ON THE LATTICE OF SUBGROUPS OF FINITE GROUPS

BY

MICHIO SUZUKI

Let G be a group. We shall denote by $L(G)$ the lattice formed by all subgroups of G. Two groups G and H are said to be lattice-isomorphic, or in short L-isomorphic, when their lattices $L(G)$ and $L(H)$ are isomorphic to each other. In this case the isomorphism from $L(G)$ onto $L(H)$ is called the L-isomorphism from G onto H. If G and H are isomorphic as groups, then G is clearly L-isomorphic to H. The converse of this statement is not always true. So the question arises: To what extent is a group determined by its lattice of subgroups? This paper is concerned with this question.

The content of this paper is as follows. In §1 we give some lemmas on groups or on lattices of subgroups. Some of these lemmas are known (cf. Ore [7](1), Iwasawa [5], and Jones [6]), others are given in generalized forms or with new demonstrations, and they are arranged so as to facilitate the following study. In §2 we consider the number of types of groups L-isomorphic to a given group G, and prove that this number is finite if $L(G)$ has no chain as its direct component. In §3 we determine the structure of groups L-isomorphic to a p-group(2). In §4 we consider the image $\phi(S)$ of a p-Sylow subgroup S of a group G by an L-isomorphism ϕ from G onto a group H, and treat the case when $\phi(S)$ is not a p-Sylow subgroup of H. In §5 we deal with the similar problem for the image of normal subgroups by an L-isomorphism. After these considerations we prove that groups L-isomorphic to a solvable group are solvable, too, and that a group G L-isomorphic to a perfect group H is also a perfect group with the same order as H and that, in this case, the modular lattice formed by all normal subgroups of G and H are isomorphic to each other. Moreover, we prove in §6 that a simple group G is isomorphic to a group H if and only if $L(G\times G)$ is isomorphic to $L(H\times H)$.

The two remaining sections 7 and 8 are devoted to the study of dualisms in the sense of Baer [2]. In these sections the structures of general nilpotent (finite solvable) groups with duals are completely determined.

In this paper we deal chiefly with L-isomorphisms of finite groups. Some of our results may be generalized, however, to the case of L-homomorphisms of finite groups. This will be treated in the next paper.

In preparing this paper the author is greatly indebted to Prof. S. Iyanaga, who made many useful comments. The author wishes to express his sincere

Received by the editors April 8, 1950.

(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) A group is called a p-group when its order is a power of a prime number p.

thanks to him. The author expresses also his best thanks to Dr. Iwasawa for his kind encouragement and advice throughout this work.

1. Preliminaries

1. **Fundamental lemmas.** Let G be a group. In this section we do not assume G to be finite. We shall mention particularly when G should be a finite group. We shall denote by $L(G)$ the lattice of subgroups of G. The subgroups of G and the corresponding elements of $L(G)$ will be denoted by the same letters as long as no confusion will arise. Letters A, B, and C, with or without suffixes, are reserved for the elements of groups, and all the other letters are used for subgroups.

The following two known theorems on $L(G)$, together with our Lemma 3 below, play a fundamental role in our study.

Lemma 1. *If $L(G)$ is a distributive lattice, any finite set of elements from G generates a cyclic subgroup and vice versa* (*Ore* [7]).

Lemma 2. *The lattice $L(G)$ of a finite group G satisfies the Jordan-Dedekind chain condition*(3) *if and only if G has a principal series all of whose factor groups are of prime orders* (*Iwasawa* [5]).

We shall call such a group a J-group and remark that the Sylow subgroup of a J-group G which belongs to the greatest prime factor of the order of G is self-conjugate (cf. Iwasawa [5]). The following propositions are obtained as corollaries of these lemmas. They are often used in the course of this paper.

If G is finite and nilpotent, then $L(G)$ is a lower semi-modular lattice, whose dimension is equal to the number of prime factors of the order of G.

$L(G)$ is a chain if and only if G is a cyclic group of prime power order (Baer [1]).

Let L be a finite lattice of dimension two, isomorphic to the $L(G)$ of a certain finite group G. Such a lattice is modular and is characterized by the number of its atoms. By Iwasawa [5], G must then be of order p^2 or pq (p and q are two primes and $p>q$). It is important that the number of atoms of L is $p+1$ and thus the greatest prime factor of the order of G is determined by L if L is not distributive, that is, if G is not cyclic.

A group G is said to be L-decomposable if $L(G)$ is decomposed into a direct product of two or more lattices none of which is a one-element lattice. We shall now prove the following lemma, due essentially to Iwasawa [5] and Jones [6].

Lemma 3. *If a group G is L-decomposable and its subgroup lattice $L(G)$ is isomorphic to the direct product of lattices L_λ ($\lambda\in\Lambda$), then G is isomorphic to a (restricted) direct product of groups G_λ: $G=\prod G_\lambda$, where $L(G_\lambda)\cong L_\lambda$ ($\lambda\in\Lambda$),*

(3) For general lattice theory, see Birkhoff [3].

and the order of any element of G_λ $(\lambda \in \Lambda)$ *is finite and relatively prime to the order of any element of* G_μ $(\mu \neq \lambda)$. *The converse of this lemma is also true.*

Proof. From our assumption we have $L(G) = \prod L_\lambda$ $(\lambda \in \Lambda)$. Let ϕ be this isomorphism from $\prod L_\lambda$ to $L(G)$. $L(G)$ has both greatest and least elements, namely G and e, so that each L_λ has them also. We shall denote the greatest (least) element of L_λ by I_λ (0_λ). Put $G_\lambda = \phi(\cdots, 0_\mu, \cdots, I_\lambda, \cdots, 0_{\mu'}, \cdots)$, where the only λ-component of $(\cdots, 0_\mu, \cdots, I_\lambda, \cdots)$ is I_λ and the others are 0_μ $(\mu \neq \lambda)$. Suppose that G has an element A of infinite order. Take the subgroup $Z = \{A\}$ generated by A, and an element $(\cdots, a_\lambda, \cdots)$ of $\prod L_\lambda$ such that $\phi(\cdots, a_\lambda, \cdots) = Z$. If $a_\lambda \neq 0_\lambda$ and $a_\mu \neq 0_\mu$, for two suffixes λ, μ $(\lambda \neq \mu)$, then we would have $Z \supseteq H_\lambda$, H_μ and $H_\lambda \cap H_\mu = e$, where $H_\lambda = \phi(0, \cdots, a_\lambda, \cdots, 0)$ and $H_\mu = \phi(0, \cdots, a_\mu, \cdots, 0)$ (only the λ- or μ-component different from 0). Since Z would be an infinite cyclic group, this implies that $H_\lambda = e$ or $H_\mu = e$, in contradiction with our assumptions. If only the λ-component of $(\cdots, a_\lambda, \cdots)$ were not equal to 0, take an element B from G_μ $(\mu \neq \lambda)$ and put $\{B\} = Z_1$. Then we would have $L(Z \cup Z_1) = L(Z) \times L(Z_1)$. Since $L(Z)$ and $L(Z_1)$ are both distributive, $Z \cup Z_1$ would be an infinite cyclic group, which is a contradiction. Thus we see that all elements of G are of finite order.

Let A_λ and A_μ be two elements of the groups G_λ and G_μ, respectively $(\lambda \neq \mu)$. By the same argument as above, $\{A_\lambda, A_\mu\} = \{A_\lambda\} \cup \{A_\mu\}$ is cyclic. This implies that each G_λ is a normal subgroup of G and that the orders of two elements A_λ and A_μ are relatively prime. This proves Lemma 3.

Conversely, let $G = \prod G_\lambda$ as above. Each subgroup H of G is a direct product of $H_\lambda = H \cap G_\lambda$. If $H = \prod H_\lambda$ and $K = \prod K_\lambda$, then we have $H \cup K = \prod (H_\lambda \cup K_\lambda)$, and similarly for meets. This shows that $L(G) = \prod L(G_\lambda)$.

If a group G is L-decomposable, G is a direct product: $G = \prod G_\lambda$. We shall call each group G_λ its "direct factor." In what follows the term "direct factor" of a group is used only in this sense.

2. **Lemmas on the Φ-subgroup of a finite group.** The Φ-subgroup of a group G is the intersection of all its maximal subgroups (cf. Zassenhaus [9, p. 44]). We shall denote it by $\Phi(G)$ or simply by Φ. Φ is a nilpotent group and $G = H\Phi$ implies $G = H$ for any subgroup H of G (cf. Zassenhaus [9, pp. 115, 45]). We shall now prove two lemmas on Φ-subgroups of finite groups.

LEMMA 4. *A prime number* p *divides the order of* G *if and only if it divides the order of* G/Φ.

Proof. If a prime number p divided the order of G but not that of G/Φ, a p-Sylow subgroup S of G would be a p-Sylow subgroup of Φ. Since Φ is nilpotent, there would exist a subgroup H of G such that $G = HS$ and $H \cap S = e$ (Schur's Theorem [9, p. 125]). We would have $G = H\Phi$ and hence $H = G$. This is a contradiction.

LEMMA 5. *A finite group G is L-decomposable if and only if G/Φ is L-decomposable.*

Proof. If a group G is L-decomposable, $G=G_1\times G_2$ by Lemma 3, and we can easily prove that $\Phi(G)=\Phi(G_1)\times\Phi(G_2)$ and $G/\Phi(G)=G_1/\Phi(G_1)\times G_2/\Phi(G_2)$. Hence G/Φ is L-decomposable by the converse of Lemma 3.

Suppose, conversely, $\overline{G}=G/\Phi$ is L-decomposable. Then we have $\overline{G}=\overline{H}_1\times\overline{H}_2$, the orders of $\overline{H}_1$ and $\overline{H}_2$ being relatively prime. We take subgroups H_1 and H_2 corresponding to $\overline{H}_1$ and $\overline{H}_2$ respectively by the natural homomorphism from G onto $\overline{G}$. Since Φ is nilpotent, it is decomposed into a direct product of its Sylow subgroups, that is, $\Phi=S_1^1\times\cdots\times S_r^1\times S_1^2\times\cdots\times S_t^2$, where S_j^i is a p_j^i-Sylow subgroup. By Lemma 4, p_j^i divides the order of $\overline{G}$ and hence we can arrange the p_j^i in an order such that p_i^1 divides the order of $\overline{H}_1$ and p_k^2 divides the order of $\overline{H}_2$. Put $D_1=\prod S_i^1$ and $D_2=\prod S_j^2$. D_1 and D_2 are clearly normal subgroups of G. Since the orders of D_1 and H_2/D_1 are relatively prime, there exists by Schur's Theorem a subgroup G_2 of H_2 such that $H_2=G_2D_1$ and $G_2\cap D_1=e$.

Moreover, since D_1 is nilpotent, any subgroup G_2^* of H_2, isomorphic to G_2, is conjugate to G_2 in H_2 (cf. Zassenhaus [9, p. 126]). Hence the normalizer T_2 of G_2 in G satisfies $T_2H_2=G$. T_2 contains G_2 and hence $T_2D_1=T_2\cup G_2\cup D_1=T_2\cup H_2=G$. It follows then that $T_2\Phi\supseteq T_2D_1=G$, which implies that $T_2=G$ and that G_2 is a normal subgroup of G. Similarly, G has a normal subgroup G_1 isomorphic to H_1/D_2. G is thus decomposed into a direct product of two groups G_1 and G_2: $G=G_1\times G_2$, and, by the converse of Lemma 3, G is L-decomposable.

3. **Remarks on P-groups.** A finite group P is called a P-group if it is:

(1) an elementary abelian group of prime power order, which we call an abelian P-group, or

(2) a group of the following structure: $P=S_pS_q$, where S_p is a p-Sylow subgroup which is an abelian P-group, S_q is a cyclic q-Sylow subgroup of order q: $S_q=\{B\}$, and for any element A of S_p we have

$$BAB^{-1}=A^r,\qquad r\not\equiv 1,\qquad r^q\equiv 1 \pmod p.$$

P-groups are the only groups whose lattices of subgroups are irreducible(4), complemented modular lattices (Iwasawa [5]). We shall prove this fact in a rather generalized form.

LEMMA 6. *If $L(G)$ of a group G is a lower semi-modular lattice all of whose intervals are irreducible, and if $\Phi(G)=e$, then G is a P-group.*

Proof(5). Suppose the lattice $L(G)$ is a lower semi-modular lattice all of

(4) We say that a lattice L is reducible when it is directly decomposable. Otherwise we call it irreducible.

(5) The author must give his hearty thanks to Mr. N. Ito, who suggested this proof to him. The author's original proof was more complicated.

whose intervals are irreducible. The commutator subgroup $C(G)$ of G is nilpotent by a theorem of Ore [8], so that it is a p-group, where p is the greatest prime factor of the order of G. If G is not a p-group, $G/C(G)$ is of the order q^n, where q is a prime and $p>q$. Hence the order of G is of the form $p^m q^n$. If $n>1$, a subgroup covering a q-Sylow subgroup Q of G has an L-decomposable subgroup, since G is a lower semi-modular group. Hence G is of order $p^m q$.

Now we consider $\Phi(G)=e$. If G is a group of prime power order, G must be an abelian P-group. If not, the Φ-subgroup of a p-Sylow subgroup S of G is equal to e, as $S\cap M$ is either equal to S or maximal in S for any maximal subgroup M of G. Hence S is an abelian P-group. S is decomposed into a direct product of simple representation modules S_i of Q: $S=S_1\times\cdots\times S_t$, as S is a representation module of a q-Sylow subgroup Q of G. The S_i are simple and hence are minimal normal subgroups of G. As G is lower semi-modular, each S_i is one-dimensional. Put $S_i=\{A_i\}$ $(i=1, 2, \cdots, t)$ and $Q=\{B\}$. We have

$$BA_iB^{-1} = A_i^{r_i}, \qquad r_i^q \equiv 1 \pmod p.$$

Since all intervals are irreducible, $r_i\not\equiv 1 \pmod p$ and moreover $r_i\equiv r \pmod p$ $(i=1, 2, \cdots, t)$. For, if $r_i\not\equiv r_j$, then the interval $\{A_i, A_j, B\}/Q$ is reducible contrary to our assumption. Hence G must be a P-group.

A P-group is a very peculiar type of group when we deal with the subgroup lattices. In fact, if G is an abelian P-group of order p^n and if H is a nonabelian P-group of order $p^{n-1}q$ $(p>q)$, then $L(G)$ and $L(H)$ are isomorphic to each other, though G and H have very different structures. This was already pointed out by Baer and others. Baer [1, Theorem 11.2] has proved that the nonabelian P-groups are the only type of groups which, although not themselves p-groups, are L-isomorphic to abelian, noncyclic p-groups. We shall generalize this result later on (see Theorem 3 below).

4. **Remarks.** 1. Let ϕ be an isomorphism (anti-isomorphism) from a lattice L to a lattice L', and let V be a sublattice of L. The set $\phi(V)$ of elements $\phi(a)$, $a\in V$, forms a sublattice of L'. ϕ induces an isomorphism (anti-isomorphism) $\bar{\phi}$ from V onto $\phi(V)$. $\bar{\phi}$ is a contraction of ϕ onto V, but we shall not distinguish between ϕ and its contraction $\bar{\phi}$, if no confusion arises, and denote it by the same letter.

2. A group is called an M-group if its subgroup lattice is modular. The structure of such groups has been determined by Iwasawa [5]. It was also proved by him that direct factors of any finite M-group are either p-groups or groups of order $p^n q^m$. We can prove this fact from our standpoint in a simpler way as follows:

Let G be an M-group of finite order. The subgroup lattice of the factor group G/Φ is a complemented modular lattice, and hence it is decomposed

into a direct product of irreducible components. By Lemma 6, direct factors of G/Φ are of orders p^n or p^nq. Hence by Lemmas 4 and 5 direct factors of G are of orders p^m or p^nq^m.

It is also proved easily that non-normal Sylow subgroups of an M-group are cyclic, by noticing that the factor group of a non-nilpotent M-group of order p^nq^m modulo its Φ-subgroup is a P-group (by Lemma 4).

2. The number of types of L-isomorphic groups

Let L be a given lattice. The number of types of groups whose subgroups form lattices isomorphic to L may vary from 0 to infinity. We prove, however, the following theorem.

Theorem 1. *Let L be a lattice*(6). *If L has no chain as its direct component, then the number of types of groups whose lattices of subgroups are isomorphic to L is finite (if any).*

Proof. If there is no group with a subgroup lattice isomorphic to L, there is nothing to prove. Thus we shall assume that such groups exist and take one of these groups G. G is clearly a finite group. By Lemma 3, we can assume that L is not a chain and that it is irreducible.

Then we shall show the order of such a group G does not exceed $m^{m(n-1)}$, where m is the number of elements of L and n is its dimension.

If G is a p-group, G is not cyclic. Hence its Φ-subgroup Φ is contained in a normal subgroup N of index p^2 (N may be equal to Φ). Then the factor group G/N is not cyclic, and the number of atoms of $L(G/N)$ is equal to $p+1$. As $L(G/N)$ is clearly isomorphic to an interval of L, we have $p<m$. If, on the other hand, the order of G is p^n, n is equal to the dimension of L. Hence the order of G is smaller than m^n.

If the order of G is not a prime power, we consider a prime factor p of it, and denote by S a p-Sylow subgroup of G. If S is not self-conjugate, the number of conjugate subgroups of S is greater than 1 and $\equiv 1 \pmod{p}$. Hence $p<m$. If S is self-conjugate, there exists, by Schur's Theorem, a subgroup H of G such that $SH=G$ and $S\cap H=e$. If H were a normal subgroup, G would, by Lemma 3, be L-decomposable, contrary to our assumption. Hence H is not self-conjugate and the number of conjugate subgroups of H is greater than 1 and is a power of p. Hence again we have $p<m$. The highest exponent of p which divides the order of G is clearly smaller than the dimension of $L(G)$, and the number of distinct prime factors of the order of G is smaller than m. Hence the order of G is smaller than $m^{m(n-1)}$. Our theorem is thus proved.

3. Groups which are L-isomorphic to a p-group

Let G be a p-group. Then $L(G)$ is a lower semi-modular lattice, all of

(6) We shall assume, of course, L to be a finite lattice.

whose intervals are irreducible. This is easily proved using the facts that any maximal subgroup of a nilpotent group is self-conjugate and that the lattices of groups of order p^2 are irreducible. Now we shall prove the following theorem.

THEOREM 2. *The lattice $L(G)$ of a group G is a lower semi-modular lattice all of whose intervals are irreducible if and only if G is either a p-group or a group of the following structure: $G=\{S, \sigma\}$, where S is an abelian p-group of the exponent p^m, and σ is an automorphism of S such that for any element A of S we have*

$$A^\sigma = A^r, \qquad r \not\equiv 1, \qquad r^q \equiv 1 \pmod{p^m}.$$

Proof. If $L(G)$ satisfies the above conditions, G is either a p-group or a group of order p^nq $(p>q)$ by Lemma 6. Suppose that the order of G is p^nq. Then a p-Sylow subgroup S of G is maximal. If $\Phi(G)\neq\Phi(S)$, then by Lemma 2 there would exist a normal subgroup N of G which contains $\Phi(S)$ and is covered by $\Phi(G)$. As S/N is completely decomposable as a G/S-module, there would exist a normal subgroup N_1 of G such that $N_1\cup\Phi(G)=S$ and $N_1\cap\Phi(G)=N$. G/N_1 would be of order pq and have only one proper subgroup. This is a contradiction. Hence we must have $\Phi(G)=\Phi(S)$.

We shall prove our theorem by induction on n. Suppose that the theorem is proved for the maximal subgroups of G. As all intervals of $L(G)$ are irreducible, there are at least two maximal subgroups M_1 and M_2 such that $S\cap M_1=S\cap M_2$, and hence all subgroups of S are self-conjugate in G. This implies that S is an abelian group (note that $p>q\geqq 2$).

Let A_1 be one of the elements of the greatest order p^m and let σ be an automorphism of S such that $G=\{S, \sigma\}$. Then we have

$$A_1^\sigma = A_1^r, \qquad r \not\equiv 1, \qquad r^q \equiv 1 \pmod{p^m}.$$

If S is cyclic, our theorem is proved. Otherwise, we can decompose S into a direct product of $\{A_1\}$ and another group S_1(7). Take any element B of S_1, then we have $B^\sigma=B^t$ and $(A_1B)^\sigma=(A_1B)^u$. Hence we have

$$A_1^uB^u = A_1^rB^t,$$

or $u\equiv r \pmod{p^m}$ and $u\equiv t$ (mod the order of B). We have, therefore, for any element A of S

$$A^\sigma = A^r, \qquad r \not\equiv 1, \qquad r^q \equiv 1 \pmod{p^m}.$$

The converse statement is obvious. q.e.d.

Let G be a group of the type $\{S, \sigma\}$ of this theorem. Then $G/\Phi(G)$ is clearly a P-group, and hence the prime factor p of the order of S is de-

(7) We always have such a decomposition. Cf. Burnside [4, p. 102].

termined by $L(G)$ as remarked in §1. Suppose for a time that S has an element A of order p^2. As the subgroup $\{A\}$ generated by A is self-conjugate, $H=\{A, \sigma\}$ is a subgroup of G whose order is p^2q. $L(H)$ has clearly $1+p^2$ atoms and only one chain of dimension 2 by the structure of H. Hence we have the following proposition: A group G is a p-group or a P-group if $L(G)$ is a lower semi-modular lattice, all of whose intervals are irreducible, and if one of the following two conditions is satisfied in every ideal of $L(G)$ which is not a chain: (1) the number of elements of the same dimension is $\equiv 1+p \pmod{p^2}$, or (2) the number of chains of the same dimension is divible by p when this dimension is greater than 1. (In these conditions we mean by p the prime number determined by $L(G)$ as remarked above under the first assumption of this proposition.) If $p>2$, the converse statement of this proposition is also valid. Conditions in this proposition on the number of subgroups of G are indeed theorems of Kulakoff and Miller (cf. Zassenhaus [9, p. 117]). On the other hand, it is clearly a necessary and sufficient condition for a noncyclic group G to be a 2-group that $L(G/\Phi(G))$ be a projective geometry over the field $GF(2)$.

From the above considerations we obtain the following theorem, which gives a generalization of a theorem of Baer [1].

THEOREM 3. *A group H, L-isomorphic to a p-group G, is also a p-group, except in the following two cases*:

(1) *G is cyclic and H is also a cyclic group of prime power order, or*

(2) *G is an abelian P-group and H is a nonabelian P-group.*

REMARK. An L-isomorphism ϕ from G onto H is called index-preserving when for any two cyclic subgroups U and V of G with $U \supseteq V$,

$$(*) \qquad [U:V] = [\phi(U):\phi(V)],$$

and ϕ is called strictly index-preserving when (*) holds for any two subgroups U, V (with $U \supseteq V$) of G. We shall prove the following theorem.

THEOREM 4. *An index-preserving L-isomorphism between finite groups is always strictly index-preserving .*

Proof. Let ϕ be an index-preserving L-isomorphism from a finite group G onto a group H. It is sufficient to prove that G and H have the same orders. Put $[G:e]=g$ and $[H:e]=h$. Let S be a p-Sylow subgroup of G, and let p^n be its order. Then the order of $\phi(S)$ is also p^n by our assumption and by Theorem 3, and $g \mid h$. Similarly we consider the L-isomorphism ϕ^{-1} from H onto G and have $h \mid g$. Hence we have $g=h$.

The concepts of index-preserving and strictly index-preserving L-isomorphisms were introduced by Baer [1], who distinguished between these two concepts. It is, however, not necessary to distinguish them when we deal with finite groups. For infinite groups it is an open question whether there

exists an index-preserving L-isomorphism which is not strictly index-preserving.

4. Groups which admit singular L-isomorphisms

We shall call an L-isomorphism singular, when it is not index-preserving. We say that a group G admits a singular L-isomorphism if there exists a group G' which is L-isomorphic to G, and the L-isomorphism ϕ from G onto G' is singular.

If G admits a singular L-isomorphism ϕ, the image of a p-Sylow subgroup of G by ϕ is not a p-group for some p.

Let S be such a group, that is, a p-Sylow subgroup of G such that the image $\phi(S)$ is not a p-group. (S is then a cyclic group or a P-group by Theorem 3.) This notation will be fixed throughout this section, and we shall denote hereafter the image $\phi(H)$ of any subgroup H of G by the same letter with an accent, that is, by H'.

Lemma 7. *If S is self-conjugate and maximal, then G is a nilpotent group or a nonabelian P-group.*

Proof. (a) Suppose first that S is cyclic. Then S covers $\Phi(G)=N$. If G/N is cyclic, G is nilpotent by Lemma 5. Otherwise, G/N is of order pq (q is also a prime and less than p). $N'=\phi(N)$ is clearly the Φ-subgroup of G', and hence self-conjugate in G'. G'/N' is, therefore, a P-group of order pr (r is also a prime). One of the maximal subgroups of G', say H', is self-conjugate by the structure of a P-group. H'/N' is clearly of order p. By the definition of S we have $H'\neq S'=\phi(S)$. Take another maximal subgroup M' of G', different from H' and S', then M' is conjugate to S'. This implies by Lemma 1 that M' and the subgroup M of G such that $\phi(M)=M'$ are cyclic subgroups of prime power order. Hence we have $N=e$ and G is a P-group. q.e.d.

(b) Next we assume that S is a P-group. Regarded as a representation module of a q-Sylow subgroup Q of G, S is decomposed into a direct product of simple Q-modules S_i ($i=1, 2, \cdots, t$): $S=S_1\times\cdots\times S_t$. If all $\phi(S_i)$ are p-groups, $\phi(S)$ is also a p-group contrary to our assumption. Hence one of $\phi(S_i)$, say $\phi(S_1)=S_1'$, is not a p-group. As S_1 is a simple Q-module, S_1' contains no proper normal subgroup of $\phi(S_1\cup Q)$. Hence the order of S_1' is a prime, say r ($r\neq p$), and the order of S_1 is p. Since any r-Sylow subgroup of S' is conjugate to S_1' by the structure of a nonabelian P-group, any subgroup U of S whose image $\phi(U)$ is an r-Sylow subgroup of S' is regarded as a Q-module, which implies that U is self-conjugate in G. Hence all subgroups of S are self-conjugate in G and G is nilpotent or a P-group. q.e.d.

Lemma 8. *If S is self-conjugate and not equal to G, and if no P-group contains S properly, then G is L-decomposable and S is its direct factor.*

Proof. We shall prove this by induction on the order of the factor group G/S. If we assume our lemma to be proved for maximal subgroups, S is

contained in the center of any maximal subgroup M containing S. If there exist two such maximal subgroups, our lemma follows immediately. Otherwise, G/S is a cyclic group of order q^n (q is a prime and not equal to p). Take a q-Sylow subgroup Q of G and put $D=M\cap Q$. Then we have $M=S\times D$. D and $D'=\phi(D)$ are normal subgroups of G and G' respectively; the latter follows from Lemma 3. We can apply Lemma 7 to the groups G/D and G'/D'. If G/D is a P-group, $\phi(Q_1)=Q_1'$ is not of prime power order for some q-Sylow subgroup Q_1 of G. This is, however, a contradiction. G/D is thus nilpotent and G itself is L-decomposable by Lemma 5. q.e.d.

THEOREM 5. *If no P-group contains S properly, there exists a normal subgroup N of G with the following properties*:

(1) *$NS=G$ and $N\cap S=e$*;

(2) *$\phi(N)=N'$ is also a normal subgroup of G'*;

(3) *the groups G'/N' and N' have mutually prime orders*; *and*

(4) *if N is not invariant under L-automorphisms of G, the factor group G/D is a cyclic group of order $(pq\cdots r)^n$, where $D=\bigcap_\sigma\sigma(N)$ (σ runs through all L-automorphisms of G and $p, q, \cdots, r$ are mutually distinct primes*).

Proof. S lies in the center of its normalizer by Lemma 8. Hence the Sylow p-complement[8] N of G exists and is self-conjugate by Burnside's Theorem (cf. Burnside [4, p. 327]). N satisfies condition (1) of this theorem. We shall prove that N satisfies the other properties.

Suppose that N is not invariant by an L-automorphism σ of G; $\sigma(N)\neq N$. The order of $\sigma(N)$ is divisible by p and hence $\sigma(N)\cap S_1\neq e$ for some p-Sylow subgroup S_1 of G. Now we have $T=N\cap\sigma^{-1}(S_1)\neq e$, and T is self-conjugate in $\sigma^{-1}(S_1)$. Since the order of T is prime to p, $\sigma^{-1}(S_1)$ is cyclic by Theorem 3. This implies that $T=\sigma^{-1}(S_1)$ or $\sigma^{-1}(S_1)\subseteq N$. Take a q-Sylow subgroup Q of G, containing $\sigma^{-1}(S_1)$, then $\sigma(Q)$ contains S_1. By our assumption and by Theorem 3 we have $\sigma(Q)=S_1$, or $\sigma^{-1}(S_1)$ is a q-Sylow subgroup of G. σ maps a q-Sylow subgroup of G onto a p-Sylow subgroup, so that the Sylow q-complement of G must be self-conjugate (note that there is no P-group containing S). Suppose that a p-Sylow subgroup is mapped onto $q, \cdots, r$-Sylow subgroups of G under all L-automorphisms of G, and denote the corresponding Sylow complements by $N_q, \cdots, N_r$ respectively. Put $N=N_p$ and $D=N_p\cap\cdots\cap N_r$. D is clearly invariant under L-automorphisms of G, and equal to $\bigcap_\sigma\sigma(N)$. G/D is a cyclic group of order $(pq\cdots r)^n$. Thus N satisfies property (4). Now property (2) is obvious.

If the orders of the groups G'/N' and N' had a common prime factor q, $P'\cap N'$ would be neither e nor P' for some q-Sylow subgroup P' of G'. We consider a subgroup P of G such that $\phi(P)=P'$. We would have $P\cap N\neq e$,

(8) When the order of G is $g=p^\alpha g'$, $(p, g')=1$, we call subgroups of G, whose indexes are p^α, Sylow p-complements of G. Such a subgroup does not always exist. Cf. papers of P. Hall, Proc. London Math. Soc. vol. 3 (1928), vol. 12 (1937), and (2) vol. 43 (1937).

$P \cap N \neq P$ and thus P would be a nonabelian P-group. q would be, therefore, the greatest prime factor of the order of P. Since p divides the order of P, we would have $S_2 \cap P \neq e$ for some p-Sylow subgroup S_2 of G and $q > p$. $\phi(S_2)$ would not be a p-group. S_2 would not be contained in P by our assumption and this implies that $\phi(S_2)$ would not be cyclic, but a P-group. Hence p would be the greatest prime factor of the order of $\phi(S_2)$. On the other hand, since $\phi(S_2) \cap P' \neq e$, the order of $\phi(S_2)$ would be divisible by q. Hence we have $p > q$ in contradiction with the above inequality $p < q$. Theorem 5 is thus proved.

A group H is called an S-group when it is either a P-group or L-decomposable with a P-group as its direct factor.

THEOREM 6. *If S is contained in some P-group T as a proper normal subgroup, then G is an S-group and T is its direct factor.*

Proof. We shall assume that $G \neq T$ and prove that G is L-decomposable. Let the order of T be $p^m q$ (q is a prime and less than p). We shall denote by Q one of the q-Sylow subgroups of T which are mapped by ϕ onto subgroups of order p. Certainly such subgroups exist. First we shall prove Q is a q-Sylow subgroup of G and is contained in the center of its normalizer. For this purpose take a q-Sylow subgroup Q^* of G, containing Q. As $\phi(Q^*)$ is not a q-group, Q^* is cyclic or a P-group. We shall show that Q^* is cyclic. In fact, if Q^* were not cyclic, q would be the greatest prime factor of the order of $\phi(Q^*)$, which is not the case. We shall show further that the order of Q^* is a prime. In fact, otherwise, $\phi(Q^*)$ would be of prime power order and so of order p^m and $m > 1$. p-Sylow subgroups of G would contain elements of order p^2. This implies that p-Sylow subgroups of G' would be cyclic and that the order of T would be pq. Take now a q-Sylow subgroup V of T, different from Q. V is conjugate to Q in T and $\phi(V)$ conjugate to $\phi(S)$ in $\phi(T)$. We see, therefore, that there would exist a cyclic subgroup S_0 of prime power order, containing S properly, which is clearly impossible. Hence we have $Q = Q^*$. If Q were contained in a P-group Q_0 as a proper normal subgroup, q would be the greatest prime factor of the order of $\phi(Q_0)$, which is again not the case. Hence Q is contained in the center of its normalizer by Lemma 6. G contains, by Burnside's Theorem, a normal subgroup N_0 such that $N_0 Q = G$ and $N_0 \cap Q = e$. Note that $N_0 \cap T = S$.

Denote by K the normalizer of S in G. Then we have $T \subseteq K$. In K any subgroup L-isomorphic to T contains S.

If $T' = \phi(T)$ were not self-conjugate in $K' = \phi(K)$, all conjugate subgroups of T' in K' would contain $S' = \phi(S)$. Hence S' would be self-conjugate in K', and a fortiori in T'. But this contradicts our assumption. T' is, therefore, self-conjugate in K'. This implies that T is the only P-group which contains S as a proper normal subgroup. The normalizer of S in N_0 is $K \cap N_0$ and satisfies the same assumptions of Lemma 8. It follows then that N_0 contains a

characteristic subgroup N such that $NS=N_0$ and $N\cap S=e$. N is a normal subgroup of G and we have $NT=G$ and $N\cap T=e$.

We shall prove now that T is also self-conjugate in G, by induction on the dimensions of intervals G/T and T/e. If T is self-conjugate, our theorem follows immediately.

Assume that the theorem is proved in the case that the order of T is p^nq. Then the same holds for T of order $p^{n+1}q$. For, there are at least two maximal subgroups T_1 and T_2 of T, which do not contain $S\cap S^*$, where S^* is a subgroup of T such that $\phi(S^*)$ is a p-Sylow subgroup of $\phi(T)$. We have $T_i\cap S\neq T_i\cap S^*$ $(i=1, 2)$ and hence by the hypothesis of induction $T_1\cup N=T_1\times N$ and $T_2\cup N=T_2\times N$. Hence T is elementwise permutable with N and we have $G=T\times N$.

We can assume thus that the order of T is p·q. Normalizers of Sylow subgroups of N contain some conjugate subgroup of T. We can therefore assume that N is an r-group (r is a prime and not equal to p or q). Suppose $\phi(N)$ is not an r-group. If there is no P-group containing N properly, G is L-decomposable by Theorem 5. If a P-group N_1 contained N properly, there would exist, as proved above, a normal subgroup H such that $HN_1=G$ and $H\cap N_1=e$, and N_1 would be self-conjugate in the normalizer of N. G/N would then be cyclic, which is not the case. We can, therefore, assume that $\phi(N)=N'$ is also an r-group. Consider now a maximal subgroup $M(\supseteq T)$. Then by the hypothesis of induction we have $M=T\times(M\cap N)$ and $G/M\cap N\cong M/M\cap N\times N/M\cap N$ (note that $\phi(M\cap N)$ is also normal subgroup of G'). Hence we can reduce our theorem to the case that T is maximal. N is then clearly an abelian P-group.

T is not cyclic, and some element of T is permutable with an element not equal to e of N by a lemma of Zassenhaus([9]). Any proper subgroup of T is mapped onto all the other proper subgroups of T by L-automorphisms of G. Hence we see that any proper subgroup of T differs from its centralizer, and that one of the proper subgroups of T, say U, is thereby a normal subgroup of G. U is elementwise permutable with N, and thus $L(U\cup N)=L(U)\times L(N)$. Take another proper subgroup V of T. Then $V\cup N$ is mapped onto $U\cup N$ by an L-automorphism of G so that $L(V\cup N)$ is also L-decomposable. This implies, by Lemma 3, that the centralizer of N is G and that $G=T\times N$. q.e.d.

THEOREM 7. *If S is contained in some P-group as a non-normal subgroup, there is a normal subgroup N of G with the following properties*:

(1) *$NS=G$ and $N\cap S=e$*;

(2) *if N is not invariant under L-automorphisms of G, put $D=\bigcap_\sigma\sigma(N)$ (σ runs through all L-automorphisms of G). Then G/D is either a cyclic group of order $pq\cdots r$ or a P-group, where $p, q, \cdots, r$ are distinct primes. When*

([9]) H. Zassenhaus, Abh. Math. Sem. Hamburgischen Univ. vol. 11 (1936). Cf. Burnside [4, §248].

G/D is a P-group, G is an S-group;

(3) *G' contains a normal subgroup N_0' in the subgroup $N'=\phi(N)$. The factor group G'/N_0' is a P-group, and the orders of the groups G'/N_0' and N_0' have no common prime factor. Moreover, G contains a subgroup N_0 such that $\phi(N_0)=N_0'$, N_0 is self-conjugate in G, and the orders of the groups G/N_0 and N_0 have also no common prime factor.*

Proof. No P-group contains S as a normal subgroup by Theorem 6. Hence there is, by Theorem 5 and by Burnside's Theorem, a normal subgroup N which satisfies the condition (1) of this theorem.

If N is not invariant under an L-automorphism σ: $\sigma(N)\neq N$, $\sigma(N)$ contains some p-Sylow subgroup S_1 of G. Let the order of $\sigma^{-1}(S_1)$ be q. Then q is a prime, not equal to p. If $\sigma^{-1}(S_1)$ is not a Sylow subgroup, then there is a q-Sylow subgroup of G which contains $\sigma^{-1}(S_1)$, is not of order q, and is mapped by σ onto a subgroup whose order is not a power of q. Hence Theorems 5 and 6 are applicable to this case and show that G is an S-group. We have $G=D\times H$, where H is a P-group. H contains $\sigma^{-1}(S_1)$ and S_1. N contains D, and this proves (2) in this case. If $\sigma^{-1}(S_1)$ is a Sylow subgroup for every L-automorphism σ of G, property (2) is proved analogously to the proof of Theorem 5.

If $N'=\phi(N)$ is not self-conjugate, G is by (2) an S-group. The property (3) follows then immediately. If N' is self-conjugate and if the orders of two groups G'/N' and N' have a common prime factor, say r, an r-Sylow subgroup of G' is mapped by ϕ^{-1} onto a subgroup of G, whose order is not a power of r. Property (3) is a consequence of Theorems 5 and 6. q.e.d.

Theorems 5, 6, and 7 give us the structures of groups with singular L-isomorphisms in considerable detail. In all cases there are normal subgroups N, N' of G, G' respectively, and one of the "exceptional" Sylow subgroups is excluded from these normal subgroups. Hence we have the following theorems on a general L-isomorphism.

THEOREM 8. *Let G and G' be two groups, L-isomorphic to each other, and ϕ be the L-isomorphism between G and G'. Then there exists a normal subgroup N of G which satisfies the following conditions:*

(1) *$N'=\phi(N)$ is also self-conjugate in G';*

(2) *N has the same order as N';*

(3) *direct factors of factor groups G/N and G'/N' are cyclic or P-groups;*

(4) *the orders of N and N' are relatively prime to those of G/N and G'/N' respectively.*

THEOREM 9. *Under the same assumptions as above, there exists also a normal subgroup N_0 which satisfies the following conditions:*

(1′) *and* (2′) *are the same as* (1) *and* (2) *of Theorem* 8;

(3′) *the factor group G/N_0 is cyclic;*

(4′) *extensions of N_0 by G/N_0 and of $N_0'=\phi(N_0)$ by G'/N_0' both split.*

This normal subgroup N_0 clearly contains the group N of Theorem 8. ϕ induces an index-preserving L-isomorphism in N_0. The following two interesting theorems now follow from the above theorems.

Putting $G=G'$ in Theorems 5, 6, and 7, we obtain:

THEOREM 10. *If a Sylow subgroup S of G is mapped by an L-automorphism onto a subgroup which is not a Sylow subgroup, then G is an S-group, and S is contained properly in the direct factor of G which is a P-group.*

A group is said to be perfect if it coincides with its commutator subgroup. Applying Theorem 9 to such a group, we obtain:

THEOREM 11. *An L-isomorphism from a perfect group onto another group is always index-preserving.*

This theorem is valid for a wider class of groups. In fact, Therorems 5, 6, and 7 show that if G admits a singular L-isomorphism, at least one of its Sylow complements is a normal subgroup of G. Hence Theorem 11 is also valid for a group none of whose Sylow complements is self-conjugate, for example, for the symmetric group of n letters ($n>3$).

5. GROUPS WHICH ADMIT NON-NORMAL L-ISOMORPHISMS

Following Baer [1], we shall call an L-isomorphism normal if it maps normal subgroups onto normal subgroups and its inverse L-isomorphism does the same. Otherwise we call it non-normal. We say, as in the preceding section, that a group G admits a non-normal L-isomorphism if there exists a group G', L-isomorphic to G, and the L-isomorphism from G onto G' is non-normal.

If G admits a non-normal L-isomorphism ϕ, the image of some normal subgroup of G (or G') by ϕ (or ϕ^{-1}) is not self-conjugate in G' (or G). We may assume, choosing suitable notation, that a normal subgroup N of G is mapped onto a non-normal subgroup $\phi(N)$ of G'. This notation will be fixed throughout this section.

LEMMA 9. *If N is a maximal subgroup of G, there exists a normal subgroup D of G with the following properties:*

(1) *D is contained in N, that is, $D\subseteq N$;*

(2) *D is invariant under all L-automorphisms of G;*

(3) *G/D is a P-group, and its order is prime to the order of D;*

(4) *the order of $D'=\phi(D)$ is also prime to the order of G'/D';*

(5) *if G/D is not abelian, G is an S-group. In this case D is a direct factor of G if $D\neq e$.*

Proof. We shall prove our lemma by induction on the order of G. Since $N'=\phi(N)$ is not self-conjugate, it is mapped onto another subgroup N_1' of G' by an L-automorphism σ of G' which is induced by an inner automorphism

of G'. σ induces an L-automorphism of G, which we shall denote by the same letter σ. Take a subgroup N_1 of G such that $\phi(N_1)=N_1'$. Then $N\cap N_1$ is a normal subgroup of N_1 and it is maximal in N_1.

Suppose $N'\cap N_1'$ is not self-conjugate. Then there is, by the hypothesis of induction, a normal subgroup D_1' of N_1' with properties (1)–(5) of our lemma. Since σ is induced by an (inner) automorphism of G', $\sigma^{-1}(D_1')$ has the same order as D_1' and is self-conjugate in N'. We have $N'\supseteq N'\cap N_1'\supseteq D_1'$ by (1), and thus the order of $D_1'\cup D'/D'$ is equal to that of $D_1'/D_1'\cap D'$, where $D'=\sigma^{-1}(D_1')$. On the other hand, the two groups N_1'/D_1' and D_1' have relatively prime orders by property (4), and thus the orders of N'/D' and D_1' are also relatively prime. Hence $D_1'\cup D'$ must coincide with D', or $D_1'=D'$. D' is, therefore, self-conjugate in G'.

The subgroup D of G such that $\phi(D)=D'$ is also self-conjugate by (2), and by (3) N_1/D and N/D are both P-groups whose orders are prime to that of D. Hence $N\cap N_1/D$ is of prime power order. It is, however, clear that the order of $N'\cap N_1'/D'$ is different from that of $N\cap N_1/D$, and hence ϕ induces a singular L-isomorphism between G/D and G'/D'. By theorems of the preceding section, G/D is a P-group, and since ϕ itself is singular, G contains a normal subgroup D^* with properties (2)–(5) of our lemma. D^* clearly coincides with D. This proves our lemma.

Suppose next $N_1'\cap N'$ is self-conjugate in N_1'. Then there is another maximal subgroup N_2' of G' different from N' and N_1', which is conjugate to N' and satisfies the relations $N_1'\cap N_2'=N_2'\cap N'=N'\cap N_1'$. We may assume that $N_2'\cap N'$ is also self-conjugate in N_2'. By this assumption $N_1'\cap N_2'$ is a normal subgroup of G'. Take a subgroup N_2 of G such that $\phi(N_2)=N_2'$. Then we have $N_1\cap N_2=N_2\cap N=N\cap N_1$, and $N_1\cap N_2$ is also a normal subgroup of G. The factor group $G/N_1\cap N_2$ is clearly a P-group and ϕ is thereby again a singular L-isomorphism. Our lemma follows from theorems of §4.

Using induction on the order of G, we obtain the following theorem.

THEOREM 12. *Let G and G' be two groups, L-isomorphic to each other. If one of these groups, say G, is solvable or perfect, then G' is also solvable or perfect.*

This theorem gives an affirmative answer to the Problem 40 of Birkhoff, described in his book [3*], for finite groups. As he suggests, the first half of this theorem can also be proved by Hall's Theorem([10]) together with our Theorem 8 (without this Lemma 9).

Now we shall continue our study of the correspondence of normal subgroups.

THEOREM 13. *If the factor group G/N is a P-group, there exists a normal subgroup D of G with the properties enumerated in Lemma 9.*

Proof. G contains a normal and maximal subgroup M containing N. Since

([10]) Cf. P. Hall, Proc. London Math. Soc. vol. 12 (1937).

G/N is a P-group, the interval G'/N' is isomorphic to some projective geometry. $M'=\phi(M)$ clearly corresponds to a hyperplane $\overline{M}$ of this projective geometry. Take any one of the subgroups of G', say U', corresponding to points not contained in $\overline{M}$. Then we have $M'\cap U'=N'$ and $M'\cup U'=G'$, and the join of all such subgroups is G'. Now, if M' were self-conjugate, N' would be self-conjugate in such subgroups and hence in G', which contradicts our assumption. Hence M' is not self-conjugate and by Lemma 9, G contains a normal subgroup D with properties (2)–(5) of Lemma 9. We must prove property (1). Put $D'=\phi(D)$ and take such a subgroup U' as defined above. Then we have $D'\cap U'=D'\cap M'\cap U'=D'\cap N'$. Since D' is self-conjugate, $D'\cap N'$ is self-conjugate in all subgroups such as U', and hence in G'. If $D\not\subseteq N$, we would have $D\cup N=M$ by property (3). Hence $M/D\cap N$ would be L-decomposable, and the centralizer of $D/D\cap N$ would contain $M/D\cap N$. Since N' is not self-conjugate, some L-automorphism of $G/D\cap N$ would move $M/D\cap N$ and fix $D/D\cap N$. Hence $D/D\cap N$ would be contained in the center of $G/D\cap N$, which is clearly not the case. Hence $D\subseteq N$, and our theorem is thus proved.

Remark. We can also obtain a more general proposition: Let H be any subgroup of G, and K a normal subgroup of H. If H/K is a P-group and if the image $\phi(K)$ of K by some L-isomorphism ϕ from G onto another group is not self-conjugate in $\phi(H)$, there is a normal subgroup D^* of G with the properties similar to (2)–(5) in Lemma 9 and the following property:

(1′) $D^*\cap H$ is contained in K.

We shall omit the proof of this proposition. Note that ϕ is singular.

Theorem 14. *If the factor group G/N is a p-group, and not a P-group, there exists a normal subgroup D of G with the properties* (1) *and* (2) *of Lemma* 9 *and the following property*:

(3′) *the factor group G/D is also a p-group.*

Proof. The intersection of all normal subgroups of G whose factor groups are p-groups will be denoted by $D_p(G)$. (This notation will be used hereafter throughout this paper.) Put here $D_p(G)=H$. Using induction on the order of G, we shall prove that if G/H is not a P-group, $\phi(H)$ is also self-conjugate for any L-isomorphism ϕ.

(a) Take a maximal subgroup M of G containing H. M is then self-conjugate in G. If $\phi(M)=M'$ were not self-conjugate in G', p-Sylow subgroups of G would be P-groups by Lemma 9, which contradicts our assumptions. Hence M' is self-conjugate.

(b) Suppose G/H is a cyclic group of order p^2, and put $H'=\phi(H)$. If H' were not self-conjugate in M', p-Sylow subgroups of G would be again P-groups by the remark given above, which is not the case. Hence H' is self-conjugate in M'. If H' were not self-conjugate in G', H' would be conjugate to a subgroup H_1' of G' $(H'\neq H_1')$. Take a subgroup H_1 of G such that

$\phi(H_1)=H_1'$. Then H_1 would be self-conjugate in M by the same reason as above. Hence the index $[M:H_1]$ would differ from $[M:H]$ because of the definition of H. G would, therefore, be an S-group, which is again not the case. Hence H' is self-conjugate in G'.

(c) Suppose that G/H is of the following type:

$$G/H = \{\overline{A}, \overline{B}\}, \qquad \overline{A}^p = \overline{B}^p = (\overline{A}, \overline{A}, \overline{B}) = (\overline{B}, \overline{A}, \overline{B}) = 1^{(11)}.$$

Take elements A and B of G which correspond to $\overline{A}$ and $\overline{B}$ respectively by the natural homomorphism from G onto G/H, and put $M_1=\{H, A, (A, B)\}$, $M_2=\{H, B, (A, B)\}$, $H_1=\{H, A\}$ and $H_2=\{H_1, B\}$. Then we have clearly $H_1\cup H_2=G$, $M_1\cap H_2=H$ and $M_2\cap H_1=H$. By (a), $\phi(M_1)$ and $\phi(M_2)$ are both self-conjugate. Hence H' is self-conjugate in both $\phi(H_1)$ and $\phi(H_2)$, and thus in G'.

(d) Let P be a p-group whose proper subgroups are all P-groups. Then P is either a P-group or a group of the types discussed above in (b) and (c). For, if P has two maximal subgroups U_1 and U_2, $U_1\cap U_2$ is contained in the center of P. Take two elements A_1 and A_2 such that $U_1=\{U_1\cap U_2, A_1\}$ and $U_2=\{U_1\cap U_2, A_2\}$. Then the commutator (A_1, A_2) is contained in $U_1\cap U_2$. Hence we have $\{A_1, A_2\}=P$ or $(A_1, A_2)=e$. This proves our proposition.

(e) Suppose that G/H is not of the types discussed in (b) and (c). Then G contains a maximal subgroup M such that M/H is not a P-group. Because $H=D_p(M)$, H' is self-conjugate in $M'=\phi(M)$ by the hypothesis of induction. M'/H' is, by Theorem 3, of prime power order, say q^m. We have $H_0'=D_q(M')\subseteq H'$. Let H_0 be a subgroup of G such that $\phi(H_0)=H_0'$. Then by the hypothesis of induction, H_0 is self-conjugate in M and, since M/H is not a P-group, M/H_0 is also a prime power order. Hence by the definition of H, H_0 must coincide with H, and $H'=D_q(M')$. This implies that H' is a normal subgroup of G'.

REMARKS. 1. In this case the condition (4) of Lemma 9 does not hold in general. Example: $G=G_1\times G_2$ and $G'=G_1\times G_3$, where G_1 is the symmetric group of three letters, G_2 is the direct product of two cyclic groups of order 9 and 3, and $G_2=\{A, B\}$, $A^9=B^3=1$, $B\cdot A\cdot B^{-1}=A^4$.

2. We shall denote by $D(H)$ the intersection of all $D_p(H)$ for all p. Then we obtain the following proposition as a corollary of the above two theorems: If $\phi(D(H))$ is not a characteristic subgroup of $\phi(H)$, then G is an S-group. We use this remark in the proof of the following theorem.

THEOREM 15. *If G is not an S-group, there are normal subgroups $\tilde{N}$, H, and D with the following properties*:

(1) $H\supset N\supset\tilde{N}$, $D\supset\tilde{N}$, *and* $D\cup N\neq G$;

(2) *the images* $\tilde{N}'=\phi(\tilde{N})$, $D'=\phi(D)$, *and* $H'=\phi(H)$ *are all self-conjugate in* G'; *and*

(11) Here we mean by (A, B, C) the commutator form $(A, (B, C))$. (A, B) is the usual commutator of A and B, that is, $(A, B)=A\cdot B\cdot A^{-1}\cdot B^{-1}$.

(3) *the groups G/D and $H/\tilde{N}$ are both nilpotent.*

Proof. Put $D(N)=N^*$. If $\phi(N^*)$ were not a characteristic subgroup of N', G would be an S-group as remarked above. Hence $\phi(N^*)$ is a characteristic subgroup of N'. Take one of the subgroups T corresponding to Sylow subgroups of G/N by the natural homomorphism from G onto G/N. If N' is self-conjugate in $\phi(T)$, $\phi(N^*)$ is clearly self-conjugate in $\phi(T)$. If N' is not self-conjugate in $\phi(T)$, Theorems 13 and 14 show that $\phi(N^*)$ is again self-conjugate in $\phi(T)$. Hence $\phi(N^*)$ is a normal subgroup of G'. We shall denote by $\tilde{N}$ a normal subgroup of G such that $\tilde{N}$ is contained in the interval N/N^*, $\tilde{N}'=\phi(\tilde{N})$ is self-conjugate in G' and is maximal in such subgroups. Then $N/\tilde{N}$ is clearly nilpotent.

In the following proof, we have only to consider factor groups $G/\tilde{N}$ and $G'/\tilde{N}'$, namely assume that $\tilde{N}=e$ and N is nilpotent.

Take a p-Sylow subgroup U of N, then $U'=\phi(U)$ is self-conjugate in $V'=\phi(V)$, where V is any subgroup of G corresponding to a q-Sylow subgroup $(q\neq p)$ of G/N by the natural homomorphism from G onto G/N. For, otherwise, G would be an S-group by Theorem 10. Hence we have $N\cup D_p(G)\neq G$ because of the definition of $\tilde{N}$. Put $D=\bigcap_p D_p(G)$, where p runs through all prime factors of the order of N. If $D'=\phi(D)$ were not self-conjugate, G would be an S-group contrary to our assumption. Hence D' is self-conjugate in G', and $D\cup N$ clearly differs from G.

Let S be any p-Sylow subgroup of G (we mean by p any prime factor of the order of N). Then S contains the p-Sylow subgroup U of N. Because of the definition of N, U is not self-conjugate in $S'=\phi(S)$. If S' were not a p-group, $D_p(G)$ would be a Sylow p-complement of G, and $D_p(G)\cup U=D_p(G)\times U$. Hence G would be an S-group, which contradicts our assumption. Hence S' is a p-group.

Let W be a subgroup of G such that $W'=\phi(W)$ is a p-Sylow subgroup of G'. W is clearly not cyclic. If W were not a p-group, $D_p(G)$ would coincide with G by theorems of §4, which is a contradiction. Hence W is a p-group.

We denote by H_p the intersection of all p-Sylow subgroups of G, and put $H=\bigcup_p H_p$, where p runs through all prime factors of the order of N. Then H is nilpotent, and H_p contains U and $H\supseteq N$. Now consider the subgroup $H_p'=\phi(H_p)$ of G'. As proved above, H_p' is the intersection of all p-Sylow subgroups of G' and therefore self-conjugate in G'. Hence $H'=\phi(H)$ is also a normal subgroup of G'.

As consequences of this theorem we obtain the following theorems.

THEOREM 16. *Let G and G' be two groups L-isomorphic to each other, let ϕ be this L-isomorphism, and let K be a normal subgroup of G. The subgroup $\phi(K)$ of G' is also self-conjugate in G' if one of the following conditions is satisfied:*

(1) *G/N has no solvable normal subgroup,*

(2) *G/N is perfect, or*

(3) *N is perfect.*

THEOREM 17. *Let $G=G_0\supset G_1\supset G_2\supset \cdots \supset G_r=e$ be a composition series of a group G, and ϕ an L-automorphism of G. If G is not an S-group, $G=\phi(G_0)\supset\phi(G_1)\supset \cdots \supset\phi(G_r)=e$ is also a composition series of G.*

Proof. If the factor group G_{i-1}/G_i is a nonabelian simple group, $\phi(G_i)$ is self-conjugate in $\phi(G_{i-1})$ by Theorem 16. Hence if $\phi(G_i)$ were not self-conjugate in $\phi(G_{i-1})$, G_{i-1}/G_i would be an abelian P-group. Moreover Theorem 13 shows that a p-Sylow subgroup of G would be mapped onto a group not of prime power order. By Theorem 10, G would be an S-group, contrary to our assumption. Hence $\phi(G_i)$ is a normal subgroup of $\phi(G_{i-1})$. q.e.d.

We shall call the maximal solvable normal subgroup of G its radical. Then we have the following theorem.

THEOREM 18. *Let R be the radical of a group G, and ϕ an L-isomorphism from G onto another group H. $\phi(R)$ is then the radical of H.*

Proof. $\phi(R)$ is self-conjugate by Theorem 16 and solvable by Theorem 12. Hence $\phi(R)$ is contained in the radical R^* of H. Similarly we have $\phi^{-1}(R^*)\subseteq R$, or $R^*\subseteq\phi(R)$. Hence we have $\phi(R)=R^*$. q.e.d.

Again let G be a group L-isomorphic to H. Suppose G is a direct product of two groups G_1 and G_2. H is not always directly decomposable. There are, however, some interesting cases in which H is directly decomposable. One such case, the most important in our discussions, is that of Lemma 3. Now we shall have some other cases as consequences of Theorem 16.

THEOREM 19. *Let G and H be two groups L-isomorphic to each other and ϕ the L-isomorphism from G onto H. If G is a direct product of two groups G_1 and G_2, and if one of the groups G_1 and G_2 is perfect, then H is directly decomposable and $H=\phi(G_1)\times\phi(G_2)$. The same holds if the radical of G coincides with e.*

For perfect groups, we obtain from Theorem 16 the following theorem.

THEOREM 20. *Any L-isomorphism from a perfect group onto another group is always normal.*

Remembering Theorems 11 and 12, we have the following interesting proposition. Let G be a perfect group. If G is L-isomorphic to another group H, H is also a perfect group of the same order as G. Moreover the modular lattice formed by all normal subgroups of H is isomorphic to that of G. In particular if G is a nonabelian simple group, then H is also a nonabelian simple group of the same order as G.

In this case the author has not yet been able to decide whether G is isomorphic to H in the sense of group theory, even with G assumed to be simple. We can, however, give a solution of this question in a very spe-

cial case, which we shall deal with in the next section. In this connection we shall give here the following theorem, which gives an affirmative answer to the first part of Birkhoff's Problem 41(a) (cf. Birkhoff [3*, p. 99]) in a generalized form. The last part of this problem will be solved by Theorem 22 in the next section.

THEOREM 21. *Let G be a perfect group, and ϕ an L-isomorphism from G onto another group H. If we denote by Z the center of G, $\phi(Z)$ is also the center of H.*

Proof. Take any element A of Z with prime power order, say p^n, and put $\{A\}=U$. Take any q-Sylow subgroup S_q of G $(q\neq p)$, then $U\cup S_q=U\times S_q$ and this group is L-decomposable. Hence $\phi(U)$ is elementwise permutable with $\phi(S_q)$ by Lemma 3. As G is perfect, we have $\bigcup_{q\neq p} S_q=G$, where q runs through all prime factors of the order of G except p. Hence the centralizer of $\phi(U)$ in H contains $\bigcup_{q\neq p}\phi(S_q)=H$. This implies that $\phi(U)$ is contained in the center Z^* of H and that $\phi(Z)\subseteq Z^*$.

As H is also perfect by Theorem 12, we have similarly $\phi^{-1}(Z^*)\subseteq Z$ or $Z^*\subseteq\phi(Z)$. Hence we have $\phi(Z)=Z^*$, which proves our theorem.

REMARK. Theorem 20 is valid for wider class of groups, for example, the symmetric groups of n letters $(n>3)$. Hence we have the following proposition (cf. Baer [1]).

Let G be the symmetric group of n letters $(n>3)$. If G is L-isomorphic to H, H is also the symmetric group of n letters.

6. THE LATTICE OF SUBGROUPS OF A DIRECT PRODUCT OF TWO ISOMORPHIC SIMPLE GROUPS

First we shall give some remarks on L-automorphisms of a group G.

The set of all L-automorphism of G clearly forms a group $\mathfrak{A}$. Any automorphism σ of G induces an L-automorphism ϕ_σ and this mapping $\sigma\rightarrow\phi_\sigma$ is a homomorphism from the group $A(G)$ of all automorphisms of G into $\mathfrak{A}$. We shall denote by K the kernel of this homomorphism. K consists of automorphisms of G such that ϕ_σ are the identical L-automorphism.

If the center of G coincides with e, the group $I(G)$ of inner automorphisms of G forms a subgroup of $A(G)$ isomorphic to G. In such a case $N=I(G)\cap K$ consists of elements of G, which are permutable with every subgroup of G. Baer calls such a subgroup the "norm" (Kern) of G(12). We now prove the following lemma.

LEMMA 10. *The norm of a (finite) group G coincides with e if the center of G coincides with e.*

Proof. We shall prove that if a prime number p divides the order of the norm, then p divides the order of the center. Let S be a p-Sylow subgroup of the norm. As S is a normal subgroup of G, S contains an element A of the

(12) Cf. R. Baer, Compositio Math. vol. 6 (1939).

center of a p-Sylow subgroup of G. We shall show that A is contained in the center of G. Take any q-Sylow subgroup S_q of G, where q is any prime factor of the order of G other than p. By the definition of the norm, S_q is self-conjugate in $\{S_q, A\}$, and hence $\{A\}$ is a p-Sylow subgroup of $\{S_q, A\}$. If $\{A\}$ were not self-conjugate in $\{S_q, A\}$, any conjugate subgroup of $\{A\}$ would be permutable with $\{A\}$. This is, however, a contradiction, since $\{A\}$ is a p-Sylow subgroup of $\{S_q, A\}$. Hence we have $\{S_q, A\} = S_q \times \{A\}$, which implies that A is contained in the center of G. q.e.d.

From this lemma it follows that if the center of G coincides with e, then $N = I(G) \cap K = e$, that is, K is elementwise permutable with $I(G)$. On the other hand, if the center of G is e, the centralizer of $I(G)$ in $A(G)$ coincides with e. Hence we have $K = e$ and the homomorphism $\sigma \to \phi_\sigma$ from $A(G)$ onto $\mathfrak{A}$ is an isomorphism. Hence we obtain:

THEOREM 22. *If the center of G coincides with e, any L-automorphism is induced by at most one automorphism of G, that is, the group of all automorphisms of G is isomorphic to a subgroup of the group of all L-automorphisms of G.*

In the same notation as above, we may therefore write $\mathfrak{A} \supseteq A(G) \supseteq G$, identifying the isomorphic groups, if the center of G is e. $A(G)$ is not always self-conjugate in $\mathfrak{A}$ (for example, when G is a P-group or the alternating group of 4 letters). But we have $\mathfrak{A} = A(G)$ if G is the symmetric group of n letters ($n > 3$). The study of the relation between $\mathfrak{A}$ and $A(G)$ seems to the author to be one of the most interesting problems concerning subgroup lattices. The clarification of this relation may throw light not only upon the study of subgroup lattices, but also upon the structure theory of finite groups. Theorems 10 and 17 show interesting facts concerning this problem.

We prove now the following theorem.

THEOREM 23. *Let G be a nonabelian simple group. If a group H is L-isomorphic to $G \times G$, then H is isomorphic to $G \times G$ in the sense of group theory.*

Proof. In order to avoid confusion we shall denote $G \times G$ by $\mathfrak{G} = G_1 \times G_2$, where $G_1 \cong G_2 \cong G$. Let ϕ be the L-isomorphism from $\mathfrak{G}$ onto H. We have then $H = \phi(G_1) \times \phi(G_2)$ by Theorem 19. Because $G_1 \cong G_2$, $\mathfrak{G}$ contains a subgroup U such that $G_i U = \mathfrak{G}$ and $G_i \cap U = e$ ($i = 1, 2$). Such a subgroup consists of products of corresponding elements under some isomorphism between G_1 and G_2. We shall call such subgroups C-subgroups of $\mathfrak{G}$.

Let U be any C-subgroup of $\mathfrak{G}$. Then we have $\phi(G_i) \cup \phi(U) = H$ and $\phi(G_i) \cap \phi(U) = e$, which implies that $H_1 = \phi(G_1) \cong \phi(G_2)$ and that $\phi(U)$ is a C-subgroup of H. Hence H is also a direct product of two isomorphic nonabelian simple groups.

Take two C-subgroups U and V of $\mathfrak{G}$, and suppose that U consists of products $A_1 A_1^\sigma$ where σ is an isomorphism from G_1 onto G_2 and V consists of products $A_1 A_1^\tau$, where τ has the same meaning as σ. $\sigma\tau^{-1}$ is then an auto-

morphism of G_1 which is determined by U and V. We shall denote it by $\sigma(U; V)$. Then any automorphism of G_1 has such a form, even if we take a fixed "first component" U, and, for any three C-subgroups U, V, and W

$$(*) \qquad \sigma(U; V)\sigma(V; W) = \sigma(U; W).$$

An automorphism $\sigma = \sigma(U; V)$ may have another form $\sigma = \sigma(W; T)$, where T is also a C-subgroup of $\mathfrak{G}$. We shall prove that, if $\sigma(U; V) = \sigma(W; T)$, then T is determined by U, V, W, and the lattice $L(\mathfrak{G})$.

Together with the identical automorphism of G_2, $\sigma = \sigma(U; V)$ induces an automorphism $\bar{\sigma}$ of $\mathfrak{G}$, which induces an L-automorphism ϕ_σ of $\mathfrak{G}$. ϕ_σ maps U onto V, W onto T, and fixes G_1 and all subgroups of G_2. We prove that if an L-automorphism ϕ of $\mathfrak{G}$ maps U onto V and fixes G_1 and all subgroups of G_2, then ϕ maps W onto T. Take another L-automorphism ψ of $\mathfrak{G}$ satisfying the same conditions as ϕ. Then $\pi = \phi^{-1}\psi$ fixes U, G_1, and all subgroups of G_2. Take any subgroup K of U, then $K = K \cdot G_1 \cap U$ and $K \cdot G_1 = G_1 \cup (K \cdot G_1 \cap G_2)$, which shows that $\pi(K) = K$. Similarly for any subgroup S of G_1, we have $\pi(S) = S$, because $S = G_1 \cap SG_2$ and $SG_2 = G_2 \cup (SG_2 \cap U)$. π induces thus the identical L-automorphism in G_1. Take any C-subgroup Y of $\mathfrak{G}$, and consider the automorphism $\tau = \sigma(\pi(Y); Y)$ of G_1. Together with the identical automorphism of G_2, τ induces an automorphism $\bar{\tau}$ of $\mathfrak{G}$, which induces an L-automorphism ϕ_τ of $\mathfrak{G}$. ϕ_τ maps $\pi(Y)$ onto Y, and fixes G_1 and all subgroups of G_2. Hence $\phi_\tau\pi$ fixes Y, G_1, and all subgroups of G_2. Thus $\phi_\tau\pi$ must induce the identical L-automorphism on G_1. Hence ϕ_τ itself induces the identical L-automorphism on G_1. By Theorem 22, τ must be the identical automorphism, which implies that $\pi(Y) = Y$ for any C-subgroup Y of $\mathfrak{G}$. This proves our assertion.

We shall denote the image $\phi(U)$ of a C-subgroup U of $\mathfrak{G}$ by the same letter with an accent, that is, by U'. U' is clearly a C-subgroup of H, and therefore $\sigma(U'; V')$ is also an automorphism of H_1. We consider the mapping $f: \sigma(U; V) \to \sigma(U'; V')$. Our assertion proved above shows that $\sigma(U; V) = \sigma(W; T)$ holds in $\mathfrak{G}$ if $\sigma(U'; V') = \sigma(W'; T')$ in H. Hence f is a one-to-one mapping from the group $A(G_1)$ of automorphisms of G_1 onto that of H_1. Moreover the relation (*) shows that f is surely an isomorphism, that is, we have $A(G_1) \cong A(H_1)$.

$f(I(G_1))$ is self-conjugate in $A(H_1)$. As H_1 is simple, $I(H_1) \cap f(I(G_1)) = I(H_1)$ or e. If $I(H_1) \cap f(I(G_1)) = e$, $f(I(G_1))$ would be elementwise permutable with $I(H_1)$. This is a contradiction. Hence we have $I(H_1) \subseteq f(I(G_1))$ and therefore $I(H_1) = f(I(G_1))$. This proves our theorem.

This theorem is surely valid for a wider class of groups, but the author has not yet been able to determine such a class more precisely.

The following theorem is a direct consequence of the above theorem.

THEOREM 24. *Let G be a simple group. G is isomorphic to a group H if and only if the lattices of subgroups $L(G \times G)$ and $L(H \times H)$ are isomorphic.*

REMARK. An element of any lattice is called characteristic when it is left invariant by all automorphisms and is not equal to the greatest or least element.

We consider now groups whose lattices of subgroups have no characteristic element. Let G be such a group. Then G must be solvable, or its radical coincides with e by Theorem 18. If it is solvable, G is by Lemma 9 either a P-group or a B-group (that is, a group whose lattice of subgroups is a Boolean lattice). If it is not solvable, G is a direct product of nonabelian simple groups by Theorem 16, that is, $G=G_1\times\cdots\times G_t$, where each G_i is a simple group. If, for instance, $G_1\cong G_2$, then $\sigma(G_1\times G_2)\cong G_1\times G_2$ for every L-automorphism σ of G by Theorem 23. Hence we have $G_1\cong\cdots\cong G_t$. Hence we obtain the following proposition.

If $L(G)$ of a group G has no characteristic element, G is one of the following types: (1) a P-group, (2) a B-group, (3) a simple group, (4) a direct product of isomorphic simple groups, or (5) a direct product of mutually L-ismorphic, but not isomorphic, nonabelian simple groups.

The author has, however, not been able to decide whether groups of the last type actually exist.

7. NILPOTENT GROUPS WITH DUALS

Let G and H be two groups. In this section we do not assume G to be finite, except when we mention it particularly. G and H are said to be duals to each other if there is an anti-isomorphism from $L(G)$ onto $L(H)$, and this anti-isomorphism is called a dualism from G onto H. The dualisms for abelian groups were treated by Baer [2], who determined completely the structure of abelian groups with duals. Here we shall consider dualisms for nilpotent groups.

First we shall prove the following lemma.

LEMMA 11. *A nilpotent group G is finite if $G/C(G)$ is finite, where $C(G)$ is a commutator subgroup of G.*

Proof. Take a descending central series

$$G = H_1 \supset H_2 \supset \cdots \supset H_c \supset H_{c+1} = e$$

of G, where $H_{i+1}=(G, H_i)$ $(i=1, 2, \cdots, c)$.

We prove this lemma by induction on the length c. Suppose our lemma is proved in the case that the length is smaller than c. Then H_1/H_c is a group of finite order, say of order n.

H_c is contained in the center of G and is generated by the commutators (A, B) of $A\in G$ and $B\in H_{c-1}$. As A^n is contained in H_c, we have $(A, B)^n=(A\cdot B\cdot A^{-1}\cdot B^{-1})^n=A^nBA^{-n}B^{-1}=e$. Hence the order of any element of H_c is at most n.

Select one element from each coset modulo H_c. We shall denote these ele-

ments by $A_1, \cdots, A_n$ and put $U=\{A_1, \cdots, A_n\}$. U is self-conjugate because $G=U\cup H_c$, and $G/U\cong H_c/H_c\cap U$ is abelian. Hence we have $U\supseteq H_2 \supseteq H_c$ and $U=G$. G has thus a finite set of generators. By the method of Reidemeister, H_c has also a finite set of generators, and hence H_c is a finite group. q.e.d.

THEOREM 25. *Let G be a nilpotent group. G has a dual if and only if G has the following two properties*:

(1) *G has no element of infinite order, and*

(2) *every primary component of G is a finite M-group which is not a 2-Hamiltonian group.*

Proof. We shall assume that G has a dual H and denote by ϕ the dualism from G onto H. The first condition (1) was proved by Baer [2] for general groups. Hence, as G is nilpotent, G is a direct product of its primary components G_p, that is, $G=\prod G_p$. We have then $L(G)\cong\prod L(G_p)$ by Lemma 3. G has a dual if all G_p have duals. Hence we can assume G itself to be primary. Denote the commutator subgroup of G by $C(G)$ and put $\phi(C(G))=K$. $G/C(G)$ and K are duals to each other by ϕ. Since $G/C(G)$ is a primary abelian group, $G/C(G)$ must be finite by a theorem of Baer [2]. Hence, by Lemma 11, G itself is finite.

We shall prove next that if a finite p-group G has a dual, G is an M-group.

Take the Φ-subgroup Φ of G. Then $\phi(\Phi)=N$ is clearly self-conjugate in H and is a P-group. If N is a p-group, H is also a p-group. $L(H)$ is therefore lower semi-modular, and at the same time upper semi-modular as an image of $L(G)$ by ϕ. $L(H)$ is then a modular lattice by a theorem of Birkhoff (cf. Birkhoff [3, p. 43]). Hence C is an M-group.

If N is not a p-group, its order is p^nq, where p and q are primes ($p>q$). Since two groups of order q do not generate a q-group, a q-Sylow subgroup Q of H is either a cyclic group or a generalized quaternion group([13]). As H is a J-group, the p-Sylow subgroup S of H is self-conjugate and covers a normal subgroup T of H by Lemma 2. Take two subgroups U and V of G such that $\phi(U)=S$ and $\phi(V)=T$. It is clear that V and H/T are duals to each other and that U and H/S are also duals to each other. Hence Q, isomorphic to H/S, is cyclic. U is then cyclic and V is an M-group. (Note that $p>q\geqq 2$.) Hence $L(V)$ is clearly self-dual, and V and H/T are L-isomorphic. By Theorem 3, H/T must be a P-group. This implies that Q is contained in N. Since $L(H)$ has no reducible interval, we have $H=N$. Hence H is again an M-group. We obtain thus the first part of our theorem.

If G is a finite modular p-group which is not a 2-Hamiltonian group, G is L-isomorphic to an abelian group by a theorem of Jones [6]. Hence $L(G)$ is

([13]) A generalized quaternion gurop is a group of the following structure: $G=\{A, B\}$, $A^{2^n}=1$, $B^2=A^{2^{n-1}}$ and $B\cdot A\cdot B^{-1}=A^{-1}$. A group which has only one minimal subgroup is either cyclic or a generalized quaternion group. Cf. Zassenhaus [9, p. 112].

clearly self-dual. This completes our proof.

By this theorem we obtain the following two theorems, one of which gives an affirmative answer to a conjecture of Baer [2].

THEOREM 26. *Let G be a nilpotent group. Two groups G and H are duals to each other if and only if $L(G)$ is self-dual and $L(H) \cong L(G)$.*

THEOREM 27. *If a nilpotent group has a dual, it is L-isomorphic to some abelian group.*

8. SOLVABLE GROUPS WITH DUALS

In this section we consider only finite solvable groups.

LEMMA 12. *If a solvable group has a dual, it is a J-group.*

Proof. Let G be a solvable group with a dual H, and ϕ be the dualism from G onto H. We shall prove this lemma by induction on the order of G.

As G is solvable, G contains a normal subgroup N with prime index, say p. $N^* = \phi(N)$ is then a minimal subgroup of H. Suppose N^* is self-conjugate. Then N and H/N^* are duals to each other and hence by the hypothesis of induction H/N^* has a principal series all of whose factor groups are of prime orders. H itself clearly has such a principal series. Lemma 12 follows now from Lemma 2.

If N^* is not self-conjugate, some L-automorphism σ of G maps N onto another group N_1, different from N. If N_1 is not self-conjugate, G is an S-group by Theorem 17. Hence we have $G = P \times G_1$ and $L(G) = L(P) \times L(G_1)$, where P is a P-group. G_1 has clearly a dual and therefore by the hypothesis of induction it is a J-group. Since a P-group is clearly a J-group, G is also a J-group.

Now we shall assume that N_1 is self-conjugate. If the index $(G:N_1) = q$ is not equal to p, a p-Sylow subgroup of G is mapped by σ onto a q-Sylow subgroup. Hence by Theorem 5, G contains a normal subgroup D such that D is invariant under all L-automorphisms of G and the factor group G/D is cyclic. $D^* = \phi(D)$ is then self-conjugate in H and is cyclic. Clearly D and H/D^* are duals to each other, and hence H/D^* has a principal series, all of whose factor groups are of prime orders. Since D^* is cyclic, H itself has such a principal series, and this proves our lemma.

It remains to prove our lemma in the case $(G:N_1) = p$. Put $K = \bigcap_\sigma \sigma(N)$, where σ runs through all L-automorphisms of G. We may assume that each $\sigma(N)$ is a normal subgroup of G with index p. Under this assumption G/K is an abelian P-group. $K^* = \phi(K)$ is clearly self-conjugate in H. Since K and H/K^* are duals to each other, K is a J-group by the hypothesis of induction. Let r be the greatest prime factor of the order of K. Then an r-Sylow subgroup N_0 of K is a normal subgroup of G. If N_0 is cyclic, we can prove our lemma in a way similar to the above. Suppose that N_0 is not cyclic. If $N_0^* = \phi(N_0)$ is not self-conjugate in H, G is an S-group by Theorem 10, and our lemma follows by induction.

Now we may assume that N_0^* to be self-conjugate. Then H/N_0^* is a p-group or a P-group by Theorem 25. If $r \neq p$, the extension of N_0 by G/N_0 splits, that is, there is a subgroup Q of G such that $QN_0 = G$ and $Q \cap N_0 = e$. Since H/N_0^* is a p-group or a P-group, there exists a normal subgroup U of N_0 such that $\phi(U)$ is a normal subgroup of H with prime index. Then $\phi(Q)$ covers $\phi(Q) \cap \phi(U)$. Hence $U \cup Q$ covers Q. We have therefore $U = (U \cup Q) \cap N_0$ which implies that U is a normal subgroup of prime order. Our lemma follows then immediately. If $r = p$, a p-Sylow subgroup P of G is self-conjugate by the hypothesis of induction. We can prove our lemma in a way similar to the above, replacing N_0 by P.

LEMMA 13. *If a solvable group has a dual, it is an M-group.*

Proof. Let G be a solvable group with a dual H, and ϕ the dualism from G onto H. We shall again use induction on the order of G. Let p be the greatest prime factor of the order of G and P its p-Sylow subgroup. P is self-conjugate by Lemma 12. If $P^* = \phi(P)$ is not self-conjugate, G is an S-group by theorems of §4, that is, $G = T \times G_1$, where T is a P-group. As G_1 has a dual, it is an M-group by the hypothesis of induction, which implies that G is an M-group.

Now we may assume that P^* is self-conjugate in H. By the hypothesis of induction $\overline{G} = G/P$ is an M-group, which is a direct product of groups $\overline{U}_i$ $(i = 1, 2, \cdots, s)$, having mutually prime orders(14). $\overline{U}_i$ is either a P-group or of prime power order. Take a subgroup U_i $(i = 1, 2, \cdots, s)$ of G, corresponding to $\overline{U}_i$ by the natural homomorphism from G onto $\overline{G}$.

First we shall assume that $s = 1$. In order to prove $\Phi(G) \supseteq \Phi(P)$, take any maximal subgroup M of G. $T = M \cap P$ is clearly a normal subgroup of G, which implies that $T\Phi(P) = P$ or T. The former equality implies that $T = P$ and hence we have $T \supseteq \Phi(P)$ or $\Phi(G) \supseteq \Phi(P)$.

Assume now $\Phi(P) = e$. There is a subgroup V of G such that $VP = G$ and $V \cap P = e$. Take any minimal subgroup U of P. Then we have $\phi(U) = (\phi(U) \cap \phi(V)) \cup P^*$ and hence $U = (U \cup V) \cap P$, which implies that U is self-conjugate in G. Applying the hypothesis of induction to the groups G/U and $\phi(U)$, we see that G/U is a P-group or a direct product of P/U and UV/U, which implies that G itself is a P-group or a direct product of P and V. This proves our lemma in this case.

If $\Phi(P) \neq e$, $G/\Phi(P)$ is a P-group or L-decomposable as proved above. If it is L-decomposable, $G/\Phi(G)$ is clearly also L-decomposable. Hence by Lemma 5, G is L-decomposable, too, which proves our lemma. Suppose $G/\Phi(P)$ is a P-group. If P is cyclic, our lemma follows immediately. If P were neither cyclic nor a P-group, H/P^* would be a P-group. By our assumption $\phi(\Phi(P))$ is a P-group, and hence P^* is also of order p, since P^* is self-conjugate. Hence H would be a p-group and G would be a P-group

(14) Cf. Iwasawa [5], or §§1, 4 of this paper.

contrary to our assumption. Our lemma is thus proved when $s=1$.

If $s \geqq 2$, $U_i^* = \phi(U_i)$ are all self-conjugate in H. Since U_i and H/U_i^* are duals to each other, U_i is a P-group or an L-decomposable group having P as its direct factor. If two groups U_i and U_j were both P-groups, P^*/U_i^* would have the same order as P^*/U_j^*, which is clearly a contradiction. Hence at most one of the U_i is a P-group. This completes our proof.

By this lemma we obtain the following theorems.

THEOREM 28. *A (finite) solvable group G has a dual if and only if it is a direct product of groups G_i of mutually prime orders, each direct factor G_i being either a P-group or an M-group of prime power order which is not a 2-Hamiltonian group.*

THEOREM 29. *If a solvable group has a dual, then it is L-isomorphic to an abelian group.*

The author has not yet been able to decide whether the assumptions of the finiteness or the solvability are necessary for the validity of these theorems. If we could prove that a finite nonabelian simple group had no dual, the above two theorems would be valid for any finite group.

BIBLIOGRAPHY

1. R. Baer, *The significance of the system of subgroups for the structure of the group*, Amer. J. Math. vol. 61 (1939) pp. 1–44.

2. ———, *Duality and commutativity of groups*, Duke Math. J. vol. 5 (1939) pp. 824–838.

3. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940.

3*. ———, *Lattice theory*, rev. ed., 1949.

4. W. Burnside, *Theory of groups of finite order*, 2d ed., 1914.

5. K. Iwasawa, *Über die Gruppen und die Verbände ihrer Untergruppen*, J. Fac. Sci. Imp. Univ. Tokyo Sec. I, IV-3 (1941) pp. 171–199.

6. A. W. Jones, *The lattice isomorphisms of certain finite groups*, Duke Math. J. vol. 12 (1945) pp. 541–560.

7. O. Ore, *Structures and group theory* II. Duke Math. J. vol. 4 (1938) pp. 247–269.

8. ———, *Contributions to the theory of groups of finite order*, Duke Math. J. vol. 5 (1939) pp. 431–460.

9. H. Zassenhaus, *Lehrbuch der Gruppentheorie* I, 1937.

TOKYO UNIVERSITY,
TOKYO, JAPAN.