ON THE $L$-HOMOMORPHISMS OF FINITE GROUPS

BY

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Let $G$ be a finite group. We shall denote by $L(G)$ the lattice formed by all subgroups of $G$. A homomorphic mapping from $L(G)$ onto a lattice $L$ is called an $L$-homomorphism from $G$ onto $L$.

In his previous paper (Suzuki [5]'), dealing with $L$-isomorphisms of finite groups, the author determined the structure of groups, $L$-isomorphic to a $p$-group, and proved that groups $L$-isomorphic to a solvable or a perfect group are also solvable or perfect respectively. In this paper we shall generalize these results to the case of $L$-homomorphisms and study the relations between $L$-homomorphisms and $L$-isomorphisms. In particular, we shall determine all $L$-homomorphisms from a perfect group, and as an application, we shall also determine the neutral elements of $L(G)$.

$L$-homomorphisms of finite groups were first considered by P. Whitman [6], who dealt with the case when $L$ is the subgroup lattice of a cyclic group. His result will be sharpened to Theorem 1 in §1 which will play a fundamental rôle in our study.

1. SOME REMARKS ON $L$-HOMOMORPHISMS

Let $G$ be a group and $\phi$ be an $L$-homomorphism from $G$ onto a lattice $L$. A set of elements of $L(G)$, which is mapped onto a fixed element of $L$, forms a convex sublattice (2) of $L(G)$, and in particular elements mapped to the least (greatest) element $0$ ($I$) (3) of $L$, form a (dual) ideal of $L(G)$. The greatest (least) element of such a (dual) ideal is called the "lower (upper) kernel," or shortly "$l$- (u-) kernel" of $\phi$ in $G$.

First we shall prove the following lemma.

**Lemma 1.** [Cf. 6]. Let $G$ be a group and $\phi$ be an $L$-homomorphism from $G$ onto a chain $C_n$ of dimension $n$. Then there are two subgroups $N$ and $G_0$ of $G$ and a prime number $p$ with the following properties:

1. $N$ is a Sylow $p$-complement (4) of $G$,
2. a $p$-Sylow subgroup $S_p$ contains $G_0$ and is cyclic or a generalized quaternion group (g. q. group),

Received by the editors April 8, 1950.

(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) For general lattice theory, see Birkhoff [2].

(3) In the following we always denote by $0$ ($I$) the least (greatest) elements of various lattices and do not mention it particularly, if there is no risk of misunderstanding.

(4) Sylow $p$-complements of a group of order $p^g$, $(p, g) = 1$, are subgroups of index $p^a$. Cf. Suzuki [5, footnote 8].
(3) If the order of $G_0$ is $p^m$, we have $m \geq n$, and
(4) If $S_p$ is a $g. q.$ group, the order of $G_0$ is 2.
Conversely if there are normal subgroups $N$ and $G_0$ of $G$ and a prime number $p$ with the properties (1)–(4), then $L(G)$ is homomorphic to a chain $C_n$ of dimension $n$.

Proof. Denote by $G_0$ the $u$-kernel of $\phi$. $G_0$ has only one maximal subgroup and hence $G_0$ is a cyclic group of prime power order. Let $p^m$ be this order. Take a Sylow subgroup $S_p$ of $G$ containing $G_0$. If there were a noncyclic subgroup $V$ of $S_p$ covering $G_0$, $V$ would be $L$-homomorphic to $C_n$. Since the factor group $V/\Phi(V)$ \((\dagger)\) is a $P$-group, there would exist at least two maximal subgroups $M_1$ and $M_2$ of $V$, different from $G_0$. Both $\phi(M_1)$ and $\phi(M_2)$ would be maximal elements of $C_n$, and we should, therefore, have $\phi(V) = \phi(M_1 \cup M_2) = \phi(M_1) \cup \phi(M_2) \neq I$, which is clearly a contradiction. Hence all subgroups of $S_p$ covering $G_0$ are cyclic and $S_p$ has only one subgroup of order $p$. $S_p$ must be cyclic or a $g. q.$ group \([\text{cf. 7, p. 112}]\).

Take a $q$-Sylow subgroup $S_q$, where $q$ is any prime factor of the order of $G$ other than $p$. We have $\phi(S_q) \cap \phi(S_p) = 0$ because $S_q \cap S_p = e$. This implies that $\phi(S_q) = 0$. Put $N = \bigcup_{q \neq p} S_q$, where $q$ runs through all prime factors of the order of $G$ except $p$. Then $N$ is clearly self-conjugate. Take a normalizer $N_q$ of $S_q$ in $G$, then we have $N_q \cdot N = G$. Hence $N_q$ contains a $p$-Sylow subgroup of $G$. Choosing a suitable $q$-Sylow subgroup $S_q$ we may assume that $N_q \supset S_p \supset G_0$. We shall prove that $G_0$ is self-conjugate in $H = G_0 \cdot S_q$, using induction on the dimension of the interval $H/G_0$. We take a maximal subgroup $M$ of $H$ containing $G_0$, then $M \cap S_q$ is self-conjugate in $H$. $H/M \cap S_q$ is $L$-homomorphic to $C_n$ because $\phi(M \cap S_q) = 0$ and $\phi(H) = I$. Hence we have only to prove our assertion in the case where $G_0$ is maximal. If $G_0$ were not self-conjugate in such a case, there would be at least two subgroups $G_1$ and $G_2$ of $H$, conjugate to and different from $G_0$. We should then have $\phi(G_1) = \phi(G_2) \neq I$, which gives the contradiction that $\phi(H) = \phi(G_1) \cup \phi(G_2) \neq I$. Hence $G_0$ is self-conjugate in $H$. Since $q$ is an arbitrary prime factor other than $p$, this implies that $G_0$ is self-conjugate in $G$ and that $G_0$ is elementwise permutable with $N$. By the definition of $N$ this implies that $N \cap G_0 = e$ and $N \cdot S_p = G$. The former part of our lemma now follows immediately.

Conversely, suppose $G$ to have such a structure. Then $G$ is proved to be $L$-homomorphic to a chain as follows.

When $S_p$ is a $g. q.$ group, the mapping $\phi$ from $L(G)$ onto the two-element lattice $C_2$ defined by
\[
\phi(V) = \begin{cases} 
I & \text{if the order of } V \text{ is even}, \\
0 & \text{if the order of } V \text{ is odd}, 
\end{cases}
\]
(\dagger) We mean by $\Phi(V)$ the $\Phi$-subgroup of $V$, which is defined to be the intersection of all maximal subgroups of $V$. Cf. Zassenhaus \([\text{7, p. 44}]\).
is an $L$-homomorphism from $G$ onto $C_2$. For subgroups of even order contain $G_0$ and those of odd order are contained in $N$.

When $S_2$ is cyclic, the mapping $\phi$ from $L(G)$ onto the chain $C_m$ of dimension $m$ defined by

$$\phi(V) = a_v \quad (v = \min (m, \lambda), \ p^{|V:e|})$$

is an $L$-homomorphism from $G$ onto $C_m$, where $a_v$ is the element of $C_m$ with dimension $v$, and $\lambda$ is the exact power of $p$ dividing the order of $V$. For $G_0 \cap N$ is $L$-decomposable, and subgroups of order $p^m \cdot g$ with $\mu \geq m ((p, g) = 1)$ contain $G_0$. Hence $G$ is clearly $L$-homomorphic to a chain $C_n$ with $n \leq m$. Note that the mapping $\phi$ defined above is equivalent to the mapping $U \rightarrow G_0 \cap U$ from $L(G)$ onto a chain $L(G_0)$.

By this lemma we can easily generalize Whitman’s theorem as follows.

**Theorem 1.** A group $G$ is $L$-homomorphic to a cyclic group $G'$ of order $\prod_{i=1}^{n} q_i^k$ if and only if there exist prime numbers $p_i (i=1, 2, \ldots, n)$ and two normal subgroups $G_0$ and $N$ with the following properties:

1. $p_i \neq p_j (i \neq j)$,
2. the order of $G$ is $\prod_{i=1}^{n} p_i^k \cdot g \cdot (p_i, g) = 1 (i=1, 2, \ldots, n)$,
3. the order of $G_0$ is $\prod_{i=1}^{n} p_i^{k_i}$ with $f_i \equiv a_i (i=1, 2, \ldots, n)$,
4. $N$ is of order $g$ and the factor group $G/N$ is a nilpotent group whose $p$-Sylow subgroups are cyclic, or a $g \cdot q$ group, and
5. if $p_i = 2$ and if a 2-Sylow subgroup is a $g \cdot q$ group, then $a_i = e_i = 1$.

**Proof.** The subgroup lattice $L(G')$ of a cyclic group $G'$ is a direct product of chains, so that there are natural homomorphisms $\psi_i (i=1, 2, \ldots, n)$ from $L(G')$ onto its direct components. Let $\phi$ be the homomorphism from $L(G)$ onto $L(G')$. Then $\psi_i \phi$ is clearly a homomorphism from $L(G)$ onto a chain. Hence $G$ has a prime factor $p_i$ and two normal subgroups $G_0$ and $N_i$ with the properties given in Lemma 1. Now we have clearly $p_i \neq p_j (i \neq j)$. Put $G_0 = \bigcup G_i$ and $N = \bigcap N_i$, then $G_0$ and $N$ satisfy the properties of Theorem 1.

Conversely, suppose that $G$ has prime factors $p_i (i=1, 2, \ldots, n)$ and two normal subgroups with the above properties. Then $G$ has the Sylow $p_i$-complement $N_i$ and $G_0$ is nilpotent. Let $G_i$ be a $p_i$-Sylow subgroup of $G_0$. Then both $N_i$ and $G_i$ are self-conjugate in $G$. By Lemma 1, $G$ is $L$-homomorphic to $L(G_i)$. We shall denote by $\phi_i$ this $L$-homomorphism from $G$ onto $L(G_i)$. We have then

\[ \phi_i(G_i) = 0 \quad (i \neq j). \]

Let $\phi_0$ be a mapping from $L(G)$ into a direct product $L = L(G_1) \times \cdots \times L(G_n)$ defined by

$$\phi_0(V) = (\phi_1(V), \ldots, \phi_n(V)).$$

$\phi_0$ is clearly an $L$-homomorphism from $G$ into $L$, and in virtue of (*) it is surely onto $L$. As is easily proved, there exists a homomorphism $\psi$ from $L$
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onto $L(G')$ of a cyclic group $G'$ of order $\prod q_i$. $\psi \phi$ is clearly an $L$-homomorphism from $G$ onto $L(G')$. q.e.d.

REMARK. The $l$-kernel and the $u$-kernel of $\phi$ are both self-conjugate, if $L$ is a chain.

We obtain now the following two theorems.

**Theorem 2.** Let $G$ be a group, and $\phi$ be an $L$-homomorphism from $G$ onto a lattice $L$. Then the $l$-kernel of $\phi$ is self-conjugate in $G$.

**Proof.** The greatest element of $L$ is represented as a join of elements $l_i$ such that the intervals $l_i/0$ are chains. Let $l_1, \ldots, l_n$ be all such elements of $L$. Take a subgroup $V_i$ of $G$ such that $\phi(V_i) = l_i$ ($i = 1, 2, \ldots, n$) and let $V_i$ be maximal under this condition. Then we have $\bigvee_{i=1}^n V_i = G$. Let $E$ be the $l$-kernel of $\phi$. Then we have $\phi(V_i \cup E) = \phi(V_i) \cup \phi(E) = \phi(V_i) = l_i$, which implies that $V_i \cup E = V_i$ or $V_i \supseteq E$. Hence $E$ is self-conjugate in $V_i$, as the $l$-kernel of $\phi$ between $V_i$ and $l_i/0$. $E$ is, therefore, self-conjugate in $G$.

**Theorem 3.** Under the same assumptions as in Theorem 2, the $u$-kernel $G_0$ of $\phi$ is also self-conjugate in $G$.

**Proof.** We shall prove our theorem by induction on the dimension of $L$. Since the greatest element of the interval $G/G_0$ is represented as a join of join-irreducible (that is, covering only one element) elements, we may assume that $G$ has only one maximal subgroup containing $G_0$. If no other maximal subgroup exists, $G$ is cyclic and our theorem is obvious. If there exists another maximal subgroup $M$, $\phi(M)$ must be a dual atom of $L$. By the hypothesis of induction, the $u$-kernel $G_0$ of $\phi$ in $M$ is self-conjugate in $M$. Since $\phi(M \cap G_0) = \phi(G_0) \cap \phi(M) = \phi(M)$, we have $M \cap G_0 \supseteq G_0$. Take any element $a$ of $M$, then $a \cdot G_0 \cdot a^{-1} \cup M = G$. Hence we have $\phi(a \cdot G_0 \cdot a^{-1}) \cup \phi(M) = I$. On the other hand, we have $\phi(a \cdot G_0 \cdot a^{-1}) \supseteq \phi(a \cdot M_0 \cdot a^{-1}) = \phi(M_0) = \phi(M)$. Hence we have $I = \phi(a \cdot G_0 \cdot a^{-1})$ which implies that $a \cdot G_0 \cdot a^{-1} \supseteq G_0$ and hence $a \cdot G_0 \cdot a^{-1} = G_0$. $G_0$ is therefore self-conjugate in $G$. q.e.d.

2. Groups which admit proper $L$-homomorphisms

An $L$-homomorphism is called proper if it is neither an $L$-isomorphism nor a trivial $L$-homomorphism. Otherwise we call it improper. We shall say that a group $G$ admits a proper $L$-homomorphism when there exists a lattice $L$ and an $L$-homomorphism from $G$ onto $L$ which is proper. In this section we shall consider the structure of groups which admit proper $L$-homomorphisms. First we shall prove the following lemma.

**Lemma 2.** If a $p$-group $G$ admits a proper $L$-homomorphism, $G$ is either a cyclic group or a g. q. group.

(*) Strictly speaking, it is a contraction of $\phi$ onto $U$. We shall, in this paper, not distinguish a contraction of $\phi$ from $\phi$, as long as no confusion arises.
Proof. Let $\phi$ be a proper $L$-homomorphism from $G$ onto a lattice $L$. If the $u$-kernel $G_0$ of $\phi$ differs from $G$, we can prove our lemma in a similar way as in the proof of Lemma 1. In the following we shall assume that $G_0=G$, and prove our lemma by induction on the order of $G$. Since $G$ is a $p$-group, $L$ satisfies the Jordan-Dedekind chain condition. Since $\phi$ is a proper $L$-homomorphism, the dimension of $L$ is different from that of $L(G)$. Hence every maximal subgroup of $G$ admits a proper $L$-homomorphism, that is, that induced by $\phi$. By the hypothesis of induction, every maximal subgroup of $G$ contains only one subgroup of order $p$. Hence $G$ is either a $P$-group of order $p^2$, or one of the types stated in Lemma 2. On the other hand, $P$-groups admit no proper $L$-homomorphism. Hence we have our lemma.

Let $\phi$ be again a proper $L$-homomorphism from $G$ onto $L$. We shall denote by $E$ the $l$-kernel and by $G_0$ the $u$-kernel of $\phi$ and put $E_0=G_0\cap E$ and $G_1=G_0\cup E$. Then these four subgroups $E$, $G_0$, $E_0$, and $G_1$ are all self-conjugate. Hence we may consider the factor group $G_1/G_0$ which is clearly a direct product of $G_0=E_0/E_0$ and $E=E/E_0$. These notations will be fixed throughout this section.

We shall prove the following propositions.

(a) The groups $G_0$ and $E$ have mutually prime orders.

Proof. If the orders of $G_0$ and $E$ had a common prime factor $p$, there would exist two subgroups $V_1$ and $V_2$ of $G_0$ and $E$ respectively whose orders are $p$. Hence $V_1 \cup V_2$ would contain another subgroup $V$ such that $G_0 \cap V = e$ and $E \cap V = e$. The first equality implies that $\phi(V) = 0$ and $V \subseteq E$, but the second equality implies that $E \nmid V$. This is a contradiction. q.e.d.

(b) $\Phi(G_0)$ contains $E_0$.

Proof. Take any maximal subgroup $M$ of $G_0$. $\phi(M)$ must be a dual atom of $L$. We have $\phi(M \cup E_0) = \phi(M) \cup \phi(E_0) = \phi(M \cup 0) = \phi(M)$ and hence $M \cup E_0 = M$. This implies that $M \supseteq E_0$ and that $\phi(G_0) \supseteq E_0$. q.e.d.

(b') (Cf. [5, Lemma 4].) $E_0$ is nilpotent, and if a prime number $p$ divides the order of $E_0$, $p$ divides also that of $G_0$.

(c) $G_1$ is a direct product of $G_0$ and another group $N$. $N$ is isomorphic to $E$ and its order is relatively prime to that of $G_0$.

Proof. By (b') and (a) the order of $E_0$ is relatively prime to that of $E/E_0$. Hence by a theorem of Schur (cf. [7, p. 125]) there exists a subgroup $N$ of $E$ such that $N \cup E_0 = E$ and $N \cap E_0 = e$. Take the normalizer $N^*$ of $N$ in $G$. Then we have $N^* \cup E = G$, since $E_0$ is nilpotent by (b') (cf. [7, p. 125]). Hence we have $I = \phi(G) = \phi(N^* \cup E) = \phi(N^*) \cup \phi(E) = \phi(N^*)$. This implies that $N^* \supseteq G_0$. Hence $N^* \supseteq G_0 \cup N = G_0 \cup E_0 \cup N = G_0 \cup E = G_1$. It follows then that $N$ is a normal subgroup of $G$. $G_1$ is clearly a direct product of $G_0$ and $N$, and $N$ is isomorphic to $E$.

(d) If a prime number $p$ divides the order of $G/G_1$, then $p$ divides that of $G_1/E$. Hence the groups $G/N$ and $N$ have mutually prime orders.

Proof. Take any prime factor $p$ of the order of $G/G_1$. If $p$ did not divide
the order of $G_1/E$, a $p$-Sylow subgroup $S$ of $G/E$ would satisfy the condition $S \cap G_1/E = e$. We mean by $S$ a subgroup of $G$ corresponding to $S$ by the natural homomorphism from $G$ onto $G/E$. Then we should have $S \cap G_1 = E$ and $\phi(S) = \phi(S \cap G_1) = \phi(E) = 0$. This implies that $S \subset E$, which gives a contradiction. Hence $p$ divides the order of $G_1/E$. q.e.d.

Hence again by Schur's theorem, $G$ contains a subgroup $H$ such that $G = H \cdot N$, $H \cap N = e$, and $H \supset G_0$. Now we have, in a similar way as for (b),

(e) $\Phi(H)$ contains $E_0$.

Next we shall prove the following proposition.

(f) If $\phi$ induces an improper $L$-homomorphism of every Sylow subgroup of $G$ into $L$, then $H$ is mapped isomorphically onto $L$ by $\phi$ and we have $G = G_0 \times E$.

**Proof.** By the assumption of this proposition and by propositions (b') and (d), we have $E_0 = e$ and $H = G_0$. Our proposition follows then immediately.

By means of proposition (f) we shall deal with a Sylow subgroup in which $\phi$ induces a proper $L$-homomorphism. We shall prove the following propositions.

(g) If a g. q. group $Q$ is mapped by $\phi$ onto a chain of dimension two, $H$ is a direct product of its 2-Sylow subgroup $S_2$ and its Sylow 2-complement $K$. In this case, $L$ is also a direct product of $\phi(S_2)$ and $\phi(K)$.

**Proof.** First, using induction on the order of $G$, we prove that $G$ has a self-conjugate Sylow 2-complement. By Lemma 2, 2-Sylow subgroups of $G$ are g.q. groups. Take any proper subgroup $V$ of $G$. If its 2-Sylow subgroup is cyclic, $V$ has a self-conjugate Sylow 2-complement by a theorem of Burnside (cf. [7, p. 131]). The same holds from the hypothesis of induction if its 2-Sylow subgroup is a g.q. group. Hence every proper subgroup of $G$ has a self-conjugate Sylow 2-complement. By a theorem of Ito(7), $G$ has also a self-conjugate Sylow 2-complement, or all proper subgroups of $G$ are nilpotent. In the latter case, if its Sylow 2-complement were not self-conjugate, $G$ would be of order $p^n 2^8$ ($p$ is a prime greater than 2). The structure of such a group has been completely determined by Iwasawa(8). We can prove by direct examinations that our assumption does not hold in this case. Hence $G$ has a self-conjugate Sylow 2-complement.

Next using again induction on the order of $H$, we prove that $H$ is a direct product of its 2-Sylow subgroup and the Sylow 2-complement. We shall denote by $K$ the Sylow 2-complement of $H$ and assume for a while that the $l$-kernel of $\phi$ coincides with $e$. Considering normalizers of Sylow subgroups

(?) Cf. N. Ito, Zenkoku Sizyô Sûgaku-Danwa-Kai 2-93 (1948) (In Japanese). His theorem asserts that if all proper subgroups of a finite group $G$ have the self-conjugate Sylow $p$-complement, then $G$ has also a self-conjugate Sylow $p$-complement except when all proper subgroups are nilpotent. His proof is a slight modification of the proof given in K. Iwasawa, Proc. of P-M. Soc. of Japan, 3-23 (1941).

of $K$, we can assume $K$ to be a $p$-group ($p > 2$). If $K$ is cyclic, the centralizer of $K$ contains the center $Z$ of a 2-Sylow subgroup $S_2$. Since $\phi(K \cup Z) = \phi(K) \cup \phi(Z) = \phi(H)$, $KZ$ contains the $u$-kernel of $\phi$ and it is a direct product of $K$ and $Z$. Hence we have $L = (\phi(K)/0) \times (\phi(Z)/0)$. Let $\psi$ be the natural homomorphism from $L$ onto $\phi(K)/0$. Then $\psi\phi$ is an $L$-homomorphism from $H$ onto $\phi(K)/0$ and $S_2$ is the $l$-kernel of $\psi\phi$, since we assumed the $l$-kernel of $\phi$ to coincide with $e$. Hence by Theorem 2, $S_2$ is self-conjugate in $H$ and we have $H = K \times S_2$.

If $K$ is not cyclic, $\phi$ induces an $L$-isomorphism from $K$ into $L$ by Lemma 2. We can, therefore, assume also that $S_2$ is maximal. If the center $Z$ of $S_2$ is self-conjugate in $H$, $S_2$ is self-conjugate in the same way as above. If $Z$ were not self-conjugate in $H$, $Z$ would be conjugate to another group $Z_1$. $Z_1$ would be the center of a 2-Sylow subgroup $Q$ and $Q \neq S_2$. Then we should have $\phi(Z \cup Z_1) = \phi(Z) \cup \phi(Z_1) = \phi(S_2) \cup \phi(Q) = \phi(H)$ and hence $Z \cup Z_1 \supseteq K$. This implies that $K$ would be cyclic, which gives a contradiction.

If the $l$-kernel $E_0$ of $\phi$ in $H$ differs from $e$, the $l$-kernel of $\phi$ in $H/E_0$ coincides with $e$. Hence the 2-Sylow subgroup $V$ of $H/E_0$ is self-conjugate. Let $V$ be a subgroup of $H$ corresponding to $V$ by the natural homomorphism from $H$ onto $H/E_0$. Then $V$ is self-conjugate in $H$. Take the normalizer $N_2$ of a 2-Sylow subgroup $S_2$ of $H$. Then we have $N_2V = H$ because $S_2 \subseteq V$. On the other hand, we have $N_2V = N_2 \cup S_2 \cup E_0 = N_2 \cup E_0$. Hence we have $N_2E_0 = H$, which implies that $H = N_2$ by (e)(4). Hence $S_2$ is self-conjugate and we have $H = S_2 \times K$. q.e.d.

(h) If $\phi$ induces a proper $L$-homomorphism from a cyclic $p$-Sylow subgroup $S$ into $L$, then $G$ has a self-conjugate Sylow $p$-complement.

**Proof.** We shall prove that $S$ is contained in the center of its normalizer. If this is done, our proposition follows from a theorem of Burnside (cf. [7, p. 131]). Choosing a suitable subgroup of $G$, we may assume $S$ to be self-conjugate. We shall then prove that $G$ is a direct product of $S$ and the Sylow $p$-complement $K$. Using induction on the order of $G$ we have only to prove our assertion assuming $K$ to be a cyclic group of prime power order, that is, $K = \{b\}$. Put $S = \{a\}$. Then we have $b \cdot a \cdot b^{-1} = a^r$. If $r \neq 1$ (mod the order of $a$), $G$ would admit no proper $L$-homomorphism, against our assumption. Hence we have $r = 1$ and $G = K \times S$. q.e.d.

By propositions (g) and (h), we get the following propositions.

(i) The factor group $H/G_0$ is a nilpotent group each of whose Sylow subgroups is either cyclic or a dihedral group.

(j) If $\phi$ induces a proper $L$-homomorphism of $G_0/E_0$, $G_0$ contains a normal subgroup $G_2$ of $G$ such that the factor group $G_0/G_2$ is cyclic and $\phi$ induces an $L$-isomorphism of $G_2/E_0$. Moreover the order of $G_0/G_2$ is relatively prime to that of $G_2/E_0$.

(*) Let $\Phi$ be the $\Phi$-subgroup of $G$, then $\Phi H = G$ implies $H = G$ for any subgroup $H$ of $G$. Cf. Zassenhaus [7, p. 45].
Remark. If the center $Z$ of a g.q. group $Q$ is mapped onto $0$ by $\phi$, and if $\phi(Q) \neq 0$, $Z$ is clearly self-conjugate by (b'), since $Z \subseteq E_0$. Hence $Z$ is contained in the center of $G$. Conversely if a 2-Sylow subgroup $Q$ of $G$ is a g.q. group and if the center $Z$ of $Q$ is self-conjugate in $G$, then the natural homomorphism from $G$ onto $G/Z$ induces an $L$-homomorphism from $G$ onto $G/Z$ (see Lemma 4 below).

From (b'), (h) and the remark given above we obtain:

(k) $E_0$ is a cyclic group contained in the center of $G$.

Proof. By (b') and Lemma 2, $E_0$ is cyclic. Let $T$ be a $p$-Sylow subgroup of $E_0$, and $S$ be a $p$-Sylow subgroup of $G$. $S$ is then cyclic or a g.q. group. If it is a g.q. group, $T$ is contained in the center of $G$ as remarked above. If $S$ is cyclic, $\phi$ induces a proper $L$-homomorphism of $S$. Hence by (h), $G$ has a self-conjugate Sylow $p$-complement $K$. As $T$ is self-conjugate by (b'), $K \cap T$ is a direct product of $K$ and $T$, which implies that $T$ is contained in the center of $G$. This proves proposition (k).

These propositions may be summarized as follows.

Theorem 4. If $G$ admits a proper $L$-homomorphism $\phi$, then $G$ contains a normal subgroup $N$ and a subgroup $H$ such that

(1) $NH = G$ and $N \cap H = e$,
(2) The orders of $N$ and $H$ are relatively prime,
(3) $H$ contains the $u$-kernel $G_0$ of $\phi$, and
(4) $N$ is contained in the $l$-kernel $E$ of $\phi$.

Moreover putting $E_0 = E \cap G_0$ we have

(5) $E_0$ is a cyclic group, contained in the center of $G$.

The factor group $H/G_0$ is a nilpotent group, each of whose Sylow subgroups is either cyclic or a dihedral group. If $H/G_0$ contains a dihedral group, $H$ is a direct product of its 2-Sylow subgroup and the Sylow 2-complement. If, moreover, $\phi$ induces a proper $L$-homomorphism of $G_0 = G_0/E_0$, $G_0$ contains a normal subgroup $G_2$ such that

(6) $G_0/G_2$ is cyclic,
(7) the order of $G_0/G_2$ is relatively prime to that of $G_2$, and
(8) $\phi$ induces an $L$-isomorphism from $G_2$ into $L$.

As special cases of this theorem we obtain the following theorem.

Theorem 5. If none of the Sylow complements of a group $G$ is self-conjugate, any $L$-homomorphism from $G$ onto a lattice $L$ is either one of the natural homomorphisms from $L(G)$ onto its direct components, or the $L$-homomorphism from $G$ onto $G/Z$, where $Z$ is the center of a 2-Sylow subgroup, which is a g.q. group, or combinations of these $L$-homomorphisms. Hence $L$ is isomorphic to the subgroup lattice of some group.

Since a group $L$-isomorphic to a perfect group is also perfect (cf. [5, Theorem 12]) we obtain the following theorem.
Theorem 6. Let $G$ be a perfect group. If $G$ is $L$-homomorphic to the subgroup lattice $L(H)$ of a group $H$, then $H$ is perfect.

3. Groups $L$-homomorphic to a nilpotent group

In the following two sections we shall consider a homomorphism from the subgroup lattice $L(G)$ of a group $G$ onto $L(G')$ of another group $G'$. We shall call this homomorphism the $L$-homomorphism from $G$ onto $G'$. In this section we assume in particular $G'$ to be nilpotent, then we can obtain more precise results than those of the preceding section.

Let $G$ be a group and $\phi$ be an $L$-homomorphism from $G$ onto a lattice $L$. Then by Theorem 4, $G$ has a normal subgroup $N$ and a subgroup $H$ with properties (1)–(4) of Theorem 4, and if we denote by $E$ or $G_0$ the $l$-kernel or the $u$-kernel of $\phi$ respectively, these groups are self-conjugate in $G$. Put $E_0 = E \cap G_0$. These notations will be fixed throughout this section.

Lemma 3. $L(H)$ is directly decomposable if and only if $L$ is directly decomposable.

Proof. If $L(H)$ is directly decomposable, $L$ is clearly decomposable. Assume conversely that $L$ is directly decomposable: $L = L_1 \times L_2$. Then there is a natural homomorphism $\psi_i$ from $L$ onto $L_i$ ($i = 1, 2$). $\psi_i \phi$ is clearly an $L$-homomorphism from $G$ onto $L_i$. We shall denote the $l$-kernel of $\psi_i \phi$ by $E_i$. By Theorem 2, $E_i$ is self-conjugate in $G$. We have clearly $E_1 \cap E_2 = E$ and $E_1 \cup E_2 = G$. When we regard $\psi_i \phi$ as an $L$-homomorphism from $G/E$ onto $L_i$, the $u$-kernel of $\psi_i \phi$ is contained in $E_2/E$, and therefore the order of $E_1/E$ is relatively prime to that of $E_2/E$ by Theorem 4. Hence $L(G/E)$ is directly decomposable. Since $G/E \cong H/E_0$ and since $E_0 \subseteq \Phi(H)$ by proposition (e) of §2, $L(H)$ is also directly decomposable (cf. [5, Lemma 5]). q.e.d.

In the following we shall assume that $L$ is the subgroup lattice of a nilpotent group $G'$ and determine the structure of the group $H$. In virtue of Lemma 3, we can assume $G'$ to be a $p$-group.

Theorem 7. Let $G$ be a group, and $\phi$ be an $L$-homomorphism from $G$ onto a $p$-group $G'$. If $G'$ is neither cyclic nor a $P$-group, $H$ is also a $p$-group and coincides with $G_0$. $G$ is therefore a direct product of $N$ and $G_0$. If $G'$ is a $P$-group, $H$ is either a $p$-group or an upper semimodular group of order $p^m q^n (10)$, where $q$ is a prime number and $p > q$, and $G_0$ is its maximal self-conjugate $M$-group.

Proof. We shall assume that $G'$ is not cyclic. Since $L(G')$ has no irreducible interval, $H/G_0$ is cyclic by Theorem 4 and Lemma 3. If $\phi$ induces a proper $L$-homomorphism from $G_0/E_0$, $G_0$ has a normal subgroup $G_2$ and $\phi$ (

\textsuperscript{(10)} Such a group $G$ has been completely determined by Sato [4]. According to him, a group of order $p^m q^n (p > q)$ is an upper semimodular group if and only if its $p$-Sylow subgroup $P$ is a $P$-group, a $q$-Sylow subgroup $Q$ is cyclic, $Q = \{b\}$, and for any element $a$ of $P$, $bab^{-1} = a^s$, $x^s \equiv 1 \pmod{p}$.

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induces an L-isomorphism from $G_2/E_0$ into $G'$. Hence by Theorem 3 of Suzuki [5], $G_2/E_0$ is a $p$-group or a $P$-group. If $G_2/E_0$ were a nonabelian $P$-group, $\phi$ would induce an L-isomorphism from a group $V/E_0$, where $V$ is a subgroup of $G_0$, covering $G_2$. Since the order of $V/E_0$ would be divisible by three distinct primes, this is a contradiction. Hence by proposition (b') and (d) of §2, we see that $H$ is a $p$-group or a group of order $p^aq^n \ (p > q)$. If $H$ is a $p$-group, by Lemma 2 we have $H = G_0$. We have now only to prove that if the order of $H$ is $p^aq^n$, $H$ is an upper semi-modular group, and $G$ is a $P$-group.

$G_0/E_0$ is a group of order $p^aq^n$ and its $p$-Sylow subgroup $S$ is self-conjugate by Theorem 3 of Suzuki [5] and our Theorem 4. $\phi$ induces an L-isomorphism from $S$ into $G'$. Take a subgroup $\overline{T}$ of $G_0/E_0$ covering $S'$, then $\phi$ induces also an L-isomorphism in $\overline{T}$. Hence $T$ is a $P$-group. Next take a $q$-Sylow subgroup $\overline{Q}$ of $G_0/E_0$ and a subgroup $\overline{V}$ covering $\overline{Q}$; then $\overline{Q}$ is cyclic. Since $G'$ is a $p$-group, $\phi(\overline{V}) \cap \phi(\overline{S})$ is of prime order. Hence $\overline{V} \cap \overline{S}$ is a normal subgroup of $G_0/E_0$ of order $p$. By direct examination we see that $\phi(\overline{V})$ is a $P$-group. This implies that $G' = \phi(\overline{T})$ and $G_0/E_0 = \overline{T}$. Hence we see that $G_0/E_0$ and $G'$ are both $P$-groups.

Since Sylow $p$-complements of $H$ are not self-conjugate, the orders of $H/G_0$ and $E_0$ are both powers of $q$ by proposition (h) of §2. The $p$-Sylow subgroup $S$ of $H$ is clearly self-conjugate in $H$ and $H$ induces an L-isomorphism from $S$ into $G'$. Take any subgroup $V$ of order $p$ and any $q$-Sylow subgroup $Q$ of $H$. Then $\phi(V \cup Q)$ is a $P$-group of order $p^2$. Hence $(V \cup Q) \cap S$ is of prime order and hence coincides with $V$; $(V \cup Q) \cap S = V$. This implies that $V$ is a normal subgroup of $H$. Put $Q = \{b\}$; then for any element $a$ of $S$ we have

$$b \cdot a \cdot b^{-1} = a^x, \quad x \not= 1, \quad x^{a_t} = 1 \pmod{p}.$$ 

Hence $H$ is an upper semi-modular group and $G_0$ is its maximal self-conjugate $M$-group. q.e.d.

In order to prove the converse of this theorem we shall first prove the following lemma.

**Lemma 4.** Let $Z$ be a cyclic subgroup of prime power order contained in the center of a group $G$. If Sylow subgroups containing $Z$ are cyclic or $g.q.$ groups, the natural homomorphism from $G$ onto $G/Z$ induces an $L$-homomorphism.

**Proof.** We can assume that $Z$ is of prime order. We have only to prove $(U \cap V) \cup Z = (U \cup Z) \cap (V \cup Z)$ for any two subgroups $U$ and $V$ of $G$. If $U \supseteq Z$ and $V \supseteq Z$, we have clearly this equality. If $U \nparallel Z$, the order of $U$ is prime to $p$. Hence we have $L(U \cup Z) = L(Z) \times L(U)$ (cf. [3]). If moreover $V \supseteq Z$, we have $(U \cup Z) \cap V = Z \cup ((\overline{U} \cup Z) \cap U) = Z \cup (U \cap V)$. If $V \nparallel Z$, $(U \cup Z) \cap (V \cup Z) = Z \cup W$ for some subgroup $W$. We have then $U \cap V \supseteq W$. Hence we have $(U \cup Z) \cap (V \cup Z) \subseteq (U \cap V) \cup Z$. On the other hand, we have

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clearly \((U \cap V) \cup Z \subseteq (U \cup Z) \cap (V \cup Z)\). Hence \((U \cap V) \cup Z = (U \cup Z) \cap (V \cup Z)\). q.e.d.

If a group \(G\) is a direct product of two groups \(G_0\) and \(N\) (having relatively prime orders), and if \(G_0\) is a \(p\)-group, \(G\) is clearly \(L\)-homomorphic to \(G_0\). If \(H\) is an upper semi-modular group and \(G_0\) is its maximal self-conjugate \(M\)-group, \(G\) is \(L\)-homomorphic to a \(P\)-group as follows. First the mapping \(U \mapsto U \cap E_0\) from \(L(G)\) onto \(L(G/E_0)\) is surely an \(L\)-homomorphism by Lemma 4. Hence we may assume that \(E_0 = e\). As \(H\) is an upper semi-modular group, the mapping \(U \mapsto U \cap G_0\) from \(L(H)\) onto \(L(G_0)\) is an \(L\)-homomorphism. We shall prove that the mapping \(U \mapsto U \cap G_0\) is also an \(L\)-homomorphism from \(G\) onto \(G_0\). First we shall show that \((U \cap G_0) \cup N = (U \cap N) \cap (G_0 \cup N)\) for any subgroup \(U\) of \(G\). Suppose that \(P \cap N = G_0\). If \(\beta = 0\), \(U\) is contained in \(S \cup N\), where \(S\) is a \(p\)-Sylow subgroup of \(G\). Since \(L(S \cup N) = L(S) \times L(N)\), we have easily \((U \cap G_0) \cup N = (U \cap N) \cap (G_0 \cup N)\). If \(\beta \neq 0\), the index \([[(U \cap G_0) \cup N : N]\) is equal to \(p^a q^g\), and \([[(U \cap N) \cap (G_0 \cup N) : N]\) is also equal to \(p^a q^g\). On the other hand, we have \((U \cap G_0) \cup N \subseteq (U \cap N) \cap (G_0 \cup N)\). Hence we have \((U \cap G_0) \cup N = (U \cap N) \cap (G_0 \cup N)\).

Now we shall show that \((U \cup V) \cap G_0 = (U \cup G_0) \cap (V \cup G_0)\). In fact, we have

\[
N \cup ((U \cup V) \cap G_0) = (U \cup V \cup N) \cap (G_0 \cup N).
\]

On the other hand, as \(G/N\) is an upper semi-modular group,

\[
((U \cup N) \cup (V \cup N)) \cap (G_0 \cup N) = (U \cup G_0) \cap (V \cup G_0) \cup N \cap G_0 = ((U \cap G_0) \cup (V \cap G_0)) \cup N.
\]

Since \(G_0 \cap N = e\), we have

\[
(U \cup V) \cap G_0 \cong N \cup ((U \cup V) \cap G_0) / N \cong ((U \cap G_0) \cup (V \cap G_0)) \cup N / N \cong (U \cap G_0) \cup (V \cap G_0).
\]

Hence we have \((U \cup V) \cap G_0 = (U \cap G_0) \cup (V \cap G_0)\). The mapping \(U \mapsto U \cap G_0\) is thus an \(L\)-homomorphism from \(G\) onto a \(P\)-group \(G_0\).

From Lemmas 1 and 3, Theorem 7, and the remark given above we obtain:

**Theorem 8.** Let \(G\) be a group. There exists an \(L\)-homomorphism \(\phi\) from \(G\) onto a nilpotent group \(G' = \prod_{i=1}^{l} S_i\), where \(S_i\) is a \(p_i\)-Sylow subgroup of \(G'\), if and only if \(G\) has a normal subgroup \(N\) and a subgroup \(H\) with the following properties:

1. \(NH = G\) and \(N \cap H = e\).
2. the order of \(N\) is relatively prime to that of \(H\),
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(3) \( H \) is a direct product of groups \( H_i \) \((i = 1, 2, \cdots, t)\) having mutually prime orders: \( H = \prod_{i=1}^{t} H_i \).

(4) \( \phi(H_i) = S_i \) \((i = 1, 2, \cdots, t)\),

(5) if \( S_i \) is cyclic, \( H_i \) is a cyclic group of prime power order or a g. q. group, and \( H_i \) contains a normal subgroup \( K_i \) of \( G \) such that \( \phi(K_i) = S_i \),

(6) if \( S_k \) is a \( P \)-group of order \( p_k^{n+1} \) \((n \geq 1)\), \( H_k \) is either isomorphic to \( S_k \), or a quaternion group \((n = 1, p_k = 2)\), or an upper semi-modular group of order \( p_k^m q^n \) \((q \text{ is a prime and } p_k > q)\), and its maximal self-conjugate \( M \)-group is a normal subgroup of \( G \),

(7) if \( S_1 \) is neither cyclic nor a \( P \)-group, \( H_1 \) is also a \( p_1 \)-group and self-conjugate in \( G \). In this case if \( H_1 \) is not \( L \)-isomorphic to \( S_1 \), \( H_1 \) is a g.q. group and \( S_1 \) is isomorphic to the factor group \( H_1/Z_1 \) of \( H_1 \) modulo its center \( Z_1 \).

We shall omit the proof of this theorem, since it runs along similar lines as the proof of Theorem 1.

4. THE \( L \)-HOMOMORPHIC IMAGE OF A SOLVABLE GROUP

In this section we shall prove the following theorem.

Theorem 9. Let \( G \) be a solvable group, and \( \phi \) be an \( L \)-homomorphism from \( G \) onto another group \( G' \). Then \( G' \) is also solvable.

Denote by \( E \) or \( G_0 \) the \( l \)-kernel or the \( u \)-kernel of \( \phi \) respectively. Then by Theorems 2 and 3, \( E \) and \( G_0 \) are self-conjugate. Put \( E_0 = E \cap G_0 \). \( \phi \) induces an \( L \)-homomorphism \( \phi \) from \( G_0/E_0 \) onto \( G' \). If \( \phi \) is an \( L \)-isomorphism, our theorem follows from a theorem on the \( L \)-isomorphism which asserts that groups \( L \)-isomorphic to a solvable group are also solvable (cf. [5, Theorem 12]). If \( \phi \) is a proper \( L \)-homomorphism, \( G_0/E_0 \) contains a normal subgroup \( G_2/E_0 \) such that \( G_0/G_2 \) is cyclic and \( \phi \) induces an \( L \)-isomorphism from \( G_2/E_0 \) into \( G' \). Hence in order to prove our Theorem 9, it is sufficient to prove the following theorem.

Theorem 10. Assume \( L \) to be a lattice of subgroups of a group \( G' \). Then under the same notations as in Theorem 4, \( \phi(G_2) \) is self-conjugate in \( G' \).

Proof. In changing the notations, we shall assume that the \( u \)-kernel of \( \phi \) coincides with \( G \) and that the \( l \)-kernel of \( \phi \) coincides with \( e \). Take a \( p \)-Sylow subgroup \( S \) of \( G \) in which \( \phi \) induces a proper \( L \)-homomorphism. By Lemma 3 and proposition (g) of \( \S 2 \), \( S \) must be cyclic, and by proposition (h) of \( \S 2 \), \( G \) has a Sylow \( p \)-complement \( N \). We shall first prove that \( \phi(S) \) is also a Sylow subgroup of \( G \).

Since \( \phi(S) \) is a cyclic group of prime power order, it is contained in some Sylow subgroup \( S' \) of \( G' \). Take the greatest subgroup \( U \) of \( G \) such that \( \phi(U) = S' \). Then \( U \) clearly contains \( S \). If \( S' \) were a \( P \)-group, \( \phi(S) \) would be of prime order. On the other hand, taking the maximal subgroup \( M \) of \( S \),
we have \( \phi(M) \neq \phi(S) \), as the \( u \)-kernel of \( \phi \) coincides with \( G \). Hence we would have \( \phi(M) = e \), that is, \( M \) would be contained in the \( l \)-kernel of \( \phi \) and by our assumption \( M = e \). Hence \( S \) is mapped \( L \)-isomorphically onto \( \phi(S) \), contrary to our assumption. Hence \( S' \) is not a \( P \)-group and \( U \) is also of prime power order by Theorem 8. Hence \( U \) must coincide with \( S \), that is, \( S' = \phi(S) \).

Next we shall prove that \( S' = \phi(S) \) is contained in the center of its normalizer. Take a subgroup \( V' \) of \( G' \) such that \( S' \) is self-conjugate in \( V' \), and \( V'/S' \) is of prime power order, say, of order \( q^n \) (\( q \) is a prime number). Take a subgroup \( V \) of \( G \) such that \( \phi(V) = V' \); then \( \phi(V \cap N) \) is a \( q \)-Sylow subgroup \( Q' \) of \( V' \). If \( V \cap N \) is cyclic and not \( L \)-isomorphic to \( Q' \), \( S \) is self-conjugate in \( V \) by proposition (h) of §2, and hence \( V \) and also \( V' \) are directly decomposable.

We can then assume \( V \cap N \) to be \( L \)-isomorphic to \( Q' \). Since the \( k \)-kernel of \( \phi \) coincides with \( e \), a subgroup \( T \) of \( V \), covering \( N \cap V \), is \( L \)-isomorphic to \( \phi(T) = T' \), and \( \phi \) induces an \( L \)-isomorphism from \( T \) onto \( T' \). By our assumption, \( T' \cap S' \) is self-conjugate in \( T' \). If \( T \cap S \) were not self-conjugate in \( T \), \( T \) would be a \( P \)-group (cf. [5, Theorems 13 and 14]) which would imply that \( Q' \) has prime order. Hence \( V \cap N \) would also be of prime order. Since \( \phi(S) \) is self-conjugate in \( V' \), \( V' \) is a \( P \)-group, which leads us to the same contradiction as above. Hence \( T \cap S \) is self-conjugate in \( T \) and so \( T \) is a direct product of \( N \cap V \) and \( T \cap S \). This implies that \( T \cap S \) is self-conjugate in \( V \). If \( S \) were not self-conjugate in \( V \), there would be another \( p \)-Sylow subgroup \( S^* \) of \( V \). \( S^* \) would also contain \( T \cap S \). Hence we would have \( \phi(S^*) \cap S' \neq e \). Since \( \phi(S^*) \) is a cyclic group of prime power order, this gives a contradiction. Hence we have \( V = (N \cap V) \times S \) and \( V' = Q' \times S' \). \( S' \) is thus contained in the center of its normalizer and \( G' \) contains a normal subgroup \( A' \) such that \( N \cap S' = G \) and \( N' \cap S' = e \).

We shall now prove that \( \phi(N) = N' \). Take all \( p \)-Sylow subgroups \( S = S_1, S_2, \ldots, S_t \) of \( G \). Then \( \phi \) induces a proper \( L \)-homomorphism in every \( S_i \). Hence the \( \phi(S_i) \) are Sylow subgroups of \( G' \) and are contained in centers of their normalizers, as proved above. \( G \) then has Sylow complements \( N' = N'_1, N'_2, \ldots, N'_t \). Put \( D' = \cap_{i=1}^{t} N'_i \). Take a subgroup \( D \) of \( G \) such that \( \phi(D) = D' \). Since \( D' \cap \phi(S_i) = e \) (\( i = 1, 2, \ldots, t \)), we have \( D \cap S_i = e \) (\( i = 1, 2, \ldots, t \)), which implies that the order of \( D \) is prime to \( p \), or \( D \subseteq N \). Since \( \phi(N) \cap D' = e \) and \( \phi(N) \cup \phi(S) = G' \), we have \( \phi(N) = N' \). This proves our theorem.

5. Neutral elements of \( L(G) \)

An element \( l \) of a lattice \( L \) is called neutral if every triple \( \{ l, x, y \} \) of elements of \( L \) generates a distributive sublattice of \( L \). An element \( l \) of \( L \) is neutral if and only if the mappings \( x \mapsto x \cup l \) and \( x \mapsto x \land l \) are homomorphisms, and \( x \cup l = y \cup l \) and \( x \land l = y \land l \) imply \( x = y \) for any two elements \( x, y \) of \( L \).


If \( L \) is directly decomposable, an element is neutral if and only if all its components are neutral.

In this section we shall determine the neutral elements of a subgroup lattice \( L(G) \) of a group \( G \). Because of the above remark we may assume \( L(G) \) to be irreducible.

Let \( K \) be a neutral element of \( L(G) \). Then the mapping \( \phi: U \to U \cup K \) is an \( L \)-homomorphism from \( G \) onto an interval \( G/K \). As \( K \) is the \( l \)-kernel of \( \phi \), it is self-conjugate in \( G \) by Theorem 2. Denote the \( u \)-kernel of \( \phi \) by \( G_0 \); then we have \( G_0 \cup K = G \). By proposition (c) of §2, we have either \( G_0 \supseteq K \) or \( L(G) \) is directly decomposable. Hence from our assumptions we have \( G_0 \supseteq K \), so \( G_0 = G \). By Theorem 4, \( K \) is a cyclic group contained in the center of \( G \). On the other hand, the mapping \( U \to U \setminus K \) is also an \( L \)-homomorphism from \( G \) onto \( K \). Since \( K \) is cyclic, the structure of \( G \) is determined by Theorem 1.

Let \( K = \prod_{i=1}^{r} K_i \) be the decomposition of \( K \) into a direct product of its Sylow subgroups \( K_i \). Then \( G \) has a normal subgroup \( N \) and a subgroup \( H \) with the following properties:

1. \( NH = G \), \( N \cap H = e \), and \( H \supseteq K \),
2. the order of \( N \) is prime to that of \( H \),
3. \( H \) is a direct product \( \prod_{i=1}^{r} H_i \) of its Sylow subgroups \( H_i \), and
4. \( H_i \) is either cyclic or a g.q. group.

Conversely suppose that a subgroup \( K \) of a group \( G \) is contained in the center of \( G \) and \( G \) has a normal subgroup \( N \) and a subgroup \( H \) with the properties (1)–(4) given above. Then \( K \) is a neutral element of \( L(G) \).

**Proof.** By (4), \( K \) is cyclic. Let \( K_i \) be a \( p_i \)-Sylow subgroup of \( K \). We shall show that \( K_i \) is neutral. By Lemma 4, the mapping \( U \to U \cup K_i \) is an \( L \)-homomorphism from \( G \) onto \( G/K_i \). By Lemma 1, the mapping \( U \to U \cap K_i \) is also an \( L \)-homomorphism from \( G \) onto \( K_i \). We have only to prove that \( U \cup K_i = V \cup K_i \) and \( U \cap K_i = V \cap K_i \) imply \( U = V \) for any two subgroups \( U \), \( V \) of \( G \). \( G \) has a Sylow \( p_i \)-complement \( N_i \). We have \( U \supseteq K_i \), or \( U \subseteq K_i ; N_i \) for any subgroup \( U \) of \( G \). Suppose now that \( U \cup K_i = V \cup K_i \) and \( U \cap K_i = V \cap K_i \). If \( U \supseteq K_i \), we have \( U \cap K_i = K_i \). Hence we have \( V \cap K_i = K_i \), or \( V \supseteq K_i \). We have, therefore, \( U = U \cup K_i = V \). If \( U \supseteq K_i \), we have also \( V \supseteq K_i \), that is, \( N_i K_i \) contains both \( U \) and \( V \). On the other hand, \( N_i K_i \) is a direct product of \( N_i \) and \( K_i \), and we have \( L(N_i, K_i) = L(N_i) \times L(K_i) \). Hence we have clearly

\[
U = (U \cap K_i) \cup (U \cap N_i) = (U \cap K_i) \cup ((U \cap K_i) \cap N_i)
= (V \cap K_i) \cup ((V \cap K_i) \cap N_i) = V.
\]

Since the join of neutral elements is also neutral, \( K = \bigcup_{i=1}^{r} K_i \) is neutral. Thus we obtain the following theorem, which gives an answer to a problem of Birkhoff (13).

**Theorem 11.** Assume that the subgroup lattice \( L(G) \) of a group \( G \) is irreducible.

reducible. A subgroup $K$ of $G$ is a neutral element of $L(G)$ if and only if $K$ is contained in the center of $G$, and $G$ has a normal subgroup $N$ and subgroup $H$ with the properties (1)–(4) given above.

*Added in proof.* After writing this paper, the author learned that G. Zappa has obtained some theorems concerning $L$-homomorphisms of finite groups, in particular Theorem 1 of this paper: Cf. G. Zappa, *Determinazione dei gruppi finiti in omomorfismo strutturale con un gruppo ciclico*, Rendiconti del seminario Matematico, Univ. di Padova (1949) pp. 140–162, and *Sulla condizione perche un omomorfismo ordinario sia anche un omomorfismo strutturale*, Giornale di Matematiche vol. 78 (1949) pp. 182–192.

For the detailed proof of a theorem of N. Ito, cited in footnote 7 of this paper, see his forthcoming paper: *Note on (LM)-groups of finite orders*, Kôdai Mathematical Seminar Reports.

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