THE STRUCTURE OF VALUATIONS OF THE RATIONAL FUNCTION FIELD $K(x)(^1)$

BY

B. N. MOYLS

1. Introduction. The following problem was suggested to the writer by Professor Saunders MacLane: Given a valuation $(V_0K = \Gamma_0, \mathcal{K})$ of a field $K$ with value group $\Gamma_0$ and residue class field $\mathcal{K}$; (A) to determine the nature of $\Gamma$ and $\mathcal{L}$ for any extension $(VL = \Gamma, \mathcal{L})$ of $V_0$ from $K$ to $L/K$; and conversely (B) to construct valuations of an extension $L/K$ with value groups and residue class fields which conform to the requirements of (A). The present paper considers this problem in the case when $L$ is a simple transcendental extension $K(x)$ of $K$. The valuations are of arbitrary rank (cf. [2](^2)).

It is well known that (1) the sum of the transcendence degree $T[\mathcal{L}/\mathcal{K}]$ of $\mathcal{L}$ over $\mathcal{K}$ and the rational rank (cf. [6, footnote 3]) $T[\Gamma/\Gamma_0]$ of the factor group $\Gamma/\Gamma_0$ cannot exceed $T[L/K]$, here equal to 1. Also, (2) if $T[\mathcal{L}/\mathcal{K}] + R[\Gamma/\Gamma_0] = T[L/K]$, then $\mathcal{L}$ and $\Gamma$ are finitely generated over $\mathcal{K}$ and $\Gamma_0$, respectively. To these conditions we add (3) $\mathcal{L}$ and $\Gamma$ must be at most denumerably generated over $\mathcal{K}$ and $\Gamma_0$; and (4) if $T[\mathcal{L}/\mathcal{K}] = 1$, then $\mathcal{L}$ must be a rational function field in one variable over a finite algebraic extension of $\mathcal{K}$. The possible forms for $\Gamma$ and $\mathcal{L}$ are given explicitly in Theorems 7.1 and 8.1.

The construction of extensions $(VK(x) = \Gamma, \mathcal{L}) \supseteq (V_0K = \Gamma_0, \mathcal{K})$ with $\Gamma$ and $\mathcal{L}$ satisfying conditions (1) to (4) is given in §9, except for the case where $\Gamma/\Gamma_0$ is finite and $\mathcal{L}$ is a finite algebraic extension of $\mathcal{K}$. §12 contains a note on the extension of these results to finitely generated purely transcendental extensions of $K$.

Two approaches have been made to the study of rank 1 valuations of $K(x)$. One, used by Ostrowski [7], represents $x$ as the limit of a pseudo-convergent sequence in the algebraic completion of $K$. The other, used by MacLane [3] and [4] and based on work of Rella [8], represents each discrete valuation of $K(x)$ by a simple sequence of approximating subvaluations of $K(x)$, in which each approximant is derived from the preceding by a certain "key" polynomial. It is an exploitation of Gauss' Lemma.

Following the latter method, we show (§6) that every valuation $V$ of

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(^1) This paper includes some of the results given in the author's doctoral dissertation (Harvard, 1947).

(^2) Numbers in brackets refer to the bibliography at the end of the paper.
K(x) (of arbitrary rank) can be approximated by a well-ordered system of “inductive” valuations. This yields MacLane’s structure theory [3] for $V$, and the results given above(3).

The existence of extensions $(VK(x)=\Gamma, L)$ of $V_0$ for which $\Gamma/\Gamma_0$ is finite and $L$ is a finite algebraic extension of $K$ depends on the presence of certain transcendental pseudo-convergent sets in $K$ or an algebraic extension of $K$. Some incomplete conditions for this case are given in §10. They are analogous to Kaplansky’s conditions [1] for the special case of immediate extensions.

2. Augmented valuations. Let $V_0$ be a general valuation of a base field $K$ with value group $\Gamma_0$; and let $\Gamma$ be any ordered abelian group containing $\Gamma_0$. If $f(x) = \sum_{i=0}^{n} c_i x^i$, $c_i \in K$, then the function

$$\begin{align*}
V_1(f) &= \min \{ V_0(c_i) + i\gamma \},
\end{align*}$$

where $\gamma$ is any element in $\Gamma$, defines a “first stage” valuation [3, §3; 7, p. 363; 8, pp. 35–36] of the polynomial ring $K[x]$; $V_1 = V_0$ on $K$. $V_1$ is denoted by $[V_0, V_1(x) = \gamma]$. For a valuation of this type, any linear polynomial in $x$ can be used in the role of $x$.

Any valuation $W$ of $K[x]$ can be augmented to other valuations of $K[x]$ by means of certain key polynomials which are monic, equivalence-irreducible, and equivalence-minimal in the following sense. Two polynomials $f$ and $g$ are equivalent in $W$, or $f \sim g$, if $w(f-g) > w(f)$; $f$ equivalence-divides $g$ if there exists $h$ in $K[x]$ such that $fh \sim g$; $f$ is equivalence-irreducible if the equivalence-divisibility of a product by $f$ implies that of a factor. The polynomial $f$ is equivalence-minimal if the degree (in $x$) of every polynomial equivalence-divisible by $f$ is not less than the degree of $f$.

Let $W$ be a valuation of $K[x]$ with value group $\Gamma' \subseteq \Gamma$, and let $\phi = \phi(x)$ be a key polynomial over $W$. If we write the polynomial $f$ in the form

$$\sum_{i=0}^{m} f_i \phi^i, \text{ where } f_i \in K[x] \text{ and } \deg f_i < \deg \phi$$

then the function $V$

$$\begin{align*}
V(f) &= \min \{ W(f_i) + i\gamma \},
\end{align*}$$

where $\gamma \in \Gamma, \gamma > W(\phi)$, is an augmented valuation [3, §4], and is denoted by $V = [W, V(\phi) = \gamma]$. It has the following property(4):

I. For $f \neq 0$, $W(f) \leq V(f)$; $W(f) < V(f)$ if and only if $f$ is equivalence-divisible by $\phi$ in $W$. In particular $W(f) = V(f)$ if $\deg f < \deg \phi$.

If we build a finite sequence $\{ V_\mu \}$ of augmented valuations(5),

$$\begin{align*}
V_\mu &= \left[ V_{\mu-1}, V_\mu(\phi_\mu) = \gamma_\mu \right], \quad \mu = 2, 3, \cdots, k,
\end{align*}$$

(4) Many of MacLane’s proofs carry over to the general case with only minor modifications. In such instances his results are quoted without proof.

(5) $\deg f$ will always mean the degree of $f(x)$ in $x$.

(6) Here $V_1$ may be any valuation of $K[x]$. The subscript 1 is not reserved for first stage valuations.
with the conditions
\begin{align}
(2.3) & \quad \deg \phi_\mu \geq \deg \phi_{\mu-1}, \\
(2.4) & \quad \phi_\mu \sim \phi_{\mu-1} \text{ in } V_{\mu-1} \text{ is false,
}
\end{align}
then \{V_\mu\} has the following properties:

II. For each \( f \in K[x] \), \( V_\mu(f) \leq V_\lambda(f) \) for \( \mu < \lambda \). If \( V_\beta(f) = V_{\beta+1}(f) \), then \( V_\beta(f) = V_\omega(f) \) for all \( \omega > \nu \).

III. If \( \deg \phi_\mu = \deg \phi_\lambda \) for \( 1 \leq \eta < \mu \) and all \( \lambda > \mu \), then:
\begin{align}
(a) & \quad V_\eta(\phi_\lambda - \phi_\mu) = \gamma_\mu < \gamma_\lambda; \\
(b) & \quad V_\eta(\phi_\mu) = V_\eta(\phi_\lambda); \ V_\mu(\phi_\lambda) = \gamma_\mu; \\
(c) & \quad V_\lambda = [V_\mu, V(\phi_\lambda) = \gamma_\lambda].
\end{align}

3. Limit valuations. Another type of valuation of \( K[x] \) can be obtained as follows: Suppose that a well-ordered set of valuations \{V_\mu\} has been defined for all \( \mu < \sigma \), and that \{V_\mu\} has property II.

If, for each \( f \in K[x] \), there is an ordinal \( \nu \) such that \( V_\nu(f) = V_{\nu+1}(f) \), let \( \nu(f) \) be the first such. The function \( W_\nu \):
\begin{align}
(3.1) & \quad W_\nu(f) = V_{\nu(f)}(f)
\end{align}
defines a valuation of \( K[x] \); we denote it by \( W_\sigma = \{V_\mu\}, \mu < \sigma \).

Otherwise, there must exist a polynomial \( g \) such that \( V_\mu(g) < V_\lambda(g) \) for all \( \mu < \lambda \). A monic polynomial of minimum degree with this property will be called a pseudo-key for \{V_\mu\}. A pseudo-key is irreducible in \( K[x] \). Expanding any \( f \) in terms of such a pseudo-key \( s \),
\begin{align}
(3.2) & \quad f = \sum_{i=0}^{m} f_is_i, \quad \deg f_i < \deg s,
\end{align}
we can define the function \( V_\sigma \),
\begin{align}
(3.3) & \quad V_\sigma(f) = \min_i [W_\sigma(f_i) + i\gamma_\sigma],
\end{align}
where \( \gamma_\sigma > V_\mu(s) \) for all \( \mu \).

**Theorem 3.1.** The function \( V_\sigma \) defined by (3.2) and (3.3) is a valuation of \( K[x] \); it is denoted by \( V_\sigma = \{V_\mu\}, \mu < \sigma, V_\sigma(s) = \gamma_\sigma \).

**Proof.** For the triangle and product laws to hold for \( V_\sigma \), it is sufficient that (cf. [3, Theorem 4.2] or [8])
(A) the triangle law hold for polynomials of degree \( < \deg s \), and
(B) if \( f \) and \( g \) are polynomials of degree less than \( \deg s \) with the expansion (3.2), \( fg = qs + r \), then
\begin{align}
V_\sigma(f) + V_\sigma(g) = V_\sigma(r) < V_\sigma(q) + \gamma_\sigma.
\end{align}
It is necessary only to verify (B). For some ordinal \( \nu \), \( V_\nu(r) = V_{\nu+1}(r) = V_\omega(r) \) and \( V_\nu(fg) = V_{\nu+1}(fg) = V_{\nu}(f) + V_\omega(g) \). Now \( V_{\nu+1}(qs) > V_\nu(qs) \geq \min \{ V_\nu(fg), V_\nu(r) \} = V_{\nu+1}(fg) = V_{\nu+1}(r) \). Hence \( V_\nu(g) + \gamma_\nu \nu > V_{\nu+1}(qs) > V_\nu(r) = V_\nu(f) + V_\omega(g) \). Q.E.D.

Both \( W_\nu \) and \( V_\nu \) are called \textit{limit valuations}.

To show that properties I and II hold for \( V_\nu \), we need Ostrowski's Lemma [7, p. 371, III; 1, p. 306, Lemma 4].

**Lemma 3.2.** Let \( \beta_0, \beta_1, \ldots, \beta_m \) be any elements of an ordered Abelian group \( \Gamma \), and let \( \{ \alpha_\mu \} \) be a well-ordered set of elements of \( \Gamma \) (without a last element) such that \( \alpha_\sigma < \alpha_\lambda \) for all \( \sigma < \lambda \). Then there exist an integer \( e \) (\( 0 \leq e \leq m \)) and an ordinal \( \eta \) such that \( \beta_i + \alpha_\mu > \beta_\nu + e \alpha_\nu \) for all \( i \neq e \) and \( \mu > \eta \).

**Theorem 3.3.** Given the limit valuation \( V_\sigma = \{ \{ V_\mu \}, \mu < \sigma \), \( V_\sigma(\phi_\sigma) = \gamma_\sigma \) with the pseudo-key \( \phi_\sigma \). For \( f \neq 0 \), \( V_\sigma(f) \leq V_\nu(f) \) for all \( \mu \). The following statements are equivalent:

(i) \( V_\mu(f) < V_\lambda(f) \) for all \( \mu < \lambda < \sigma \);
(ii) \( V_\mu(f) < V_\sigma(f) \) for all \( \mu < \sigma \);
(iii) \( \phi_\sigma \) equivalence-divides \( f \) in all \( V_\mu \) for \( \mu \) greater than some ordinal \( \eta \).

**Proof.** Let \( f = \sum_{i=0}^{m} f_i \phi_\mu^i \) be the expansion (3.2) for \( f \). By II and Lemma 3.2, there exist an integer \( e \) and an ordinal \( \eta \) such that \( V_\mu(f \phi_\mu^e) < V_\mu(f \phi_\mu^i) \) for all \( \mu > \eta \) and all \( i \neq e \). Thus, for \( \mu > \eta \), \( V_\mu(f) = V_\mu(f \phi_\mu^e) = \min \{ V_\mu(f \phi_\mu^i) \} \leq \min \{ V_\nu(f \phi_\nu^i) \} = V_\nu(f) \). Moreover, the inequality sign holds if and only if \( e \neq 0 \), which in turn is true if and only if \( V_\mu(f) < V_\lambda(f) \) for all \( \eta < \mu < \lambda \) (or for all \( \mu < \lambda \), by II). If \( e \neq 0 \), then \( \phi_\sigma \) equivalence-divides \( f \) in \( V_\mu \), \( \mu > \eta \). Conversely if \( V_\mu(f - q \phi_\sigma) > V_\mu(f) = V_\mu(q \phi_\sigma) \) for some \( q \in K[x] \), then for \( \lambda > \mu \), \( V_\lambda(f) \geq \min \{ V_\lambda(f - q \phi_\sigma), V_\lambda(q \phi_\sigma) \} \geq \min \{ V_\mu(q \phi_\sigma), V_\mu(q \phi_\sigma) \} = V_\mu(f) \). Q.E.D.

**Note.** Theorem 3.3 proves that II holds for the set \( \{ V_\mu \}, \mu \leq \sigma \). It further shows that I holds for \( V_\sigma \) if we make the convention that \( W \) is to be interpreted as representing all \( V_\mu \) for \( \mu \) greater than some ordinal \( \eta \); \( \eta \) depends on \( f \). The pseudo-key \( \phi_\sigma \) takes the place of a key for \( V_\sigma \). The next theorem shows that augmenting a limit-valuation with a key of sufficiently high degree preserves II.

**Theorem 3.4.** If \( \deg \phi_\sigma \leq \deg \phi_{\sigma+1} \) in the valuation \( V_{\sigma+1} = \{ \{ V_\mu \}, \mu < \sigma \), \( V_\sigma(\phi_\sigma) = \gamma_\sigma \), \( V_{\sigma+1}(\phi_{\sigma+1}) = \gamma_{\sigma+1} \), then \( V_\mu(f) = V_\nu(f) \) for some \( \mu < \sigma \) implies \( V_\sigma(f) = V_{\sigma+1}(f) \).

**Proof.** If \( V_\mu(f) = V_\sigma(f) \), then in the expansion (3.2) in terms of \( \phi_\sigma \), \( V_\sigma(f) = V_\nu(f) < V_\sigma(f - f_0) \) (cf. the preceding proof). But \( V_{\sigma+1}(f - f_0) \geq V_\sigma(f - f_0) \), and \( V_{\sigma+1}(f_0) = V_\sigma(f_0) \), by I. Therefore \( V_{\sigma+1}(f - f_0) > V_{\sigma+1}(f_0) \), which implies \( V_{\sigma+1}(f) = V_{\sigma+1}(f_0) = V_\sigma(f) \).

4. **Inductive valuations.**

**Definition 4.1.** A \( p \)th stage inductive valuation \( V_\rho \) of \( K[x] \) is any valuation obtained by a well-ordered sequence of valuations \( \{ V_\sigma \}, \sigma \leq \rho \), where
(i) \( V_\alpha = [V_0, V_1(\phi_1) = \gamma_1], \phi_1 \text{ linear}; \)
(ii) if \( \sigma \) is not a limit-ordinal, \( \sigma > 1, V_\sigma = [V_{\sigma-1}, V_\sigma(\phi_\sigma) = \gamma_\sigma]; \)
(iii) if \( \sigma \) is a limit-ordinal, then \( V_\sigma \) is the limit valuation \([\{V_\mu\}, \mu < \sigma], \) or \([\{V_\mu\}, \mu < \sigma, V_\sigma(\phi_\sigma) = \gamma_\sigma], \) where \( \phi_\sigma \) is a pseudo-key for \([V_\mu]; \)
(iv) \( \deg \phi_\mu \leq \deg \phi_\lambda \) for all ordinals \( \mu < \lambda \leq \rho; \)
(v) if \( \deg \phi_\mu = \deg \phi_\lambda \) in \( V_\mu, \) \( \phi_\mu \sim \phi_\lambda \)

If \( \rho \) is a limit ordinal, \( V_\rho \) is called a constant degree limit valuation when the set \([\deg \phi_\sigma], \sigma < \rho, \) is bounded; otherwise, increasing degree.

An inductive valuation \( V_\rho \) has property I, and the subvaluations \([V_\sigma], \sigma \leq \rho, \) have properties II and III. Any augmented valuation \( V_{\rho+1} \) is an inductive valuation, provided that the key \( \phi_{\rho+1} \) satisfies conditions (iv) and (v). However, we have the following theorem.

**Theorem 4.2.** The limit valuation \( W_\rho = [\{V_\mu\}, \mu < \rho] \) cannot be augmented to an inductive valuation \( V. \)

**Proof.** Let \( \phi \) be a prospective key for \( V. \) We write \( \phi = g\phi_{\rho}(\phi_{\rho}+1) + r \) (cf. (3.1)), where \( \deg r < \deg \phi_{\rho}(\phi_{\rho}+1). \) By I and II, we have \( W_\rho(r) = V_{\rho}(r) < V_{\rho}(\phi_{\rho}(\phi_{\rho}+1)) \leq W_\rho(g\phi_{\rho}(\phi_{\rho}+1)) \leq V(g\phi_{\rho}(\phi_{\rho}+1)). \) By condition (iv), \( \deg \phi > \deg r; \) hence \( W_\rho(r) = V(r), \) and \( V(\phi) = V(r) = W_\rho(\phi). \) This contradicts the requirement that \( V(\phi) > W_\rho(\phi). \)

5. Conditions for limit valuations. If \( \rho \) is a limit ordinal, and \([V_\sigma], \sigma < \rho, \) is a well-ordered set of inductive valuations

\[
(5.1) \quad V_\sigma = [V_{\sigma-1}, V_\sigma(\phi_\sigma) = \gamma_\sigma] \quad \text{or} \quad V_\sigma = [\{V_\mu\}, \mu < \sigma, V_\sigma(\phi_\sigma) = \gamma_\sigma],
\]

then \([V_\sigma]\) has property II. If \([V_\sigma]\) has a pseudo-key \( s, \) there exists an integer \( d \) such that \( \deg \phi_\sigma = d \) for all \( \sigma \) not less than some ordinal \( \omega. \) Moreover, the value group \( \Gamma_\sigma \) of \( V_\sigma \) with respect to \( K(x) \) equals \( \Gamma_\sigma, \) for \( \sigma > \omega, \) by III. Now the inductive valuation \([V_\sigma], \sigma < \rho, V_\rho(s) = \gamma_\rho\) can be constructed if \( \gamma_\rho \) can be chosen greater than all \( V_\sigma(s). \) This can be done (without increasing the rank of the valuation) if and only if the set \([V_\sigma(s)]\) is bounded in \( \Gamma_\omega. \)

**Theorem 5.1(\(^*\)).** In the set \([V_\sigma], \sigma < \rho, \) of valuations defined by (5.1), the set \([V_\sigma(s)]\) is bounded by some element of \( \Gamma_\omega \) if and only if the same is true for \([\gamma_\sigma], \sigma > \omega.\)

**Proof.** We expand \( s \) in terms of each \( \phi_\sigma, \sigma > \omega, \)

\(^*\) Henceforth the term limit valuation refers only to inductive valuations.

\(^\dagger\) This theorem has particular relevance to the rank 1 case. In this case the set \([V_\sigma],\) \( \omega < \sigma < \rho, \) can always be replaced by a cofinal denumerable sequence \([V_\mu]\) (2.2). A limit value can be defined (as in [3]) on \( K(x) \) by the function \( V: V(f) = \lim_{\mu \to \rho} V_\mu(f). \) This function may be nonfinite in the sense that it assigns to some nonzero polynomials the value \( \infty. \) Our Theorem 5.1 and MacLane's Theorem 7.1 [3] together give a NAS condition for the finiteness of \( V. \)

On the other hand, the latter theorem gives a NAS condition for the existence of a pseudo-key for \([V_\mu]\) and hence for \([V_\sigma], \) when \( \lim_{\mu \to \rho} \gamma_\mu = \infty. \)

\(^*\) Note that by III, \( \gamma_\omega < \gamma_\lambda \) for \( \omega < \sigma < \lambda. \)
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$$s = \sum_{i=0}^{m} b_{i} \phi_{i}, \quad \text{deg } b_{i} < d.$$  

By III, $V_{\sigma} = [V_{\omega}, V_{\sigma}(\phi_{\sigma}) = \gamma_{\sigma}]$; and $V_{\omega}(\phi_{\omega}) = \gamma_{\omega}$. As noted in the proof of Lemma 3.4 of [4],

$$V_{\omega}(s) = \min_{i} \{V_{\omega}(b_{i} + i \gamma_{\omega})\}.$$  

For some index $e$ there exists a well-ordered set $\{\sigma(\alpha)\}$ of ordinals cofinal in the set $\{\sigma\}$, $\omega < \sigma < \rho$, such that

$$V_{\omega}(s) = V_{\omega}(b_{e,\sigma(\alpha)}) + e \gamma_{\omega} \leq V_{\omega}(b_{i,\sigma(\alpha)}) + i \gamma_{\omega},$$

for all $i$. Thus, for each $\alpha$, $V_{\omega}(b_{e,\sigma(\alpha)}) = \delta$, and, for all $i$ and $\alpha$, $V_{\sigma}(b_{i,\sigma(\alpha)}) \geq \xi$. By II, each $V_{\sigma}(s) \leq V_{\sigma}(a)(s) \leq \delta + e \gamma_{\sigma(a)}$ for some $\alpha$.

On the other hand

$$V_{\sigma(a)}(s) = \min_{i} \{V_{\omega}(b_{i,\sigma(a+1)}) + i \gamma_{\sigma(a)}\}$$

$$= V_{\omega}(b_{e(\alpha),\sigma(\alpha+1)}) + e(\alpha) \gamma_{\sigma(a)} + \xi + e(\alpha) \gamma_{\sigma(a)},$$

for all $\alpha$ and some index $e(\alpha)$, depending on $\alpha$. Moreover $e(\alpha) \neq 0$; for otherwise $V_{\sigma(a+1)}(s) = V_{\sigma(a)}(s)$. Thus each $\gamma_{\sigma} \leq \gamma_{\sigma(\beta)} \leq (V_{\sigma(\beta)}(s) - \xi)/e(\beta)$, for some ordinal $\beta$. This completes the proof.

6. The sufficiency of inductive valuations. From the proof of Theorem 8.1 of [3] we borrow the following result:

**Lemma 6.1.** Let $W$ be any valuation of $K[x]$. Let $V_{\sigma}$ be an inductive valuation $[V_{\sigma-1}, V_{\sigma}(\phi_{\sigma}) = W(\phi_{\sigma}) = \gamma_{\sigma}]$ or $[V_{\sigma}, \mu < \sigma, V_{\sigma}(\phi_{\sigma}) = W(\phi_{\sigma}) = \gamma_{\sigma}]$ such that

IV.  
(a) $W(f) \geq V_{\sigma}(f)$ for all $f$ in $K[x]$,  
(b) $W(f) = V_{\sigma}(f)$ if $\text{deg } f < \text{deg } \phi_{\sigma}$.

Then any monic polynomial $\phi$ of minimum degree such that $W(\phi) > V_{\sigma}(\phi)$ defines an inductive valuation $V = [V_{\sigma}, V(\phi) = W(\phi) = \gamma]$ which satisfies IV. Moreover, $V_{\sigma}(f) = W(f)$ implies $V_{\sigma}(f) = W(f)$.

**Theorem 6.2.** Every valuation $W$ of $K[x]$ can be represented as an inductive valuation.

**Proof.** First, $V_{1} = [V_{0}, V_{1}(x) = \gamma_{1} = W(x)]$ is an inductive valuation satisfying IV.

Now suppose that $V_{\sigma}$ is an inductive valuation with property IV and such that $V_{\sigma}(f) = W(f)$ for all $f$ of degree less than $n$. We proceed by induction on $n$. If there exists a polynomial $h$ of degree $n$ such that $V_{\sigma}(h) < W(h)$, we
let \( \mathcal{N}_n \) be the set of all monic polynomials of degree \( n \) with this property, and let \( \mathcal{M}_n \) be the corresponding set of \( W \)-values.

Case (1). If \( \mathcal{M}_n \) has a maximum element \( \gamma \), we choose a member of \( \mathcal{N}_n \) with value \( \gamma \), call it \( \phi_{n+1} \) and define \( V_{\phi_{n+1}} = \{ V_\phi, V_{\phi_{n+1}}(\phi_{n+1}) = \gamma \} \). By Lemma 6.1, this is an inductive valuation satisfying IV. Moreover, if \( \deg f = n \), \( V_{\phi_{n+1}}(f) = W(f) ; f = c\phi_{n+1} + f_0 \), where \( c \in K \), \( \deg f_0 < n \); and \( W(f) > V_{\phi_{n+1}}(f) \) implies \( W(f/c) > W(\phi_{n+1}) \), contradicting the choice of \( \phi_{n+1} \).

Case (2). If \( \mathcal{M}_n \) has no maximum element, we choose from \( \mathcal{N}_n \) a set of polynomials whose values form a well-ordered cofinal subset \( \{ \gamma_{n+1}, \gamma_{n+2}, \ldots; \mu < \lambda \} \). Using transfinite induction and Lemma 6.1, we construct a well-ordered set \( \{ V_{\phi_{n+1}} \} \), \( \mu < \lambda \) of inductive valuations, each with property IV. If \( \mu \) is not a limit-ordinal, \( V_{\phi_{n+1}} = \{ V_{\phi_{n+1}} \}, \mu < \lambda \), or \( V_{\phi_{n+1}} = \{ V_{\phi_{n+1}} \}, \mu < \lambda \), \( V_{\phi_{n+1}}(s) = W(s) \), where \( s \) is a pseudo-key for \( \{ V_{\phi_{n+1}} \} \). Now \( W_{\phi_{n+1}} = W \), or \( V_{\phi_{n+1}} \) possesses IV, by the last statement of Lemma 6.1. If \( \deg f = n \), \( f = c\phi_{n+1} + f_\mu \) for each \( \mu ; c \in K \), \( \deg f_\mu < n \). Now \( W(f) > V_{\phi_{n+1}}(f) \) implies \( W(f/c) > W(\phi_{n+1}) \) for all \( \mu < \lambda \), contradicting the choice of the set \( \{ \phi_{n+1} \} \). Hence \( W(f) = V_{\phi_{n+1}}(f) \) for all \( f \) of degree not greater than \( n \).

This completes the induction on \( n \). If the process does not stop at some finite degree, it will go on indefinitely to give an increasing degree limit valuation equal to \( W \).

7. The value group. If \( V_\phi \) is an inductive valuation \( \{ V_{\phi_{n+1}}, V_\phi(\phi_\phi) = \gamma_\phi \} \), and if \( \Gamma_\phi \) is the value group of \( V_\phi \) with respect to \( K(x) \), then \( \Gamma_\phi = \Gamma_{\phi_{n+1}}(\gamma_\phi) \), that is, all elements of the form \( \gamma + m\gamma_\phi \), where \( \gamma \in \Gamma_{\phi_{n+1}} \) and \( m \) is an integer. If \( V_\phi \) is a limit valuation \( \{ V_\phi \}, \mu < \sigma, V_\phi(\phi_\phi) = \gamma_\phi \), then by III, \( \Gamma_\phi = \Gamma_\sigma(\gamma_\phi), \) where \( \phi_\sigma \) is the first key in the set \( \{ \phi_\mu \}, \mu < \sigma, \) of highest degree. Moreover, if \( \gamma_\phi \) is incommensurable with \( \Gamma_0 \), that is, no multiple of \( \gamma_\phi \) is in \( \Gamma_0 \), then \( V_\phi \) cannot be augmented to a new inductive valuation \( [3, \text{Theorem 6.7}] \). An induction argument gives the following theorem.

Theorem 7.1. Let \( V_0 \) be a valuation of \( K \) with value group \( \Gamma_0 \), and let \( V_\phi \) be an inductive valuation of \( K(x) \) with keys and pseudo-keys \( \{ \phi_\phi \} \); then \( \Gamma_\phi \) has one of the forms:

(a) \( \Gamma_\phi = \Gamma_0(\gamma_1, \gamma_2, \ldots, \gamma_n) \),
(b) \( \Gamma_\phi = \Gamma_0(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \xi_n) \),
(c) \( \Gamma_\phi = \Gamma_0(\gamma_1, \gamma_2, \ldots), \) not reducible to form (a),

where \( \gamma_i \) or \( \xi_i \) is the \( V_\phi \)-value of the last key or pseudo-key of degree \( i \) (when such actually is present); \( \gamma_i \) is commensurable with \( \Gamma_0; \xi_i \) is not.

8. The residue class field. The structure theorems in §§9–14 of [3] may be verified for the more general inductive valuations considered here. The proofs are not sufficiently different from MacLane’s to warrant their repetition here. The pertinent results are as follows:
Let $\mathcal{K}$ be the residue class field of $K$ with respect to $V_0$; let $\mathcal{L}_\sigma$ be the residue field of $L = K(x)$ with respect to $V_\sigma$; and let $H_\sigma$ be the corresponding homomorphism mapping the valuation ring in $K(x)$ onto $\mathcal{L}_\sigma$. If $V_\sigma$ is commensurable, then $\mathcal{L}_\sigma = F_\sigma(y)$, where $F_\sigma$ is an algebraic extension of the field $K_\sigma$ and $y$ is transcendental over $F_\sigma$; if $V_\sigma$ is incommensurable, $\mathcal{L}_\sigma = F_\sigma$. If $V_\sigma$ is augmented to $V$, the resulting residue field $\mathcal{L}_\sigma$ is $F(z) = F_\sigma(\theta, z)$, where $\theta$ and $z$ are algebraic and transcendental over $F_\sigma$, respectively, and are determined by the augmenting key $\phi$. Corresponding to $\phi$ there exists a polynomial $p(x)$ such that $V_\mu(p) = V_\sigma(p) = -V_\sigma(\phi)$ for some $\mu < \sigma$; $p(x)$ will be called a $V_\sigma$-deflater of $\phi$. Then $\theta$ is a root of $H_\sigma(p\phi)$, a polynomial of degree $[\deg \phi/(\tau_\sigma \deg \phi_\sigma)]$ in the ring $F_\sigma[y]$; where $\tau_\sigma$ is the commensurability number of $V_\sigma$, that is, the order of $\Gamma_\sigma/\Gamma_{\sigma-1}$ or $\Gamma_\sigma/\Gamma_\omega$ (cf. §7); and $z = H(q\phi^r)$, where $q$ is a $V$-deflater of $\phi$.

Analogous to property III we have the condition that if $\deg \phi_\sigma = \deg \phi$, then $F_\sigma = F$. If $V_\sigma$ is a limit valuation $\{V_\mu\}, \mu < \sigma$ without a pseudo-key, $\mathcal{L}_\sigma$ is the union $\bigcup_{\mu<\sigma} F_\mu$ of the fields $F_\mu$. If $V_\sigma$ is a limit valuation with pseudo-key $s$, then $\mathcal{L}_\sigma$ is $F_\sigma(z)$ or $F_\sigma$, as before, where now $F_\sigma = U_{\mu<\sigma} F_\mu$.

**Theorem 8.1.** Let $(V_\rho K(x) = \Gamma_\rho, \mathcal{L}_\rho)$ be an extension of $(V_0 K = \Gamma_0, \mathcal{K})$ with keys and pseudo-keys $\{\phi_\sigma\}$; then $\mathcal{L}_\rho$ has one of the forms:

(i) $\mathcal{L}_\rho = \mathcal{K}(\alpha_1, \alpha_2, \ldots, \alpha_m)$;
(ii) $\mathcal{L}_\rho = \mathcal{K}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, y)$;
(iii) $\mathcal{L}_\rho = \mathcal{K}(\alpha_1, \alpha_2, \ldots)$;

where the $\alpha_i$ are algebraic over $\mathcal{K}$; $y$ is transcendental. If $\Gamma_\rho$ is not commensurable with $\Gamma_0$, $\mathcal{L}_\rho$ must have form (i). If $\mathcal{L}_\rho$ has form (ii), $\Gamma_\rho/\Gamma_0$ must be finite. The number of adjoined elements is not greater than the number of degrees represented in the set $\{\phi_\sigma\}$.

From Theorem 7.1 and 8.1 the possible combinations of $\Gamma_\rho$ and $\mathcal{L}_\rho$ are (a)(i), (a)(ii), (a)(iii), (b)(i), (c)(i), and (c)(iii).

9. The existence of inductive valuations with a prescribed structure. Given $(V_0 K = \Gamma_0, \mathcal{K})$, the construction of $(V K(x) = \Gamma_\rho, \mathcal{L}_\rho)$ with $\Gamma_\rho$ and $\mathcal{L}_\rho$ satisfying the conditions of Theorems 7.1 and 8.1 is given in most cases by the following theorem of MacLane [3, Theorem 13.1].

**Lemma 9.1.** In a given inductive valuation $(V_\sigma K(x) = \Gamma_\sigma, \mathcal{L}_\sigma)$, let $\psi(y) \neq y$ be a monic polynomial of degree $m > 0$, irreducible in $F_\sigma[y]$. Then there is one and, except for equivalent polynomials in $V_\sigma$, only one $\phi(x)$ which is a key over $V_\sigma$ and which has $H_\sigma(p\phi) = \psi(y)$ for a suitable $V_\sigma$-deflater $p$ of $\phi$.

**Theorem 9.2.** Given $(V_0 K = \Gamma_0, \mathcal{K})$; let $\Gamma \not= 0$ and $\mathcal{L}$ be extensions of $\Gamma_0$ and $\mathcal{K}$, respectively, such that (schematically) $\Gamma$ and $\mathcal{L}$ occur in any of the following combinations (cf. Theorems 7.1 and 8.1):

(*) The excluded case is the combination (a)(i).
Then there exists an extension \((VK(x) = \Gamma, L)\) of \(V_0\).

10. A special case. Further conditions are needed for the existence of type (ai), that is, an extension \((VK(x) = \Gamma, L)\) of \((V_0K = \Gamma_0, K)\) with \(\Gamma/\Gamma_0\) finite and \(L\) a finite algebraic extension of \(K\).

First let \(K\) be algebraically closed. Then \(\Gamma = \Gamma_0\) and \(L = K\); that is, the extension must be immediate. Any inductive representation \(V_0\) of \(V\) must have an infinite number of linear keys (§8) and none of higher degree. Suppose \(V_0\) is defined by the well-ordered set \(\{V_{(\alpha)}\}, \sigma < \rho\), where \(V_{(\alpha)} = [V_{(\alpha-1)}, V_\sigma(x-a_\sigma) = \gamma_\sigma]\). For \(\sigma < \lambda\), \(V_0(a_\lambda - a_\sigma) = \gamma_\sigma\), hence \(V_0(a_\lambda - a_\sigma) > V_0(a_\sigma - a_\rho)\) for \(\mu < \sigma < \lambda\), that is, \(\{a_\sigma\}\) is a pseudo-convergent set in \(K\). For every \(a \in K\), the set \(\{a - a_\sigma\}\) ultimately attains a constant value; otherwise the valuation \(V_0\) would equal the first stage valuation \(W_1 = [V_0, W_1(x-a) = V_0(x-a)]\). This is to say that \(\{a_\sigma\}\) has no limit in \(K\). Since \(K\) is closed, it further implies that \(\{a_\sigma\}\) is of transcendental type. Conversely, any transcendental pseudo-convergent set in \(K\) without a limit in \(K\) \((t.p.c.s.w.l.)\) defines a valuation \((V_0K(x) = \Gamma_0, K)\). Hence, if \(K\) is algebraically closed, there exists an immediate \((the only type (ai)) extension to \(K(x)\) if and only if there exists a \(t.p.c.s.w.l.\) in \(K\).

If \(K\) is arbitrary and \(A\) is its algebraic closure, then any type (ai) valuation of \(K(x)\) can always be extended (cf. [7, p. 300, 11]) to an (ai) valuation of \(A(x)\). Hence, for each \((ai)\) extension of \((V_0K = \Gamma_0, K)\) to \(K(x)\) there is a \(t.p.c.s.w.l.\) in \(A\) \((with respect to some extension of \(V_0\) to \(A)\).

A partial converse is given by the following theorem.

**Theorem 10.1.** Let \((V_0K = \Gamma_0, K)\) be any valuation of \(K\). Let \(\Gamma\) be a finite commensurable extension of \(\Gamma_0\) and \(L\) a finite algebraic extension of \(K\). Let \(M\) be any algebraic extension of \(K\) with a valuation \((V_0M = \Gamma, L)\) which is an extension of \(V_0\). If (1) \(M\) is a simple extension of \(K\), and (2) \(M\) contains a \(t.p.c.s.w.l.\), then there exists an extension \((VK(x) = \Gamma, L)\).

**Note.** \(M\) must always exist, but it is not always a simple extension of \(K\). The latter is true in the important case when \(M/K\) is separable; in particular, when \(K\) has characteristic 0.

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\(^{(10)}\) For these definitions cf. [1, §2]. If \(\{a_\sigma\}\) is pseudo-convergent, then for each \(f \in K[x]\), eventually either (1) \(V_0(f(a_\sigma)) = V_0(f(a_\rho))\) or (2) \(V_0(f(a_\sigma)) > V_0(f(a_\rho))\) for all \(\lambda > \sigma\). The set \(\{a_\sigma\}\) is of transcendental or algebraic type according as (1) does or does not hold for all \(f\).

\(^{(11)}\) This follows from Theorems 1, 2, and 3 of [1]. In fact Kaplansky proves this result under his hypothesis A, a weaker condition than closure. It is of interest to note that \(V_0\) is equal to the valuation which assigns to \(f(x)\) the ultimately constant value of \(f(a_\sigma)\) (cf. [1, Theorem 2] and [7, §65]).
Proof. Since $M$ contains a t.p.c.s.w.l. there exists an extension $(V'M(x_1) = \Gamma, \mathcal{L})$ of $(V_0'M, \Gamma, \mathcal{L})$ to $M(x_1)$; $x_1$ transcendental over $M$ [1, Theorem 2]. Moreover, there exists in $M(x_1)$ an element $x_2$, transcendental over $M$, such that $V'(x_2-1)$ is arbitrarily large.

Suppose $\gamma = \Gamma_0(\gamma_1, \ldots, \gamma_m), \mathcal{L}_0 = K(\alpha_{m+1}, \ldots, \alpha_n)$ and $M = K(v)$. From $M$ we select $u_i$ such that $V_0'(u_i) = \gamma_i$, $i = 1, \ldots, m$, and $H_0'(u_i) = \alpha_i$, $i = m+1, \ldots, n$. Suppose $u_i = \sum_{j=0}^{\gamma_j} a_{ij}v^j$, $a_{ij} \in K$. We set $u_i' = \sum_{j=0}^{\gamma_j} a_{ij}v^j x_2^j$, $i = 1, 2, \ldots, n$. Then we have $u_i' - u_i = (x_2-1) \left[ a_{i1}v + a_{i2}v^2(x_2+1) + \cdots + a_{in}v^n(x_2^m+1+\cdots+1) \right]$. Since $V'(x_2) = 0$, $V'(x_2^m+1+\cdots+1) \geq 0$ for all positive integers $j$. Hence $V'(u_i' - u_i) \geq V'(x_2-1) + \min_{j=1,\ldots,r}[V_0'(a_{ij}v^j)]$ for $i = 1, 2, \ldots, n$. If we choose $x_2$ so that $V'(x_2-1) > \max_i \{ V'(u_i) - \min_j \{ V_0'(a_{ij}v^j) \} \}$, then $V'(u_i' - u_i) > V'(u_i)$, for $i = 1, \ldots, m, \ldots, n$. It follows that $V'(u_i') = \gamma_i$ for $i = 1, \ldots, m$, and $H'(u_i') = H_0'(u_i) = \alpha_i$ for $i = m+1, \ldots, n$. Now $K(x)$, where $x = v x_2$, is a transcendental extension of $K$ with the desired valuation.

11. The existence of limit valuations with pseudo-keys. In constructing extensions $(VK(x), \Gamma, \mathcal{L})$ with prescribed $\Gamma$ and $\mathcal{L}$ (Theorem 9.2), it is not necessary to use limit valuations with pseudo-keys. One might therefore ask if there actually exist such valuations which can not be represented by a finite set of keys. Are pseudo-keys really necessary? The answer is yes, even in the rank 1 case.

Let $(V_0K = \Gamma_0, K)$ be of rank 1; and let $\{a_j\}$ be an algebraic (10) pseudo-convergent sequence in $K$ without a limit in $K$ such that $\lim_{j+\alpha} V_0(a_{j+1} - a_j) < \infty$ (12). We construct the inductive valuations $V_j = [V_{j-1}, V_j(x - a_j) = \gamma_j]$, $j = 1, 2, \ldots$, where $\gamma_j = V_0(a_{j+1} - a_j)$. Let $q(x)$ be any monic polynomial in $K[x]$ for which $V_0(q(a_j)) < V_0(q(a_{j+1}))$ for $j$ greater than some integer $j_0$. For each $j$ we expand

$$q(x) = \sum_{i=0}^{m} q_i(a_j)(x - a_j)^i,$$

where $q_0 = q$ and $q_n = 1$. Now $V_j(q(x)) = \min_i \{ V_0(q_i(a_j)) + i \gamma_j \}$. For $j$ greater than some $j_0$, the value of each term of (11.1) increases with $j$. It follows that $V_j(q(x)) < V_{j+1}(q(x))$ for all $j$. This implies that $\{ V_j \}$ has a pseudo-key $s$. Furthermore, $s$ cannot be of degree 1, namely, $x - a$, $a \in K$; for then $a$ would be a limit of $\{a_j\}$. Finally the $V_j$-values of $s$ are bounded, by Theorem 5.1. Hence the valuation $[\{ V_j \}], V(s) = \lim_{j+\alpha} V_j(s)$ is the desired valuation.

12. Extensions to $L_n = K(x_1, x_2, \ldots, x_n)$. The results of §§7–10 can be extended at once to the case of a purely transcendental extension $L_n = K(x_1, \ldots, x_n)$ of degree $n$. When $T[\mathcal{L}/K] + R[\Gamma/T_0] < n$ (§1), our results are not complete, but for rank 1 valuations they can be improved by the following lemma.

(11) The existence of a field with such a sequence is implied by the first counterexample in §5 of [1].
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Lemma 12.1. Let \( V_1 = [V_0, V_1(x_1) = \epsilon > 0] \) be a first stage (rank 1) valuation of \( K(x_1) \); then, for any positive integer \( q \), there exists an immediate extension of \( V_1 \) to the field \( K(x_1, \ldots, x_q) \), where the \( x_i \) are algebraically independent over \( K \).

Proof. In the power series field \( K\{\{x_1\}\} \) it is possible to select \( (q-1) \) series which are algebraically independent over \( K(x_1) \) and in which the non-zero coefficients have zero \( V_0 \)-value (cf. [6, §3], especially method II). But these series lie in an immediate extension of \( K(x_1) \).

Theorem 12.2. Let \( (V_0K = \Gamma_0, K) \) be a rank 1 valuation, and let \( \mathcal{L}, \Gamma \) be at most denumerably generated extensions of \( \mathcal{K}, \Gamma_0 \) with \( U = T[\mathcal{L}/\mathcal{K}] + R[\Gamma/\Gamma_0] \) \( < n \). There exists an extension \( (V\mathcal{L}_n = \Gamma, \mathcal{L}) \) if

(i) when \( \mathcal{L} \) and \( \Gamma \) can be finitely generated over \( K_0 \) and \( \Gamma_0 \), \( U \geq 2 \) and, whenever \( R[\Gamma/\Gamma_0] = 0 \), \( \mathcal{L} \) is a rational function field over an extension of \( K \);

(ii) otherwise, \( U \geq 1 \).

Bibliography


University of British Columbia,
Vancouver, B.C.