ON A CLASS OF MARKOV PROCESSES

BY

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1. Introduction. Let $x_k(t), x_k(0) = 0, 0 \leq t < \infty \ (k = 1, \ldots, n)$ be elements of $n$ independent Wiener spaces. Let $\tilde{x}(t)$ be an element of the product space of the $n$ independent Wiener spaces. Throughout the paper we shall assume $V(t, \tilde{x})$ to be a Borel measurable function of $t$ and the $n$-dimensional vector $\tilde{x} = (x_1, \ldots, x_n)$ that is bounded in every finite Euclidean sphere of the $(t, \tilde{x})$ space, $0 \leq t < \infty$. These assumptions will not be restated in the rest of the paper. We shall often assume that $V(t, \tilde{x})$ satisfies additional regularity conditions specified later on in the paper.

Let

$$y(t) = \int_0^t V(\tau, \tilde{x}(\tau))d\tau.$$  \hfill (1.1)

We wish to relate the study of the Markov process

$$(\tilde{x}(t), y(t))$$  \hfill (1.2)

to the study of certain differential and integral equations. Such a study is of interest for the following reasons. We obtain information about Markov processes of type (1.2). For certain choices of the function $V(t, \tilde{x})$, the study yields interesting information about the problem of the absorbing barrier in diffusion theory. Moreover, if we look at the problem from a different point of view, we obtain a general method of getting limit theorems of a certain type $[2]^{(2)}$.

Consider the distribution function

$$\sigma(\alpha; t) = \Pr \left\{ y(t) \leq \alpha \right\}. \hfill (1.3)$$

Let $\bar{G}_m = (G_{m}^{(1)}, \ldots, G_{m}^{(n)})$, where $G_{1}^{(k)}, G_{2}^{(k)}, \ldots (k = 1, \ldots, n)$ are independent, normally distributed random variables with mean 0 and variance 1. The distribution function of

$$\frac{1}{m} \sum_{k/m \leq t} V \left( \frac{k}{m}, \frac{\bar{G}_1 + \cdots + \bar{G}_k}{m^{1/2}} \right)$$

is the same as that of
by definition of Wiener measure. The distribution function of (1.4) approaches
the distribution function \( \sigma(\alpha; t) \) at every continuity point of the latter as
\( m \to \infty \), given that \( V(t, \bar{x}) \) is a limit of continuous functions.

In many instances, it is possible to show that \( \sigma(\alpha; t) \) is not only the limiting
distribution of (1.4) but also that of

\[
\frac{1}{m} \sum_{k \leq t} V\left( \frac{k}{m}, \bar{x}\left( \frac{k}{m} \right) \right)
\]

where \( X^{(k)}_1, X^{(k)}_2, \ldots (k = 1, \ldots, n) \) are general, independent, identically
distributed random variables with mean 0 and variance 1 and \( \bar{X}_m = (X^{(1)}_m, \ldots, X^{(n)}_m) \). In the last section we shall consider the case of a
function \( V(t, \bar{x}) = V(\bar{x}) \) for which such an invariance is easily proved.

Let \( V(t, \bar{x}) \) be bounded below by the constant \( M \) in \((t, \bar{x})\)-space. Then

\[
Q(t, \bar{x}) = E\left\{ \exp \left( -uy(t) \right) \mid \bar{x}(t) = \bar{x} \right\} \exp \left( -\frac{\bar{x}^2}{2t} \right) \frac{(2\pi t)^{n/2}}{\Gamma(n/2)}, \quad u \geq 0,
\]

exists since

\[
Q(t, \bar{x}) \leq E\left\{ \exp \left( u \mid M \mid t \right) \mid \bar{x}(t) = \bar{x} \right\} \exp \left( -\frac{\bar{x}^2}{2t} \right) \frac{(2\pi t)^{n/2}}{\Gamma(n/2)}.
\]

Set

\[
C(\xi, \tau; \bar{x}, t) = \exp \left( -\frac{\bar{x}^2}{2(t - \tau)} \right) \frac{(2\pi (t - \tau))^{n/2}}{\Gamma(n/2)}.
\]

If \( V(t, \bar{x}) \) is bounded in \((t, \bar{x})\)-space, \( Q(t, \bar{x}) \) satisfies the following integral equation

\[
Q(t, \bar{x}) + u \int_0^t \int_{-\infty}^{\infty} C(\xi, \tau; \bar{x}, t)V(\tau, \xi)Q(\tau, \xi)d\xi d\tau = C(\bar{0}, 0; \bar{x}, t)
\]

where the integration from \(-\infty\) to \( \infty \) denotes integration over all of \( \bar{x} \) space
and \( \bar{0} \) denotes the origin of the \( \bar{x} \)-space. M. Kac proved this result in the 1-
dimensional case and the proof for the \( n \)-dimensional case follows in a completely analogous manner [1].

Definition 1. The function \( g(t, \bar{x}) \) is said to satisfy a Hölder condition at
\((t', \bar{x}')\) if there are numbers \( K, \alpha > 0 \) such that
in a neighborhood of \((t', \bar{x}')\).

**Definition 2.** The function \(g(t, \bar{x})\) is said to satisfy a uniform Hölder condition at \((t', \bar{x}')\) if there are numbers \(K, \alpha, \delta > 0\) such that

\[
| g(t + \Delta t, \bar{x} + \Delta \bar{x}) - g(t, \bar{x}) | \leq K \left( | \Delta t |^\alpha + | \Delta \bar{x} |^\alpha \right)
\]

when

\[
| \Delta t | + | \Delta \bar{x} | < \delta
\]

for all points \((t, \bar{x})\) in a neighborhood of \((t', \bar{x}')\).

The boundedness of \(V(t, \bar{x})\) in \((t, \bar{x})\)-space is assumed throughout §2. The boundedness of \(V(t, \bar{x})\) and a set of basic estimates imply that \(Q(t, \bar{x})\) satisfies a uniform Hölder condition at all \((t, \bar{x}) \neq (0, \bar{0})\). \(Q(t, \bar{x})\) is then shown to satisfy the differential equation

\[
\frac{1}{2} \Delta Q - \frac{\partial Q}{\partial t} - uV(t, \bar{x})Q = 0
\]

at every point \((t, \bar{x}) \neq (0, \bar{0})\) at which \(V(t, \bar{x})\) satisfies a Hölder condition. \(\Delta Q\) denotes the Laplacian of \(Q\) in \(\bar{x}\)-space. Continuity properties of the derivatives of \(Q\) are then examined.

Stronger conditions are assumed in §3 to insure the applicability of Green's theorem in obtaining further results. We assume \(V(t, \bar{x})\) satisfies a uniform Hölder condition everywhere except in a regular set \(S\). The definition of a regular set is given in §3. We require that \(V(t, \bar{x})\) be bounded below in \((t, \bar{x})\)-space. \(Q(t, \bar{x})\) is then a solution of equation (1.11) at all points \((t, \bar{x})\) not in the set \(S\) and satisfies

\[
\lim_{t \to 0} \int_{|\bar{x}| < \epsilon} Q(t, \bar{x}) d\bar{x} = 1 \quad \text{for all } \epsilon > 0
\]

and a few other auxiliary conditions. \(Q(t, \bar{x})\) is the unique solution of equation (1.11), satisfying these conditions if \(V(t, \bar{x}) \geq 0\). Additional remarks are made indicating that an analogous theorem holds for

\[
F(t, \bar{x}) = E \{ \exp (iuy(t)) \mid \bar{x}(t) = \bar{x} \} \frac{\exp \left( -\frac{\bar{x}^2}{2t} \right)}{(2\pi t)^{n/2}}.
\]

The transform

\[
q(\bar{x}, s) = \int_0^\infty e^{-st}Q(t, \bar{x})dt, \quad s \geq 0,
\]

is considered in §4 when \(V(t, \bar{x}) = V(\bar{x}) \geq 0\). We again assume that \(V(\bar{x})\) satisfies a uniform Hölder condition everywhere except in a regular set \(S\). \(q(\bar{x}, s)\) then is the unique solution of
at all points \( \bar{x} \neq \bar{0} \) not in \( S \), satisfying
\[
\lim_{\varepsilon \to 0} \int_{|x| = \varepsilon} \frac{\partial q}{\partial n} ds = -2 \quad \text{for all } \varepsilon > 0
\]
and a few other auxiliary conditions. \( \frac{\partial q}{\partial n} \) is the derivative of \( q \) normal to the sphere \( |x| = \varepsilon \). M. Kac proved this theorem in the 1-dimensional case \([2]\).

Let
\[
\sigma(\bar{x}, \alpha, t) = \Pr \{ y(t) \leq \alpha \mid \bar{x}(t) = \bar{x} \}.
\]
The integral equation satisfied by \( \sigma(\bar{x}, \alpha, t) \) when \( V(t, \bar{x}) \) is bounded is derived in \S 5.

Note that
\[
\int_0^\infty e^{-u \alpha} d\alpha \sigma(\alpha; t) = \int_{-\infty}^\infty Q(t, \bar{x}) d\bar{x}
\]
when \( V(t, \bar{x}) \geq 0 \). Moreover, if \( V(t, \bar{x}) = V(\bar{x}) \geq 0 \),
\[
\int_0^\infty \int_0^\infty e^{-u \alpha - s} d\alpha \sigma(\alpha; t) dt = \int_{-\infty}^\infty q(\bar{x}, s) d\bar{x}.
\]

2. The parabolic differential equation.

Lemma 1. Let \( |V(t, \bar{x})| \leq M \). Then
\[
\lim_{t \to 0} \int_{|\bar{x}| \leq \varepsilon} Q(t, \bar{x}) d\bar{x} = 1
\]
for all \( \varepsilon > 0 \). Moreover, \( Q(t, \bar{x}) \) satisfies the uniform Hölder condition
\[
|Q(t + \Delta t, \bar{x} + \Delta \bar{x}) - Q(t, \bar{x})| \leq M(t, \bar{x}) \{ |\Delta \bar{x}| + |\Delta t| |\lg \Delta t| \}
\]
at all \( (t, \bar{x}) \neq (0, \bar{0}) \).

Equation (1.9) is basic for the required estimates. The function \( C(\bar{0}, 0; \bar{x}, t) \) is infinitely differentiable in \( t \) and the components of \( \bar{x} \) at all points \( (t, \bar{x}) \neq (0, \bar{0}) \). Moreover \( \lim_{t \to 0} \int_{|\bar{x}| \leq \varepsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x} = 1 \) for all \( \varepsilon > 0 \). Hence we need only consider
\[
G(t, \bar{x}) = \int_0^t \int_{-\infty}^\infty C(\xi, \tau; \bar{x}, t) V(\tau, \xi) Q(\tau, \xi) d\xi d\tau.
\]
Inequality (1.7) implies that
\[
\int_{|\bar{x}| \leq \varepsilon} G(t, \bar{x}) d\bar{x} \leq Me^{u |M| t} \int_{|\bar{x}| \leq \varepsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x}
\]
so that equation (2.1) is easily verified.

The derivatives

\( \frac{\partial G(t, \tilde{x})}{\partial x_k} = \int_0^t \int_{-\infty}^{\infty} \frac{\partial C(\xi, \tau; \tilde{x}, t)}{\partial x_k} V(\tau, \xi)Q(\tau, \xi) d\xi d\tau \)  

\( k = 1, \ldots, n \) exist and are continuous everywhere. Hence, the derivatives \( \partial Q(t, \tilde{x})/\partial x_k (k = 1, \ldots, n) \) exist and are continuous at all \((t, \tilde{x}) \neq (0, 0)\). This in turn implies that \( Q(t, \tilde{x}) \) satisfies the uniform Hölder condition

\[ |Q(t, \tilde{x} + \Delta \tilde{x}) - Q(t, \tilde{x})| \leq M(t, \tilde{x}) |\Delta \tilde{x}| \tag{2.5} \]

at all \((t, \tilde{x}) \neq (0, 0)\). We derive the Hölder condition in \( t \) again making use of inequality (1.7).

\[ G(t + \Delta t, \tilde{x}) - G(t, \tilde{x}) \]

\[ = \int_0^{t + \Delta t} \int_{-\infty}^{\infty} C(\xi, \tau; \tilde{x}, \tilde{t} + \Delta t)V(\tau, \xi)Q(\tau, \xi) d\xi d\tau \]

\[ + \int_0^t \int_{-\infty}^{\infty} \left[ C(\xi, \tau; \tilde{x}, \tilde{t} + \Delta t) - C(\xi, \tau, \tilde{x}, t) \right] V(\tau, \xi)Q(\tau, \xi) d\xi d\tau \]

\[ = I_1 + I_2, \quad \Delta t > 0. \]

Now

\[ |I_1| \leq C(\bar{0}, 0; \tilde{x}, t + \Delta t)M e^{uM'\Delta t}, \]

\[ I_2 = \int_0^{t - \Delta t} \int_{-\infty}^{\infty} + \int_{-\infty}^{\Delta t} \int_{-\infty}^{\infty} \int_{-\infty}^{t - \Delta t} \int_{-\infty}^{\infty} \left[ C(\xi, \tau; \tilde{x}, \tilde{t} + \Delta t) - C(\xi, \tau, \tilde{x}, t) \right] V(\tau, \xi)Q(\tau, \xi) d\xi d\tau \]

\[ = I_3 + I_4 + I_5. \]

Clearly

\[ |I_3 + I_4| \leq 2 \left( C(\bar{0}, 0; \tilde{x}, t + \Delta t) + C(\bar{0}, 0; \tilde{x}, t) \right) M e^{uM'\Delta t} \]

while

\[ |I_5| \leq \Delta t M e^{uM'\Delta t} \int_{\Delta t}^{t - \Delta t} \int_{-\infty}^{\infty} \left| \frac{\partial C(\xi, \tau; \tilde{x}, \tilde{t} + \theta \Delta t)}{\partial t} \right| C(\bar{0}, 0; \xi, \tau) d\xi d\tau \]

\[ \leq \Delta t M e^{uM'\Delta t} \int_{\Delta t}^{t - \Delta t} \int_{-\infty}^{\infty} \frac{n}{2} \frac{(t - \bar{\tau} + \Delta t)^{n/2}}{(t - \tau)^{(n+1)/2}} C(\xi, \tau; \tilde{x}, \tilde{t} + \Delta t)C(\bar{0}, 0; \xi, \tau) d\xi d\tau \]

\[ + \Delta t M e^{uM'\Delta t} \int_{\Delta t}^{t - \Delta t} \int_{-\infty}^{\infty} \frac{1}{2} \frac{|\tilde{x} - \xi|^2(t - \bar{\tau} + \Delta t)^{n/2}}{(t - \tau)^{(n+1)/2}} C(\xi, \tau; \tilde{x}, \tilde{t} + \Delta t) \]

\[ \cdot C(\bar{0}, 0; \xi, \tau) d\xi d\tau = I_6 + I_7, \quad 0 \leq \theta \leq 1. \]
Now

\[ |I_6| \leq \Delta t M e^{\mu M t} C(0, 0; \tilde{x}, t + \Delta t) \int_{\Delta t}^{t-\Delta t} \frac{(t + \Delta t - \tau)^{n/2}}{(t - \tau)^{(n+2)/2}} d\tau \]

\[ \leq M(t, \tilde{x}) \Delta t \left| \log \Delta t \right|, \]

\[ |I_7| \leq \frac{4 \Delta t M e^{\mu M t}}{(\pi t)^{(n+4)/2}} \int_{\Delta t}^{t/2} \int_{-\infty}^{\infty} \frac{x - \xi}{|\xi|} C(0, 0; \xi, \xi) d\xi d\tau \]

\[ + \frac{\Delta t M e^{\mu M t}}{(\pi t)^{n/2}} \int_{t/2}^{1-\Delta t} \frac{(t + \Delta t - \tau)^{(n+2)/2}}{(t - \tau)^{(n+4)/2}} d\tau \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du \]

\[ \leq M(t, \tilde{x}) \Delta t \left| \log \Delta t \right|. \]

Therefore

(2.6) \[ |G(t + \Delta t, \tilde{x}) - G(t, \tilde{x})| \leq M(t, \tilde{x}) \Delta t \left| \log \Delta t \right|. \]

Inequalities (2.5), (2.6) imply that \(Q(t, \tilde{x})\) satisfies the uniform Hölder condition (2.2) at all \((t, \tilde{x}) \neq (0, \tilde{0})\). Lemma 1 is thereby proved.

**Lemma 2.** Let \(V(t, \tilde{x})\) be a bounded function satisfying a Hölder condition at \((t, \tilde{x}) \neq (0, \tilde{0})\). Then the function

\[ H(t, \tilde{x}) = \int_{0}^{t} \int_{|\tilde{x} - \xi| \leq a} V(\tau, \xi) C(\xi, \tau; \tilde{x}, t) d\xi d\tau, \quad a > 0, \]

satisfies the differential equation

\[ \frac{1}{2} \Delta H - \frac{\partial H}{\partial t} + V(t, \tilde{x}) = 0 \]

at \((t, \tilde{x})\). The derivatives \(\partial^3 H/\partial x^k \ (k = 1, \cdots, n)\) are given by

\[ \frac{\partial^3 H}{\partial x^k} = \int_{0}^{t} \int_{|\tilde{x} - \xi| \leq a} \frac{\partial^3 C(\xi, \tau; \tilde{x}, t)}{\partial x^k} (V(\tau, \xi) - V(t, \tilde{x})) d\xi d\tau. \]

This lemma is proved on p. 229 of Levi [4] when \(\tilde{x}\) is 1-dimensional. The proof in the \(n\)-dimensional case completely parallels the proof of Levi.

**Lemma 3.** Let \(V(t, \tilde{x})\) be a bounded function satisfying a Hölder condition at a point \((t, \tilde{x}) \neq (0, \tilde{0})\). \(Q(t, \tilde{x})\) then satisfies the differential equation (1.11) at \((t, \tilde{x})\).

\(V(t, \tilde{x}) Q(t, \tilde{x})\) satisfies a Hölder condition at \((t, \tilde{x})\) by Lemma 1. The function \(C(0, 0; \tilde{x}, t)\) satisfies the differential equation

\[ \frac{1}{2} \Delta C - \frac{\partial C}{\partial t} = 0. \]
Equation (1.9) and Lemma 2 then imply that $Q(t, \bar{x})$ satisfies equation (1.11) at $(t, \bar{x})$.

**Lemma 4.** Let $V(t, \bar{x})$ be a bounded function satisfying a uniform Hölder condition at a point $(t, \bar{x}) \neq (0, 0)$. The derivatives $\frac{\partial^2 Q}{\partial x_k^2}$ $(k = 1, \cdots, n)$, $\frac{\partial Q}{\partial t}$ exist in a neighborhood of $(t, \bar{x})$ and are continuous at $(t, \bar{x})$.

The derivatives $\frac{\partial^2 Q}{\partial x_k^2}$ $(k = 1, \cdots, n)$, $\frac{\partial Q}{\partial t}$ exist in a neighborhood of $(t, \bar{x})$ by Lemma 3. Now

$$\frac{\partial Q}{\partial t} = \frac{1}{2} \Delta Q - uVQ$$

so that we need only verify the continuity of $\frac{\partial^2 Q}{\partial x_k^2}$ $(k = 1, \cdots, n)$ at $(t, \bar{x})$.

$$\frac{\partial^2 G(t, \bar{x})}{\partial x_k^2} - \frac{\partial^2 G(t + \Delta t, \bar{x} + \Delta \bar{x})}{\partial x_k^2} = \left\{ \int \int_{|\xi - \bar{x}| + |\tau - \bar{t}| < \epsilon} + \int \int_{|\xi - \bar{x}| + |\tau - \bar{t}| \geq \epsilon} \right\} \left( \frac{\partial^2 C(\xi, \tau; \bar{x}, \bar{t})}{\partial x_k^2} \right) (V(\tau, \xi)Q(\tau, \xi)) - V(t, \bar{x})Q(t, \bar{x}) - \frac{\partial^2 C(\xi, \tau; \bar{x} + \Delta \bar{x}, t + \Delta t)}{\partial x_k^2} (V(\tau, \xi)Q(\tau, \xi)) - V(t + \Delta t, \bar{x} + \Delta \bar{x})Q(t + \Delta t, \bar{x} + \Delta \bar{x}))d\xi d\tau$$

$$= I_8 + I_9.$$ 

$I_9$ vanishes as $|\Delta t|, |\Delta \bar{x}| \to 0$. The uniform Hölder condition allows us to obtain the same estimate for each of the two terms of $I_8$. We find that

$$|I_8| \leq 2 \int \int_{|\xi - \bar{x}| + |\tau - \bar{t}| < 2\delta} \frac{\partial^2 C(\xi, \tau; \bar{x}, \bar{t})}{\partial x_k^2} (V(\tau, \xi)Q(\tau, \xi) - V(t, \bar{x})Q(t, \bar{x}))d\xi d\tau$$

$$\leq 2K \int_{|t - \tau| < 2\delta} \int_{-\infty}^{\infty} e^{-u^2} (1 + u_k^2) \left\{ |t - \tau|^{\alpha - 1} + \sum_{k} u_k^\alpha |t - \tau|^{\alpha/2 - 1} \right\} d\tau d\xi$$

$$\leq \frac{M\delta^{\alpha/2}}{\alpha}.$$ 

The continuity is verified on letting $\delta \to 0$.

3. **Unbounded** $V(t, \bar{x})$.

**Definition 3.** We shall say that the set $S$ is regular if the following conditions are satisfied for every pair $(T, \gamma)$, $T \geq 0$, and all $b > b(T, \gamma) > 0$:

1. For every $\epsilon > 0$ there is a set $R(\epsilon)$, the sum of a finite number of nonoverlapping closed parallelepipeds each of diameter less than $\epsilon$ with sides parallel to the coordinate axes, in $\{0 \leq t \leq T, |\bar{x} - \gamma| \leq b\}$ and covering $\{0 \leq t \leq T, |\bar{x} - \gamma| \leq b\} \cap S$. 

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The volume of \( R(\epsilon) \) is less than \( \epsilon \).

(3) The distance between the complement of \( R(\epsilon) \) in \( \{0 \leq \tau \leq T, |\tilde{x} - \tilde{y}| \leq \epsilon \} \) and \( S \) is positive.

(4) The total surface area of the parallelepipeds of \( R(\epsilon) \) is bounded for all \( \epsilon \).

**Theorem 1.** Let \( V(t, \tilde{x}) \) be a function bounded below and satisfying a uniform Hölder condition everywhere except in a regular set \( S \). Then \( Q(t, \tilde{x}) \) is a solution of equation (1.11) at all points \( (t, \tilde{x}) \neq (0, \vec{0}) \) not in \( S \) and satisfies the following conditions:

1. \( Q(t, \tilde{x}) \to 0 \) as \( |\tilde{x}| \to \infty \).
2. \( \lim_{t \to 0} \int_{|\tilde{x}| = \epsilon} Q(t, \tilde{x}) \, d\tilde{x} = 1 \) for all \( \epsilon > 0 \).
3. \( \frac{\partial Q}{\partial x_k} \) \((k = 1, \ldots, n)\) are continuous at all \( (t, \tilde{x}) \neq (0, \vec{0}) \). \( \frac{\partial^2 Q}{\partial x_k^2} \) \((k = 1, \ldots, n)\), \( \frac{\partial Q}{\partial t} \) are continuous at all \( (t, \tilde{x}) \neq (0, \vec{0}) \) not in \( S \).

If \( V(t, \tilde{x}) \geq 0 \), \( Q(t, \tilde{x}) \) is the unique solution of equation (1.11), satisfying conditions (1)–(3).

Let

\[
V_N(t, \tilde{x}) = \begin{cases} V(t, \tilde{x}) & \text{if } |V(t, \tilde{x})| \leq N, \\ N \text{ otherwise} & \end{cases}
\]

and

\[
Q_N(t, \tilde{x}) = E \left\{ \exp \left( -u \int_0^t V_N(\tau, \tilde{x}(\tau)) \, d\tau \right) \right| \tilde{x}(t) = \tilde{x} \} \frac{\exp(-\tilde{x}^2/2t)}{(2\pi t)^{n/2}}.
\]

Lemma 3 implies that \( Q_N(t, \tilde{x}) \) satisfies

\[
\frac{1}{2} \Delta Q_N - \frac{\partial Q_N}{\partial t} - uV_N(t, \tilde{x})Q_N = 0
\]

at all points \( (t, \tilde{x}) \neq (0, \vec{0}) \) not in \( S \). Inequality (1.7) and Lemmas 1 and 4 imply that \( Q_N(t, \tilde{x}) \) satisfies conditions (1) to (3). Let

\[
f(\xi, \tau; \tilde{x}, t) = C(\xi, \tau; \tilde{x}, t) - \int_0^{t-r} \frac{e^{-b^2/2(t-\tau-z)}}{(2\pi(t-\tau-z))^{n/2}} v_\alpha(|\tilde{x} - \xi|, z) \, dz
\]

where

\[
v_\alpha(r, t) = \sum_{j=1}^{\infty} 2 \exp(-\alpha_j^2 t/2b^2)J_{(n-2)/2}(\alpha_j^2 r/b) / J_{(n-2)/2}(\alpha_j^2)
\]

(see §6). The numbers \( \alpha_j^2 \) \((j = 1, \ldots)\) are the positive zeros of the Bessel function \( J_{(n-2)/2}(x) \). \( f(\xi, \tau; \tilde{x}, t) \) is the fundamental solution of

\[
\frac{1}{2} \Delta f(\cdot, \cdot; \tilde{x}, t) - \frac{\partial f}{\partial t} = 0, \quad \frac{1}{2} \Delta f(\xi, \tau; \cdot, \cdot) + \frac{\partial f}{\partial \tau} = 0
\]
with boundary value zero at the spheres of radius \( b \) about \( \xi, \bar{x} \) respectively.

Let \( b > 0 \) be such that we can find sets \( R(\epsilon) \) satisfying conditions (1) to (4) of definition 3 covering

\[
\{ 0 \leq \tau \leq t, \ | \bar{x} - \xi | \leq b \} \cap S.
\]

Let

\[
R(\epsilon, \delta) = R(\epsilon) \cap \{ \delta \leq \tau \leq t - \delta, \ | \bar{x} - \xi | \leq b \}, \quad \delta > 0.
\]

Call the surfaces consisting of the upper and lower faces respectively of the parallelepipeds of \( R(\epsilon, \delta) \) perpendicular to the \( t \) axis, \( U(\epsilon) \) and \( L(\epsilon) \). Call the surface consisting of the faces of the parallelepipeds of \( R(\epsilon, \delta) \) parallel to the \( t \) axis \( P(\epsilon) \): Let

\[
B(W) = \frac{1}{2} \Delta W - \frac{\partial W}{\partial \tau} - Nu W W,
\]

\[
C(W) = \frac{1}{2} \Delta W + \frac{\partial W}{\partial \tau}.
\]

We apply Green's theorem making use of the continuity of \( \partial^2 Q_N / \partial x_k^2 \) \((k = 1, \ldots, n)\), \( \partial Q / \partial t \) away from the set \( S \) and obtain

\[
0 = \left\{ \int_{t-\delta}^{t} \int_{|\bar{x} - \eta| \leq b} - \int_{\int_{R(\epsilon, \delta)}} \right\} (f(\xi, \tau; \bar{x}, t) B(Q_N(\tau, \xi))
\]

\[ - Q_N(\tau, \xi) C(f(\xi, \tau; \cdot, \cdot, \cdot))) \, d\xi \, d\tau
\]

\[ = - \frac{1}{2} \int_{t-\delta}^{t} \int_{|\bar{x} - \eta| \leq b} \frac{\partial f(\xi, \cdot; \cdot, \cdot, \cdot)}{\partial \eta} Q_N d\xi \, d\tau
\]

\[ - \left\{ \int_{t-\delta}^{t} \int_{|\bar{x} - \eta| \leq b} - \int_{\int_{R(\epsilon, \delta)}} \right\} \omega_f V_N Q_N Q_N d\xi \, d\tau
\]

\[ + \frac{1}{2} \int_{P(\epsilon)} \left( \frac{\partial f(\xi, \tau; \cdot, \cdot)}{\partial \eta} Q_N - \frac{\partial Q_N(\tau, \xi)}{\partial \eta} f \right) ds
\]

\[ - \left\{ \int_{U(\epsilon)} - \int_{L(\epsilon)} \right\} f(\xi, \tau; \cdot, \cdot, \cdot) Q_N ds
\]

\[ + \int_{|\bar{x} - \eta| \leq b} (f(\xi, \delta; \bar{x}, t) Q_N(\delta, \xi) - f(\xi, t - \delta; \bar{x}, t) Q_N(t - \delta, \xi)) d\xi.
\]

Let \( \epsilon \to 0 \). Condition (2) of Definition 3 and the boundedness of \( f V_N Q_N \) over the \( R(\epsilon, \delta) \) imply that the volume integral over \( R(\epsilon, \delta) \) vanishes in the limit. The bounded surface area of \( P(\epsilon), U(\epsilon), \) and \( L(\epsilon) \), the symmetry of the parallelepipeds, and the continuity of \( \partial f / \partial \eta, \partial Q_N / \partial \eta, Q_N, f \) imply that the surface
integrals over \( P(\epsilon), U(\epsilon), L(\epsilon) \) vanish in the limit. We then let \( \delta \to 0 \) and obtain

\[
Q_N(t, \bar{x}) + u \int_0^t \int_{|\bar{x} - \xi| < a} f(\xi, \tau; \bar{x}, t)Q_N(\tau, \xi)V_N(\tau, \xi) d\xi d\tau
= f(0, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x} - \xi| < a} \frac{\partial f(\xi, \tau; \bar{x}, t)}{\partial n} Q_N(\tau, \xi) d\tau d\sigma.
\]

Now

\[
\lim_{N \to \infty} Q_N(t, \bar{x}) = Q(t, \bar{x})
\]

which is finite since \( V(t, \bar{x}) \) is bounded below. On taking the limit of equation (3.3) as \( N \to \infty \), we have

\[
Q(t, \bar{x}) + u \int_0^t \int_{|\bar{x} - \xi| < a} Q(\tau, \xi)f(\xi, \tau; \bar{x}, t)V(\tau, \xi) d\xi d\tau
= f(0, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x} - \xi| < a} \frac{\partial f(\xi, \tau; \bar{x}, t)}{\partial n} Q(\tau, \xi) d\tau d\sigma.
\]

This integral equation plays the same role that equation (1.9) did in §2. The surface integral of equation (3.4) is infinitely differentiable in \( t \) and the components of \( \bar{x} \). It satisfies

\[
\frac{1}{2} \Delta C - \frac{\partial C}{\partial t} = 0.
\]

The counterparts of Lemmas 1 to 4 with \( f(\xi, \tau; \bar{x}, t) \) in place of \( C(\xi, \tau; \bar{x}, t) \) are proved in exactly the same manner as before. Hence \( Q(t, \bar{x}) \) satisfies equation (1.11) and conditions (1) to (3) of the theorem.

A uniqueness argument for \( Q(t, \bar{x}) \) as a solution of equation (1.11) can be carried out if \( V(t, \bar{x}) \) is non-negative. We shall give an example of such a uniqueness argument in the proof of Theorem 2.

**Theorem 2.** Let \( V(t, \bar{x}) \) satisfy a uniform Hölder condition at all points \((t, \bar{x})\) not in a regular set \( S \). Then \( F(t, \bar{x}) \) is the unique solution of equation

\[
\frac{1}{2} \Delta F - \frac{\partial F}{\partial t} + iuV(t, \bar{x})F = 0
\]

at all points \((t, \bar{x}) \neq (0, 0)\) not in \( S \) with \( F(t, \bar{x}) \) satisfying the following conditions:

1. \( F(t, \bar{x}) \to \infty \) as \( |\bar{x}| \to \infty \).
2. \( \lim_{t \to 0} \int_{|\bar{x}| < \epsilon} F(t, \bar{x}) d\bar{x} = 1 \) for all \( \epsilon > 0 \).
3. \( \partial F/\partial x_k \) \( (k = 1, \cdots, n) \) are continuous at all \((t, \bar{x}) \neq (0, 0). \partial^2 F/\partial x_k^2 \)
(k = 1, \cdots, n), \partial F/\partial t are continuous at all \((t, \bar{x}) \neq (0, \bar{0})\) not in \(S\).

The proof that \(F(t, \bar{x})\) is a solution of (3.5) and satisfies conditions (1) to (3) parallels the proof of Theorem 1. One need not bound \(V(t, \bar{x})\) below since

\[ |F(t, \bar{x})| \leq \frac{\exp \left( -\bar{x}^2/2t \right)}{(2\pi t)^{n/2}}. \]

The analogue of equation (3.4) is

\[
F(t, \bar{x}) - iu \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| \leq a} f(\bar{\xi}, \tau; \bar{x}, t) F(\tau, \bar{\xi}) V(\tau, \bar{\xi}) d\bar{\xi} d\tau
\]

\[ = f(0, 0; \bar{x}, t) - \frac{1}{2} \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| = a} F(\tau, \bar{\xi}) f(\bar{\xi}, \tau; \bar{x}, t) ds d\tau. \]

We now prove that \(F(t, \bar{x})\) is the unique solution of equation (3.5) satisfying conditions (1) to (3). Let \(F = F_1 - F_2\) be the difference of two such solutions. Let \(\bar{F}\) be the complex conjugate of \(F\). Clearly

\[
\frac{1}{2} \Delta \bar{F} = - \frac{\partial \bar{F}}{\partial t} - iu V(t, \bar{x}) \bar{F} = 0.
\]

Let

\[
D(W) = \frac{1}{2} \Delta W - \frac{\partial W}{\partial t} + iu V(t, \bar{x}) W.
\]

We apply Green’s theorem as in the proof of Theorem 1 and obtain

\[
\int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| \leq b} \{ f(\bar{\xi}, \tau; \bar{x}, t) D(|F(\tau, \bar{\xi})|^2) - |F(\tau, \bar{\xi})|^2 C(f(\bar{\xi}, \tau; \bar{x}, t)) \} d\bar{\xi} d\tau
\]

\[ = - \frac{1}{2} \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| = b} |F(\tau, \bar{\xi})|^2 \frac{\partial f(\bar{\xi}, \tau, \bar{x}, t)}{\partial n} ds d\tau - |F(t, \bar{x})|^2 ds d\tau
\]

\[ + iu \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| \leq b} V(\tau, \bar{\xi}) |F(\tau, \bar{\xi})|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau
\]

\[ = \frac{1}{2} \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| \leq b} f(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^{n} \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau
\]

\[ + iu \int_{0}^{t} \int_{|\bar{x} - \bar{\xi}| \leq b} V(\tau, \bar{\xi}) |F(\tau, \bar{\xi})|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau.
\]

Since \(|F(t, \bar{x})| \leq M\), on letting \(b \to \infty\) in equation (3.7), we obtain

\[ - |F(t, \bar{x})|^2 = \frac{1}{2} \int_{0}^{t} \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^{n} \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau.
\]
This cannot be so unless \( F(t, \bar{x}) = F_1 - F_2 \equiv 0 \). The uniqueness proof is complete. The theorem is thereby proved.

Consider

\[
Q(\tau, \bar{x}; t, \bar{x}) = E\left\{ \exp \left( -u(y(t) - y(\tau)) \right) \mid \bar{x}(t) = \bar{x}, \bar{x}(\tau) = \bar{\xi} \right\}
\]

\[
\times \frac{\exp \left( -\left| \bar{x} - \bar{\xi} \right|^2/2(t - \tau) \right)}{(2\pi(t - \tau))^{n/2}}
\]

\( 0 \leq \tau \leq t, 0 \leq V(t, \bar{x}) \). We again assume \( V(t, \bar{x}) \) satisfies a uniform Hölder condition at all points \( (t, \bar{x}) \) not in a regular set \( \mathcal{S} \). \( Q(\tau, \bar{\xi}; t, \bar{x}) \) exists and satisfies equation (1.11). Moreover \( Q(\tau, \bar{\xi}; t, \bar{x}) \) satisfies the following conditions:

1. \( Q(\tau, \bar{\xi}; t, \bar{x}) \to 0 \) as \( |\bar{x}| \to \infty \).
2. \( \lim_{t-\tau^+} \int_{|\bar{x}-\bar{\xi}| \leq \varepsilon} Q(\tau, \bar{\xi}; t, \bar{x}) d\bar{x} = 1 \) for all \( \varepsilon > 0 \).

\( Q(\tau, \bar{\xi}; t, \bar{x}) \) satisfies the adjoint differential equation and the corresponding conditions in the backward variables \( \tau, \bar{\xi} \). The proof is essentially the proof of Theorem 1. Note that \( Q(\tau, \bar{\xi}; t, \bar{x}) \) satisfies the Chapman-Kolmogorov equation. It does not, however, have the norming of a probability density.

4. The elliptic differential equation. Let

\[
\psi(s, \bar{r}) = \frac{1}{((2s)^{1/2}r)^{(n-2)/2}} K(n-2)/2((2s)^{1/2}r)
\]

(4.1)

\[
\frac{I_{(n-2)/2}((2s)^{1/2}r)K_{(n-2)/2}((2s)^{1/2}b)}{I_{(n-2)/2}((2s)^{1/2}b)}
\]

(see §6). \( \psi(s, |\bar{x}|) \) is the fundamental solution of

\[
\frac{1}{2} \Delta \psi - s\psi = 0
\]

with boundary value zero at \( |\bar{x}| = b \). Note that

\[
\lim_{\varepsilon \to 0} \int_{|\bar{x}| = \varepsilon} \frac{\partial \psi}{\partial n} ds = -2.
\]

**Lemma 5.** Let \( V(\bar{x}) \) be a bounded function satisfying a Hölder condition at the point \( \bar{x} \neq 0 \). Then the function

\[
H(\bar{x}) = \int_{|\bar{x}-\bar{\xi}| \leq a} V(\bar{\xi})\psi(s, |\bar{x} - \bar{\xi}|) d\bar{\xi}
\]

satisfies the differential equation

\[
\frac{1}{2} \Delta H - sH + V(\bar{x}) = 0
\]

at \( \bar{x} \).
The proof parallels an argument of Kellogg on p. 153 [3] proving that

\[ h(\bar{x}) = \int_{|\xi - \bar{\xi}| \leq e} \frac{V(\bar{\xi})}{|\bar{x} - \bar{\xi}|} d\bar{\xi}, \quad \bar{x} = (x_1, x_2, x_3), \]

satisfies

\[ \frac{1}{2} \Delta h + V(\bar{x}) = 0 \]

at \( \bar{x} \neq 0 \) if \( V(\bar{x}) \) satisfies a Hölder condition at \( \bar{x} \).

**Theorem 3.** Let \( V(\bar{x}) \geq 0 \) satisfy a uniform Hölder condition at all \( \bar{x} \) not in a regular set \( S \). Then \( q(\bar{x}, s) \) is the unique solution of equation (1.14), satisfying the following conditions:

1. \( q(\bar{x}, s) \to 0 \) as \( |\bar{x}| \to \infty \).
2. \( \lim_{s \to 0} \int_{|\xi| \leq e} (\partial q/\partial n) d\xi = -2 \).
3. \( \partial q/\partial x_k \) (\( k = 1, \ldots, n \)) are continuous at all \( \bar{x} \neq 0 \). \( \partial^2 q/\partial x_k^2 \) (\( k = 1, \ldots, n \)) are continuous at all \( \bar{x} \neq 0 \) not in \( S \).

We obtain the integral equation

\[ q(s, \bar{x}) + u \int_{|\xi - \bar{\xi}| \leq b} \psi(s, |\bar{x} - \bar{\xi}|) V(\bar{\xi}) q(s, \bar{\xi}) d\bar{\xi} = \psi(s, |\bar{x}|) - \frac{1}{2} \int_{|\xi - \bar{\xi}| \leq b} q(s, \bar{\xi}) \frac{\partial \psi(s, |\bar{x} - \bar{\xi}|)}{\partial n} d\xi \]

(4.3)

by Laplace transforming equation (3.4) with respect to \( t \). \( \partial q/\partial x_k \) (\( k = 1, \ldots, n \)) exist and are continuous at all \( \bar{x} \neq 0 \) as can be seen by differentiating equation (4.3). Hence \( q(s, \bar{x}) V(\bar{x}) \) satisfies a Hölder condition at \( \bar{x} \neq 0 \) when \( V(\bar{x}) \) does. Lemma 5 implies that \( q(s, \bar{x}) \) satisfies equation (1.14) at all \( \bar{x} \neq 0 \) not in \( S \). Equations (4.2) and (4.3) imply that

\[ \lim_{e \to 0} \int_{|\xi| = e} \frac{\partial q}{\partial n} ds = -2. \]

Now

\[ 0 \leq q(s, \bar{x}) \leq \frac{K_{(n-2)/2}((2s)^{1/2}r)}{((2s)^{1/2}r)^{(n-2)/2}} \]

since \( V(\bar{x}) \geq 0 \). Hence

\[ q(s, \bar{x}) \to 0 \quad \text{as} \quad |\bar{x}| \to \infty. \]

The proof of the continuity of \( \partial^2 q/\partial x_k^2 \) (\( k = 1, \ldots, n \)) proceeds as in Lemma 4. The uniqueness argument for \( q(s, \bar{x}) \) satisfying conditions (1) to (3) is analogous to the uniqueness argument for \( F(t, \bar{x}) \) carried out in §3.
5. The integral equation.

**Theorem 4.** Let \( V(t, \tilde{x}) \) be bounded. Then \( \sigma(\tilde{x}, \alpha, t) \) satisfies the following integral equation

\[
C(\tilde{0}, 0; \tilde{x}, t) \int_{y}^{y+\lambda} (J(\alpha) - \sigma(\tilde{x}, \alpha, t))d\alpha
\]

\[
= \int_{0}^{t} \int_{-\infty}^{\infty} C(\xi, \tau; \tilde{x}, t)C(\tilde{0}, 0; \xi, \tau)V(\tau, \xi) \{ \sigma(\xi, y + \lambda, \tau) - \sigma(\xi, y, \tau) \} d\xi d\tau
\]

where

\[
J(x) = \frac{1 + \text{sgn} x}{2}
\]

It is clear that

\[
F(t, \tilde{x})/C(\tilde{0}, 0; \tilde{x}, t) = \int_{-\infty}^{\infty} e^{iu\alpha}d\alpha \sigma(\tilde{x}, \alpha, t).
\]

Since \( V(t, \tilde{x}) \) is bounded, on letting \( a \to \infty \) in equation (3.6) we obtain

\[
F(t, \tilde{x}) - C(\tilde{0}, 0; \tilde{x}, t) - iu \int_{0}^{t} \int_{-\infty}^{\infty} C(\xi, \tau; \tilde{x}, t)F(\tau, \xi)V(\tau, \xi)d\xi d\tau = 0.
\]

The boundedness of \( V(t, \tilde{x}) \) implies that

\[
\int_{-\infty}^{\infty} \alpha d\alpha \sigma(\tilde{x}, \alpha, t)
\]

exists. The modified form of the P. Lévy inversion formula [5] used requires the existence of the first moment. Multiply equation (5.1) by \((1-e^{-\tilde{\alpha}u}/(iu)^{2}) \cdot (1/2\pi)e^{-iu\tilde{y}}\) and integrate with respect to \( y \) from \(-T\) to \( T\). The interchange of order of integration goes through readily. In the limit as \( T \to \infty \) we obtain equation (5.1).

Interest in the integral equation (5.1) arises for several reasons. The equation holds without any strong regularity conditions on \( V(t, \tilde{x}) \). Moreover, the density

\[
\Pr \{ y(t) = y, \tilde{x}(t) = \tilde{x} \} = \frac{\partial \sigma}{\partial y}(\tilde{x}, y, t)
\]

need not exist even when very strong regularity conditions are imposed on \( V(t, \tilde{x}) \) so that it makes no sense to speak of the density satisfying the differential equation.
\[
\frac{\partial P}{\partial t} = \frac{1}{2} \Delta P - V(t, x) \frac{\partial P}{\partial y}.
\]

In particular this is so when \( V(t, x) = 1 \).

6. An example. We illustrate the theory in \( n \) dimensions by considering the function

\[
V(x) = \begin{cases} 
1, & |x| \geq b, \\
0, & |x| < b.
\end{cases}
\]

The invariance proof for the distributions of the Wiener functionals \( \int_0^T V(x(t)) \, dt \) considered closely parallels that given in [2] for \( V(x) = (1 + \text{sgn } x)/2 \). The computations for the case \( n = 1 \) have been carried out in [2] and elsewhere. We solve the equation

\[
\Delta q - 2(s + uV(r))q = 0, \quad V(r) = \begin{cases} 
0, & r < b, \\
1, & r \geq b,
\end{cases}
\]

where \( r = |x| \). The solution of the differential equation is given in terms of the Bessel functions \( I_{(n-2)/2}, K_{(n-2)/2} \) by

\[
q(s, r) = \begin{cases} 
\frac{1}{((2s)^{(1/2)r})^{(n-2)/2}} \left( \alpha K_{(n-2)/2}((2s)^{(1/2)r}) + \beta I_{(n-2)/2}((2s)^{(1/2)r}) \right), & r < b, \\
\gamma \left( \frac{\gamma}{((2s + u)^{(1/2)r})^{(n-2)/2}} \right) K_{(n-2)/2}((2(s + u))^{1/2}), & r \geq b,
\end{cases}
\]

where \( \alpha, \beta, \gamma \) do not depend on \( r \). We evaluate \( \alpha, \beta, \gamma \) by making use of the auxiliary conditions. The continuity of \( q \) and \( \partial q/\partial r \) at \( b \) implies that

\[
\alpha K_{(n-2)/2}((2s)^{1/2}) + \beta I_{(n-2)/2}((2s)^{1/2}) = \gamma \left( \frac{s}{s + u} \right)^{(n-4)/4} K_{(n-2)/2}((2(s + u))^{1/2}),
\]

\[
- \alpha K_{n/2}((2s)^{1/2}) + \beta I_{n/2}((2s)^{1/2}) = - \gamma \left( \frac{s}{s + u} \right)^{(n-4)/4} K_{n/2}((2(s + u))^{1/2}).
\]

We solve the equations above and find that

\[
\beta = \frac{- K_{(n-2)/2}((2s)^{1/2}) - \left( \frac{s}{s + u} \right)^{(n-4)/4} K_{(n-2)/2}((2(s + u))^{1/2})}{D}
\]

and

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\[ \gamma = \frac{\alpha}{(2s)^{1/2}bD} \]

where

\[ D = \begin{vmatrix} I_{(n-2)/2}(2s^{1/2}b) & \left(\frac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}(2(s+u)^{1/2}b) \\ I_{n/2}(2s^{1/2}b) & \left(\frac{s}{s+u}\right)^{(n-4)/4} K_{n/2}(2(s+u)^{1/2}b) \end{vmatrix} \]

The condition

\[ \lim_{\epsilon \to 0} \int_{|s|=\epsilon} \frac{\partial q}{\partial n} ds = -2 \]

implies that \( \alpha = 1 \). On letting \( u \to \infty \), \( q(s, r) \) tends to limit \( \psi(s, r) \) given by (4.1) for \( r < b \). \( q(s, r) \) tends to zero for \( r \geq b \). The limiting expression is the Laplace transform of the probability density of diffusion from \( 0 \) to a point \( \bar{x}, r \) units away from \( 0 \), when there is an absorbing barrier at the sphere of radius \( b \) about \( 0 \). We invert

\[ I_{(n-2)/2}(2s^{1/2}r)/I_{(n-2)/2}(2s^{1/2}b) \]

and obtain \( v_n(r, t) \) given by (3.6). Hence the probability density of the diffusion is \( f(\bar{0}, 0; \bar{x}, t) \) for \( |\bar{x}| \leq b \) and zero for \( |\bar{x}| \geq b \).

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