ON THE DEGREE OF VARIATION IN CONFORMAL MAPPING OF VARIABLE REGIONS

BY
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1. Introduction. Suppose C is a closed Jordan curve and the function \( w = f(z) \) maps the circle \( |z| < 1 \) conformally onto the interior \( R \) of \( C; f(z) \) is analytic in \( |z| < 1 \) and continuous in \( |z| \leq 1 \). It is well known that the function \( f(z) \) varies continuously in the closed circle \( |z| \leq 1 \) under a suitable continuous deformation(2) of \( C \). It is of interest, however, to go beyond this merely “qualitative” statement and to estimate the degree of variation of \( f(z) \) in this dependence upon the change in \( C \). If \( C_1 \) is a “neighboring” closed Jordan curve and if \( w = f_1(z) \), normalized in the same manner as \( f(z) \), maps \( |z| < 1 \) conformally onto the interior of \( C \), it is desired to find an upper bound for \( |f(z) - f_1(z)| \) for \( |z| \leq 1 \) which measures the effect of the deformation of \( C \).

This problem has been treated with some degree of completeness in the case of “nearly circular” regions, that is, in the special case in which \( C \) is a circle. The principal contributions here were made by L. Bieberbach [1], 1924, A. R. Marchenko [10], 1935, and Jacqueline Ferrand [4], 1945. Marchenko’s and Ferrand’s estimates are, in a certain sense, best possible results.

The general problem of two arbitrary regions presents a more diversified aspect because of the various degrees of smoothness which may be imposed upon one or both of the boundary curves. Under suitable differentiability assumptions regarding the curve \( C \) one can reduce the problem to the “nearly circular” case by a conformal transformation. In this way A. Markoushevitch [11], 1936, extended Marchenko’s result to the more general configuration of two Jordan curves. A different approach is used by J. Ferrand in [4], where a theorem on “nearly polygonal” curves(3) is presented. The method indicated here is based on appraisals of the change of the harmonic measure

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(1) A part of this work was done while the author was connected with the Institute for Numerical Analysis, National Bureau of Standards, and it was sponsored (in part) by the Office of Naval Research.

(2) See, for example, Gattegno and Ostrowski [5, section 14]. Numbers in brackets refer to the bibliography at the end of the paper.

(3) That is, one of the curves lies between two simple closed polygons, one of which is obtained from the other by dilation from an interior point, the ratio being a number close to 1, and the other curve is a polygon.

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of a side of a polygon under certain deformations (displacement of the sides) of the polygon.

The main object of the present paper is the study of the problem in more general cases in which either no restrictions or only weak conditions are placed on the boundary; in particular, the regions considered need not be bounded by Jordan curves. The principal tools in this paper are: (i) estimates for the oscillation of the mapping function of the unit circle onto a bounded region, which are valid in the neighborhood of the boundary (see §2); (ii) some auxiliary theorems, in which the mapping function of two regions, one of which is contained in the other, are compared with each other in any circle \( |z| \leq \rho < 1 \) (see Lemmas 3 and 6).

As a first result, Marchenko’s theorem is extended to a “nearly circular” region which is not necessarily bounded by a Jordan curve (§3). §§4 and 5 deal with the general situation of two arbitrary regions. §6 contains a result on regions bounded by Jordan curves with continuously turning tangents which is easily obtained by the method developed in this paper.

2. The oscillation of the mapping function at the boundary. Suppose that \( R \) is a simply connected bounded region and that the function \( w = f(z) \) maps the circle \( |z| < 1 \) conformally onto \( R \). Let \( |z_0| = 1 \). Then we define for \( 0 < r < 1 \)

\[
\omega(r; z_0) = \sup_{|z_2 - z_1| \leq r} |f(z_1) - f(z_2)| \quad (|z_1| < 1, \ |z_2| < 1)
\]

and

\[
\omega(r) = \sup_{|z_0| = 1} \omega(r; z_0).
\]

We call \( \omega(r) \) the oscillation of \( f(z) \) at the boundary, and we are interested in obtaining estimates for \( \omega(r) \). Related questions were investigated by J. Wolff [16, pp. 217–218], J. Ferrand [3, pp. 150–154], and M. Lavrentieff [7](4). Their results, however, do not entail direct estimates for \( \omega(r) \). In order to obtain such estimates we shall introduce in the following a function \( \eta(\delta) \), associated with the boundary of \( R \), in terms of which we shall express our bounds for \( \omega(r) \). We shall establish two theorems (Theorems I and II), and in the proof of the first of these we shall make use of the above mentioned result by Wolff and Ferrand (Lemma 1).

(4) J. Ferrand (1942) proved the following: If \( z_1 \) and \( z_2 \) are points in \([z - z_0] < r, \ |z| < 1\), \( |z_0| = 1, \ 0 < r < 1 \), and if \( f(0) = w_0 \), then \( w_1 = f(z_1) \) and \( w_2 = f(z_2) \) may be separated from \( w_0 \) by a cross-cut in \( R \) of length \( l \leq (2\pi A / \log (1/r))^{1/3} \), where \( A \) is the area of \( R \). The earlier result of J. Wolff (1934) is similar but not stated quite in this way (see Lemma 1 below). Lavrentieff introduced, for \( w_1, \ w_2 \) in \( R \), a “distance \( \rho[w_1, \ w_2] \) with respect to \( R \) and \( w_0 \)” \( \rho = f(z_0) \) and showed that for \( |z_1| < 1, \ |z_2| < 1, \ \rho[f(z_1); f(z_2)] \leq \text{const.}/(\log |z_1 - z_2|)^{1/3} \). The “distance” \( \rho[w_1, \ w_2] \) is defined as \( \min(\rho_1, \ rho_2) \) where \( \rho_1 \) is the greatest lower bound of the lengths of all arcs in \( R \), which connect \( w_1 \) and \( w_2 \), and \( \rho_2 \) is the greatest lower bound of the lengths of all cross-cuts of \( R \) which separate \( w_1 \) and \( w_2 \) from \( w_0 \).
Several estimates have been given in the literature for the oscillation of the inverse function, that is, the function which maps $R$ onto $|z| < 1$. We shall make use of one of these results (see Theorem III).

2.1. Estimate of $\omega(r)$ in the general case. We begin with the following definition.

**Definition.** Suppose $R$ is a simply connected bounded region which contains the origin $O$. Let $c$ be a cross-cut of $R$ which does not pass through $O$, and let $T$ be the one of the two subregions of $R$ which does not contain $O$. Denote by $\lambda$ the diameter of $c$ and by $\Lambda$ the diameter of $T$. For any $\delta > 0$ consider all possible cross-cuts $c$ of $R$ with $\lambda \leq \delta$ and define

$$\eta(\delta) = \sup_{|\lambda| \leq \delta} \Lambda.$$

The function $\eta(\delta)$ is in a certain sense a measure for the “irregularity” of the boundary of $R$. If the boundary of $R$ is a simple closed curve, then it is easily seen that $\lim_{\delta \to 0} \eta(\delta) = 0$. The converse, however, is not necessarily true (consider, for example, the region obtained by removing one radius from the interior of the unit circle). We shall call $\eta(\delta)$ the *structure modulus of the boundary of $R$.*

**Theorem I.** Suppose that $R$ is a simply connected bounded region which contains the origin and that $w = f(z)$ maps the circle $|z| < 1$ conformally onto $R$ such that $f(0) = 0$. If $A$ denotes the area of $R$, then the oscillation of $f(z)$ at the boundary,

$$\omega(r) \leq \eta \left( \left( \frac{2\pi A}{\log 1/r} \right)^{1/2} \right), \quad 0 < r < 1.$$

This theorem is easily proved by means of the following lemma.

**Lemma 1.** Suppose that the circle $|z| < 1$ is mapped conformally onto a simply connected region $R$ of finite area $A$. Let $z_0$ be a point on $|z| = 1$ and $k$, the part of the circle $|z - z_0| = r$ which is contained in $|z| < 1$. Then for every $r$, $0 < r < 1$, there exists a $\rho_1, r \leq \rho_1 \leq r^{1/2}$, such that the image of $k_\rho$ is a cross-cut of $R$ of length

$$l_\rho \leq \left( \frac{2\pi A}{\log 1/r} \right)^{1/2}.$$

**Proof.** Let $w = f(z)$ map $|z| < 1$ conformally onto $R$. We introduce polar coordinates about $z_0$ and write, for $0 < \rho < 1$,

$$l_\rho = \int_{k_\rho} |f'(z)| dz = \int_{k_\rho} \left| f'(z_0 + \rho e^{i\theta}) \right| \rho d\theta.$$
Here \( l_0 \leq +\infty \). By the inequality of Schwarz:

\[
(2.13) \quad l^2 \leq \int_{k_p} \int_{k_p} |f'(z_0 + pe^{i\theta})|^2 p d\theta \leq \pi \int_{k_p} |f'(z_0 + pe^{i\theta})|^2 p d\theta.
\]

Let \( 0 < r < 1 \). Integrating with respect to \( \rho \) from \( r \) to \( r^{1/2} \) we obtain

\[
\int_r^{r^{1/2}} \frac{l^2}{\rho} d\rho < \int_0^{r^{1/2}} \frac{l^2}{\rho} d\rho \leq \pi \int_{k_p}^{r^{1/2}} \int_{k_p} |f'(z_0 + pe^{i\theta})|^2 p d\theta < \pi A.
\]

Hence there exists a \( r_1, r_1 \leq r_1 \leq r^{1/2} \), such that

\[
\frac{l^2}{r_1} \int_r^{r_1} \frac{dp}{p} = \frac{1}{2} \frac{l^2}{r_1} \log \frac{1}{r} < \pi A.
\]

Since the image of \( k_{p_1} \) has finite length \( l_{p_1} \), it is easily seen that it forms a cross-cut of \( R \).

**Proof of Theorem 1.** Let \( T_r \) denote the image of the region \( \{ |z - z_0| < r, |z| < 1 \} \) in the \( w \)-plane, and let \( p_1 \) be determined by Lemma 1 so that (2.12) holds, \( r \leq r_1 \leq r^{1/2} \). Then \( T_r \) is contained in \( T_{p_1} \). Now \( k_{p_1} \) (Lemma 1) is mapped onto cross-cut \( c_{p_1} \) of \( R \) whose diameter is not greater than \( l_{p_1} \). Since \( T_{p_1} \) does not contain the origin, it follows from the definition of \( \eta(\delta) \) that the diameter of \( T_{p_1} \) and hence that of \( T_r \) does not exceed \( \eta(l_{p_1}) \leq \eta((2\pi A/\log (1/r))^{1/2}) \).

Now if \( z_1 \) and \( z_2 \) are points of \( |z| < 1 \) which are in \( |z - z_0| < r \), then \( f(z_1) \) and \( f(z_2) \) are in \( T_r \), and therefore \( |f(z_1) - f(z_2)| \leq \eta(l_{p_1}) \). This proves (2.11).

2.2. Estimates of \( \omega(r) \) for linear \( \eta(\delta) \). A more accurate estimate for \( \omega(r) \) may be obtained if some information regarding the order of magnitude of \( \eta(\delta) \) is available. A particularly simple form is

\[
(2.21) \quad \eta(\delta) \leq \kappa \delta + \eta_0 \quad \text{for} \quad \delta \leq \delta_0,
\]

for some \( \delta_0 > 0 \); here \( \kappa \) and \( \eta_0 \) are constants(7), \( \kappa > 0, \eta_0 \geq 0 \). For this case we prove the following theorem.

**Theorem II.** Suppose that \( R, w = f(z), \) and \( \omega(r) \) are defined as in Theorem 1. Suppose furthermore that the structure modulus of the boundary of \( R \) satisfies (2.21) and that \( D \) is the diameter of \( R \). Then for every \( \mu \geq 1 \),

\[
(2.22) \quad \omega(r) \leq mr^\alpha + \left(1 + \frac{2}{\mu^{1/2}}\right) \eta_0 \quad \text{for} \quad \alpha = \frac{2}{\pi^2 \kappa^2},
\]

where \( m \) is a constant which depends only on \( \kappa, \mu, \delta_0, \) and \( D \). In fact, if \( h = \min(\delta_0, D/\kappa) \) and \( \rho_0 = \exp \left[-\pi^2 D^2/2h^2\right] \), one may take

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(7) The case \( \eta_0 = 0 \) occurs, for example, when the boundary of \( R \) consists of a finite number of Jordan arcs which possess continuously turning tangents and which form a finite number of corners (that is, angles different from 0 and \( 2\pi \)), such as the interior of the unit circle with a finite number of slits: \( \arg w = \delta_i, 1/2 \leq |w| \leq 1, i = 1, 2, \ldots, n \).
\[ m = \begin{cases} 
2he\mu_0^{a}(\frac{e}{\mu})^{1/2}, & \text{when } 0 < r \leq r_0 = \rho_0 \exp \left[ -\frac{\pi^2k^2}{4}\mu \right] \\
D\mu_0^{a}(\frac{e}{\mu})^{1/2}, & \text{when } r_0 < r < 1^{(9)}.
\end{cases} \]

Remarks. 1. The value of \( m \) is least when \( \mu = 1 \).
2. One has, from (2.22), \( \lim_{r \to 0} \omega(r) \leq (1 + 2/\mu^{1/2})\eta_0 \) and since \( \mu \) is arbitrary one may let \( \mu \to \infty \) and one obtains \( \lim_{r \to 0} \omega(r) \leq \eta_0 \).

In the proof of this theorem we shall use the following lemma.

Lemma 2. Suppose that the hypotheses of Theorem II are satisfied. Let \( z_0 \) be a fixed point, \( |z_0| = 1 \), and let \( T_\rho \) denote the image of the region \( \{ |z - z_0| < \rho, |z| < 1 \} \) under the transformation \( w = f(z) \). Let \( A_\rho \) be the diameter of \( T_\rho \) and \( l_\rho \) the length of the image of the circular arc \( k_\rho : \{ |z - z_0| = \rho, |z| < 1 \} \). Then for \( \rho \leq \rho_0 = \exp \left[ -\pi^2D^2/2\delta_0^2 \right] \)
\[ (2.23) A_\rho \leq kl_\rho + \eta_0^{(9)}. \]

Proof. If \( l_\rho \leq \delta_0 \), then (2.23) follows from the hypothesis (2.21). If, however, \( l_\rho > \delta_0 \), then an application of Lemma 1 shows that for every \( \rho \) there exists a \( \rho_1 \), such that \( \rho \leq \rho_1 \leq \rho^{1/2} \) and \( (A_\rho \text{ is the area of } R) \)
\[ l_\rho \leq \left( \frac{2\pi A}{\log 1/\rho} \right)^{1/2} \leq \left( \frac{\pi^2D^2}{2\log 1/\rho_0} \right)^{1/2} = \delta_0. \]

Hence
\[ A_\rho \leq kl_\rho + \eta_0. \]

But since \( T_\rho \subset T_{\rho_1} \), we have \( A_\rho \leq A_{\rho_1} \) and on the other hand \( l_{\rho_1} \leq \delta_0 < l_\rho \). Hence
\[ A_\rho \leq A_{\rho_1} \leq kl_{\rho_1} + \eta_0 \leq kl_\rho + \eta_0. \]

Proof of Theorem II. We use the same notation as in Lemma 1, and repeat the argument leading to inequality (2.13), or
\[ \frac{l_\rho^2}{\rho} \leq \pi \int_{k_\rho} |f'(z)|^2\rho d\theta \quad (z = z_0 + \rho e^{i\theta}). \]

Let
\[ g(\rho) = \rho \int_0^\infty \frac{l_\rho^2}{\rho} \rho d\rho. \]

Then we have

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\(^{(9)} \) By the method of Theorem II one may obtain bounds for \( \omega(r) \) (better than those resulting from Theorem I) for other choices of \( \eta(\delta) \), for example, \( \eta(\delta) = \delta^\beta, 0 < \beta < 1. \)

\(^{(9)} \) If \( \delta_0 \geq D \), then we may let \( \delta_0 \to \infty \) (since \( \eta(\delta) = \eta(D) \) for \( \delta \geq D \)) and take \( \rho_0 = 1 \). Then Lemma 2 is trivial and the proof of Theorem II is somewhat simplified.
\[ g(x) \leq \pi \int_0^x \int_{k\rho} \left| f'(z) \right|^2 \rho d\theta dp. \]

The integral on the right (apart from the factor \( \pi \)) represents the area of \( T_x \), the image of the region \( \{ |z - z_0| < x, |z| < 1 \} \) in the \( w \)-plane, and it does not exceed therefore the value \( \pi \left( \frac{\Lambda_x}{2} \right)^2 \), where \( \Lambda_x \) is the diameter of \( T_x \). For \( x \leq \rho_0 \) we have, by Lemma 2, \( \Lambda_x \leq \kappa \rho_x + \eta_0 \), and therefore

\[ g(x) \leq \frac{\pi^2}{4} (\kappa \rho_x + \eta_0)^2, \quad x \leq \rho_0. \]

The function \( g(x) \) is positive, monotone increasing, and continuous for \( 0 \leq x \leq 1 \), and, for almost all these \( x \): \( \int_0^x = xg'(x) \). Hence

\[ g(x) \leq \frac{\pi^2}{4} (\kappa \rho(x))^{1/2} + \eta_0)^2. \]

If we set

\[ \kappa \eta_1 = \eta_0, \quad \gamma = \frac{2}{\pi \kappa}, \]

then we may write this inequality in the form

\[ \gamma (g(x))^{1/2} \leq (xg'(x))^{1/2} + \eta_1. \]

Suppose that there exists a number \( \xi, 0 \leq \xi < \rho_0 \), such that

\[ \gamma (g(x))^{1/2} \leq 2 \eta_1 \quad \text{for } x \geq \xi \] (\( \xi \) is the least such number).

Then we have, from (2.25), \( (\gamma (g(x))^{1/2} - \eta_1)^2 \leq xg'(x), x \geq \xi \), or

\[ \frac{1}{x} \geq \frac{g'(x)}{(\gamma (g(x))^{1/2} - \eta_1)^2}. \]

Integration over the interval \( \rho \leq x \leq \rho_0 \), where \( \rho \geq \xi \), yields

\[ \log \frac{\rho_0}{\rho} \leq \int_\rho^{\rho_0} \frac{g'(x)dx}{(\gamma (g(x))^{1/2} - \eta_1)^2} \]

or

\[ \frac{\gamma^2}{2} \log \frac{\rho_0}{\rho} \leq \log \frac{(g(\rho_0))^{1/2} - \eta_1}{\gamma (g(\rho))^{1/2} - \eta_1} + \eta_1 \left[ \frac{1}{(\gamma (g(\rho))^{1/2} - \eta_1) \gamma (g(\rho))^{1/2} - \eta_1} \right]. \]

Because of (2.26), the second term on the right does not exceed 1. Hence

\[ \left( \frac{\rho_0}{\rho} \right)^{\gamma/2} \leq \frac{\gamma (g(\rho_0))^{1/2} - \eta_1}{\gamma (g(\rho))^{1/2} - \eta_1} e. \]
The assumption (2.26) implies that

\[(2.28) \quad \gamma(g(\rho))^{1/2} - \eta_1 \geq \frac{\gamma}{2} (g(\rho))^{1/2}, \quad \rho \geq \xi.\]

Furthermore, by Lemma 2,

\[\frac{1}{2} \Lambda_{\rho_0} \leq (g(\rho_0))^{1/2} \leq \frac{1}{\gamma} (l_{\rho_0} + \eta_1).\]

If \(l_{\rho_0} \leq \delta_0\), we have

\[(2.29) \quad \gamma(g(\rho_0))^{1/2} - \eta_1 \leq \delta_0.\]

However, if \(l_{\rho_0} > \delta_0\), then there exists, by Lemma 1, a number \(\rho_1 \geq \rho_0\) such that \(l_{\rho_1} \leq \delta_0\). Then we have

\[\frac{1}{2} \Lambda_{\rho_1} \leq (g(\rho_1))^{1/2} \leq \frac{1}{\gamma} (l_{\rho_1} + \eta_1) \leq \frac{1}{\gamma} \delta_0 \leq \frac{1}{\gamma} \delta_0 + \eta_1,\]

by hypothesis (2.23), and (2.29) is again true.

Thus we find from (2.27) by use of (2.28) and (2.29)

\[(2.210) \quad \gamma(g(\rho))^{1/2} \leq 2\delta_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma/2}, \quad \xi \leq \rho \leq \rho_0.\]

We also note: it follows from (2.27), (2.28), and the inequality \(\gamma(g(\rho_0))^{1/2} \leq \gamma \cdot (\pi/2)D = D/\kappa\) that we may replace, in (2.210), \(2\delta_0\) by \(2\delta_0\delta_0\). Thus, we may write, in any case, \(2\kappa h\) in place of \(2\delta_0\), where \(h = \min (\delta_0, D/\kappa)\).—The inequality (2.210) presupposes \(\rho \geq \xi\). If \(\rho < \xi\) then

\[(2.211) \quad (g(\rho))^{1/2} \leq \frac{2\eta_1}{\gamma}.\]

Hence for every \(\rho\), \(0 < \rho \leq \rho_0\),

\[(2.212) \quad (g(\rho))^{1/2} \leq \frac{2\kappa h}{\gamma} \left(\frac{\rho}{\rho_0}\right)^{\gamma/2} + \frac{2\eta_1}{\gamma}.\]

This inequality has been derived under the assumption that there exists a \(\xi\) such that (2.26) holds. However, if there is no such \(\xi\) then (2.211) is true for all \(\rho\), \(0 < \rho \leq \rho_0\), and the inequality (2.212) is again valid for all these \(\rho\).

Now let \(\rho > 1\). Then

\[\int_{\rho/p}^\rho 2x^2 \, dx < g(\rho)\]

and hence, for a suitable \(\rho_1\), \(\rho/p \leq \rho_1 \leq \rho\),
We obtain therefore from (2.212), substituting the values of $\gamma$ and $\eta_1$ from (2.24),

$$l_{p_1} \leq \frac{e h \pi}{(\log p)^{1/2}} \left( \frac{p}{p_0} \right)^{1/2} + \frac{\pi \eta_0}{(\log p)^{1/2}}.$$ 

We write now $r = p/p_0$ and take $p = e^{\mu r^2}$ where $\mu$ is not less than 1 and a constant. Then

$$l_{p_1} \leq 2 e h \left( \frac{e^{\mu r^2}}{\mu} \right)^{1/2} \left( \frac{r}{p_0} \right)^{1/2} + \frac{2 \eta_0}{\kappa \mu^{1/2}}.$$ 

Since $\rho_1 \leq \rho_0$ we have, by Lemma 2, $\Delta_{p_1} \leq k \rho_1 + \eta_0$. Furthermore, since $r \leq \rho_1$, it follows that $\omega(r) \leq \Delta_r \leq \Delta_{p_1}$. Thus, for $r \leq \rho_0 e^{-\mu r^2}$,

$$\omega(r) \leq k \rho_1 + \eta_0 \leq 2 e h \left( \frac{e^{\mu r^2}}{\mu} \right)^{1/2} \left( \frac{r}{p_0} \right)^{1/2} + \eta_0 \left( 1 + \frac{2}{\mu^{1/2}} \right).$$

This proves (2.22) with the value of $m$ as given for $r \leq r_0$. If $r > r_0$ the value of $m$ is so chosen that $m r_0 \geq D$ and (2.22) is obviously true.

2.3. Oscillation of the inverse mapping function. In §5 we shall need an estimate for the oscillation of the inverse mapping function. We shall use the following theorem\(^{(19)}\).

**Theorem III.** Suppose that $R$ is a simply connected bounded region which contains the origin $O$ and that $\sigma$ is the distance of $O$ from the boundary $B$ of $R$. Let $z = \phi(w)$ map $R$ conformally onto the circle $|z| < 1$ so that $\phi(0) = 0$. If $w_1$ and $w_2$ are points in $R$ which are separated from $O$ by a circular cross-cut of $R$ of radius $r$, $0 < r < \sigma/2$, whose center lies on $B$, then

$$|\phi(w_1) - \phi(w_2)| \leq M r^{1/2}.$$ 

Here $M$ is a constant which depends only on $\sigma$ and the diameter of $R$.

3. Application to nearly circular regions. In 1935 A. R. Marchenko \(^{10}\) established the following theorem:

Suppose $C$ is a closed Jordan curve which lies in the ring $1 \leq |w| \leq 1 + \epsilon$ for some $\epsilon$, $0 < \epsilon < 1$. Consider any arc of $C$ which subtends a chord whose length does not exceed $\epsilon$. Let $\lambda$ be the least upper bound of the diameters of all such arcs, whereby, in each case, the arc with the smaller diameter is chosen. If $w = f(z)$ maps the circle $|z| < 1$ conformally onto the interior of $C$ such that $f(0) = 0$ and $f'(0) > 0$, then there exist two absolute constants $K$ and $K_1$ such that for $|z| \leq 1$:

\(^{(19)}\) See J. Ferrand [3 (1942) pp. 166-171]. A result which contains this theorem was given earlier (1936) by Lavrientieff [7, formula (a)].
This inequality is the best possible as to the order of magnitude(11) in $\epsilon$ and in $\lambda$.

As an application of Theorem II we shall extend this theorem to the case of a "nearly circular" region whose boundary is not necessarily a Jordan curve.

3.1. Statement of the result. We shall prove the following

**Theorem IV.** Suppose $R$ is a simply connected region which contains the origin and whose boundary is contained in the ring

$$1 < |w| < 1 + \epsilon$$

for some $\epsilon$, $0 < \epsilon < \log(8/\pi)$. Let $\lambda$ be a number with the following property: Any two points in $R$ whose distance is less than $\epsilon$ may be connected by an arc in $R$ whose diameter does not exceed $\lambda$. If $w = f(z)$ maps the circle $|z| < 1$ conformally onto $R$ such that $f(0) = 0$, $f'(0) > 0$, then

$$|f(z) - z| \leq K \epsilon \log \frac{1}{\epsilon} + K_1 \lambda,$$

(3.11)

where $a = (e^{\pi})^{1/9}$ and $k(\epsilon)$ is bounded(12) for $0 < \epsilon < 1$.

Since $\epsilon \leq \lambda$ and $k(\epsilon)$ is bounded for $0 < \epsilon < 1$, the right-hand side of (3.11) is clearly of the form $K \epsilon \log (1/\epsilon) + K_1 \lambda$ (13).

3.2. Two lemmas. The following lemma will be used in the proofs of several theorems in this paper.

**Lemma 3.** Suppose that $H$ is a simply connected region which contains the origin and whose boundary lies in the ring

$$1 < |w| < 1 + \epsilon$$

if $h(z)$ maps $|z| < 1$ conformally onto $H$ such that $h(0) = 0$, $h'(0) > 0$, then for $|z| \leq \rho < 1$

$$|h(z) - z| \leq \rho \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \right) e^{\epsilon}.$$

(3.22)

(11) In [4], 1945, J. Ferrand announced a new proof of Marchenko's Theorem according to which $2/\pi$ is the best possible value for $K$.

(12) In fact $k(\epsilon) = 1 + \epsilon^2 + \epsilon^2 + \pi \epsilon(1 + \log 4 + 4(1 + \epsilon)^2 + \epsilon \log 1/\epsilon)$.

(13) The factor $\pi$ of $\epsilon \log (1/\epsilon)$ is larger than the value announced by J. Ferrand for Mar-chenko's theorem. However, the factor of $2\lambda$ is "asymptotically" (as $\epsilon \to 0$) the best possible in the sense that it approaches 1 when $\epsilon/\lambda \to 0$; simple examples show that it cannot be less than 1.
Proof. The function \( h(z)/z \) is regular for \( |z| < 1 \) if defined as \( f'(0) \) for \( z = 0 \). Since the boundary of \( H \) lies in (3.21),

\[
\frac{1}{1 + \epsilon} \leq \frac{|h(z)|}{z} \leq 1 + \epsilon \quad (|z| < 1)
\]

and therefore

\[
|\log \frac{h(z)}{z}| \leq \epsilon.
\]

The branch of \( \log (h(z)/z) = \log (h(z)/z) + i \arg (h(z)/z) \) for which \( \arg h'(0) = 0 \) is single-valued and regular for \( |z| < 1 \). Hence by an inequality of Carathéodory (see, for example, [2, p. 45, (76.3)]):

\[
\left| \log \frac{h(z)}{z} \right| \leq \epsilon \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \right) \quad \text{for } |z| \leq \rho < 1.
\]

Since for any complex \( a \)

\[
|e^a - 1| \leq |a| e^{|\Re a|},
\]

we have (taking \( a = \log (h(z)/z) \)) the inequality (3.22).

For the proof of Theorem IV we shall need one more lemma.

**Lemma 4.** If \( R \) is a region which satisfies the hypotheses of Theorem IV, then the structure modulus of the boundary of \( R \),

\[
\eta(\delta) \leq \delta + 2\lambda \quad \text{for } \delta < 1.
\]

**Proof.** Let \( c \) be a cross-cut of \( R \) whose diameter \( \delta < 1 \). Then \( c \) does not pass through the origin \( O \); \( c \) decomposes \( R \) into two subregions and we denote by \( T \) the one which does not contain \( O \). The arc \( c \) may or may not intersect the circle \( |w| = 1 \). If it does, let \( \gamma \) denote the set of all arcs of \( |w| = 1 \) which are in \( T \) and whose end points are points of \( c \). Since \( \delta < 1 \), the set \( \gamma \) is contained in an arc of the unit circle which is less than \( \pi \). Let \( \Gamma \) be the union of \( c \) and \( \gamma \), \( \Gamma = c + \gamma \). Then the diameter of \( \Gamma \) is equal to \( \delta \). To see this consider two points \( w \) and \( w' \) on \( \Gamma \). If both are on \( c \), then clearly \( |w - w'| \leq \delta \); if both are on \( \gamma \), then \( |w - w'| \) is smaller than the distance of suitable end points of the two arcs on which \( w \) and \( w' \) lie, and since these end points are on \( c \), we have again \( |w - w'| \leq \delta \). Suppose now that \( w \) is on \( c \), \( w' \) on \( \gamma \); let \( a, b \) be the end points of the arc of \( \gamma \) on which \( w' \) lies. Then the circle about \( w \) with radius \( r \) equal to the larger of the two numbers \( |w - a| \) and \( |w - b| \) must contain the (open) arc \( [ab] \) \((< \pi)\) of the unit circle, as is easily seen from the fact that \( r \leq \delta < 1 \) and the radius of \( [ab] \) is equal to one. Hence again \( |w - w'| \leq \delta \). Thus the diameter of \( \Gamma \) does not exceed \( \delta \); but since \( \Gamma \) contains \( c \) as subset, it must be equal to \( \delta \).
Now the proof of the lemma is easily completed. Let \( w_0 \) be a boundary point of \( T \) not on \( c \). Then there is, in any neighborhood of \( w_0 \) a point \( w_1 \) in \( T \) such that \( 1 < |w_1| < 1 + \epsilon \). Draw the radius of the unit circle through \( w_1 \) and denote by \( w_2 \) its intersection with the circle. Since \( |w_1 - w_2| < \epsilon \) it is possible, by hypothesis, to connect \( w_1 \) and \( w_2 \) by an arc \( \beta \) in \( R \) whose diameter does not exceed \( \lambda \). This arc may or may not intersect \( c \). If it does intersect \( c \) at a point \( w_3 \), say, then \( |w_1 - w_3| \leq \lambda \). If \( \beta \) does not intersect \( c \), then \( w_2 \) is a point of \( \gamma \), and \( |w_1 - w_2| \leq \epsilon \leq \lambda \). Thus, it is always possible to find a point \( \omega_0 \) of \( \Gamma \) (and \( \omega_0 \) is either \( w_2 \) or \( w_3 \)) such that

\[ |w_1 - \omega_0| \leq \lambda. \]

If \( \omega_0' \) is another boundary point of \( T \), not on \( c \), then there exists in every neighborhood of \( \omega_0' \) a point \( w_1 \) in \( T \) and a point \( \omega_0' \) on \( T \) such that

\[ |w_1 - \omega_0'| \leq |\omega_0 - \omega_0'| + 2\lambda \leq \delta + 2\lambda \]

and therefore also

\[ |w_0 - \omega_0'| \leq \delta + 2\lambda. \]

If \( \omega_0' \) is a point on \( c \), then

\[ |w_1 - \omega_0'| \leq |w_1 - \omega_0| + |\omega_0 - \omega_0'| \leq \lambda + \delta, \]

and hence

\[ |w_0 - \omega_0'| \leq \lambda + \delta. \]

Finally, if \( \omega_0 \) and \( \omega_0' \) are points of \( c \) then clearly \( |w_0 - \omega_0'| \leq \delta \). Thus, the diameter of the boundary of \( T \) and hence that of \( T \) itself does not exceed \( \delta + 2\lambda \). This completes the proof.

3.3. Proof of Theorem IV. We apply Theorem II to the function \( w = f(z) \) of Theorem IV. By Lemma 4, we may take \( k = 1, \delta = 1, \eta = 2\lambda \); furthermore \( D \leq 2(1 + \epsilon), \alpha = 2/\pi^2 \). Since \( 2\epsilon > D \), we have for \( 0 < r < 1 \):

\[ (3.31) \quad m \leq 2e^{4(1+\epsilon)^2(\epsilon^2)^{1/2}} = 2 \exp \left[ 1 + 4(1 + \epsilon)^2 + \frac{\mu}{2} \right]. \]

Let \( z = \rho e^{i\theta}, z_1 = \rho_1 e^{i\theta}, r = 1 - \rho_1 \). We choose \( r \) so that \( mr \lambda = \epsilon \), or

\[ (3.32) \quad r = 1 - \rho_1 = \left( \frac{\epsilon}{m} \right)^{1/\alpha} < \epsilon^\epsilon. \]

Then we have by Theorem II, if \( \rho_1 \leq \rho < 1 \):

\[ (3.33) \quad |f(z) - f(z_1)| \leq \epsilon + \left( 1 + \frac{2}{\mu^{1/2}} \right) 2\lambda. \]

Now
Here \(|z - z_1| \leq 1 - \rho_1\). Furthermore, by Lemma 3,

\[
\left| f(z_1) - z_1 \right| \leq \varepsilon \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} \right) \leq \varepsilon \left( 1 + \frac{2}{\pi} \frac{2m^{1/\alpha}}{e^{1/\alpha}} \right).
\]

Since \(\varepsilon \leq 1 + \varepsilon^t\) we have, using (3.31),

\[
\left| f(z_1) - z_1 \right| \leq \varepsilon \pi \log \frac{1}{\varepsilon} + \varepsilon^t \pi \log \frac{1}{\varepsilon} + \varepsilon^t \pi \log (2m) + \varepsilon^t \pi \log \frac{1}{\varepsilon}.
\]

Thus we find from (3.34) by use of (3.33), (3.35), and (3.32)

\[
\left| f(z) - z \right| \leq \varepsilon \pi \log \frac{1}{\varepsilon} + \varepsilon^t \pi \log \frac{1}{\varepsilon} + \left( 1 + \frac{2}{\mu^{1/\alpha}} \right) 2\lambda,
\]

where \(k(\varepsilon) = 1 + \varepsilon^2 + \varepsilon^t + \pi \varepsilon^t (1 + \log 4 + 4(1 + \varepsilon)^2 + \varepsilon \log (1/\varepsilon)).\) The sum of the two terms involving \(\mu\) will be least if \(\varepsilon^t \pi \mu / 2 = (2/\mu^{1/\alpha}) 2\lambda\), which gives \(\mu = ((4/\varepsilon^t \pi) \cdot (2\lambda/\varepsilon))^{2/3} \geq 1.\) Substituting this value for \(\mu\) into the right-hand side of (3.36) we obtain (3.11). This proves the theorem for \(\rho \leq |z| < 1\), and by the principle of the maximum modulus it is therefore true for \(|z| < 1\).

4. Arbitrary regions. We consider now the general case in which the mapping functions of two arbitrary regions are compared with each other.

4.1. Statement and discussion of results. Let \(R_1\) and \(R_2\) be two simply connected bounded regions and let \(B_1\) and \(B_2\) denote their boundaries. We define first the “inner distance” \(D_\epsilon(B_1, B_2)\) of \(B_1\) and \(B_2\): Let \(P\) be a point of \(B_1\) which lies in \(R_2\) and let \(d(P, B_2)\) denote the (shortest) distance of \(P\) from \(B_2\). Then we set \(d_1 = \max_{P \in B_1 \cap R_2} d(P, B_2)\) Similarly, let \(Q\) be any point on \(B_2\), which lies in \(R_1\), \(d(Q, B_1)\) the (shortest) distance of \(Q\) from \(B_1\) and \(d_2 = \max_{Q \in B_2} d(Q, B_2)\). (If \(B_2\) is contained in \(R_1\), then \(d_2 = 0\), and similarly if \(B_1\) lies in \(R_3\), then \(d_1 = 0\).) We define now(14)

\[
D_\epsilon(B_1, B_2) = D_\epsilon(B_2, B_1) = \max(d_1, d_2).
\]

Using this definition we state first the following theorem:

**Theorem V.** Suppose \(R_1\) and \(R_2\) are simply connected bounded regions which contain the origin. Suppose furthermore that the inner distance of their bound-

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(14) For this definition see [12]. It should be noted that \(D_\epsilon(B_1, B_2)\) does not necessarily satisfy the “triangle inequality.”
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aries, $D_i(B_1, B_2) < \epsilon$ for some $\epsilon$, $0 < \epsilon < 1$, $0 < \epsilon < \sigma/64$, where $\sigma$ is the distance of $O$ from $B_1$ and $B_2$. Let $\eta_1(\delta)$ and $\eta_2(\delta)$ denote the structure moduli of $B_1$ and $B_2$, respectively.

If $w = f_1(z)$ and $w = f_2(z)$ map the circle $|z| < 1$ conformally onto $R_1$ and $R_2$, respectively, such that $f_1(0) = f_2(0) = 0$, $f_1'(0) > 0$, $f_2'(0) > 0$, then for $|z| < 1$

$$|f_1(z) - f_2(z)| \leq \left(1 + \frac{k}{\sigma^{1/4}} e^{1/4} \log \frac{4}{\epsilon} \right) \left\{ \eta_1 \left( \frac{8\pi A_1}{(\log (1/\epsilon))^{1/2}} \right) + \eta_2 \left( \frac{8\pi A_2}{(\log (1/\epsilon))^{1/2}} \right) \right\}. \tag{4.11}$$

Here $A_1$ and $A_2$ are the areas of $R_1$ and $R_2$, respectively, and $k$ is an absolute constant ($\leq 16\epsilon$).

As a corollary one obtains at once the following result: Suppose $R_\infty$ ($n = 1, 2, \ldots$) and $R_0$ are simply connected bounded regions, all of which contain the origin $O$. Denote by $B_n$ the boundary of $R_n$ ($n = 0, 1, 2, \ldots$). Suppose that (i) $D_i(B_0, B_\infty) \to 0$ as $n \to \infty$; (ii) the structure moduli of $B_n$, $\eta_n(\delta) \to 0$ as $\delta \to 0$, uniformly for all $n = 0, 1, 2, \ldots$; (iii) the areas $A_n$ of $R_n$ are uniformly bounded. If $w = f_n(z)$, normalized by the conditions $f_n(0) = 0$, $f_n'(0) > 0$, maps $|z| < 1$ conformally onto $R_n$, then

$$f_n(z) \to f_0(z) \quad \text{as } n \to \infty, \text{ uniformly in } |z| \leq 1. \tag{4.12}$$

If the $B_n$ ($n = 0, 1, 2, \ldots$) are closed Jordan curves, these three conditions are also necessary for (4.12).

We must prove the necessity. Suppose that $w_n(t)$, $0 \leq t \leq b$, are parametric representations of $B_n$ ($n = 0, 1, 2, \ldots$) and that

$$w_n(t) \to w(t) \quad \text{as } n \to \infty, \text{ uniformly for } 0 \leq t \leq b \tag{4.13}$$

(for example, $w_n(t) = f_n(e^{it})$, $0 \leq t \leq 2\pi$). Then it is clear that conditions (i) and (iii) are satisfied. To prove (ii) we must show that for every $\epsilon > 0$ there exists a $\delta_0 > 0$ such that

$$\eta_n(\delta) < \epsilon \quad \text{if } 0 < \delta \leq \delta_0 \text{ for all } n = 0, 1, 2, \ldots.$$

Suppose this were not true. Then there would exist an $\epsilon_0 > 0$ and, for every $k = 1, 2, \cdots$, a curve $B_{nk}$ of the sequence with the following property: there is a cross-cut $c_{nk}$ of $R_{nk}$ (of the interiors of $B_{nk}$), whose diameter is less than $1/k$, such that the subregion $T_{nk}$ of $R_{nk}$ formed by $c_{nk}$ and $B_{nk}$ which does not contain the origin has the diameter

$$\Lambda_{nk} \geq \epsilon_0. \tag{4.14}$$

(15) A somewhat more general sufficient condition for (4.12) is obtained if the hypothesis (ii) is replaced by the following two assumptions: $\lim_{\delta \to 0} \eta_n(\delta) = 0$ and $\limsup_{\delta \to 0} \eta_n(\delta) \leq \overline{\eta}(\delta)$ where $\lim_{\delta \to 0} \overline{\eta}(\delta) = 0.
Suppose that the end points of $c_{nk}$ have the parameter values $t_k$ and $t_k'$, respectively, $0 \leq t_k < t_k' \leq b$. We may assume that

$$\lim_{k \to \infty} t_k = \tau, \quad \lim_{k \to \infty} t_k' = \tau' \quad (0 \leq \tau \leq \tau' \leq b)$$

exist. Hence

$$\lim_{k \to \infty} w_{nk}(t_k) = w_0(\tau), \quad \lim_{k \to \infty} w_{nk}(t_k') = w_0(\tau').$$

Since the diameter of $c_{nk}$ is less than $1/k$ and thus approaches 0 as $k \to \infty$, we have $w_0(\tau) = w_0(\tau')$, hence, $\tau = \tau'$ or $\tau' = \tau + b$, that is, $\tau = 0$, $\tau' = b$. By changing the origin of the $t$-scale we can avoid the second possibility. Because of (4.13), the arcs $\gamma_t : w = w_{nk}(t), t_k \leq t \leq t_k'$, will lie in any given neighborhood of $w_0(\tau)$ for sufficiently large $k$; the same is true of the arcs $c_{nk}$ and hence also of the subregions of $R_{nk}$ formed by $c_{nk}$ and $\gamma_{nk}$. For sufficiently large $k$, these subregions will not contain the origin, and hence will be the $T_{nk}$. Thus the diameter of $T_{nk}$, $\Lambda_{nk} \to 0$ as $k \to \infty$, contrary to (4.14).

Some of the known theorems on the convergence of the mapping functions of variable regions bounded by Jordan curves are easily derived from our corollary. We indicate this for T. Rado's theorem [13] (16), which states that a necessary and sufficient condition for (4.12) is that the Frechet distance $d_n$ between the boundary curves $B_n$ and $B_0$ approach 0 as $n \to \infty$. We need to show only the sufficiency. The assumption that $\lim_{n \to \infty} d_n = 0$ implies the existence of parametric representations $w_n(t)$ of $B_n$ such that (4.13) holds, and we have just shown that (4.13) implies the conditions (i), (ii), and (iii).

Next we state a result concerning a more restricted class of regions, for which we obtain a sharper estimate than (4.11).

Theorem VI. Suppose $R_1$ and $R_2$ are regions which satisfy the hypotheses of Theorem V. Suppose, furthermore, that the structure moduli of their boundaries $B_1$, $B_2$ satisfy the inequalities

$$\eta_1(\delta) \leq k\delta + \eta_1, \quad \eta_2(\delta) \leq k\delta + \eta_2, \quad \delta \leq \delta_0,$$

for some $\delta_0$, where $k$, $\eta_1$, and $\eta_2$ are constants, $k > 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$. If $f_1(z)$ and $f_2(z)$ are defined as in Theorem V, then for $|z| < 1$:

$$|f_1(z) - f_2(z)| \leq K \left( e^{1/2 \log \frac{4}{\epsilon}} + K_1(\eta_1 + \eta_2), \quad \alpha = \frac{2}{\pi^2 n^2},

$$

where $K$ and $K_1$ are constants; $K$ depends only on $\delta_0$, $\kappa$, the larger of the diameters of $B_1$ and $B_2$, and the minimal distance $\sigma$ of $O$ from $B_1$ and $B_2$; $K_1$ depends only on $\sigma$.

The proofs of Theorems V and VI will be given in sections 4.3 and 4.4;
section 4.2 contains lemmas used in this proof.

4.2. Two lemmas. We shall use the following lemma which is an immediate consequence of a lemma due to G. Szegő [14, p. 191, (11)].

**Lemma 5.** Suppose that the function \( z = \phi(w) \) maps the simply connected region \( R \) conformally onto the circle \( |z| < 1 \) so that \( \phi(0) = 0 \). If \( w \) is a point in \( R \) whose distance from the boundary of \( R \) is less than \( \epsilon \), then

\[
1 - |\phi(w)| \leq 4(\epsilon |\phi'(0)|)^{1/2},
\]

that is, the image \( \phi(w) \) lies in the ring

\[
1 - 4(\epsilon |\phi'(0)|)^{1/2} \leq |z| < 1.
\]

The following lemma has possibly some interest beyond its immediate use in the proofs of this section.

**Lemma 6.** Suppose \( R \) and \( T \) are two simply connected bounded regions; \( T \) is contained in \( R \) and contains the origin \( O \). Let \( \sigma \) be the distance of \( O \) from the boundary \( B \) of \( R \) and let every boundary point of \( T \) be within distance \( \epsilon \) from \( B \), where \( \epsilon < \sigma/64 \). Suppose that \( w = f(z) \) and \( w = g(z) \) map \( |z| < 1 \) conformally onto \( R \) and \( T \), respectively, so that \( f(0) = g(0) = 0 \) and \( f'(0) > 0 \), \( g'(0) > 0 \). Then for all \( |z| \leq \rho \), \( 1/2 < \rho < 1 \):

\[
|f(z) - g(z)| \leq 4\epsilon \left( \frac{\epsilon}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \right) \frac{\omega(2(1 - \rho))}{1 - \rho},
\]

where \( \omega(r) \) is the oscillation of \( f(z) \) at the boundary.

**Proof.** Let \( \phi(w) \) denote the inverse function of \( f(z) \). Since \( R \) contains the circle \( |w| < \sigma \), we have

\[
0 < \phi'(0) \leq 1/\sigma.
\]

The function \( \xi = \phi(w) \) carries \( T \) into a subregion \( H \) of \( |\xi| < 1 \) which contains \( O \), and, by Lemma 5 and (4.22), the boundary of \( H \) lies in the ring

\[
1 - 4\left( \frac{\epsilon}{\sigma} \right)^{1/2} \leq |\xi| < 1.
\]

Let \( \xi = h(z) \) map \( |z| < 1 \) conformally onto \( H \) such that \( h(0) = 0 \), \( h'(0) > 0 \). We apply Lemma 3 to \( h(z) \) (since \( 4(\epsilon/\sigma)^{1/2} < 1/2 \) we notice that \( 1 - 4(\epsilon/\sigma)^{1/2} > 1/(1 + 8(\epsilon/\sigma)^{1/2}) \) and we may replace the \( \epsilon \) of Lemma 3 by \( 8(\epsilon/\sigma)^{1/2} \)). Thus we find:

\[
|h(z) - z| \leq 8\epsilon \left( \frac{\epsilon}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \right), \quad |z| \leq \rho.
\]
Now we have \( g(z) = f(h(z)) \) and therefore, for \( |z| = \rho < 1 \)

\[(4.25) \quad f(z) - g(z) = f(z) - f(h(z)) = \int_{h(z)}^{z} f'(t) \, dt, \]
the integration being taken along the straight line segment \( s \) from \( h(z) \) to \( z \).

Since, by the lemma of Schwarz, \( |h(z)| \leq |z| \), \( s \) lies in the circle \( |z| \leq \rho \). Hence

\[ \max_{t \in s} |f'(t)| \leq \max_{|t| \leq \rho} |f'(t)| = |f'(\zeta_1)| \]
for some \( \zeta_1 \) with \( |\zeta_1| = \rho \). Let \( r = 1 - \rho \). Then

\[ \pi r^2 \left| f'(\zeta_1) \right|^2 \leq \int_{0}^{r} \int_{0}^{2\pi} \left| f'(\zeta + Re^{i\theta}) \right|^2 R \, d\theta \, dR, \]
and the last integral represents the area of the image of the circle \( |z - \zeta_1| < r \) under the transformation \( w = f(z) \). If \( \zeta_1 = pe^{i\theta} \), then this circle is contained in the region \( \Delta : \{ |z - e^{i\theta}| < 2r, |z| < 1 \} \), and the double integral is smaller than the area of the image of \( \Delta \). Since the diameter of this image does not exceed \( \omega(2r) \), we obtain finally

\[ \pi r^2 \left| f'(\zeta_1) \right|^2 \leq \frac{\pi \omega^2(2r)}{4} \]
or

\[(4.26) \quad \left| f'(\zeta_1) \right| \leq \frac{\omega(2(1 - \rho))}{2(1 - \rho)}. \]

If we note from (4.25) that

\[ |f(z) - g(z)| \leq |h(z) - z| \left| f'(\zeta_1) \right|, \]
we find (4.21) from (4.24) and (4.26).

**Remark.** Suppose \( R \) and \( T \) are two regions as in Lemma 6: \( T \) is contained in \( R \) and contains the origin \( O \), and every boundary point of \( T \) is within a distance of \( \epsilon \) from the boundary \( B \) of \( R \). Suppose that any function \( z = \phi(w) \) which maps \( R \) onto \( |z| < 1 \) such that \( \phi(0) = 0 \) satisfies the following condition:

If \( w \) is a point in \( R \), \( w_0 \) a point on \( B \) nearest to \( R \), then

\[(4.27) \quad 1 - |\phi(w)| \leq K |w - w_0|^\gamma \]
where \( K \) and \( \gamma \) are constants, \( 0 < \gamma \leq 1 \). If \( f(z) \) and \( g(z) \) are defined as in the lemma, and if \( 2K\epsilon^r < 1 \), then for \( |z| \leq \rho \), \( 1/2 < \rho < 1 \),

\[ |f(z) - g(z)| \leq Ke^\epsilon \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \right) \frac{\omega(2(1 - \rho))}{1 - \rho}. \]
For the condition (4.27) implies that $\xi = \phi(w)$ carries $T$ into a region $H$
whose boundary lies in the ring

$$1 - Ke^{-\gamma} \leq |\xi| < 1,$$

and if this inequality is used in place of (4.23), the proof of this remark is
merely a repetition of that of Lemma 6.

4.3. Proof of V. Let $T$ be the largest subregion of the intersection of $R_1$
and $R_2$ which contains the origin $O$; $T$ is simply connected and every bound-
ary point of $T$ is within distance $e$ from the boundary of $R_1$ and of $R_2$. Let
$w = g(z)$, normalized by the condition $g(0) = 0$, $g'(0) > 0$, map the circle $|z| < 1$
conformally onto $T$. Then by Lemma 6, for $|z_i| \leq \rho$, $1/2 < \rho < 1$,

$$|f_k(z_i) - g(z_i)| \leq 4e \left( \frac{e}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho}{1 - \rho} \frac{\omega_k(2(1 - \rho))}{1 - \rho} \right)$$

where $\omega_k(r)$ denotes the boundary oscillation of $f_k(z)$.

Let $z_1$ be a fixed point, $|z_1| = \rho$, and let $z$ be such that $\rho < |z| < 1$, arg $z$
= arg $z_1$. Then, by Theorem I, for $r = 1 - \rho$,

$$|f_k(z) - f_k(z_1)| \leq \omega_k(r) \leq \eta_k \left( \frac{2\pi A_k}{\log (1/r)} \right)^{1/2}$$

Now

$$|f_1(z) - f_2(z)| \leq |f_1(z) - f_1(z_1)| + |f_1(z_1) - g(z_1)| + |g(z_1) - f_2(z_1)| + |f_2(z_1) - f_2(z)|.$$

By choosing $r = 1 - \rho = e^{1/4}/2$ and using the estimates (4.31) and (4.32), we
obtain easily the desired inequality (4.11) for all $z$ in $\rho \leq |z| < 1$. By the prin-
ciple of the maximum modulus it holds then also for all $|z| \leq \rho$.

4.4. Proof of VI. This proof differs from the preceding one only insofar
as Theorem II will be used in order to estimate $\omega_k(r)$ and the relation be-
tween $r = 1 - \rho$ and $e$ will be changed. If $g(z)$ has the same meaning as in §4.3
we obtain from (4.31) and Theorem II, applied with $\mu = 1$: For $|z_1| \leq \rho$

$$|f_k(z_1) - g(z_1)| \leq 8e \left( \frac{e}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{2}{r} \right) \frac{1}{2r} (m(2r)^a + 3\eta_k), \quad k = 1, 2,$$

where $m$ is a constant which depends only on $\delta_0$, $\kappa$, and the larger of the di-
ameters of $B_1$ and $B_2$. We choose now $2r = e^{1/2} \log (4/e) > e$. Then a short
calculation yields the inequality:

$$|f_k(z_1) - g(z_1)| \leq \frac{14e}{\sigma^{1/2}} \left[ m \left( e^{1/2} \log \frac{4}{e} \right)^a + 3\eta_k \right].$$
Let $z$ be a point in $\rho < |z| < 1$ such that $\arg z = \arg z_1$. Then, again by Theorem II,

\[(4.42) \quad |f_k(z) - f_k(z_1)| \leq m \eta_k + 3\eta_k = \frac{m}{2\alpha} \left( e^{1/2} \frac{4}{\epsilon} \right) \alpha + 3\eta_k.\]

Using (4.33) in conjunction with (4.41) and (4.42) we obtain the desired inequality (4.15) with $K = 2m(1+\epsilon/e^{1/2})$, $K_1 = 3(1+\epsilon/e^{1/2})$. It holds for $|z| \geq \rho$, and hence by the principle of the maximum modulus for all $|z| \leq \rho$.

5. Arbitrary regions: inverse mapping function. We consider now the analogous problem for the inverse function. As is to be expected the result obtained is sharper than the one for the direct mapping function.

**Theorem VII.** Suppose that $R$ and $S$ are two simply connected bounded regions such that $S \subset R$ and $w = 0$ lies in $S$. Let $\eta(\delta)$ denote the structure modulus of the boundary $B$ of $S$ and $\sigma$ the distance of $O$ from $B$. If $B$ is the boundary of $R$, suppose that $D(B, B_1) < \epsilon$, $0 < \epsilon < 1$, $\epsilon < \sigma/64$.

Let $z = \phi(w)$ and $z = \psi(w)$, normalized by the condition $\phi(0) = \psi(0) = 0$, $\phi'(0) > 0$, $\psi'(0) > 0$ map $R$ and $S$ conformally onto $|z| < 1$.

(a) If $\eta(\delta) \leq \kappa \delta$, then for $w \in S$:

\[(5.1) \quad |\phi(w) - \psi(w)| \leq K\epsilon^{1/2} \log \frac{2}{\epsilon},\]

where $K$ is a constant which depends only on $\kappa$, $\sigma$, and the diameter of $R$.

(b) In the general case, for $w \in S$,

\[(5.2) \quad |\phi(w) - \psi(w)| \leq L(\eta(\epsilon^{1/2}))^{1/2}\]

where $L$ is a constant which depends only on $\sigma$ and the diameter of $R$.

**Proof.** (i) Let $C_{\rho_0}$ denote the level curve $|\psi(w)| = \rho_0 < 1$ of $S$ where $D(C_{\rho_0}, B) < \epsilon$. Let $w_0$ be a point on $C_{\rho_0}$, $z_0 = \psi(w_0) = \rho_0 e^{i\theta_0}$. Let $z_1 = \rho_1 e^{i\theta_1}$, $0 < \rho_1 < \rho_0 < 1$. If $z_1 = \psi(w_1)$, then

\[(5.3) \quad |\psi(w_0) - \psi(w_1)| = \rho_0 - \rho_1 < 1 - \rho_1.\]

Since $z_0$ and $z_1$ lie in the region $\{ |z - e^{i\theta_0}| \leq 1 - \rho_1, |z| < 1 \}$ it follows that $w_0$ and $w_1$ are within a subregion $T$ of $S$ which does not contain $O$ and whose diameter is not greater than $\eta_1$, where in case (a), by Theorem II,

\[\eta_1 = m(1 - \rho_1)^{\alpha}, \quad (\alpha = \frac{2}{\pi^2 \kappa^2})\]

and in case (b), by Theorem I,

\[(5.4) \quad \eta_1 = \eta \left( \frac{2\pi A}{\log (1/(1 - \rho_1))} \right)^{1/2}.\]

Here $m$ is a constant which depends only on $\kappa$ and the diameter $D(\epsilon)$ of $R$, $A$ is the area of $S$. Let $w_2$ be a point on $B$ at a distance not greater than $\epsilon$
from \( w_0 \). We assume here at first that \( \eta_1 + \epsilon < \sigma \). Then there exists a subarc of the circle \( |w - w_2| = \eta_1 + \epsilon \), which forms a cross-cut \( c \) of \( R \) and separates \( w_0 \) and \( w_1 \) from \( w = 0 \). By Theorem III we have, therefore:

\[
(5.5) \quad |\phi(w_0) - \phi(w_1)| \leq M(\epsilon + \eta_1)^{1/2} \quad (M = M(\sigma, D)).
\]

(ii) The function \( \zeta = \phi(w) \) carries \( S \) into a subregion \( H \) of the unit circle which contains the origin. From Lemma 5 and the fact that \( \phi'(0) \leq 1/\sigma \) we infer that the boundary of \( H \) is contained in the ring

\[
1 - 4 \left( \frac{e}{\sigma} \right)^{1/2} < |\zeta| < 1.
\]

Let \( \zeta = h(z) \) map \( |z| < 1 \) conformally onto \( H \) so that \( h(0) = 0 \) and \( h'(0) > 0 \). Then by Lemma 3 (applied with the \( \epsilon \) of the lemma replaced by \( \epsilon(e/\sigma)^{1/2} \))

\[
|z_1 - h(z_1)| \leq 8\epsilon \left( \frac{\epsilon}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} \right), \quad |z_1| = \rho_1.
\]

It is easily seen that \( \phi(w_1) = h(z_1) \). Hence

\[
(5.6) \quad |\psi(w_1) - \phi(w_1)| = |z_1 - h(z_1)| \leq 8\epsilon \left( \frac{\epsilon}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} \right).
\]

(iii) Now the proof is easily completed. Using the inequality

\[
|\phi(w_0) - \psi(w_0)| \leq |\phi(w_0) - \phi(w_1)| + |\phi(w_1) - \psi(w_1)| + |\psi(w_1) - \psi(w_0)|
\]

we obtain from (5.5), (5.3), and (5.6)

\[
(5.7) \quad |\phi(w_0) - \psi(w_0)| \leq M(\epsilon + \eta_1)^{1/2} + (1 - \rho_1) + 8\epsilon \left( \frac{\epsilon}{\sigma} \right)^{1/2} \left( 1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} \right).
\]

In the case (a) we choose \( m(1 - \rho_1) = \epsilon \), so that \( \eta_1 + \epsilon = 2\epsilon < \sigma \), and a simple estimate leads to (5.1) for \( w \in \mathcal{C}_{\rho_0} \). In case (b) we choose \( \rho_1 \), so that \( 2\pi A / |\log (1 - \rho_1)| = \epsilon^{2\beta} \) for some \( \beta > 0 \). Then the right-hand side of (5.7) does not exceed

\[
M(\epsilon + \eta(\epsilon^\beta))^{1/2} + \frac{\epsilon^{2\beta}}{2\pi A} + \frac{L_1}{\sigma^{1/2}} \epsilon^{1/2-2\beta} \leq M(2\eta(\epsilon^\beta))^{1/2} + \frac{\epsilon^{2\beta}}{2\pi A^{\sigma}} + \frac{L_1}{\sigma^{1/2}} \epsilon^{1/2-2\beta},
\]

where \( L_1 \) depends only on the diameter of \( R \). Here we take \( \beta = 1/5 \) and obtain (5.2) for \( w \in \mathcal{C}_{\rho_0} \), provided \( \epsilon + \eta(\epsilon^{1/5}) < \sigma \). But if \( \epsilon + \eta(\epsilon^{1/5}) \geq \sigma \), then

\[ \text{By Theorem II, } m \text{ depends on the diameter } D_s \text{ of } S, \text{ but since } D_s \leq D, \text{ the value of } m \text{ is not decreased if } D_s \text{ is replaced by } D. \]
Since \( \rho_0 \) may be taken so that \( \rho_0 \) is arbitrarily close to 1, the theorem holds for all \( w \in S \), by the principle of the maximum modulus.

6. Regions with smooth boundaries. The estimate for degree of proximity of the mapping functions of two neighboring regions may be sharpened considerably if the boundaries of these regions are Jordan curves possessing continuously turning tangents.

**Theorem VIII.** Suppose that \( C_1 \) and \( C_2 \) are closed Jordan curves which contain the origin in their interiors and satisfy the hypotheses:

(a) \( C_k \) \((k = 1, 2)\) has continuously turning tangents, and the tangent angle \( \alpha_k(s) \), considered as function of the arc length, has the modulus of continuity \( \beta(t) \), that is,

\[
| \alpha_k(s \pm t) - \alpha_k(s) | \leq \beta(t),
\]

where \( \beta(t) \) is nondecreasing and \( \lim_{t \to 0} \beta(t) = 0 \).

(b) If \( w_1 \) and \( w_2 \) are points on \( C_k \) and \( \Delta s \) is the (shorter) arc of \( C_k \) between them, then there exists a constant \( a \) such that \( \Delta s/| w_1 - w_2 | \leq a \).

(c) The diameter of \( C_k \) does not exceed \( D \) and the distance of \( 0 \) from \( C_1 \) and \( C_2 \) is at least \( \sigma \).

(d) \( D(C_1, C_2) < \varepsilon \) for some \( \varepsilon > 0 \).

If \( w = f_k(z) \) maps the circle \( |z| < 1 \) conformally onto the interior \( R_k \) of \( C_k \) and if \( f_k(0) = 0, f_k'(0) > 0 \), then there exists for every \( \delta, 0 < \delta < 1 \), a constant \( M_\delta \) which depends only on \( \delta, a, \sigma, D \), and the function \( \beta(t) \)—and in no other way upon \( C_1 \) and \( C_2 \)—such that for \( |z| \leq 1 \)

\[
| f_k(z) - f_k(z) | \leq M_\delta e^{1-\delta}.
\]

**Proof.** Under the present assumptions on \( C_k \) there exists for every \( \theta \), \( 0 < \theta < 1 \), a \( B \), which depends only on \( \theta, a, \sigma, D \), and the function \( \beta(t) \), such that for \( |z_0| = 1, |z| \leq 1 \)

\[
\frac{1}{B} | z - z_0 |^{1+\theta} \leq | f_k(z) - f_k(z_0) | \leq B | z - z_0 |^{1-\theta}.
\]

Let \( z = \phi_k(w) \) be the inverse of \( w = f_k(z) \). Then, for any \( w \) in \( R_k \), \( w_0 \) on \( C_k \)

\[
1 - | \phi(w) | \leq | \phi(w_0) - \phi(w) | \leq B' | w - w_0 |^{1/(1+\theta)} \quad (B' = B^{1/(1+\theta)}).
\]

Now let \( T \) be the largest subregion of \( R_1 \cdot R_3 \) which contains \( 0 \). \( T \) is simply connected and every boundary point of \( T \) is within distance \( \varepsilon \) from \( C_1 \) and \( C_2 \). Let \( w = g(z) \), \( g(0) = 0 \), \( g'(0) > 0 \), map \( |z| < 1 \) conformally onto \( T \).

\(^{(\star)} \) These inequalities are well known, see for example [8, p. 1408]. In [15] it is shown that \( B \) depends only on the parameters indicated. Cf. also [6, p. 35 (VIII)].
We assume at first that $2B'e^{i(1+\theta)}<1$. Then by the remark following Lemma 6, we have, because of (6.3), for $|z_1| \leq \rho, 1/2 < \rho < 1$,

$$|f_k(z_1) - g(z_1)| \leq B'e^{i(1+\theta)}e \left(1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} \omega(2(1 - \rho_1)) \right).$$

By the right-hand inequality of (6.2), $\omega(r) \leq 2B'r^{1-\theta}$ and therefore:

$$|f_k(z_1) - g(z_1)| \leq A e^{i(1+\theta)} \left(1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} (1 - \rho_1)^{-\theta} \right)$$

where $A = 4BB'e$. We choose now $1 - \rho_1 = \epsilon$. Then we note first that

$$1 + \frac{2}{\pi} \log \frac{1 + \rho_1}{1 - \rho_1} < 2 \log \frac{\epsilon}{\epsilon} = 2\epsilon^{-\theta} \left(\epsilon^{\theta} \log \frac{\epsilon}{\epsilon} \right) \leq \frac{2\epsilon^{-\theta}}{\theta}.$$

Thus we obtain from (6.4)

$$|f_k(z_1) - g(z_1)| \leq \frac{2A}{\theta} \epsilon^{(1-2\theta-2\theta^2)/(1+\theta)}.$$

Let $|z_0| = 1, \arg z_0 = \arg z_1$. Then by (6.2)

$$|f_k(z_0) - f_k(z_1)| \leq B(1 - \rho_1)^{1-\theta} = Be^{1-\theta}.$$

Given $\delta$, choose $\theta$ so small that $(1 - 2\theta - 2\theta^2)/(1 + \theta) = \delta$. Then using (4.33) and applying (6.5) and (6.6), we obtain the desired result (6.1) for the case that $2B'e^{i(1+\theta)}<1$. If $\epsilon \geq (2B')^{-(1+\theta)}$, then we have trivially for $|z| \leq 1$

$$|f_1(z) - f_2(z)| \leq [2D(2B')^{1+\theta}]\epsilon.$$

**Bibliography**


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