COVERINGS WITH CONNECTED INTERSECTIONS

BY

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If \( G \) is a collection of subsets of a set, then a subintersection of \( G \) is a non-null set which is the common part of the elements of a subcollection of \( G \).

Suppose that a space \( X \) is a compact, locally connected, metric continuum. We show that \( X \) has a countable basis whose subintersections are connected and uniformly locally connected. In fact, there is a basis for \( X \) with the additional property that the collection of closures of elements of this basis is a family of continuous curves such that each subintersection of this family is a continuous curve. This extends a result of Anderson [1](1) showing that there is a sequence \( G_1, G_2, \ldots \) such that \( G_i \) is a finite \( 1/i \)-collection of continuous curves covering \( X \) and the subintersections of \( \sum G_i \) are locally connected.

The notion of partitioning [2, 3, 4] will be used in proving these results. A partitioning of \( X \) is a finite collection of mutually exclusive connected domains whose sum is dense in \( X \). The partitioning \( U \) is a brick partitioning if each of its elements is uniformly locally connected and equal to the interior of its closure while the interior of the closure of the sum of two adjacent elements of \( U \) is connected and uniformly locally connected. If each element of \( U \) is of diameter less than \( \epsilon \), \( U \) is an \( \epsilon \)-partitioning. In general, if each element of a collection is of diameter less than \( \epsilon \), the collection is called an \( \epsilon \)-collection.

The brick partitioning \( V \) is a core refinement of the brick partitioning \( U \) if (a) \( V \) is a refinement of \( U \), (b) for each pair of adjacent element \( u', u'' \) of \( U \) there is a pair of adjacent element \( v', v'' \) of \( V \) in \( u' \) and \( u'' \) respectively such that \( \bar{u}' + \bar{u}'' \) is a subset of the interior of \( \bar{u}' + \bar{u}'' \), and (c) for each element \( u \) of \( U \), the elements of \( V \) in \( u \) may be ordered \( v_0, v_1, \ldots, v_n \) such that \( v_0 \) intersects each \( \bar{u} \) while \( v_i \) intersects the boundary of \( u \) if and only if \( i > 0 \). We call \( v_0 \) a core element and \( v_1, v_2, \ldots, v_n \) border elements.

If \( B \) is a subset of \( X \) and \( G \) is a collection of subsets of \( X \), we use \( S(B, G) \) to denote the interior of the closure of the sum of the elements of \( G \) which have limit points on \( \bar{B} \).

We shall use the following result which was proved in [3].

\textbf{Theorem 1.} For each brick partitioning \( U \) of \( X \) and each positive number \( \epsilon \), there is a brick \( \epsilon \)-partitioning \( V \) of \( X \) which refines \( U \).

Although the following result is a corollary of Theorem 6, it is given here since its proof is much simpler than that of Theorem 6.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

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Theorem 2. The space $X$ has a countable basis whose subintersections are connected and uniformly locally connected.

Proof. Let $U_1, U_2, \ldots$ be a decreasing sequence of brick partitionings of $X$ ($U_{i+1}$ refines $U_i$ and the maximum of the diameters of elements of $U_i$ approaches 0 with $1/i$). The basis is the collection of sets $S(p, U_i)$ for $p \in X$ and $i = 1, 2, \ldots$.

If $G = \{g\}$ is an infinite subcollection of this basis, $\Pi g$ is either empty or a single point. The conclusion follows in this case. If $G$ is finite, there is a largest integer $n$ such that some $S(p, U_n)$ is an element of $G$. Any two elements of $U_n$ in $\Pi g$ are adjacent; moreover, the interior of the closure of their sum is connected, uniformly locally connected, and contained in $\Pi g$. Since $\Pi g$ is the interior of the sum of the closures of adjacent pairs of elements of $U_n$, it is connected and uniformly locally connected.

Theorem 3. For each brick partitioning $U$ of $X$ and each positive number $\epsilon$ there is a brick partitioning $V$ of $X$ such that $V$ is a core refinement of $U$ and each border element of $V$ is of diameter less than $\epsilon$.

Proof. Since $U$ is a brick partitioning of $X$, there is a positive number $\epsilon'$ less than $\epsilon$ such that the common boundary of each pair of adjacent elements of $U$ contains a point that is farther than $\epsilon'$ from any other element of $U$. By Theorem 1 there is a brick $\epsilon'$-partitioning $U'$ of $X$ which refines $U$. For each element $u$ of $U$, let $T_u$ be a dendron in $u$ which intersects each element of $U'$ in $u$. There is a brick $(\delta/2)$-partitioning $U''$ of $X$ which refines $U'$ where $\delta$ is less than the distance between any $T_u$ and the corresponding $X - u$. The core element $v_0$ of $V$ in $u$ is the set which is maximal with respect to being a connected domain containing $T_u$ and being the interior of the closure of the sum of some elements of $U''$ whose closures lie in $u$. The border elements of $V$ in $u$ are components of the intersection of elements of $U''$ with $u - v_0$.

Theorem 4. For each ordering $u_1, u_2, \ldots, u_n$ of the elements of the brick partitioning $U$ of $X$ and each positive number $\epsilon$ there is a brick partitioning $V$ of $X$ such that

1. $V$ is a core refinement of $U$;
2. each border element of $V$ has a diameter of less than $\epsilon$;
3. if $v_1, v_2$ are adjacent elements of $V$, one of the sets $S(v_1, U), S(v_2, U)$ contains the other;
4. if $v_1, v_2$ are adjacent elements of $V$ and the element of $U$ containing $v_1$ precedes in $u_1, u_2, \ldots, u_n$ the element of $U$ containing $v_2$, then $S(v_1, U)$ contains $S(v_2, U)$.

Proof. We first show that there is a brick partitioning $V$ satisfying (1), (2), and (3). Suppose $N$ is a fixed positive integer. Denote by Lemma $N$ the result obtained by replacing in Theorem 4 conditions (3) and (4) by
(3') if \(v_1, v_2\) are adjacent elements of \(V\), one of the sets \(S(v_1, U)\), \(S(v_2, U)\) contains the other if each contains \(N\) or more elements of \(U\).

Lemma \(N\) holds if \(N\) is greater than the number of elements in \(U\). We show that it holds for \(N = M\) if it holds for \(N = M + 1\). Induction then establishes Lemma \(N\); Lemma \(N\) for \(N = 1\) is Theorem 4 with condition (4) deleted.

Let \(U'\) be a brick partitioning of \(X\) satisfying the conditions of Lemma \(N\) for \(N = M + 1\). Define \(A\) to be the set of all points \(p\) such that \(S(p, U)\) contains at least \(M + 1\) elements of \(U\). We define a brick partitioning \(U''\) of \(X\) whose elements are of two types; (a) each element of \(U'\) in \(W = S(A, U')\) is an element of \(U''\); (b) if \(u\) is an element of \(U\), \(u - W\) is an element of \(U''\). We note that \(U''\) is a refinement of \(U\) and a consolidation of \(U'\). However, it may not be a core refinement of \(U\).

There is a positive number \(\delta_1\) so small that if \(B\) is a subset of \(X - W\) of diameter less than \(\delta_1\), \(S(B, U)\) does not contain \(M + 1\) elements of \(U\).

Let \(\delta_2\) be the minimum of the distances between nonadjacent elements of \(U'\). We note that if \(B\) is a subset of \(X\) of diameter less than \(\delta_2\) and \(u'\) is an element of \(U''\) with a limit point on \(B\), then \(S(u', U)\) contains \(S(B, U)\).

We now describe a brick partitioning \(V\) which insures that Lemma \(N\) holds for \(N = M\). Let \(V'\) be a brick partitioning of \(X\) which is a core refinement of \(U''\) and such that each border element of \(V''\) is of diameter less than \(\min(\delta_1/2, \delta_2)\). If \(u''\) is an element of \(U''\) of type (b) in an element \(u\) of \(U\), the core element of \(V\) in \(u\) is the interior of the closure of the sum of the elements of \(V'\) in \(u''\) whose closures lie in \(u\). The other elements of \(V\) are the elements of \(U'\) in \(W\) and the elements of \(V'\) which are not in \(W\) and whose closures do not lie in any element of \(U\). We find that \(V\) is a core refinement of \(U\).

We now show that the elements of \(V\) satisfy conditions (3'). Suppose \(v_1\) and \(v_2\) are two adjacent elements of \(V\) such that each of \(S(v_1, U)\) and \(S(v_2, U)\) contains \(M\) or more elements of \(U\). We may suppose that neither \(v_1\) nor \(v_2\) is a core element, for if \(v_1\) is a core element, \(S(v_2, U)\) contains \(S(v_1, U)\). Hence, if \(v_1\) is not in \(W\), it may be supposed to be of diameter less than either \(\delta_1/2\) or \(\delta_2\).

If both \(v_1\) and \(v_2\) are subsets of \(W\), they are elements of \(U'\) and condition (3') holds for them because each of the sets \(S(v_1, U), S(v_2, U)\) contains \(M + 1\) elements of \(U\).

If \(v_1\) is a subset of \(W\) and \(v_2\) is not, then \(v_1\) is an element of \(U'\). Since the diameter of \(v_2\) is less than \(\delta_2\), \(S(v_1, U)\) contains \(S(v_2, U)\).

If neither \(v_1\) nor \(v_2\) is a subset of \(W\), \(v_1 + v_2\) is of diameter less than \(\delta_1\). Then \(S(v_1 + v_2, U)\) does not contain \(M + 1\) elements of \(U\). Hence \(S(v_1, U) = S(v_2, U)\).

By induction we find that Lemma \(N\) holds for all values \(N\). Since it holds for \(N = 1\), there is a sequence \(U = V_0, V_1, \ldots, V_n\) of brick partitionings of \(X\) such that \(V_{i+1}\) satisfies (1) and (3) where \(U\) is \(V_i\) and the diameters of the border elements of \(V_{i+1}\) are less than the distance between any nonadjacent elements of \(V_i\) and less than the \(\epsilon\) mentioned in the statement of Theorem 4.

Consider the core partitioning \(V\) of \(U\) where the core element of \(V\) in \(u_i\)
is the interior of the closure of the sum of the elements of \( V_i \) whose closures lie in \( u_i \). The border elements of \( V \) in \( u_i \) are the elements of \( V_i \) in \( u_i \) which are not in this core.

If \( v_1 \) and \( v_2 \) are two adjacent elements of \( V \) in the same element \( u_i \) of \( V \), one of the sets \( S(v_1, U) \), \( S(v_2, U) \) contains the other because if neither \( v_1 \) nor \( v_2 \) is a core element, then both are elements of \( V_i \). If \( v_1 \) and \( v_2 \) are adjacent elements, \( v_1 \) is in \( u_i \), \( v_2 \) is in \( u_j \), and \( j > i \), then \( S(v_1, U) \) contains \( S(v_2, U) \) because the diameter of \( v_2 \) is less than the distance between any two nonadjacent elements of \( V_i \). Hence, \( V \) satisfies conditions (1), (2), (3), and (4).

**Theorem 5.** Suppose \( U \) is a brick partitioning of \( X \) and \( G \) is a collection of open sets satisfying the following conditions:

(a) each element of \( G \) is the interior of the closure of the sum of the elements of a subcollection of \( U \);

(b) the subintersections of \( G \) are connected and uniformly locally connected;

(c) For each subcollection of \( G \), the intersection of the closures of the elements of the subcollection is the closure of the intersection of the elements of the subcollection.

Then for each positive number \( \epsilon \) there are a brick partitioning \( V \) of \( X \) and an \( \epsilon \)-covering \( H \) of \( X \) such that \( G + H \) satisfies the above conditions (a), (b), and (c) with \( V \) substituted for \( U \) in condition (a).

**Proof.** Let \( U' \) be a brick \((\epsilon/2)\)-partitioning of \( X \) that refines \( U \). We note the \( G \) satisfies condition (a) with \( U' \) substituted for \( U \).

Since \( U' \) has only a finite number of elements, \( G \) has only a finite number. Let \( u_1, u_2, \ldots, u_n \) be an ordering of the elements of \( U' \) such that if \( i < j \), \( u_i \) intersects as many elements of \( G \) as \( u_j \) does. It follows from condition (c) that if \( u_i \) intersects \( u_j \), then each element of \( G \) containing \( u_i \) also contains \( u_j \).

Let \( \delta \) be a positive number so small that if \( B \) is a subset of \( X \) of diameter less than \( \delta \), then there exists a point \( p \) of \( X \) such that \( S(p, U') \) contains \( S(B, U') \). Suppose \( V \) is a core refinement of \( U' \) such that \( V \) satisfies conditions (3) and (4) of Theorem 4 and the border elements of \( V \) are of diameter less than \( \delta/2 \). Let \( v_1, v_2, \ldots, v_m \) be an ordering of the elements of \( V \) such that \( v_1 \) precedes \( v_j \) provided either (1) \( v_i \) lies in an element of \( U' \) which precedes the element of \( U' \) containing \( v_j \) in the ordering \( u_1, u_2, \ldots, u_n \) or (2) \( v_i \) and \( v_j \) lie in the same element of \( U' \) and \( S(v_i, U') \) contains more elements of \( U' \) than \( S(v_j, U') \) does. We note that if \( u_i \) intersects \( u_j \) and \( i < j \), then each element of \( G \) containing \( v_i \) also contains \( v_j \).

For each point \( p \), define \( h(p) \) to be the interior of the closure of the sum of all elements of \( V \) whose closures lies in \( S(p, U') \). Let \( H \) be the collection of all such sets \( h(p) \). We prove that \( H \) is an \( \epsilon \)-covering of \( X \) and that \( G + H \) satisfies conditions (a), (b), and (c) with \( V \) substituted for \( U \) in condition (a).

To prove that \( H \) is a covering, consider a point \( q \). Since each border element of \( V \) is of diameter less than \( \delta/2 \), there exists a point \( p \) such that \( S(p, U') \)
contains $S[S(q, V), U']$. Then $h(p)$ contains $q$. As the elements of $U'$ are of diameter less than $\epsilon/2$, each $S(p, U')$ is of diameter less than $\epsilon$ and the elements of $H$ are of diameter less than $\epsilon$.

We next verify condition (b). Let $J$ be a subcollection of $G+H$ and $\pi$ be the intersection of the elements of $J$. Since $\pi$ is the interior of the closure of the sum of the elements of a subcollection of $V$, it is uniformly locally connected. If $J$ contains no element of $H$, then $\pi$ is connected by hypothesis. Suppose $J$ contains an element $h(p)$ of $H$. Let $v_1, v_j$ be elements of $V$ contained in $\pi$ and $u_r, u_s$ be the elements of $U'$ containing $v_1$ and $v_j$ respectively. Then $u_r, u_s$ is not null. Hence $\pi$ contains $v_1, v_j$, the core elements of $V$ in $u_r$ and $u_s$, and the border elements of $V$ whose boundaries intersect the closure of no elements of $U'$ except $u_r$ and $u_s$. Then $\pi$ is connected.

Finally, we check condition (c). If $i$ intersects $i$ and $i < j$, $S(v_i, U')$ contains $S(v_j, U')$. Hence $i$ lies in $S(v_i, U')$ and if $h(p)$ contains $v_i$, it also contains $v_j$. Hence each element of $G+H$ containing $v_i$ also contains $v_j$. If $p$ is a point of the intersection of the closures of the elements of $J$, the last element of $v_1, v_2, \ldots, v_m$ having $p$ on its closure is in each element of $J$. Hence $\pi$ contains $p$ and is the intersection of the closures of the elements of $J$.

**Theorem 6.** The space $X$ has a countable basis $G$ whose subintersections are connected and uniformly locally connected and such that if $G'$ is a subcollection of $G$, then the intersection of the closures of the elements of $G'$ is the closure of the intersection of the elements of $G'$.

**Proof.** Let $G_0$ be the covering of $X$ whose only element is $X$ itself, and $U_0$ be the brick partitioning of $X$ whose only element is $X$ itself. Repeated applications of Theorem 5 give a sequence $G_0, G_1, \ldots$ of coverings of $X$ such that $G_i$ ($i \geq 1$) is a $(1/i)$-covering and such that $G_i + G_2 + \cdots + G_i$ satisfies conditions (b) and (c). Define $G$ to be $\sum G_i$ and the theorem follows. The following result is a consequence of Theorem 6.

**Theorem 7.** For each positive integer $i$, $X$ is the sum of a finite $(1/i)$-collection $G_i$ of continuous curves such that each subintersection of $\sum G_i$ is a continuous curve.

**References**


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