UNIQUENESS THEORY FOR HERMITE SERIES

BY

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I. Introduction

1.1. Hermite series. We consider the system $\{\phi_n(x)\}$ of normalized Hermite functions

$$\phi_n(x) = e^{-x^2/2}H_n(x)/2^{n/2}(n!)^{1/2}\pi^{1/4} \qquad (n = 0, 1, 2, \cdots), \tag{1.1.1}$$

where $H_n(x)$ is the Hermite polynomial of degree n [8, p. 101] (1). The Hermite functions satisfy the orthogonality relations

$$\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)dx = \begin{cases} 0 & (m \neq n), \\ 1 & (m = n), \end{cases} \tag{1.1.2}$$

and the differential equations [8, p. 102]

$$\phi_n''(x) - (x^2 + 1)\phi_n(x) = -(2n + 2)\phi_n(x) \qquad (n = 0, 1, 2, \cdots). \tag{1.1.3}$$

For any real number p, we say that $f \in H_p$ if the function $f \in L$ on every finite interval, and if

$$\int_{-\infty}^{\infty} | x^p f(x) | e^{-x^2/2}dx < +\infty. \tag{1.1.4}$$

If $f \in H_p$ for every $p \geqq 0$, we say that $f \in H$.

If $f \in H$, and if

$$a_n = \int_{-\infty}^{\infty} f(x)\phi_n(x)dx \qquad (n = 0, 1, 2, \cdots), \tag{1.1.5}$$

we say that the series $\sum_0^\infty a_n\phi_n(x)$ is the Hermite series of $f(x)$, and write

$$f(x) \sim \sum_{n=0}^{\infty} a_n\phi_n(x). \tag{1.1.6}$$

1.2. The present paper is concerned with the problem of finding sufficient conditions under which a given series of Hermite functions is a Hermite series in the above sense. The results, which are summarized in III, also apply (with some evident minor changes) to series of Hermite polynomials.

In IV we give examples which show that, in a certain sense, our results are the best possible.

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

II. Notations, definitions

2.1. We put

$$\alpha(x) = \int_{-\infty}^{x} e^{-u^2} du, \qquad \beta(x) = e^{x^2/2}\alpha(x). \tag{2.1.1}$$

Then the equation

$$y''(x) - (x^2 + 1)y(x) = 0, \tag{2.1.2}$$

which arises by equating the left member of (1.1.3) to zero, has $\beta(x)$ and $\beta(-x)$ as linearly independent solutions. It should be noted here that $\alpha(-x) = \int_x^\infty e^{-u^2}du$, and that $\alpha(x)+\alpha(-x)=\alpha(+\infty)=\pi^{1/2}$.

2.2. **Generalized Hermite operators.** Consider a function $F(t)$, defined in a neighborhood of the point x. Given $h>0$, there exists a unique function $y(t)=y(t;\, F,\, h)$ which is a solution of (2.1.2) and is such that

$$y(x + h) = F(x + h), \qquad y(x - h) = F(x - h). \tag{2.2.1}$$

We put

$$\Delta_h F(x) = y(x; F,\, h) - F(x), \tag{2.2.2}$$

and define

$$\Lambda F(x) = \lim_{h\to 0} 2\Delta_h F(x)/h^2, \tag{2.2.3}$$

provided the limit exists. $\Lambda^* F(x)$ and $\Lambda_* F(x)$ are defined likewise, with lim sup and lim inf in place of lim.

From the representation $y(t;\, F,\, h)=c_1\beta(t)+c_2\beta(-t)$ we obtain the explicit formula

$$\Delta_h F(x) = e^{-h^2/2}\frac{e^{-hx}I(x - h,\, x)F(x + h) + e^{hx}I(x,\, x + h)F(x - h)}{I(x - h,\, x + h)} - F(x), \tag{2.2.4}$$

where $I(a,\, b)=\alpha(b)-\alpha(a)$. If F is bounded in a neighborhood of x, (2.2.4) can be written

$$\begin{aligned}\Delta_h F(x) = {} & (F(x + h) + F(x - h) - 2F(x))/2 \\ & - h^2(x^2 + 1)(F(x + h) + F(x - h))/4 + O(h^3) \qquad (h \to 0),\end{aligned} \tag{2.2.5}$$

from which it follows that $\Lambda F(x) = F''(x) - (x^2+1)F(x)$ if $F''(x)$ exists. Hence we call Λ a generalized Hermite operator. In particular, we note that (1.1.3) can be written

$$\Lambda\phi_n(x) = -(2n + 2)\phi_n(x) \qquad (n = 0, 1, 2, \cdots). \tag{2.2.6}$$

2.3. **The inverse operator.** We put

$$k(x, t) = \begin{cases} \pi^{-1/2}\beta(x)\beta(-t) & (x < t), \\ \pi^{-1/2}\beta(-x)\beta(t) & (t \leqq x). \end{cases} \tag{2.3.1}$$

Then $k(x, t)$ is the Green's function of (2.1.2) [3, p. 324]. We define

$$\Omega f(x) = -\int_{-\infty}^{\infty} f(t)k(x, t)dt, \tag{2.3.2}$$

provided $f \in H_0$. The operator Ω is the inverse of Λ (this statement will be made more precise in 5.4 and in Theorem 1).

2.4. **Poisson sums.** Suppose the series $\sum_0^\infty a_n\phi_n(x)r^n$ converges, for $0 \leqq r < 1$, to $f(x, r)$. Then the functions $f^*(x) = \lim\sup_{r\to 1} f(x, r)$ and $f_*(x) = \lim\inf_{r\to 1} f(x, r)$ are called the upper and lower Poisson sums respectively of the series $\sum_0^\infty a_n\phi_n(x)$.

III. Main results

THEOREM 1. *Let $p \geqq 0$ be given. Suppose*

(a) *$F \in H_p$ and F is continuous;*

(b) *$\Lambda^*F(x) > -\infty$ and $\Lambda_*F(x) < +\infty$ except possibly on countable sets E_1 and E_2;*

(c) *$\lim\sup_{h\to 0} \Delta_h F(x)/h \geqq 0$ on E_1, and $\lim\inf_{h\to 0} \Delta_h F(x)/h \leqq 0$ on E_2;*

(d) *there exists a function $y \in H_p$ such that $y(x) \leqq \Lambda^*F(x)$ for all x.*

Then $\Lambda F(x)$ exists almost everywhere, $\Lambda F \in H_p$, and $F(x) = \Omega\Lambda F(x)$ for all x.

THEOREM 2. *Suppose $f \in H$. Then $f(x) \sim \sum_0^\infty a_n\phi_n(x)$ if and only if $\Omega f(x) \sim -\sum_0^\infty a_n\phi_n(x)/(2n+2)$.*

THEOREM 3. *Suppose*

(i) *$-\sum_0^\infty a_n\phi_n(x)/(2n+2) \sim F(x)$, where F is continuous and satisfies* (b) *and* (c);

(ii) *there exists a function $y \in H$ such that $y(x) \leqq \Lambda^*F(x)$ for all x.*

Then

$$\Lambda F(x) \sim \sum_0^\infty a_n\phi_n(x).$$

THEOREM 4. *Suppose*

(i) *$-\sum_0^\infty a_n\phi_n(x)/(2n+2) \sim F(x)$;*

(ii) *at some fixed point x_0, $F(x_0)$ is finite, and*

$$-\lim_{r\to 1} \sum_{n=0}^\infty a_n\phi_n(x)r^n/(2n+2) = F(x_0);$$

(iii) *$F(x)$ is bounded above (or below) in a neighborhood of x_0.*

Then $\Lambda_*F(x_0) \leqq f^*(x_0)$ and $f_*(x_0) \leqq \Lambda^*F(x_0)$.

THEOREM 5. *Suppose*

(A) $-\sum_0^\infty a_n\phi_n(x)/(2n+2)\sim F(x)$, *where* F *is continuous*;

(B) $f_*(x)>-\infty$ *and* $f^*(x)<+\infty$ *except possibly on countable sets* E_1 *and* E_2;

(C) F *satisfies* (c);

(D) *there exists a function* $y\in H$ *such that* $y(x)\leqq f_*(x)$ *for all* x.

Then the series $\sum_0^\infty a_n\phi_n(x)$ *is Poisson summable almost everywhere, and is the Hermite series of its Poisson sum.*

THEOREM 6. *Suppose* $a_n=o(n^{1/4})$. *If there exists a function* $y\in H$ *such that*

$$-\infty < y(x) \leqq f_*(x) \leqq f^*(x) < +\infty$$

for all x, *then the conclusion of Theorem* 5 *holds.*

THEOREM 7. *If* $\lim_{n\to\infty} a_n\phi_n(x)=0$ *for* x *on a set* E *of positive measure, then* $a_n=o(n^{1/4})$.

THEOREM 8. *If the series* $\sum_0^\infty a_n\phi_n(x)$ *converges, for all* x, *to a finite function* $f(x)$, *such that* $f\in H$, *then*

$$f(x)\sim\sum_0^\infty a_n\phi_n(x).$$

Theorem 1, for which an analogue may be found in [6], is of fundamental importance in our treatment of the problem. Theorem 3 is the most general uniqueness theorem obtained in this paper. Since F is assumed to be continuous, it should be noted that the conditions $\Lambda^*F>-\infty$, $\Lambda_*F<+\infty$ can also be written $D^{*2}F>-\infty$, $D_*^2F<+\infty$, where D^2 is the generalized second derivative [12, p. 270] (by (2.2.5)). In its main features, the proof of Theorem 3 follows the method developed by Riemann for trigonometric series.

Theorem 4 furnishes an analogue to Rajchman's inequalities [12, p. 298], and is the basis for Theorems 5 and 6. Theorem 6 is implicitly contained in a result obtained by Zygmund by equiconvergence methods [13, p. 440]. We include it here since it can be deduced from Theorem 5, and because it leads to Theorem 8, which has not apparently been stated in the literature.

Uniqueness theorems of a different kind have been obtained by Dominguez [4]. For results obtained by equiconvergence methods we also refer to [11].

IV. EXAMPLES

4.1. The identity [9]

$$(4.1.1)\quad \sum_{n=0}^\infty \phi_n(x)\phi_n(t)r^n = (\pi(1-r^2))^{-1/2}\exp\left(\frac{x^2-t^2}{2}-\frac{(x-rt)^2}{1-r^2}\right)$$

$$(0\leqq r<1)$$

shows that the series

$$\sum_{n=0}^{\infty} \phi_n(0)\phi_n(x), \tag{4.1.2}$$

whose coefficients are $O(n^{-1/4})$ (see (9.1.1)), is Poisson summable to zero for all $x \neq 0$. Applying the operator Ω to the right member of (4.1.1), with $t=0$, and letting $r \to 1$, we obtain

$$-\sum_{n=0}^{\infty} \phi_n(0)\phi_n(x)/(2n+2) \sim -k(x, 0) \tag{4.1.3}$$

(compare (7.3.1)). Putting $F(x) = -k(x, 0)$, and using (2.2.5), we see that

$$\lim_{h \to 0} \Delta_h F(0)/h = \frac{1}{2} > 0, \tag{4.1.4}$$

and $\Lambda_* F(0) = +\infty$. Hence the condition $\liminf_{h \to 0} \Delta_h F(x)/h \leqq 0$ cannot be omitted in Theorems 1, 3, 5.

4.2. If we multiply (4.1.1) by $e^{x^2/2}$, put $t=0$, differentiate with respect to x, and use the identity $H_n'(x) = 2nH_{n-1}(x)$ [8, p. 102], we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} (2n+2)^{1/2}\phi_{n+1}(0)\phi_n(x)r^n \\ &\quad = -2\pi^{-1/2}r(1-r^2)^{-3/2}x \exp\left(-\frac{1+r^2}{1-r^2}\frac{x^2}{2}\right) \quad (0 \leqq r < 1). \end{aligned} \tag{4.2.1}$$

This shows that the series

$$\sum_{n=0}^{\infty} (2n+2)^{1/2}\phi_{n+1}(0)\phi_n(x), \tag{4.2.2}$$

whose coefficients are $O(n^{1/4})$, is Poisson summable to zero for all x. Hence the order $o(n^{1/4})$ in Theorem 6 is the best possible. Applying Ω to (4.2.1), and letting $r \to 1$, we obtain

$$-\sum_{n=0}^{\infty} (2n+2)^{-1/2}\phi_{n+1}(0)\phi_n(x) = \begin{cases} -\pi^{-1/2}\beta(x) & (x < 0), \\ 0 & (x = 0), \\ \pi^{-1/2}\beta(-x) & (x > 0). \end{cases} \tag{4.2.3}$$

Thus the continuity condition imposed on F in Theorems 3 and 5 is violated in this example.

V. The operator Ω

Let $f \in H_0$. Then Ωf (see 2.3) has the following properties:

5.1. *Ωf is continuous.*

5.2. *$d\Omega f(x)/dx$ exists for all x.*

5.3. $\Delta_h \Omega f(x) = -\int_{x-h}^{x+h} f(t)\Delta_h k(x, t)dt$.

5.4. *For almost all x, $\Lambda\Omega f(x) = f(x)$.*

5.5. *Let $p \geqq 0$ be given. If $f \in H_p$, then $\Omega f \in H_p$. If $f \geqq 0$ and $\Omega f \in H_p$, then $f \in H_p$.*

5.6. *If f is upper semi-continuous at x, then $\Lambda^* \Omega f(x) \leqq f(x)$.*

5.7. **Proofs of 5.1–5.6.** We begin with the asymptotic formula

$$\int_x^\infty t^p e^{-ct^2} dt = \frac{1}{2c} x^{p-1} e^{-cx^2}(1 + O(x^{-2})) \qquad (x \to +\infty), \tag{5.7.1}$$

where p and c are fixed, $c > 0$. To prove (5.7.1), we note that

$$\lim_{x \to +\infty} 2cx^{1-p} e^{cx^2} \int_x^\infty t^p e^{-ct^2} dt = 1.$$

The bound for the error term then follows from the recursion formula

$$\int_x^\infty t^p e^{-ct^2} dt = \frac{1}{2c} x^{p-1} e^{-cx^2} + \frac{p-1}{2c} \int_x^\infty t^{p-2} e^{-ct^2} dt.$$

Taking $c = 1$, $p = 0$ in (5.7.1), we obtain asymptotic formulas for $\beta(-x)$ as $x \to +\infty$, and for $\beta(x)$ as $x \to -\infty$. Substitution of these formulas into (2.3.2) shows that $\Omega f(x)$ is finite for all x.

Let x be fixed, and suppose $z > x$. Then, by (2.3.1) and (2.3.2),

$$\begin{aligned} \pi^{1/2} \frac{\Omega f(z) - \Omega f(x)}{z - x} = &- \frac{\beta(-z) - \beta(-x)}{z - x} \int_{-\infty}^x f(t)\beta(t)dt \\ &- \frac{1}{z - x} \int_x^z f(t)(k(z, t) - k(x, t))dt \\ &- \frac{\beta(z) - \beta(x)}{z - x} \int_z^\infty f(t)\beta(-t)dt. \end{aligned} \tag{5.7.2}$$

Denoting the three terms on the right of (5.7.2) by A, B, C, it is easily seen that

$$\lim_{z \to x} A = \beta'(-x) \int_{-\infty}^x f(t)\beta(t)dt,$$

$$\lim_{z \to x} C = -\beta'(x) \int_x^\infty f(t)\beta(-t)dt.$$

Since $k(z, t) - k(x, t) = O(z - x)$, we have $\lim_{z \to x} B = 0$. We thus obtain the value of the right-hand derivative of $\Omega f(x)$. Taking $z < x$, the same value is obtained for the left-hand derivative. This proves 5.2, and also 5.1.

Considering $k(x, t)$ as a function of x alone, with t fixed, $k(x, t)$ satisfies

(2.1.2) for $x \neq t$. Hence $\Delta_h k(x, t) = 0$ for $|x-t| > h$. 5.3 follows.

Next, let E be the set of points x at which

$$\frac{d}{dx}\int_{-\infty}^{x} f(t)\beta(t)dt = f(x)\beta(x) \quad \text{and} \quad \frac{d}{dx}\int_{x}^{\infty} f(t)\beta(-t)dt = -f(x)\beta(-x).$$

The complement of E is of measure zero. For x on E, we write (2.3.2) in the form

$$\pi^{1/2}\Omega f(x) = -\beta(-x)\int_{-\infty}^{x} f(t)\beta(t)dt - \beta(x)\int_{x}^{\infty} f(t)\beta(-t)dt,$$

and verify by direct differentiation that

$$\frac{d^2}{dx^2}\Omega f(x) - (x^2+1)\Omega f(x) = f(x) \qquad (x \in E).$$

5.4 follows.

Before proceeding to the proof of 5.5, let us put

$$g_p(x) = \int_{-\infty}^{\infty} k(x, t)e^{-t^2/2}|t|^p dt. \tag{5.7.3}$$

We wish to prove that, for fixed $p \geqq 0$,

$$g_p(x) = (2p+2)^{-1}e^{-x^2/2}|x|^p(1 + O(x^{-1})) \qquad (x \to \pm\infty). \tag{5.7.4}$$

Suppose $x > 0$, and split the integral in (5.7.3) into three parts: $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_0^x + \int_x^\infty$. Denoting these three integrals by A, B, C respectively, and using (5.7.1), we obtain

$$A = \pi^{-1/2}\beta(-x)\int_{-\infty}^{0} \alpha(t)|t|^p dt = O(x^{-1}e^{-x^2/2}),$$

$$C = \pi^{-1/2}\beta(x)\int_{x}^{\infty} \alpha(-t)t^p dt < \frac{1}{2}e^{x^2/2}\int_{x}^{\infty} e^{-t^2}t^{p-1}dt$$
$$= O(x^{p-2}e^{-x^2/2}).$$

The major contribution is due to

$$B = \pi^{-1/2}\beta(-x)\int_0^x \alpha(t)t^p dt$$
$$= \pi^{-1/2}\beta(-x)(p+1)^{-1}\left(\alpha(x)x^{p+1} - \int_0^x t^{p+1}e^{-t^2}dt\right)$$
$$= \pi^{-1/2}\beta(-x)(p+1)^{-1}(\pi^{1/2}x^{p+1} + O(1))$$
$$= (2p+2)^{-1}x^p e^{-x^2/2}(1 + O(x^{-1})).$$

This proves (5.7.4) for $x>0$. A similar proof applies if $x<0$.

Now suppose $f\in H_p$. By Fubini's Theorem on the change of the order of integration, we obtain, using (5.7.4),

$$\int_{-\infty}^{\infty} | x^p \Omega f(x) | e^{-x^2/2} dx \leqq \int_{-\infty}^{\infty} | x |^p e^{-x^2/2} dx \int_{-\infty}^{\infty} | f(t) | k(x, t) dt$$

$$= \int_{-\infty}^{\infty} | f(t) | g_p(t) dt < +\infty.$$

Thus $\Omega f\in H_p$. This proves the first part of 5.5.

By 5.1, $g_p(x)$ is continuous. Hence there exists a constant c such that

$$| x |^p e^{-x^2/2} < c g_p(x)$$

for all x. Thus, if $f\geqq 0$ and $\Omega f\in H_p$,

$$\int_{-\infty}^{\infty} | x |^p e^{-x^2/2} f(x) dx \leqq c \int_{-\infty}^{\infty} f(x) g_p(x) dx$$

$$= c \int_{-\infty}^{\infty} f(x) dx \int_{-\infty}^{\infty} k(x, t) e^{-t^2/2} | t |^p dt$$

$$= c \int_{-\infty}^{\infty} e^{-t^2/2} | t^p \Omega f(t) | dt < +\infty.$$

This proves the second part of 5.5.

Next, we note that 5.3 and 5.4 imply

$$\lim_{h\to 0} \frac{2}{h^2} \int_{x-h}^{x+h} \Delta_h k(x, t) dt = -1, \tag{5.7.5}$$

and that

$$\Delta_h k(x, t) < 0 \tag{5.7.6}$$

for $|x-t| < h$. Suppose f is upper semi-continuous at x. Choose $m>f(x)$. There exists $\delta>0$ such that $f(t)<m$ for $|x-t|<\delta$. By (5.7.6),

$$\Delta_h \Omega f(x) = -\int_{x-h}^{x+h} f(t) \Delta_h k(x, t) dt < -m \int_{x-h}^{x+h} \Delta_h k(x, t) dt,$$

which implies, by (5.7.5), that $\Lambda^* \Omega f(x) \leqq m$. 5.6 follows.

VI. Λ-CONVEX FUNCTIONS

6.1. DEFINITION. The function $F(x)$, defined in (a, b), is said to be Λ-convex in (a, b) if the equations $y(c)=F(c)$, $y(d)=F(d)$ $(a<c<d<b)$ imply $F(x)\leqq y(x)$ for $c<x<d$, where $y(x)$ is a solution of (2.1.2).

Generalized convex functions of this type have been studied by Becken-

bach and Bing [1], [2]. In particular, if F is Λ-convex in (a, b), then F is continuous in (a, b), and $F(a+)$, $F(b-)$ exist (as finite numbers or as $+\infty$ or $-\infty$).

6.2. LEMMA. *If F is upper semi-continuous in (a, b), and if $\Lambda^*F(x) \geqq 0$ except possibly on a countable set E, on which* $\limsup_{h\to 0} \Delta_h F(x)/h \geqq 0$, *then F is Λ-convex in (a, b).*

Proof. If F is not Λ-convex, there exists a solution $y(x)$ of (2.1.2) such that

$$y(x_1) = F(x_1), \qquad y(x_2) = F(x_2), \qquad y(x_3) < F(x_3)$$

for some x_1, x_2, x_3 $(a < x_1 < x_3 < x_2 < b)$. Let us define

$$\mu(F; x) = \limsup_{t\to x} \frac{F(t) - F(x)}{|t - x|}. \tag{6.2.1}$$

We choose $\epsilon \geqq 0$ such that, putting $z(x) = y(x) + \epsilon\beta(x)$, we have

$$z(x_1) \geqq F(x_1), \qquad z(x_2) \geqq F(x_2), \qquad z(x_3) < F(x_3),$$

and

$$\mu(z; x) \neq \mu(F; x) \qquad (x \text{ on } E). \tag{6.2.2}$$

(6.2.2) can be satisfied by proper choice of ϵ, since E is at most countable. Put $\omega(x) = F(x) - z(x)$. Then ω attains a maximum of $c > 0$ at a point x_0 $(x_1 < x_0 < x_2)$. If $x_0 \notin E$, we have $\Delta_h\omega(x_0) \leqq \Delta_h c$, by (2.2.4). Hence

$$\Lambda^*F(x_0) = \Lambda^*(\omega(x_0) + z(x_0)) = \Lambda^*\omega(x_0) \leqq \Lambda^*c = -c(x^2 + 1) < 0,$$

which is a contradiction. If $x_0 \in E$, we have

$$\limsup_{h\to 0} \Delta_h\omega(x_0)/h \geqq 0.$$

Hence $\lim \Delta_h\omega(x_0)/h \geqq 0$ as h tends to zero through a properly chosen sequence $\{h_n\}$. For this sequence, (2.2.5) holds, and we conclude that

$$\limsup_{h\to 0} \left\{\frac{\omega(x_0 + h) - \omega(x_0)}{h} + \frac{\omega(x_0 - h) - \omega(x_0)}{h}\right\} \geqq 0.$$

Since ω attains a maximum at x_0, the last inequality implies that $\mu(\omega; x_0) = 0$, which contradicts (6.2.2).

6.3. COROLLARY. *If F is continuous, and $\Lambda_*F(x) \leqq 0 \leqq \Lambda^*F(x)$ in (a, b), then F satisfies* (2.1.2) *in (a, b).*

6.4. LEMMA. *If F is Λ-convex in $(-\infty, \infty)$ and if $F \in H_0$, then $F(x) \leqq 0$ for all x.*

Proof. If there is a point x_0 such that $F(x_0)>0$, then, by continuity, there exist $x_1<x_0<x_2$ such that $F(x)>0$ for $x_1<x<x_2$. Choose c_1, c_2 such that

$$c_1\beta(x_1) + c_2\beta(-x_1) = F(x_1), \qquad c_1\beta(x_2) + c_2\beta(-x_2) = F(x_2).$$

Then $c_1>0$ or $c_2>0$ (or both), and $F(x)\geqq c_1\beta(x)+c_2\beta(-x)$ for $x<x_1$ and for $x>x_2$. Since $\beta(x)\sim\pi^{1/2}e^{x^2/2}$ as $x\to+\infty$, it follows that $F\notin H_0$.

6.5. Lemma. *Let $p\geqq 0$ be given. If $F\in H_p$ and F is Λ-convex in $(-\infty, \infty)$, then $\Lambda^*F\in H_p$ and $\Lambda_*F\in H_p$.*

Proof. Put $f(x)=\Lambda^*F(x)$. Then $f(x)\geqq 0$. There exists a sequence $\{u_n(x)\}$ of non-negative upper semi-continuous functions associated with $f(x)$ in the sense of the Vitali-Carathéodory theorem, and such that $u_n(x)=0$ for $|x|>n$ [7, p. 75]. Put

$$W_n(x) = F(x) - \Omega u_n(x) \qquad (n = 1, 2, 3, \cdots).$$

Then W_n is continuous, by 5.1, and, by 5.6,

$$\Lambda^*W_n(x) \geqq \Lambda^*F(x) - \Lambda^*\Omega u_n(x) \geqq f(x) - u_n(x) \geqq 0.$$

Hence W_n is Λ-convex in $(-\infty, \infty)$. Since u_n is bounded and vanishes outside a finite interval, $u_n\in H_p$. By 5.5, it follows that $W_n\in H_p$, which, by 6.4, implies that $W_n(x)\leqq 0$. Hence

$$F(x) \leqq \Omega u_n(x) \leqq 0 \qquad (n = 1, 2, 3, \cdots).$$

Since $u_n\to f$ p.p. monotonically as $n\to\infty$, we may pass to the limit, and obtain

$$F(x) \leqq \Omega f(x) \leqq 0,$$

which shows that $\Omega f\in H_p$. Since $f\geqq 0$, 5.5 shows that $f\in H_p$. Noting that $0\leqq\Lambda_*F(x)\leqq\Lambda^*F(x)$ for all x, the lemma follows.

VII. Proofs of Theorems 1, 2, 3

We shall use the following variation of the Vitali-Carathéodory theorem:

7.1. Lemma. *If $f\in H_p$ and if f is defined for all x, then there exists a sequence $\{u_n(x)\}$, $n=1, 2, 3, \cdots$, such that*

(a) *$u_n(x)$ is upper semi-continuous and less than $+\infty$ for all x;*
(b) *$u_1(x)\leqq u_2(x)\leqq \cdots \leqq f(x)$ for all x;*
(c) *$\lim_{n\to\infty} u_n(x)=f(x)$ p.p.;*
(d) *$u_n\in H_p$.*

To see this, put

$$h(x) = \begin{cases} e^{-1/2} & (|x| < 1), \\ |x|^p e^{-x^2/2} & (|x| \geqq 1). \end{cases}$$

Let $\{u_n^*(x)\}$ be a sequence of upper semi-continuous functions associated, in the sense of the Vitali-Carathéodory theorem, with the function $f(x)h(x)$ in $(-\infty, \infty)$. Then $u_n^* \in L$ on $(-\infty, \infty)$ for every n. The functions

$$u_n(x) = u_n^*(x)/h(x) \qquad (n = 1, 2, 3, \cdots)$$

have the desired properties.

7.2. Now, suppose $F(x)$ satisfies the hypotheses of Theorem 1. Let $u(x)$ be one of the functions associated with $y(x)$ in the sense of 7.1. Put $W(x) = F(x) - \Omega u(x)$. Then $W(x)$ is continuous, and, by 5.6,

$$\Lambda^* W(x) \geqq \Lambda^* F(x) - \Lambda^* \Omega u(x) \geqq \Lambda^* F(x) - u(x) \geqq 0$$

on the complement of E_1. By 5.2, we have

$$\limsup_{h \to 0} \Delta_h W(x)/h \geqq 0 \qquad (x \text{ on } E_1).$$

Hence W satisfies all conditions of 6.2 and is thus Λ-convex. 5.5 shows that $W \in H_p$. By 5.4 and 6.5, the obvious inequalities

$$\Lambda_* W(x) + \Lambda_* \Omega u(x) \leqq \Lambda_* F(x) \leqq \Lambda^* F(x) \leqq \Lambda^* W(x) + \Lambda^* \Omega u(x)$$

show that $\Lambda^* F \in H_p$ and $\Lambda_* F \in H_p$.

Let $f(x)$ be a measurable function, defined for all x, such that

$$\Lambda_* F(x) \leqq f(x) \leqq \Lambda^* F(x).$$

Then $f \in H_p$. Let $\{u_n(x)\}$ be a sequence associated with $f(x)$ in the sense of 7.1. Put $W_n(x) = F(x) - \Omega u_n(x)$. Proceeding as above, with W_n and u_n in place of W and u, we see that W_n is Λ-convex. Hence, for every x, n, and h $(h > 0)$, $\Delta_h F(x) \geqq \Delta_h \Omega u_n(x)$. That is, by 5.3,

$$\Delta_h F(x) \geqq -\int_{x-h}^{x+h} u_n(t) \Delta_h k(x, t) dt.$$

Since $u_n \to f$ p.p. monotonically, we may pass to the limit (taking into account (5.7.6)), and obtain

$$\Delta_h F(x) \geqq \Delta_h \Omega f(x).$$

Approximating f in a similar fashion by a monotonically decreasing sequence of lower semi-continuous functions, we obtain the last formula with the inequality reversed. Hence

$$\Delta_h F(x) = \Delta_h \Omega f(x)$$

for all x and all $h > 0$. By 6.3, $F(x) - \Omega f(x) = y(x)$, where y is a solution of (2.1.2). Since $F \in H_p$ and $\Omega f \in H_p$, it follows that $y \in H_p$, and therefore $y = 0$ (no other solution of (2.1.2) $\in H_p$). Thus $F(x) = \Omega f(x)$. By 5.4, $\Lambda F(x) = f(x)$ p.p.

This completes the proof of Theorem 1.

7.3. To prove Theorem 2, we note that (2.2.6) and Theorem 1 imply

$$\Omega\phi_n(x) = -\phi_n(x)/(2n+2). \tag{7.3.1}$$

Further,

$$\begin{aligned}\int_{-\infty}^{\infty}\phi_n(x)\Omega f(x)dx &= -\int_{-\infty}^{\infty}\phi_n(x)dx\int_{-\infty}^{\infty}f(t)k(x,t)dt\\ &= -\int_{-\infty}^{\infty}f(t)dt\int_{-\infty}^{\infty}\phi_n(x)k(x,t)dx\\ &= \int_{-\infty}^{\infty}f(t)\Omega\phi_n(t)dt.\end{aligned}$$

By (7.3.1.) and (1.1.5), this proves Theorem 2. The change in the order of integration is justified by (5.7.4), since $\phi_n(x)=O(x^ne^{-x^2/2})$.

7.4. To prove Theorem 3, we apply Theorem 1 with $p=1, 2, 3, \cdots$, and see that $F(x)=\Omega\Lambda F(x)$ and that $\Lambda F\in H$. Thus

$$\Omega\Lambda F(x) \sim -\sum_{n=0}^{\infty} a_n\phi_n(x)/(2n+2).$$

The theorem follows from Theorem 2.

VIII. Proofs of Theorems 4, 5.

8.1. Suppose $f\in H$, and $f(x)\sim\sum_0^\infty a_n\phi_n(x)$. The series

$$\sum_{n=0}^{\infty} a_n\phi_n(x)r^n \qquad (0\leqq r<1) \tag{8.1.1}$$

is formally equal to the function

$$f(x,r) = \int_{-\infty}^{\infty} f(t)P(x,t,r)dt \qquad (0\leqq r<1), \tag{8.1.2}$$

where $P(x, t, r)$, the Poisson kernel for Hermite series, is equal to the right member of (4.1.1) [4, p. 4]. At every point x on the Lebesgue set of f we have

$$\lim_{r\to 1} f(x,r) = f(x), \tag{8.1.3}$$

although the radius of convergence of (8.1.1) may be less than 1 [5, p. 423] (Hille deals with developments in Hermite *polynomials*, therefore the kernel used in [5] differs slightly from (4.1.1)). For fixed x, the function (8.1.2) is analytic in a certain ellipse in the complex r-plane which contains the segment $-1<r<1$ [5, p. 423]. Hence (8.1.2) can be differentiated under the integral sign, and we obtain the partial differential equation

$$\Lambda f(x, r) + 2 \frac{\partial}{\partial r} (rf(x, r)) = 0 \qquad (0 \leqq r < 1). \tag{8.1.4}$$

8.2. Suppose now that the hypotheses of Theorem 4 are satisfied. If $\Lambda_* F(x_0) = -\infty$, the first part of the conclusion is trivial. Suppose $\Lambda_* F(x_0) > -\infty$. If F is bounded above near x_0, then F must also be bounded below, for otherwise $\Lambda_* F(x_0) = -\infty$. Hence it suffices to consider the case in which F is bounded below.

Next, $\Lambda_* F(x_0) \leqq f^*(x_0)$ will be proved if we can show that the inequality $\Lambda_* F(x_0) > m$ implies $f^*(x_0) \geqq m$ for every finite m. By changing a_0 and a_1, if necessary, we see that we can assume without loss of generality that

$$F(x_0) = 0, \tag{8.2.1}$$

$$\Lambda_* F(x_0) > 0. \tag{8.2.2}$$

We have to prove that (8.2.2) implies

$$f^*(x_0) \geqq 0. \tag{8.2.3}$$

For $0 \leqq r < 1$, put

$$F(x_0, r) = -\sum_{n=0}^{\infty} a_n \phi_n(x_0) r^n/(2n+2) = \int_{-\infty}^{\infty} F(t) P(x_0, t, r) dt, \tag{8.2.4}$$

then

$$f(x_0, r) = \sum_{n=0}^{\infty} a_n \phi_n(x_0) r^n = \Lambda F(x_0, r), \tag{8.2.5}$$

by (8.1.4). By hypothesis, the series in (8.2.4) converges for $0 \leqq r < 1$, and so the same is true for the series in (8.2.5).

Suppose (8.2.3) is false. By (8.1.4), $f^*(x_0) < 0$ implies, by (8.2.5),

$$\liminf_{r \to 1} \frac{\partial}{\partial r} (rF(x_0, r)) > 0. \tag{8.2.6}$$

By (8.2.1) and the mean value theorem, (8.2.6) implies

$$\limsup_{r \to 1} \frac{F(x_0, r)}{1 - r} < 0. \tag{8.2.7}$$

Next, (8.2.2) implies that there exists $h_0 > 0$ such that

$$\Delta_h F(x_0) > 0 \qquad (0 < h < h_0). \tag{8.2.8}$$

Since $F(x)$ is bounded below near x_0, there is a positive number k such that $F(x) > -k$ in a neighborhood of x_0. Put

$$G(x) = \min (F(x), 2k). \tag{8.2.9}$$

Then it is easily seen that $\Delta_h G(x_0) > 0$ for small enough h. (2.2.5) can be applied to G, and gives

$$G(x_0 + h) + G(x_0 - h) > 0 \qquad (0 < h < \delta). \tag{8.2.10}$$

We now have, by (8.2.4),

$$\begin{aligned} \liminf_{r\to 1} \frac{F(x_0, r)}{1-r} &= \liminf_{r\to 1} \frac{1}{1-r}\int_{-\infty}^{\infty} F(x_0+u)P(x_0, x_0+u, r)du \\ &= \liminf_{r\to 1} \frac{1}{1-r}\int_{-\delta}^{\delta} F(x_0+u)P(x_0, x_0+u, r)du \\ &\geqq \liminf_{r\to 1} \frac{1}{1-r}\int_{-\delta}^{\delta} G(x_0+u)P(x_0, x_0+u, r)du, \end{aligned} \tag{8.2.11}$$

since the integrals $\int_{-\infty}^{-\delta}$, $\int_{\delta}^{\infty}$ tend to zero as $r\to 1$. Since

$$P(x, x+u, r) = (\pi(1-r^2))^{-1/2} \exp\left(-\frac{1-r}{1+r}x^2 - \frac{1-r}{1+r}xu - \frac{1+r^2}{1-r^2}\cdot\frac{u^2}{2}\right),$$

and

$$\exp\left(-\frac{1-r}{1+r}x_0 u\right) = 1 + g(r, u),$$

where

$$|g(r, u)| < (1-r)|x_0 u| \qquad (r_0 < r < 1,\ |u| < \delta),$$

it follows from (8.2.10) and (8.2.11) that

$$\begin{aligned} &\liminf_{r\to 1} F(x_0, r)/(1-r) \\ &\quad \geqq \liminf_{r\to 1} c_1(1-r)^{-3/2}\int_{-\delta}^{\delta} G(x_0+u)(1+g(r,u)) \exp\left(-\frac{1+r^2}{1-r^2}\cdot\frac{u^2}{2}\right)du \\ &\quad \geqq \liminf_{r\to 1} c_1(1-r)^{-3/2}\int_{-\delta}^{\delta} G(x_0+u)g(r,u) \exp\left(-\frac{1+r^2}{1-r^2}\cdot\frac{u^2}{2}\right)du \\ &\quad \geqq -c_2 \liminf_{r\to 1} (1-r)^{-1/2}\int_{-\delta}^{\delta} |u| \exp\left(-\frac{1+r^2}{1-r^2}\cdot\frac{u^2}{2}\right)du = 0. \end{aligned}$$

(Here c_1 and c_2 denote positive constants.) We have thus arrived at a contradiction to (8.2.7), which proves that $\Lambda_* F(x_0) \leqq f^*(x_0)$. The second part of the conclusion follows by a change of sign.

8.3. Suppose the hypotheses of Theorem 5 are satisfied. Since $f^*(x)$, $f_*(x)$ exist, the series $\sum_0^\infty a_n\phi_n(x)r^n$ converges for $0 \leqq r < 1$ (see 2.4). The same is true of the series $\sum_0^\infty a_n\phi_n(x)r^n/(2n+2)$. Since F is continuous, conditions (ii), (iii) of Theorem 4 holds everywhere. Thus Theorem 5 is an immediate

consequence of Theorems 3 and 4.

IX. Proofs of Theorems 6, 7, 8

9.1. **Asymptotic formulas.** For our purpose, the approximation [8, p. 194]

$$\phi_n(x) = cn^{-1/4}\{\cos((2n+1)^{1/2}x - n\pi/2) + r_n(x)\}, \tag{9.1.1}$$

where c is a constant, will be sufficient. In this formula,

$$r_n(x) = O(n^{-1/2}), \qquad r_n'(x) = O(1) \qquad (n \to \infty) \tag{9.1.2}$$

uniformly in every finite interval. We shall use the following consequences of (9.1.1) and (9.1.2):

$$\int_{-h}^{h} \phi_n(x+t)dt = O(n^{-3/4}). \tag{9.1.3}$$

$$\phi_n(x+t) - \phi_n(x) = O(tn^{1/4}). \tag{9.1.4}$$

The last formula implies

$$\int_{-h}^{h} (\phi_n(x+t) - \phi_n(x))dt = O(h^2 n^{1/4}). \tag{9.1.5}$$

Formulas (9.1.3) and (9.1.5) hold equally well with $\int_{-h}^{h}$ replaced by $\int_0^h$ or by $\int_{-h}^{0}$.

An analogue, for trigonometric series, of the next lemma may be found in [12, pp. 272–273].

9.2. Lemma. *Suppose the series $\sum_0^\infty b_n\phi_n(x)$ is Poisson summable everywhere to $F(x)$, and $b_n = o(n^{-3/4})$. Then*

(i) *putting $L(x) = \int_0^x F(t)dt$, we have*

$$L(x) = \sum_0^\infty b_n \int_0^x \phi_n(t)dt,$$

the convergence of the integrated series being uniform in every finite interval;

(ii) *$\sum_0^\infty b_n\phi_n(x)$ converges to $F(x)$ if and only if*

$$(L(x+h) - L(x-h))/2h \to F(x) \quad as \quad h \to 0;$$

(iii) *$(L(x+h)+L(x-h)-2L(x))/h \to 0$ for all x, as $h \to 0$;*

(iv) *if $F(x+)$ exists, then $\sum_0^\infty b_n\phi_n(x)$ converges to $F(x+)$; the same is true for $F(x-)$;*

(v) *if $F(x+)$ and $F(x-)$ exist, then $F(x+) = F(x-)$.*

Proof. By the Riesz-Fischer theorem, $F(x) \sim \sum_0^\infty b_n\phi_n(x)$. The sequence $\{b_n\}$ satisfies Zygmund's condition $A_{1/2}$ [13, pp. 434–435]. Hence $\sum_0^\infty b_n\phi_n(x)$ is equiconvergent with a trigonometric series, and may be integrated termwise [13, p. 437]. This proves (i). Next, put $S_N = \sum_0^N b_n\phi_n(x)$. Then, keeping

x fixed,

$$(L(x+h)-L(x-h))/2h - S_N$$
$$= \frac{1}{2h}\sum_{n=0}^{N} b_n \int_{-h}^{h} (\phi_n(x+t)-\phi_n(x))dt + \frac{1}{2h}\sum_{n=N+1}^{\infty} b_n \int_{-h}^{h} \phi_n(x+t)dt.$$

Denote the two terms on the right by P and Q, and let $N=[h^{-2}]$. By (9.1.5),

$$P = \frac{1}{2h}\sum_{n=0}^{N} o(n^{-3/4})O(h^2n^{1/4}) = h\sum_{n=0}^{N} o(n^{-1/2}) = o(hN^{1/2}) = o(1).$$

By (9.1.3),

$$Q = \frac{1}{2h}\sum_{n=N+1}^{\infty} o(n^{-3/4})O(n^{-3/4}) = \frac{1}{h}\sum_{n=N+1}^{\infty} o(n^{-3/2}) = o(h^{-1}N^{-1/2}) = o(1).$$

This proves (ii). To prove (iii), we again let $N=[h^{-2}]$. Then

$$L(x+h)+L(x-h)-2L(x) = \sum_{n=0}^{\infty} b_n \left\{ \int_0^h \phi_n(x+t)dt - \int_0^h \phi_n(x-t)dt \right\}.$$

In the last sum, write $\sum_0^\infty = \sum_0^N + \sum_{N+1}^\infty = A+B$. Applying (9.1.5) to A and (9.1.3) to B, in the same manner as to P and Q above, we obtain $A=o(h)$, $B=o(h)$. This proves (iii).

Next, if $F(x+)$ exists, then $D^+L(x)=f(x+)$. By (iii), we then have $D^-L(x)=D^+L(x)$. Hence $L'(x)=f(x+)$, and (iv), (v) follow from (ii).

9.3. Suppose now that the hypotheses of Theorem 6 are satisfied. Since $f^*(x)$ and $f_*(x)$ are finite, the function

$$f(x,r) = \sum_{n=0}^{\infty} a_n\phi_n(x)r^n \qquad (0 \leqq r < 1) \tag{9.3.1}$$

is bounded in r for each x. Thus the series

$$-\sum_{n=0}^{\infty} a_n\phi_n(x)/(2n+2) \tag{9.3.2}$$

is Poisson summable at all points x to a function $F(x)$. Moreover, every perfect set P of points x contains a portion $\Pi = J\cdot P$, where J is a segment, such that $f(x,r)$ is uniformly bounded for x on Π [12, p. 299]. Hence (9.3.2) is uniformly Poisson summable on Π. It follows that $F(x)$ is continuous on a dense open set.

By the Riesz-Fischer theorem, (9.3.2) is a Hermite series. By Theorem 5, it is therefore sufficient to show that $F(x)$ is continuous for all x.

Let $u(x)$ be a function which is associated with $y(x)$ in the sense of 7.1, such that $u\in H_0$. Put $W(x)=F(x)-\Omega u(x)$. Then, if F is bounded above in a neighborhood of x, Theorem 4, and 5.6, imply

$$\Lambda^* W(x) \geqq \Lambda^* F(x) - \Lambda^* \Omega u(x) \geqq f_*(x) - u(x) \geqq f_*(x) - y(x) \geqq 0.$$

By 6.2, $W(x)$ is thus Λ-convex in every segment in which $W(x)$ is upper semi-continuous.

Let E be the set of points x at which $F(x)$ is discontinuous. Suppose E contains an isolated point x_0. Then $W(x)$ is Λ-convex to the right and to the left of x_0, $W(x_0+)$ and $W(x_0-)$ exist, hence $F(x_0+)$ and $F(x_0-)$ exist; applying 9.2, with $b_n = a_n/(2n+2)$, we see that $F(x_0+) = F(x_0-)$, and that (9.3.2) converges to the common value. Thus x_0 is a point of continuity of F, and E contains no isolated point.

Thus, if E is not vacuous, its closure $\overline{E}$ is perfect. Let $\Pi = J \cdot \overline{E}$, where J is a segment, be a portion of $\overline{E}$ on which F is continuous. Let (a, b) be a segment contiguous to Π. Since W is Λ-convex in (a, b), $W(a+)$ and $W(b-)$ exist. Hence $F(a+)$ and $F(b-)$ exist. By 9.2, $F(a+) = F(a)$, $F(b-) = F(b)$. Hence $W(a+) = W(a)$, $W(b-) = W(b)$. It follows that, if x_0 is any point of Π, then W is upper semi-continuous at x_0. Thus W is upper semi-continuous in J, which implies that W is Λ-convex, and therefore continuous, in J.

Thus E is vacuous, and Theorem 6 follows. (The above proof is very similar to that of the analogous theorem for trigonometric series in [12, p. 301].)

9.4. Theorem 7 is a consequence of (9.1.1) and the Riemann-Lebesgue theorem [10, p. 403]. Theorem 8 follows immediately from Theorems 6 and 7.

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