ON THE MAXIMUM PARTIAL SUMS OF SEQUENCES OF INDEPENDENT RANDOM VARIABLES(*)

BY

KAI LAI CHUNG

1. Introduction. In this paper we deal with a sequence of independent random variables \(X_n, n = 1, 2, \cdots\). We write

\[
(1) \quad S_n = \sum_{r=1}^{n} X_r,
\]

\[
(2) \quad S_n^* = \max_{1 \leq r \leq n} |S_r|.
\]

Two types of fundamental limit theorems are known about \(S_n\), the one clustering around the central limit theorem and the other the law of the iterated logarithm.

In 1945 Feller \([12]\)(‡) called attention to the study of the behavior of \(S_n^*\). Since then an important result has been obtained by Erdős and Kac \([8]\), namely, the limiting distribution of \(S_n^*\) for sufficiently general sequences of \(X_n\). This corresponds to the central limit theorem for \(S_n\). Now under certain conditions when the distribution of \(S_n\) tends to the normal distribution, an estimate of the difference of the two distributions has been given by Liapounoff \([17]\), Cramér \([5]\), Berry \([3]\) and Essen \([9]\). Cramér \([6]\) and Feller \([10]\) have also obtained more precise estimates for this difference for certain domains of variation of \(S_n\), which proved essential to the general form of the law of the iterated logarithm. It is therefore of interest to make the same kind of investigations regarding \(S_n^*\). The problem is more difficult, since we have as yet no standard tools as in the case of \(S_n\). We shall prove in this direction, as consequences of a more general but less handy inequality (Lemma 7), two theorems corresponding to the two types of estimation mentioned above. In order to state them we introduce the following notations. Let \(E(X)\) denote the mathematical expectation of \(X\). We shall assume that for each \(X\), the first moment is zero, and the third absolute moment is finite. Thus we can write

\[
(3) \quad E(X_r) = 0;
\]

\[
(4) \quad E(X_r^2) = \sigma_r^2; \quad S_n^2 = \sum_{r=1}^{n} \sigma_r^2;
\]

Presented to the Society, September 4, 1947; received by the editors May 27, 1947.

(*) The present paper is the revised form of a Dissertation for the degree of Doctor of Philosophy accepted by Princeton University, 1947.

‡ Numbers in brackets refer to the references cited at the end of the paper.
(5) \[ E(|X_r|^3) = \gamma_r; \quad \Gamma_n = \sum_{r=1}^{n} \gamma_r. \]

Naturally we assume that \( s_n \to \infty \). We shall further make the following assumption:

(6) \[ \max_{1 \leq r \leq n} \gamma_r \sigma_r^{-2} = O(s_n^{1-\theta}) \]

where \( \theta \) is a fixed but arbitrarily small positive number. Then we can prove the following two theorems.

**Theorem 1.** If \( c \) is a positive constant, then we have

\[
Pr (S_n^* < cs_n) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i + 1} \exp\left( -\frac{(2i + 1)^2\pi^2}{8c^2} \right) + O\left(\frac{\log s_n}{\log s_n}\right)^{1/2}. 
\]

**Theorem 2.** If \( g_n \downarrow 0 \) and

(8) \[ g_n^{-1} = O((\log s_n)^{1/2}) \]

then we have(9)

(9) \[ Pr (S_n^* < g_ns_n) = (1 + o(1)) \exp\left( -\frac{\pi^2}{8g_n^2} \right). \]

Theorem 2 is one of a number of possible statements; we give prominence to it here because it furnishes the means of proving the next group of theorems which we now consider.

We might attempt to extend the law of the iterated logarithm to \( S_n^* \). This turns out to be illusory since the same law holds for \( S_n^* \) as for \( S_n \). More precisely, if \( \phi_n \uparrow \infty \), we have always ("i. o." standing for "infinitely often")

\[
Pr (S_n^* > \phi_n s_n \text{ i. o.}) = Pr (S_n > \phi_n s_n \text{ i. o.).} 
\]

This is obvious since both \( S_n^* \) and \( \phi_n s_n \) are monotone increasing functions of \( n \). Hence in particular three of Feller’s theorems [11] read as follows:

I. If \( \sup |X_n| = O(s_n(\log s_n)^{-3/2}) \) and \( \phi_n^2 = 2 \log s_n + 3 \log_2 s_n + 2 \log_3 s_n + \cdots + 2 \log_{p-1} s_n + (2+\delta) \log_\delta s_n \) then the probability

(10) \[ Pr (S_n^* > s_n \phi_n \text{ i. o.}) \]

(9) **Added in proof.** For the application of Theorem 2 in Lemma 9 it is important to notice that the constant in the \( o(1) \) term in (9) depends only on the constants in the \( O(1) \) terms in (6) and (8), and the \( \theta \) in (6), but otherwise is independent of the random variables considered.
is equal to zero or one according as $\delta$ is positive or not.

II. If $\phi_n \uparrow \infty$ and

$$\text{sup } |X_n| = O \left( \frac{s_n}{\phi_n^2} \right),$$

then (10) is equal to zero or one according as the series

$$\sum_n \frac{s_n}{\phi_n^2} \phi_n e^{-(1/2) \phi_n^2}$$

is convergent or divergent.

III. If $\phi(t) \uparrow \infty$ and

$$\text{sup } |X_n| = O \left( \frac{s_n}{\phi(s_n)} \right),$$

then $\Pr(S_n > s_n \phi(s_n))$ i. o. is equal to zero or one according as the integral

$$\int_0^\infty \frac{1}{t} \phi(t) e^{-(1/2) \phi^2(t)} dt$$

is convergent or divergent.

These results give very precise upper bounds for $S_n^*$, with probability one. The question naturally arises as to the precise lower bounds for $S_n^*$. (We may mention that the analogous problem for $S_n$ has been treated by Erdös and the author [4] and is radically different.) In this connection Erdös has communicated to the author the following result: there exist two constants $c_2 > c_1 > 0$ such that

$$\Pr \left( c_1 < \lim \inf \frac{S_n^*}{s_n(\log s_n)^{-1/2}} < c_2 \right) = 1.$$

His method, of an elementary nature, does not seem capable of a sharper result. Using Theorem 2 stated above we can easily prove that

$$\Pr \left( \lim \inf \frac{S_n^*}{s_n(\log s_n)^{-1/2}} = 8^{-1/2} \pi \right) = 1.$$

This corresponds to Khintchine-Kolmogoroff's original form of the law of the iterated logarithm ([14] and [16]). However, we can go much further and prove the following theorems which are the exact counterparts of Feller's theorems cited above.

**Theorem 3.** Under the assumptions (3) to (6), if
(11) \[ \phi_n^2 = \log_2 s_n + 2 \log_3 s_n + \log_4 s_n + \cdots + \log_{p-1} s_n + (1 + \delta) \log_p s_n, \]

then

(12) \[ \Pr \left( S_n^* < 8^{-1/2} \pi \phi_n^{-1} s_n \right) \text{ i. o.} \]
is equal to zero or one according as \( \delta \) is positive or not.

**Theorem 4.** Under the same assumptions, if \( \phi_n \uparrow \infty \), then (12) is equal to zero or one according as the series

(13) \[ \sum_{s_n} \frac{\phi_n^2 - \phi_n^2}{s_n} \]
is convergent or divergent.

Theorem 3 is a particular case of Theorem 4.

**Theorem 5.** Under the same assumptions, if \( \phi(s_n^2) \uparrow \infty \), then

(14) \[ \Pr \left( S_n^* < 8^{-1/2} \pi \phi^{-1} (s_n)s_n \right) \text{ i. o.} \]
is equal to zero or one according as the integral

(15) \[ \int \frac{1}{t} \phi^2(t) e^{-\phi^2(t)} dt \]
is convergent or divergent.

The similarity between these theorems and Feller's is indeed striking. It should however be noted that the condition (6) is not the best possible, although it is weaker than those considered by Cramér [5]. That condition (6) can be trivially weakened will be apparent from the proof. But no complete settlement of the question seems in sight.

We outline the methods of proof as follows. We approximate the distribution on \( S_n^* \) by that of

(16) \[ \max_{1 \leq j \leq k} |S_{n_j}| \]

where \( k \) is an integer to be determined later and \( 0 < n_1 < \cdots < n_k = n \) is a suitably chosen sequence such that \( s_{n_j}^2 \sim jk^{-1} s_n^2 \).

In §2 we study the approximate distribution of (16). It is found to approach that of a \( k \)-dimensional normal distribution with a remainder we shall estimate. The treatment in Lemma 2(6), much to be preferred to the

\(^{(6)}\) In the special case of equal components Bergström's result [2] seems to imply a better estimate than Lemma 2, replacing the factor \( 4^k \) by a fixed power of \( k \). The improvement however is annulled by Lemma 3. It becomes essential in the problem of \( \max S_n \) without the absolute value. We shall consider this elsewhere.
author's original proof using characteristic functions, is due to G. A. Hunt.

In §3 we estimate the difference between the distribution of $S_n^*$ and that of (16). This is done by a substantial improvement of the method of Erdös and Kac (8), using sharper estimates resulting from the one-dimensional Berry-Esseen estimate. To obtain the approximate distribution of $S_n^*$ it remains to evaluate the $k$-dimensional normal distribution obtained in §2. The problem appears to be one of multiple integrals but has not been worked out directly. Instead we use a quantitative refinement of the "invariance principle" of Erdös and Kac and study the simplest case of random walk. This latter problem, being almost classical, has been treated by many authors with different methods. However as we require not only the limiting distribution but also a remainder no reference seems available in the literature. We shall obtain the precise result by going back to a combinatorial formula due (apparently) to Bachelier [1]. After this we combine the results in §§2 and 3 to establish a theorem (Lemma 7) which includes Theorems 1 and 2 as particular cases.

In §4 we prove Theorems 3, 4 and 5. The proof of these theorems depends essentially on Theorem 2, which plays the role here as the theorem of Cramér-Feller does in the case of Feller's theorems cited above. Several arguments of Feller's are also used and the author's indebtedness to his previous work is considerable.

The author wishes to express his gratitude to Professor Cramér for his warm encouragement and valuable counsel. To Dr. Erdös, whose first result actually started the investigation, the author owes many heartfelt thanks for his sustained interest. To Mr. Hunt, who is responsible not only for Lemma 2 but for many corrections on the original manuscript, the author's gratitude is equally great.

2. An approximation theorem for a certain multi-dimensional distribution. We shall use $A_1, A_2, \ldots$ to denote absolute constants.

Let $n_1 < \cdots < n_k = n$ be a subsequence of $1, \cdots, n$ defined by the following:

$$s_{n_j}^2 \leq j^{-1} s_n^2 < s_{n_{j+1}}^2, \quad j = 1, \cdots, k.$$  \hfill (17)

Then $(S_{n_1}, \cdots, S_{n_k})$ is a random point in $j$-dimensional space. Let its distribution function be

$$F_j(u_1, \cdots, u_j) = \Pr (S_{n_1} \leq u_1, \cdots, S_{n_j} \leq u_j).$$  \hfill (18)

Write also

$$F_j^*(x) = \Pr (S_{n_j} - S_{n_{j-1}} \leq x), \quad S_{n_0} = 0.$$  \hfill (19)

We put

$$B_j = s_{n_j}^2 - s_{n_{j-1}}^2.$$
Lemma 1. We have
\[ F_i^*(x) = \Phi_i^*(x) + R_i^*(x), \]
where \( \Phi_i^*(x) \) is the normal distribution function with mean 0 and variance \( B_i^2 \), and \( |R_i^*(x)| \leq A_1 M_n B_i^{-1}. \)

This is a restatement of the Berry-Esseen theorem.

Lemma 2. Suppose that (6) holds and also that
\[ \max_{1 \leq r \leq n} \sigma_r^2 = o(k^{-1} s_n). \]

Then we have
\[ |F_j(u_1, \ldots, u_j) - \Phi_j(u_1, \ldots, u_j)| \leq A_2 k^{1/2} M_n s_n^{-1}, \]
where \( \Phi_j(u_1, \ldots, u_j) \) is the \( j \)-dimensional normal distribution function with the same moments of the first and second order as \( F_j(u_1, \ldots, u_j) \).

Proof. From (17), (19) and (21) it is easy to see that
\[ B_j \sim k^{-1/2} s_n. \]
Hence by Lemma 1, we have
\[ |R_j^*(x)| \leq A_2 k^{1/2} M_n s_n^{-1}. \]

For \( j = 1 \), \( R_1(x) = R_1^*(x) \); hence (22) is true for \( j = 1 \). Now we use induction on \( j \). Assume that
\[ |R_j(u_1, \ldots, u_j)| \leq A_2 k^{1/2} M_n s_n^{-1}. \]
We have, by the definition (18),
\[ F_{j+1}(u_1, \ldots, u_{j+1}) = \int_{-\infty}^\infty F_j(u_1, \ldots, u_{j-1}, \min (u_j, u_{j+1} - x)) dF_{j+1}^*(x) \]
\[ = \int_{-\infty}^\infty \{ \Phi_j(u_1, \ldots, u_{j-1}, \min (u_j, u_{j+1} - x)) + R_j(u_1, \ldots, u_{j-1}, \min (u_j, u_{j+1} - x)) \} d[\Phi_{j+1}^*(x) + R_{j+1}^*(x)] \]
\[ = \Phi_{j+1}(u_1, \ldots, u_{j+1}) + \int R_j d\Phi_{j+1} \]
\[ + \int R_j d\Phi_{j+1}^* + \int \Phi_j dR_{j+1}^*. \]
Evidently we have
\[ \left| \int R_j d\Phi^*_j \right| \leq \sup R_j, \]
\[ \left| \int R_j dR^*_j \right| \leq 2 \sup R_j. \]

Finally, using integration by parts, we have
\[ \int_{-\infty}^{\infty} \Phi_j(u_1, \ldots, u_{j-1}, \min (u_j, u_{j+1} - x)) dR^*_j(x) \]
\[ = \int_{-\infty}^{\infty} \Phi_j(u_1, \ldots, u_j) dR^*_j(x) \]
\[ + \int_{u_{j+1} - u_j}^{\infty} \Phi_j(u_1, \ldots, u_j, u_{j+1} - x) dR^*_j(x) \]
\[ = \Phi_j(u_1, \ldots, u_j) R^*_j(u_{j+1} - u_j) - \Phi_j(u_1, \ldots, u_j) R^*_j(u_{j+1} - u_j) \]
\[ + \int_{u_{j+1} - u_j}^{\infty} R^*_j(x) d\Phi_j(u_1, \ldots, u_{j-1}, u_{j+1} - x). \]

Hence the absolute value of the left-hand side is less than or equal to
\[ \sup R^*_j. \]

Substituting these estimates into (22), we obtain
\[ |F_j - \Phi_j| \leq 3(\sup R_j + \sup R^*_j). \]

From (25) and (26), we have
\[ |F_{j+1} - \Phi_{j+1}| \leq 3A 2^{1/2} M_n s_n^{-1} (4^j + 1) \]
\[ \leq A 2^{1/2} k^{1/2} M_n s_n^{-1}. \]

Thus the induction is complete.

Now we put, for non-negative u′s,

(27) \[ F_0(u_1, \ldots, u_k) = \Pr \left( |S_{u_1}| \leq s_{u_1}, \ldots, |S_{u_k}| \leq s_{u_k} \right), \]
\[ \Phi_0(u_1, \ldots, u_k) = \frac{s_{u_1}}{(2\pi)^{k/2} B_1 \cdots B_k} \int_{-u_1}^{u_1} \cdots \int_{-u_k}^{u_k} \]
\[ \exp \left( -\frac{1}{2} \sum_{j=1}^{k} \frac{s_{u_j}^2}{B_j} (x_j - x_{j-1})^2 \right) dx_1 \cdots dx_k. \]

**Lemma 3.** Under the same assumptions as in Lemma 2, we have
Proof. Taking $j = k$ in (22), we have

$$|F_k - \Phi_k| \leq A_2 k^{1/2} 4^k M_n s_n^{-1} \leq A_3 5^k M_n s_n^{-1}. \tag{30}$$

It is well known that we have

$$F_0(u_1, \ldots, u_k) = F_k(s_{nu_1}, \ldots, s_{nu_k})$$

$$- F_k(-s_{nu_1}, s_{nu_2}, \ldots, s_{nu_k}) - \cdots$$

$$- F_k(s_{nu_1}, \ldots s_{nu_{k-1}}, -s_{nu_k})$$

$$+ F_k(-s_{nu_1}, s_{nu_2}, \ldots, s_{nu_k}) + \cdots$$

$$+ (-1)^k F_k(-s_{nu_1}, \ldots, -s_{nu_k});$$

and a similar relation holds between $\Phi_k$ and $\Phi_0$. Since there are $2^k$ terms on the right-hand side, (29) follows immediately from (30). It is not hard to obtain the explicit form of $\Phi_0(u_1, \ldots, u_k)$ in (28) by considering the covariance matrix.

3. The distribution of the maximum partial sum. Let $c$ be a positive constant; $g_n$ a monotone function of $n$; $\epsilon_n = o(1)$.

**Lemma 4.** Suppose that (6) and (21) are satisfied, and also that we have

$$\epsilon_n g_n = o(k^{1/2} s_n), \tag{31}$$

$$\sigma_n = o((\epsilon_n g_n s_n)^{2/3}). \tag{32}$$

Then we have

$$\Pr(S_n^* < c g_n s_n) \geq \Pr\left( \max_{1 \leq i \leq k} |S_{n,i}| < (c - \epsilon_n) g_n s_n \right) - R_n \tag{33}$$

where

$$R_n \leq A_4 k^{-1/2} \epsilon_n g_n s_n \exp\left(-4^{-1/2} \epsilon_n g_n s_n^2 + (\epsilon_n g_n s_n)^{-2/3}\right). \tag{34}$$

**Proof.** Write

$$P_n = \Pr(S_n^* < c g_n s_n),$$

$$W_r = \Pr(S_{r-1}^* < c g_n s_n, |S_r| \geq c g_n s_n).$$

Then we have

$$\sum_{r=1}^{n} W_r = \Pr(S_n^* \geq c g_n s_n) = 1 - P_n \leq 1. \tag{35}$$

Suppose that $n_j < y \leq n_{j+1}$. We have
Let $A > 0$ be an integer to be determined later. If $n_{j+1} - r \leq B$, we have by the Tchebychef inequality,

$$\Pr \left( |S_{n_{j+1}} - S_r| \leq \varepsilon_{n_{j+1}} \right) \leq \left( \frac{2}{\varepsilon_{n_{j+1}}^2} \right)^{1/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt + O(\varepsilon_{n_{j+1}}),$$

(37) if $n_{j+1} - r = B > A$, we have by the Berry-Esseen theorem,

$$\Pr \left( |S_{n_{j+1}} - S_r| \leq \varepsilon_{n_{j+1}} \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-t^2/2} dt + O(\varepsilon_{n_{j+1}}),$$

(38) where

$$v = \varepsilon_{n_{j+1}}^2 \left( s_{n_{j+1}} - s_{n_{j+1} - A+1} \right)^{-1/2},$$

and

$$\rho = M_n \left( \sum_{r=n_{j+1}-A+1}^{n_{j+1}} \sigma_r \right)^{-1/2}.$$

Hence from (38), since $A < B \leq n_{j+1} - n$, $s_{n_{j+1}}^2 - s_{n_{j+1} - A+1}^2 \leq s_{n_{j+1}}^2 - s_{n_{j}}^2 = B_{j+1}^2$, we have

$$\Pr \left( |S_{n_{j+1}} - S_r| \leq \varepsilon_{n_{j+1}} \right) \leq A \left( \frac{B_{j+1}}{\varepsilon_{n_{j+1}}^2} \exp \left( - \frac{2}{\varepsilon_{n_{j+1}}^2} \right) + \frac{M_n}{(s_{n_{j+1}}^2 - s_{n_{j+1} - A+1}^2)^{1/2}} \right),$$

(39) We choose $A$ such that

$$M_n \sim \left( s_{n_{j+1}}^2 - s_{n_{j+1} - A+1}^2 \right) \left( \frac{2}{\varepsilon_{n_{j+1}}^2} \right)^{1/2},$$

that is,

$$\left( s_{n_{j+1}}^2 - s_{n_{j+1} - A+1}^2 \right)^{3/2} \sim \frac{2}{\varepsilon_{n_{j+1}}^2} \frac{2}{\varepsilon_{n_{j+1}}^2} \frac{2}{\varepsilon_{n_{j+1}}^2} M_n.$$
Pr\left( |S_{n_{j+1}} - S_r| \geq \epsilon_n g_n s_n \right) \leq A_6 \left( \frac{B_{j+1}}{\epsilon_n g_n s_n} \exp \left( -\frac{2}{2B_{j+1}} \right) + (\epsilon_n g_n s_n)^{-2/3} \right).$

Since $B_{j+1}^2 s_n^{-2} \sim k^{-1}$ by (23), we obtain

$$\Pr \left( |S_{n_{j+1}} - S_r| \geq \epsilon_n g_n s_n \right) \leq A_4 \left( \frac{1}{k^{1/2} \epsilon_n g_n} \exp \left( -\frac{2}{4k} \right) + (\epsilon_n g_n s_n)^{-2/3} \right).$$

If we denote the maximum of the left-hand side of this inequality for all $r$ by $R_n$, (36) becomes

$$W_r \leq R_n + \Pr \left( S_r^* < c g_n s_n, \ |S_r| \geq c g_n s_n, \ |S_{n_{j+1}} - S_r| < \epsilon_n g_n s_n \right).$$

From (35) and (40), we obtain

$$\Pr (S_n^* \geq c g_n s_n) \leq R_n + \sum_{j=0}^{k-1} \sum_{r=n_{j+1}}^{n_{j+1}} \Pr (S_r^* < c g_n s_n, \ |S_r| \geq c g_n s_n, \ |S_{n_{j+1}} - S_r| < \epsilon_n g_n s_n)$$

$$\leq R_n + \Pr \left( \max_{1 \leq j \leq k} |S_{n_j}| \geq (c - \epsilon_n) g_n s_n \right).$$

This is equivalent to (33).

If in the function $F_0(u_1, \ldots, u_k)$ of (27) all the arguments are equal to $u$ we shall use the shorter notation $F_0(u)$; similarly for $\Phi_{0k}$.

**Lemma 5.** Suppose that the condition (6) is satisfied, and also for a $\Theta > 0$ we have

$$8 \lg 10 \cdot \Theta \cdot \frac{\lg s_n}{\Theta} \leq \frac{2^2}{\lg s_n} \leq \epsilon_n g_n = o \left( \frac{s_n^\theta}{(\lg s_n)^{1/2}} \right).$$

Then if we choose

$$k \sim \frac{\theta \lg s_n}{2 \lg 10}$$

we have

$$\Phi_{0k}(c - \epsilon_n) g_n - L_n \leq \Pr (S_n^* < c g_n s_n) \leq \Phi_{0k}(c g_n) + L_n$$

where

$$L_n = O((\lg s_n)^{-\Theta}).$$

**Proof.** From (6) it follows that
\[ \sigma_n \leq \gamma_n \sigma_n^{-2} = O(s_n^{1-\theta}). \]

Hence with the \( k \) in (42) condition (21) is satisfied. Further condition (32) in Lemma 4 is satisfied with \( \epsilon_g \) satisfying (41). Condition (31) is clearly satisfied with (41) and the choice of \( k \) in (42). Hence both Lemma 3 and Lemma 4 are applicable. Taking all the \( u \)'s in (29) to be \( (c - \epsilon_n)g_n \) and recalling (27) we obtain

\[
\Pr (S_n < c g_n s_n) \geq F_{0h}((c - \epsilon_n)g_n) - R_n
\]
\[
\geq \Phi_{0h}((c - \epsilon_n)g_n) - A_3(10) s_n^{-1} M_n - R_n.
\]

On the other hand we have

\[
\Pr (S_n < c g_n s_n) \leq F_{0h}(c g_n) \leq \Phi_{0h}(c g_n) + A_3(10) s_n^{-1} M_n.
\]

We find from (42) and (46)

\[
(10)^k = O(s_n^{\theta/2}), \quad (10) s_n^{-1} M_n = O(s_n^{-\theta/2}), \]
\[
k^{-1/2} \epsilon_n g_n^{-1} \exp (-4^{-1/2} \epsilon_n k) = O((\lg s_n)^{-\theta}), \]
\[
(\epsilon_n g_n s_n)^{-\theta/3} = O(s_n^{-\theta/3}).
\]

Hence if we take

\[
L_n = R_n + A_3(10) s_n^{-1} M_n = O((\lg s_n)^{-\theta})
\]

(45) and (46) imply (43).

**Lemma 6.** Suppose that for each \( \nu \),

\[
X_\nu = \begin{cases} 
+1 & \text{with probability } 1/2, \\
-1 & \text{with probability } 1/2.
\end{cases}
\]

Then if \( g_n = o(n^{1/2}) \), we have

\[
\Pr (S_n < c g_n^{1/2}) = T(c g_n) + O(g_n^{-1/2}) + O(n^{-1/2})
\]

where \( T(x) \) is the distribution function defined by

\[
T(x) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp \left( -\frac{(2i+1)^2 \pi^2}{8x^2} \right), \quad x > 0.
\]

**Proof.** Write, for integral \( a \) and \( b \),

\[
P(a) = \Pr (S_n = a, -b < S_n < b, \text{ for } 0 < \nu \leq n).
\]

By a formula due to Bachelier [1, pp. 252–253],

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ 2^n P(a) = C_{n,(n+a)/2} + \sum_{1 \leq i \leq (n+a)/2b} (-1)^i C_{n,(n+a)/2-i b} \]
\[ + \sum_{1 \leq i \leq (n-a)/2b} (-1)^i C_{n,(n-a)/2-i b} \]

if \( n \) and \( a \) have the same parity, otherwise \( P(a) = 0 \). Without loss of generality we may assume \( n \) to be even, \( b \) odd. Then

\[ \sum_{-b < a < b, a \equiv 0 \pmod{2}} C_{n,(n+a)/2-i b} = \sum_{(-b+1)/2 \leq i \leq (b-1)/2} C_{n,n/2+j-i b}. \]

Write

\[ P_i = P_{-i} = \sum_{(-b+1)/2 \leq i \leq (b-1)/2} C_{n,n/2+j-i b} \frac{1}{2^n} \]
\[ = \sum_{n/2+1/2+(n^{1/2}/2) \mid s \mid \leq n/2-1/2+(n^{1/2}/2) \mid t} C_{n,m}; \]

where

\[ \xi_1 = -(2i + 1)bn^{-1/2}, \quad \xi_2 = -(2i - 1)bn^{-1/2}. \]

Finally we write

\[ (50) \quad P = \sum_{-b < a < b} P(a). \]

From a formula of Uspensky [16, p. 129], noticing that the limits of the range of \( m \) are integers, we deduce easily that

\[ \sum_{n/2+1/2+(n^{1/2}/2) \mid s \mid \leq n/2-1/2+(n^{1/2}/2) \mid t} C_{n,m} \frac{1}{2^n} \]
\[ = \left( \frac{1}{2\pi} \right)^{1/2} \int_{\xi_1}^{\xi_2} e^{-u^2/2}du + O\left( \frac{1}{n} \right). \]

Hence

\[ (51) \quad P_i + P_{-i} = \left( \frac{2}{\pi} \right)^{1/2} \int_{\xi_1}^{\xi_2} e^{-u^2/2}du + O\left( \frac{1}{n} \right). \]

Since \(-b < a < b, \)

\[ \frac{1}{2^n} \left[ \sum_{1 \leq i \leq (n+a)/2b} (-1)^i C_{n,(n+a)/2-i b} + \sum_{1 \leq i \leq (n-a)/2b} (-1)^i C_{n,(n-a)/2-i b} \right. \]
\[ - \sum_{1 \leq i \leq n/2b} (-1)^i [C_{n,(n+a)/2-i b} + C_{n,(n-a)/2-i b}] \right| \leq \frac{1}{2^n}. \]

Hence
Therefore from (50) to (52),

\[
P = \left( \frac{1}{2\pi} \right)^{1/2} \int_{\frac{\pi}{2}}^{\pi} e^{-u^2/2} du + O\left( \frac{1}{n} \right) + \sum_{1 \leq i \leq n/2b} (-1)^i \left( \frac{1}{2\pi} \right)^{1/2} \int_{\frac{i\pi}{2}}^{\frac{i+1\pi}{2}} e^{-u^2/2} du + O\left( \frac{1}{n^2} \right) + O\left( \frac{1}{n^2} \right)
\]

\[
= \left( \frac{1}{2\pi} \right)^{1/2} \int_{\frac{\pi}{2}}^{\pi} e^{-u^2/2} du + \sum_{1 \leq i \leq n/2b} (-1)^i \left( \frac{1}{2\pi} \right)^{1/2} \int_{\frac{i\pi}{2}}^{\frac{i+1\pi}{2}} e^{-u^2/2} du + O\left( \frac{1}{b} \right) + O\left( \frac{1}{n^2} \right) + O\left( \frac{1}{n^2} \right)
\]

Since the terms are alternating in sign and decreasing in absolute value, we have

\[
\left| \sum_{i \leq n/2b} (-1)^i \left( \frac{2}{\pi} \right)^{1/2} \int_{\frac{i\pi}{2}}^{\frac{(i+1)\pi}{2}} e^{-u^2/2} du \right| \leq \left( \frac{2}{\pi} \right)^{1/2} \int_{(n-b+1)\pi}^{(n-b+1)\pi} e^{-u^2/2} du = O\left( e^{-\pi^2/4} \right)
\]

if \( b = o(n) \).

Hence if \( b = o(n) \), we obtain from (53) and (54),

\[
P = \left( \frac{1}{2\pi} \right)^{1/2} \int_{\frac{\pi}{2}}^{\pi} e^{-u^2/2} du + \sum_{i=1}^{\infty} (-1)^i \left( \frac{2}{\pi} \right)^{1/2} \int_{i\pi}^{(i+1)\pi} e^{-u^2/2} du + O\left( \frac{1}{b} \right)
\]

\[
= \left( \frac{1}{2\pi} \right)^{1/2} \sum_{i=1}^{\infty} (-1)^i \int_{(2i-1)\pi}^{(2i+1)\pi} e^{-u^2/2} du + O\left( \frac{1}{b} \right).
\]

We shall now construct a function \( h(x) \) with period \( 2\pi \) as follows:

\[
h(x) = \begin{cases} 
1 & \text{if } 0 < x < \alpha/2, \\
-1 & \text{if } \alpha/2 < x < \alpha;
\end{cases}
\]

\[
h(x) = h(-x); \quad h(x) = h(x + 2\alpha).
\]

It is easy to find that

\[
h(x) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \cos \left( \frac{2i+1}{\alpha} \pi x \right).
\]
Taking $\alpha$ to be $2bn^{-1/2}$ in the above, we have

$$
\left( \frac{1}{2\pi} \right)^{1/2} \sum_{i=-\infty}^{\infty} \frac{(-1)^i}{2i+1} \int_{2b-1}^{2b+1} e^{-x^2/2} dx = \left( \frac{1}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} h(x)e^{-x^2/2} dx
$$

(56)

$$
= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \int_{-\infty}^{\infty} e^{-x^2/2} \cos \left( \frac{(2i+1)\pi n^{1/2}}{2b} \right) dx
$$

$$
= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp \left( - \frac{(2i+1)^2s^2n^2}{8b^2} \right)
$$

since

$$
\left( \frac{1}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} e^{-x^2/2} \cos tx dx = e^{-t^2/2}.
$$

Therefore from (55) and (56) we obtain

(57)

$$
P = T(bn^{-1/2}) + O(b^{-1}).
$$

Since by assumption $g_n = o(n^{1/2})$, $cg_n^{1/2} = o(n)$; taking $b$ successively to be the nearest odd integers to $cg_n^{1/2}$ in (54) and observing that $T(bn^{-1/2}) - T(cg_n) = o(n^{-1/2})$ we obtain (48).

**Lemma 7.** Returning to the general case, we have, if (6) and (41) are satisfied,

(58)

$$
T((c - \epsilon_n)g_n) - H_n \leq \Pr (S_n < cg_n s_n) \leq T((c + \epsilon_n)g_n) + H_n;
$$

where $T(x)$ is defined in (49) and

(59)

$$
H_n = O((\lg s_n)^{-\theta} + g_n^{-1/2}).
$$

**Proof.** For the special case (47), we have according to the general notation (4) and (5),

$$
\sigma_m = 1, \quad \gamma_m = 1, \quad s_m = m, \quad M_m = 1.
$$

Condition (6) is satisfied with $\theta = 1$. Hence by Lemma 5, if

$$
\frac{8 \log 10 \cdot 10 \cdot \log m^{1/2}}{\theta \log m^{1/2}} \leq \epsilon_m g_m^{1/2} = o \left( \frac{m^{1/2}}{(\log m)^{3/2}} \right)
$$

we have from (43),

(60)

$$
\Phi_{kl}(c - \epsilon'_m)g'_m - L_m \leq \Pr (S_m < cg'_m m^{1/2}) \leq \Phi_{kl}(cg'_m) + L_m,
$$

where, by (42), $k$ is given by

$$
k \sim \frac{\theta \log m}{4 \log 10}.
$$
and where \( \Phi_{tk} \) is obtained from \( \Phi_{tk} \) in (28) after we replace \( s^* \) by \( m \) and \( B_j \) by \( B'_j \) defined according to (17) and (19) by

\[
m_j \leq jk^{-1}m < m_j + 1, \quad B_j^2 = m_j - m_{j-1},
\]

and where

\[
L_m = O(\lg m)^{-\Theta}.
\]

On the other hand, by Lemma 6, we have for the special case in question

\[
\text{Pr}(S_m^* < c g_m^{1/2}) = T(c g_m^*) + O(g_m^{-1} m^{-1/2}) + O(m^{-1/2}).
\]

Substituting from (58) into (60) we obtain

\[
\Phi_{tk}(c g_m^* - L_m) \leq T(c g_m^*) + O(g_m^{-1} m^{-1/2} + m^{-1/2}) \leq \Phi_{tk}(c g_m^*) + L_m.
\]

From (62) we deduce

\[
\Phi_{tk}(c - \epsilon_m) g_m^* \leq T((c - \epsilon_m) g_m^*) + O(g_m^{-1} m^{-1/2} + m^{-1/2}) + O(L_m);
\]

\[
\Phi_{tk}(c g_m^*) \leq T((c + \epsilon_m) g_m^*) + O(g_m^{-1} m^{-1/2} + m^{-1/2}) + O(L_m).
\]

Now putting

\[
m = \lceil s_n^* \rceil, \quad \epsilon_m = \epsilon_n, \quad g_m^* = g_n^*,
\]

we obtain from (63) and (64) the following: if

\[
\frac{8 \lg 10 \cdot \Theta}{\theta} \cdot \frac{\lg s_n}{\lg s_n} \leq \epsilon_n g_n = o\left(\frac{s_n^\theta}{(lg s_n)^{3/2}}\right)
\]

then we have

\[
T((c - \epsilon_n) g_n - K_n) \leq \Phi_{tk}((c - \epsilon_n) g_n) \leq \Phi_{tk}(c g_n)
\]

\[
\leq T((c + \epsilon_n) g_n) + K_n,
\]

where

(42 bis)

\[
k \sim \frac{\theta \lg s_n}{2 \lg 10},
\]

(66)

\[
K_n = O((\lg s_n)^{-\Theta} + g_n^{-1} s_n^{-1}).
\]

Writing \( \lambda_j = s_n B_j^{-1}, \lambda'_j = [s_n^*]^{1/2} B'_j^{-1} \) we have from (28)

\[
\Phi_{0}(u_1, \ldots, u_k)
= \frac{\lambda_1 \cdots \lambda_k}{(2\pi)^{k/2}} \int_{-u_1}^{u_1} \cdots \int_{-u_k}^{u_k} \exp \left(-\frac{1}{2} \sum_{j=1}^{k} \lambda_j^2 (x_j - x_{j+1})^2\right) dx_1 \cdots dx_k.
\]
It is easy to verify that
\[
\left| \frac{\partial \Phi_0}{\partial \lambda_j} \right| \leq \frac{3}{2\lambda_j}.
\]
Hence
\[
| \Phi_{0k} - \Phi_{1k} | \leq \frac{3}{2} \sum_{j=1}^{k} \frac{1}{\lambda_j} | \lambda_j - \lambda'_j |. \tag{67}
\]
Since \( \sigma_n = O(s_n^{1-\theta}) \), \( B_j^2 = s_n^{2k^{-1}} + O(s_n^{2-2\theta}) \),
\[
\lambda_j = \frac{s_n^{2k^{-1}}}{B_j^2} = k \left( 1 + O\left( \frac{k}{s_n^{2\theta}} \right) \right).
\]
The same holds for \( \lambda'_j \). Thus \( |\lambda_j - \lambda'_j| = O(k^2 s_n^{-2\theta}) \) and by \((42 \text{ bis})\) and \((67)\),
\[
| \Phi_{0k} - \Phi_{1k} | = O(s_n^\theta). \tag{68}
\]
Therefore from \((65)\) we obtain
\[
T((c - \epsilon_n)g_n) - J_n \leq \Phi_{0k}(c - \epsilon_n)g_n) \leq \Phi_{0k}(cg_n)
\leq T((c + \epsilon_n)g_n) + J_n, \tag{69}
\]
where \( k \) is given by \((42 \text{ bis})\) and from \((66)\) and \((68)\) we have
\[
J_n = O(R_n) + O(s_n^\theta) = O((\lg s_n)^{-1} + s_n^{\frac{1}{2}} s_n^{-1}). \tag{70}
\]
Using \((69)\) in \((43)\), Lemma 5, we obtain
\[
T((c - \epsilon_n)g_n) - H_n \leq \Pr(S^*_n < cg_n s_n) \leq T((c + \epsilon_n)g_n) + H_n.
\]
where \( H_n = O(J_n) + O(L_n) \). Hence by \((44)\) and \((70)\) we have established \((58)\) and \((59)\).

**Proof of Theorem 1.** Taking \( g_n = 1 \) in Lemma 5, we get
\[
T(c - \epsilon_n) - H_n \leq \Pr(S^*_n < c s_n) \leq T(c + \epsilon_n) + H_n, \tag{71}
\]
where
\[
H_n = O((\lg s_n)^{-1}).
\]
Now we have, by the mean-value theorem,
\[
\exp\left( -\frac{(2i + 1)^2 \pi^2}{8(c + \epsilon_n)^2} \right) - \exp\left( -\frac{(2i + 1)^2 \pi^2}{8c^2} \right) \leq \frac{(2i + 1)^2 \pi^2}{4c^2} \exp\left( -\frac{(2i + 1)^2 \pi^2}{8(c + \epsilon_n)^2} \right) \epsilon_n.
\]
Hence
\[ T(c + \epsilon_n) - T(c) \leq \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(2i + 1)\pi^2}{4c^3} \exp \left( -\frac{(2i + 1)^2 \pi^2}{8(c + \epsilon_n)^2} \right) \epsilon_n = O(\epsilon_n). \]

Thus (71) becomes
\[ (72) \quad \Pr(S_n^* < c \epsilon_n) = T(c) + O(\epsilon_n) + O((\lg s_n)^{-\theta}). \]

Choosing, for example, \( \Theta = 1 \) and
\[ \epsilon_n = \frac{8 \lg 10 \lg_2 s_n}{\theta \lg s_n}, \]
which is permissible by (41), we obtain (7) from (72).

**Proof of Theorem 2.** We have
\[ \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8x^2} \right) - \frac{4}{3\pi} \exp \left( -\frac{9\pi^2}{8x^2} \right) \leq T(x) \leq \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8x^2} \right). \]

Since \( \epsilon_n \downarrow 0 \), we have if \( \epsilon_n < 4^{-1} \),
\[ T((1 + \epsilon_n)g_n) \leq \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} (1 + \epsilon_n)^{-2} \right) \]
\[ \leq \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} (1 - 2\epsilon_n) \right) \]
\[ = \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} \right) \exp \left( \frac{\pi^2 \epsilon_n}{4g_n^2} \right), \]
\[ T((1 - \epsilon_n)g_n) \geq \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} (1 - \epsilon_n)^{-2} \right) - \frac{4}{3\pi} \exp \left( -\frac{9\pi^2}{8g_n^2} \right) \]
\[ \geq \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} (1 + 4\epsilon_n) \right) - \frac{4}{3\pi} \exp \left( -\frac{9\pi^2}{8g_n^2} \right) \]
\[ = \frac{4}{\pi} \exp \left( -\frac{\pi^2}{8g_n^2} \right) \exp \left( -\frac{\pi^2 \epsilon_n}{2g_n^2} \right) - \frac{4}{3\pi} \exp \left( -\frac{9\pi^2}{8g_n^2} \right). \]

Choosing
\[ \epsilon_n = \frac{8 \lg 10 \cdot \Theta \lg_2 s_n}{\theta \lg_2 s_n}, \]
then (41) is satisfied, and from (8),
\[ \frac{\epsilon_n^2}{g_n^4} = \frac{8 \lg 10 \cdot \Theta \lg_2 s_n}{\theta \lg_2 s_n} = o(1). \]
Hence we have
\[
\exp \left( \frac{\pi^2 \varepsilon_n}{4 g_n^2} \right) = 1 + o(1),
\]
\[
\exp \left( - \frac{\pi^2 \varepsilon_n}{2 g_n^2} \right) = 1 + o(1).
\]

Since \( g_n \downarrow 0 \), we have
\[
\exp \left( - \frac{9\pi^2}{8 g_n^2} \right) = o \left( \exp \left( - \frac{\pi^2}{8 g_n^2} \right) \right).
\]

Thus from (73) and (74), we obtain
\[
\frac{4}{\pi} (1 + o(1)) \exp \left( - \frac{\pi^2}{8 g_n^2} \right) \leq T((1 - \varepsilon_n) g_n) \leq T((1 + \varepsilon_n) g_n)
\]
\[
\leq \frac{4}{\pi} (1 + o(1)) \exp \left( - \frac{\pi^2}{8 g_n^2} \right).
\]

Therefore (58) becomes
\[
\Pr \left( S_n < g_n s_n \right) = \frac{4}{\pi} (1 + o(1)) \exp \left( - \frac{\pi^2}{8 g_n^2} \right) + O((\log s_n)^{-\Theta}).
\]

Since we may choose \( \Theta \) arbitrarily large, (9) follows on account of (8).

4. Some strong limit theorems. Since we shall deal with indices \( n, \nu, k \) and so on, which ultimately tend to infinity, we shall often omit mention of this proviso. Thus, sometimes our statements are true only when the appropriate index is sufficiently large.

The condition (6) is assumed in this section. From (6) it follows:

\[
\sigma_n = O(s_n^{-1/3}), \quad \theta > 0.
\]

Let \( \psi_n \uparrow \infty \), and
\[
\psi_n = O((\log s_n)^{1/2}).
\]

Taking \( g_n = \psi_n^{-1} \) in Theorem 2, we have
\[
A e^{-\psi_n^2} \leq \Pr \left( S_n^* < 8^{-1/2} \pi s_n \psi_n^{-1} \right) \leq A e^{-\psi_n^2}.
\]

We shall construct a subsequence \( \{n_k\} \), \( k = 1, 2, \ldots \), as follows. Take \( a > 0 \). Put \( n_1 = 1 \). Suppose that \( n_k \) is defined already, then since \( s_n \uparrow \), there is a unique \( n_{k+1} \) such that
\[
s_{n_{k+1} - 1} \leq s_{n_k} (1 + a \psi_{n_k}^{-2}) \leq s_{n_{k+1}}.
\]
Hence (for \( k \) sufficiently large)

\[
2 \sum_{n_k} \leq s_n^2 (1 + 3a_n^{-2}).
\]

By virtue of (75) and (76), we have

\[
2 s_{n_k+1} \leq s_{n_k} + 3a_n s_{n_k}^{-2} + s_{n_k+1} \leq s_{n_k} + 4a_n s_{n_k+1} s_{n_k}^{-2};
\]

\[
2 s_{n_k+1} \leq s_{n_k} (1 - 4a_n^{-2})^{-1}.
\]

Thus there exists \( b > a \) such that

\[
(77) \quad s_{n_k} (1 + a_n^{-2}) \leq s_{n_k+1} \leq s_{n_k} (1 + b_n^{-2}).
\]

For simplicity we shall write \( k' \) for \( n_k \), \( s_k' \) for \( s_n' \), \( \psi_k' \) for \( \psi_n' \), and so on.

**Lemma 8.** Suppose that \( \psi_n \to \infty \) and (76) holds. Let \( \{ s_k \} \) be any sequence satisfying (77). Then if

\[
(78) \quad \sum_k e^{-s_k^2} < \infty
\]

we have

\[
(79) \quad \Pr (S_n^* < 8^{-1/2} \pi s_n^{-1} \text{ i. o.}) = 0.
\]

**Proof.** From (9 bis) we have

\[
\Pr \left( S_{k'}^* < \frac{\pi}{81/2} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \right) \leq A_n e^{-(\psi_k'^2 - 3b)}.
\]

Hence by (78)

\[
\sum_{k=1}^{\infty} \Pr \left( S_{k'}^* < \frac{\pi}{81/2} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \right) < \infty.
\]

By the lemma of Borel-Cantelli (see, for example [13, pp. 26–27]), we conclude that

\[
\Pr \left( S_{k'}^* < \frac{\pi}{81/2} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \text{ i. o.} \right) = 0,
\]

that is,

\[
(80) \quad \Pr \left( S_{k'}^* \geq \frac{\pi}{81/2} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \text{ for all sufficiently large } k \right) = 1.
\]

Now suppose that \( n_k < n \leq n_{k+1} \). Then if

\[
S_{k'}^* \geq \frac{\pi}{81/2} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}}
\]
we have by (77)

\[ S_n^* \leq S_k^* \leq \frac{\pi}{8^{1/2}} \frac{S_{k+1}'}{(\psi_k'^2 - 3b)^{1/2}} \frac{s_k'}{s_{k+1}} \]

\[ \geq \frac{\pi}{8^{1/2}} s_{k+1}'(\psi_k'^2 - 3b)^{-1/2} (1 + b\psi_k'^{-2})^{-1}. \]

If \( k \) is sufficiently large, we have

\[ S_n^* \leq \frac{\pi}{8^{1/2}} \psi_k^{-1} \leq \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n}. \]

Thus (80) entails

\[ \Pr \left( S_n^* > \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n} \text{ for all sufficiently large } n \right) = 1. \]

This is equivalent to (79).

**Lemma 9.** Suppose that \( \psi_n \uparrow \infty \) and (76) holds. Let \( \{n_k\} \) be any sequence satisfying (77). Then if

(81) \[ \sum \frac{e^{-\psi_n^2}}{k_w} = \infty \]

we have

(82) \[ \Pr \left( S_n^* < \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n} \text{ i. o.} \right) = 1. \]

**Proof.** By (77), given \( s_{k-1} \), there is a unique \( v \) such that

(83) \[ s_{k_v} \leq s_{k-1} \psi_{k-1}^2 < s_{k_v+1}. \]

From (9 bis) we have, if \( \epsilon > 1/8 \) is any constant,

\[ \Pr \left( S_k^* < \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 + 8C)^{1/2}} \right) \geq A e^{-\psi_k'^2 + 8C). \]

Hence by (81) we have

(84) \[ \sum_{r=1}^{\infty} \sum_{k_{r+1}}^{k_r} \Pr \left( S_k^* < \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 + 8C)^{1/2}} \right) = \infty. \]

Let \( \{\nu(\tau)\}, \tau = 1, 2, \ldots, \) denote the subsequence of \( \nu = 1, 2, \ldots \) for which

(85) \[ \psi^2(k_{\nu(\tau)} + 1) > \psi^2(k_{\nu(\tau)-1}) + 1. \]

Then we have
Theorem 1. Let $\{S_n\}$ be a sequence of independent random variables with $\mathbb{E}[S_n] = 0$ and $\mathbb{E}[S_n^2] < \infty$. If $\sum_{n=1}^{\infty} \mathbb{E}[S_n^2] < \infty$, then $\sum_{n=1}^{\infty} |S_n|$ diverges.

Proof. Let $S_n^* = S_n - \mathbb{E}[S_n]$. Then $\sum_{n=1}^{\infty} S_n^*$ is absolutely convergent, so $\sum_{n=1}^{\infty} |S_n|$ diverges.

Corollary. If $\sum_{n=1}^{\infty} \mathbb{E}[S_n^2] < \infty$, then $\sum_{n=1}^{\infty} S_n^2$ converges.

Proof. By the Borel-Cantelli Lemma, for any $\epsilon > 0$, we have

$$
\sum_{n=1}^{\infty} \mathbb{P}(|S_n| > \epsilon) < \infty.
$$

If $\sum_{n=1}^{\infty} \mathbb{E}[S_n^2] < \infty$, then $\sum_{n=1}^{\infty} S_n^2$ converges.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Then if we have the conjunction $E'_{r-1}E_{r-1,\mu}$, we have by (88)

$$\begin{align*}
S_u^* &< \frac{\pi}{8^{1/2}} \left( \frac{s_u}{(\psi_u^2 + 8C)^{1/2}} + C \sqrt{s_{k_{r-1}}} \right) < \frac{\pi}{8^{1/2}} \left( \frac{s_u}{(\psi_u^2 + 8C)^{1/2}} + \frac{2C s_u}{\psi_u^2} \right) \\
&< \frac{\pi}{8^{1/2}} \left( \frac{1}{(1 + 8C\psi_u^{-2})^{1/2}} + \frac{2C}{\psi_u^2} \right) < \frac{\pi}{8^{1/2}} \frac{s_u}{\psi_u^2}.
\end{align*}$$

Therefore if $\nu \neq \nu(r)$, the conjunction $E'_{r-1}E_{r-1,\mu}$ implies $E_{\mu}$. Writing

$$F_r = \sum_{\mu=k_r}^{k_{r+1}-1} E_{\mu}, \quad F'_r = \sum_{\mu=k_r}^{k_{r+1}-1} E_{r-1,\mu},$$

we have, if $\nu \neq \nu(r)$, $E'_{r-1}F'_r$ implies $F_r$, hence

$$\sum_{r=1}^{\infty} E'_{r-1}F'_r \text{ implies } \sum_{r=1}^{\infty} F_r,$$

(89) \hspace{1cm} $\Pr \left( \sum_{r=1}^{\infty} E'_{r-1}F'_r \right) \leq \Pr \left( \sum_{r=1}^{\infty} F_r \right).$

The events $F'_1, F'_{r+2}, F'_{r+4}, \ldots$ are independent and $F'_j$ for $j \geq r$ is independent of $E'_{r-1}$. By the Kolmogoroff inequality [15] we have

(90) \hspace{1cm} $\Pr \left( E'_{r-1} \right) \leq 1 - 1/C^2.$

We obtain, by an obvious argument and (90), for all $\nu_1 \geq \nu_0$,

$$\Pr \left( \sum_{r=1}^{\infty} E'_{r-1}F'_r \right) = \Pr \left( \sum_{r=1}^{\infty} \left( E'_{r-1}F'_r - E'_{r-1}F'_r \sum_{j=r+1}^{\infty} E'_{j-1}F'_j \right) \right) \geq \Pr \left( \sum_{r=1}^{\infty} E'_{r-1} \left( F'_r - F'_r \sum_{j=r+1}^{\infty} F'_j \right) \right) = \sum_{r=1}^{\infty} \Pr \left( E'_{r-1} \right) \Pr \left( F'_r - F'_r \sum_{j=r+1}^{\infty} F'_j \right) \geq \left( 1 - \frac{1}{C^2} \right) \Pr \left( \sum_{r=1}^{\infty} F'_r \right).$$

(91)

Hence by (89) and (91) we have

$$\Pr \left( \sum_{r=1}^{\infty} F'_r \right) \leq \left( 1 - \frac{1}{C^2} \right) \Pr \left( \sum_{r=1}^{\infty} F'_r \right) \leq \left( 1 - \frac{1}{C^2} \right) \Pr \left( \sum_{r=1, r \equiv \nu_0 \pmod{2}}^{\infty} F'_r \right).$$

(92)

Since the events $F'_{r_0}, F'_{r_0+2}, F'_{r_0+4}, \ldots$ are independent, by the lemma of
Borel-Cantelli we know that

\[ \Pr \left( \sum_{r=1, r \equiv r_0 \pmod{2}}^{\infty} F_r' \right) = 1 \]

if and only if

\[ \sum_{r=1, r \equiv r_0 \pmod{2}}^{\infty} \Pr (F_r') = \infty. \]

If we can prove (93), then from this remark and (91) we shall have for all \( \nu \geq \nu_0 \), hence in fact for all \( \nu_1 \),

\[ \Pr \left( \sum_{r=\nu_1}^{\infty} F_r \right) \geq 1 - \frac{1}{C^2}, \]

da fortiori, for all \( n_1 \),

\[ \Pr \left( \sum_{n=n_1}^{\infty} E_n \right) \geq 1 - \frac{1}{C^2}. \]

Since we may choose \( C \) arbitrarily large while the left-hand side does not depend on \( C \) we shall have proved for all \( n_1 \), \( \Pr (\sum_{n=n_1}^{\infty} E_n) = 1 \), which is equivalent to (82).

Hence to prove (82) it is sufficient to prove (93). By definition this is equivalent to

\[ \sum_{r=1, r \equiv r_0 \pmod{2}}^{\infty} \Pr \left( \sum_{\mu=k_r}^{k_{r+1}-1} E_{r-1, \mu} \right) = \infty. \]

Comparing (94) and (87) we see that in order to prove (94) it is sufficient to prove that for \( \nu \not\equiv \nu(\tau) \), there exists a constant \( A_{10} > 0 \) such that for all sufficiently large \( \nu \), the following shall hold:

\[ \Pr \left( \sum_{\mu=k_r}^{k_{r+1}-1} E_{r-1, \mu} \right) \geq A_{10} \sum_{k=k_r}^{k_{r+1}-1} \Pr \left( S_{k'} < \frac{\pi}{8^{1/2}} \frac{S_k'}{\psi_2^2 + 8C^{1/2}} \right). \]

We have for any integer \( N > 0 \),

\[ \Pr \left( \sum_{\mu=k_r}^{k_{r+1}-1} E_{r-1, \mu} \right) \geq \Pr \left( \sum_{k=k_r}^{k_{r+1}-1} E_{r-1, k'} \right) \]

\[ \geq \frac{1}{N} \sum_{k=k_r}^{k_{r+1}-1} \Pr \left( E_{r-1, k'} - E_{r-1, k'} \sum_{j=k+1}^{k_{r+1}} E_{r-1, j'} \right). \]

Now we see easily that \( E_{r-1, k'} \) implies \( E_{k', i} \) where \( E_{k', i} \) denotes the event
Since $E_{r-1,k'}$ and $E'_{k,i}$ are independent, we have
\[
\Pr \left( E_{r-1,k'} E'_{r-1,i'} \right) \leq \Pr \left( E_{r-1,k'} \right) \Pr \left( E'_{k,i} \right).
\]

If we can prove that, for a suitable $N$,
\[
\sum_{j=k+N}^{k+p+1} \Pr \left( E'_{k,j} \right) < \frac{1}{2},
\]
then from (96)
\[
\Pr \left( \sum_{j=k+p}^{k+p+1} E_{r-1,p} \right) \geq \frac{1}{N} \sum_{k=k_p}^{k+p+1} \Pr \left( E_{r-1,k'} - \sum_{j=k+N}^{k+p+1} E_{r-1,k'} E_{r-1,i'} \right)
\]
\[
\geq \frac{1}{N} \sum_{k=k_p}^{k+p+1} \Pr \left( E_{r-1,k'} \right) \left( 1 - \sum_{j=k+N}^{k+p+1} \Pr \left( E'_{k,i} \right) \right)
\]
\[
\geq \frac{1}{2N} \sum_{k=k_p}^{k+p+1} \Pr \left( E_{r-1,k'} \right).
\]

By (9 bis)(*) we have
\[
\Pr \left( E_{r-1,k'} \right) \leq \Pr \left( \max_{k-1<k<k'} | S-y - S_{k-1}^{i'} | < 8^{-1/2} \frac{(s_{k'} - s_{k+1})^{1/2}}{(\psi_{k'} + 8C)^{1/2}} \right)
\]
\[
\leq \frac{A_6}{A_7} \Pr \left( S_{k'}^{i'} < 8^{-1/2} \frac{s_{k'}}{(\psi_{k'} + 8C)^{1/2}} \right).
\]

Thus from (98) and the last inequality we shall have proved (95) with $A_{10} = A_6/2A_7$. Hence it is sufficient to prove (97).

Now we have, since $\psi_{k}^{i2} \geq \psi_{k'}^{i2} \geq \psi_{k+1}^{i2} - 1 \geq \psi_{j'}^{i2} - 1$,
\[
\frac{s_{k'}}{(\psi_{k'}^2 + 8C)^{1/2}} < \frac{s_{j'}}{(\psi_{j'}^2 - 1 + 8C)^{1/2}}.
\]

(99) \[
\Pr \left( E'_{k,i} \right) \leq \Pr \left( \max_{n_k<i<\hat{n}_j} | S_i - S_{n_k} | < \frac{\pi}{8^{1/2}} (s_{j'}^{i2} - s_{k'}^{i2})^{1/2} \right)
\]
where
\[
(100) \quad g_{i} = \frac{2s_{j'}^{i}}{(s_{j'}^{i2} - s_{k'}^{i2})^{1/2} (\psi_{j'}^2 + 8C - 1)^{1/2}}.
\]

(*) See footnote 3.
It is obvious that \( g_j \downarrow 0 \); in order to apply Theorem 2 we have to verify that

\[
\frac{(s_j'^2 - s_k'^2)^{1/2}(\psi_j'^2 + 8\gamma - 1)^{1/2}}{s_j'} \leq A_{11}(\log_2(s_j'^2 - s_k'^2))^{1/2}
\]

which is evident since

\[
\left(\frac{s_j'^2 - s_k'^2}{\log_2(s_j'^2 - s_k'^2)}\right)^{1/2} \leq A_{12} \frac{s_j'}{\psi_j'}
\]

by (76). Therefore we have from (99) and (9), Theorem 2,

(101)

\[
\Pr (E_{k,i}) \leq A_{13} \delta - g_j^{-2}.
\]

We have for sufficiently large \( k \), from (77),

(102)

\[
\frac{s_{j+1}'}{s_j'^2} \leq 1 - \frac{a}{2\psi_j'^2}, \quad \frac{s_k'^2}{s_j'} \leq \left(1 - \frac{a}{2\psi_j'^2}\right)^{j-k}.
\]

If \( hx \leq \delta \) where \( \delta > 0 \) is sufficiently small, then \((1-x)^h \leq 1 - \delta' hx \) where \( \delta' > 0 \) is another constant. Hence if \( j-k \leq \delta' \psi_j'^2 \) we have from (102)

\[
\frac{s_k'}{s_j'} \leq 1 - \frac{\delta' a(j-k)}{\psi_j'^2}, \quad 1 - \frac{s_k'^2}{s_j'^2} \geq a' j-k \psi_j'^2
\]

where \( a' > 0 \). Then from (100)

\[
g_j \leq 2(a'(j-k))^{-1/2}.
\]

Hence by (101) we have

(103)

\[
\Pr (E_{k,i}) \leq A_{13} \exp \left(-4^{-1}a'^2(j-k)\right).
\]

If \( hx > \delta \), then \((1-x)^h \leq \delta'' < 1 \), hence from (102), if \( j-k \geq \delta' \psi_j'^2 \),

\[
\frac{s_{j+1}'}{s_j'^2} \leq \delta_0 < 1, \quad 1 - \frac{s_k'^2}{s_j'^2} \geq 1 - \delta_0^2;
\]

\[
g_j \leq 2(1 - \delta_0^{-1/2})^{-1/2} \psi_j'^{-1};
\]

(104)

\[
\Pr (E_{k,i}) \leq A_{13} \exp \left(-4^{-1}(1-\delta_0^2)\psi_j'^2\right).
\]

From (103) and (104),

(105)

\[
\sum_{j=k+1}^{k+\infty} \Pr (E_{k,i}) \leq A_{13} \left(\sum_{i=k}^{\infty} e^{-a'i/4} + (k+1 - k) \exp\left(-\frac{1 - \delta_0^2}{4\psi_k'^2}\right)\right).
\]

We have by (83),

\[
\psi_k'^2 \geq \frac{s_{k+1}'}{s_k'} \geq \left(1 + \frac{a}{\psi_k'^2}\right) \cdots \left(1 + \frac{a}{\psi_k'^2}\right) \geq 1 + \frac{a(k+1 - k)}{\psi_{k+1}'^2}.
\]
Hence we have

\[ k_{r+1} - k_r \leq A_{14}\psi_{k_{r+1}}^{15}. \]

Since \( \nu \neq \nu(r) \), we have \( \psi_{k_{r+1}}^{2^2} \leq 2\psi_{r}^{2^2} \). Hence

\[ k_{r+1} - k_r \leq 6A_{14}\psi_{k_r}^{15}; \]

\[ (k_{r+1} - k_r) \exp \left( -\frac{1 - \delta_0}{4} \psi_{k_r}^{15} \right) \leq A_{14}\psi_{k_r}^{15}e^{-A_{14}\psi_{k_r}^{15}} = o(1). \]

Thus by choosing \( N \) sufficiently large we obtain from (105) the desired (97), if \( \nu \) is sufficiently large. The proof of Lemma 9 is thus complete.

**Lemma 10.** If \( \{n_k\}, k = 1, 2, \cdots, \) is defined by (77), then the series

\[ \sum_n e^{-\psi_{n_k}} \]

and

\[ \sum_n \frac{\sigma_n}{\psi_{n_k}^2} e^{-\psi_{n_k}^2} \]

converge and diverge together.

**Proof.** We have

\[ \frac{\sigma_n}{\psi_{n_k}^2} = 1 - \frac{\sigma_{n+1}}{\psi_{n_k}^2}. \]

Since \( x \leq -\log(1-x) \leq 2x \) if \( 0 < x < 1 \), we obtain

\[ \sum_{n_k < n \leq n_{k+1}} \frac{\sigma_n}{\psi_{n_k}^2} \geq \log \left( 1 - \frac{\sigma_{n+1}}{\psi_{n_k}^2} \right) \geq 2 \log \left( 1 + \frac{b}{\psi_{n_k}^2} \right) \geq \frac{2b}{\psi_{n_k}^2}. \]

Since \( \psi_{n_k}^2 e^{-\psi_{n_k}^2} \downarrow \), we have

\[ \frac{a}{2} e^{-\psi_{n_k}^2} \leq \psi_{n_k}^2 e^{-\psi_{n_k}^2} \leq \sum_{n_k < n \leq n_{k+1}} \frac{\sigma_n}{\psi_{n_k}^2} e^{-\psi_{n_k}^2} \leq \psi_{n_k}^2 e^{-\psi_{n_k}^2} \leq 2b e^{-\psi_{n_k}^2}. \]

Lemma 10 follows from this inequality.
Proof of Theorem 3. The \( \phi_n \) given in (11) is monotone increasing and \( \phi_n = O((\lg_2 s_n)^{1/2}) \). Hence Lemma 8 and Lemma 9 are applicable. Hence
\[
\Pr (s_n^* < 8^{-1/2} \pi s_n \phi_n^{-1} \text{ i. o.}) = \begin{cases} 0 \\ 1 \end{cases}
\]
according as
\[
\sum_k e^{-q_{k,n}} \left\{ \frac{a_k}{b_k} \right\} = \infty.
\]
By Lemma 10, the last series converges and diverges with (13), which in this case is
\[
\sum_n \frac{(1 + o(1))s_n^2 \lg_2 s_n}{s_n^2 \lg s_n (\lg_2 s_n)^2 \lg_3 s_n \cdots \lg_p s_n (\lg_{p+1} s_n)^{1+\delta}}.
\]
Hence a well known theorem of Abel-Dini asserts that this is convergent if and only if \( \delta \) is positive. Thus Theorem 3 is proved.

Proof of Theorem 4. Suppose that \( \psi_n \uparrow \infty \). Define
\[
\psi_n^2 = \min (\phi_n^2, 2 \lg_2 s_n).
\]
If (13) is convergent, then
\[
\sum_n \frac{\sigma_n^2 \psi_n^2 - \psi_n^2}{s_n^2} = \sum_{\psi_n = \phi_n} + \sum_{\psi_n > \phi_n} < \infty
\]
since again by the Abel-Dini theorem we have
\[
\sum_{\psi_n > \phi_n} \psi_n s_n \leq \sum_{\psi_n > \phi_n} \frac{2s_n^2 \lg_2 s_n}{s_n^2 (\lg s_n)^2} < \infty.
\]
By the definition (106) \( \psi_n \) satisfies (76), hence by Lemma 8,
\[
\Pr (s_n^* < 8^{-1/2} \pi s_n \psi_n^{-1} \text{ i. o.}) = 0.
\]
Since \( \psi_n \leq \phi_n \), a fortiori (12) is equal to zero.

If (13) is divergent, then since \( \psi_n \leq \phi_n \), we have
\[
\sum_n \frac{\sigma_n^2 \psi_n^2 - \psi_n^2}{s_n^2} = \infty.
\]
Since \( \psi_n \) satisfies (76), by Lemma 9, \( \Pr (s_n^* < 8^{-1/2} \pi s_n \psi_n^{-1} \text{ i. o.}) = 1 \). By Theorem 3, we have
\[
\Pr (s_n^* < 8^{-1/2} \pi \frac{s_n}{(2 \lg_2 s_n)^{1/2}} \text{ i. o.}) = 0.
\]
Hence there exists a subsequence \( n_i \) such that \( \psi_{n_i}^2 \leq 2 \lg_2 s_{n_i} \) and
Pr \((S^*_{n_i} < 8^{-1/2} \pi s_{n_i} \psi_{n_i}^{-1}, \ o.) = 1.

By the definition (106), we have \( \psi_{n_i} = \phi_{n_i} \). Hence (12) is equal to one. Theorem 4 is proved.

Proof of Theorem 5. By Theorem 4 it is sufficient to prove that the series

\[
\sum_n \frac{\sigma_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)}
\]

and the integral (15) converge and diverge together.

We have, since \( t^{-1} \phi^2(t)e^{-\phi^2(t)} \downarrow 0 \),

\[
\int_{s_n^2}^{s_{n+1}^2} t^{-1} \phi^2(t)e^{-\phi^2(t)} dt = \sum_{n=k+1}^{\infty} \int_{s_n^2}^{s_{n+1}^2} t^{-1} \phi^2(t)e^{-\phi^2(t)} dt \leq \sum_{n=k+1}^{\infty} \frac{s_{n+1}^2 - s_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)}.
\]

Hence if (107) diverges, (15) diverges too.

On the other hand, we have

\[
\sum_{n=N+1}^{\infty} \frac{\sigma_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)} \geq \int_{s_N^2}^{s_{\infty}^2} t^{-1} \phi^2(t)e^{-\phi^2(t)} dt.
\]

From (75) we have \( s_n^2 = s_{n-1}^2 + \sigma_n^2 \leq s_{n-1}^2 + O(s_n^2) \). Hence if \( n \) is large enough, we have

\[
\sum_{n=N+1}^{\infty} \frac{\sigma_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)} \geq \int_{s_N^2}^{s_{\infty}^2} t^{-1} \phi^2(t)e^{-\phi^2(t)} dt.
\]

(108)

Let \( n_k, k = 1, 2, \ldots \), denote the subsequence of \( n = 1, 2, \ldots \) for which

(109)

Evidently we have by (109) and (110),

\[
\sum_k \frac{\sigma_{n_k}^2}{s_{n_k}^2} \phi \left( \frac{s_{n_k}}{s_{n_k}^2} \right) e^{-\phi\left( s_{n_k}^2 \right)} \leq \sum_k \frac{\sigma_{n_k}^2}{s_{n_k}^2} \phi \left( \frac{s_{n_k}}{s_{n_k}^2} \right) e^{-\phi\left( s_{n_k}^2 \right)} \leq A_{\infty} \sum_k \frac{\sigma_{n_k}^2}{s_{n_k}^2} e^{-\phi\left( s_{n_k}^2 \right)/2} < \infty.
\]

Hence if (15) diverges, we have, by (108) and (111),

(112)

By (110) if \( n \neq n_k \), we have \( \phi^2(s_n^2) \leq \phi^2(s_{n_k}^2) + 1 \). From this and (112) we obtain

\[
\sum_n \frac{\sigma_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)} \geq \frac{1}{2} \sum_n \frac{\sigma_n^2}{s_n^2} \phi \left( \frac{s_n}{s_n^2} \right) e^{-\phi\left( s_n^2 \right)} = \infty.
\]

Theorem 5 is proved.
References


Princeton University,
Princeton, N. J.