum of all β having the properties ascribed to β. Then it is clear that $\|\phi\| = \max(|\alpha_0|, |\beta_0|)$. The norm having been thus algebraically defined, it follows that the norm of any function and hence the distance between any two functions is preserved under every algebraic isomorphism.

The second statement of the theorem may be proved at once from a theorem of Stone (see [26, p. 469, Theorem 83]), which states that an isometry between $\mathfrak{C}^*(J, R)$ and $\mathfrak{C}^*(L, R)$, where J and L are bicompact Hausdorff spaces, implies the existence of a homeomorphism between J and L. If $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(Y, R)$ are connected by an isometry σ, then $\mathfrak{C}^*(\beta X, R)$ and $\mathfrak{C}^*(\beta Y, R)$ are connected by an isometry ρ; Stone's theorem implies that βX and βY are homeomorphic; this implies that $\mathfrak{C}^*(\beta X, R)$ and $\mathfrak{C}^*(\beta Y, R)$ are algebraically isomorphic, which finally implies that $\mathfrak{C}^*(X, R)$ and $\mathfrak{C}^*(Y, R)$ are algebraically isomorphic.

A similar result may be stated for rings $\mathfrak{C}(X, R)$.

THEOREM. *Let X and Y be completely regular spaces, with the property that $\mathfrak{C}(X, R)$ is algebraically isomorphic to $\mathfrak{C}(Y, R)$ under an isomorphism τ. The isomorphism τ is necessarily a homeomorphism under the m- and the u-topologies in $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$.*

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