CURVATURE OF CLOSED HYPERSURFACES AND NON-EXISTENCE OF CLOSED MINIMAL HYPERSURFACES

BY

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1. Introduction. It is known that (1) a closed (sufficiently differentiable) surface in euclidean space $E_3$ has a point of positive Gaussian curvature, (2) a minimal surface in $E_3$ has nonpositive Gaussian curvature at every point, and (3) hence no closed minimal surface exists in $E_3$.

The object of this paper is to generalize these results to higher dimensions and noneuclidean Riemannian manifolds. The principal results appear as Theorems 1, 2 of §3 and Theorems 3, 4 of §4.

2. Riemannian manifolds in polar coordinates. Let $R_n$ be a Riemannian manifold of dimension $n \geq 2$ and class $C^r(\mathbb{R})$. Let $O$ be any point of $R_n$, let $(v^1, \cdots, v^n)$ be normal coordinates with respect to a unit orthogonal frame at $O$, so that the metric tensor in the coordinate system $(v)$ has the form

$$ds^2 = a_{pq}(v)dv^pdv^q,$$

(1)

$$a_{pq}(0) = \delta_{pq}, \quad \frac{\partial a_{pq}}{\partial v^r}(0) = (0), \quad p, q, r = 1, \cdots, n.$$  

Let $P_0$ be any point of $R_n$ which can be joined to $O$ by a geodesic $g_0$ on which there is no conjugate point to $O$. Then the normal coordinates of a point $P$ near $P_0$ are $v^p = \eta^p r$, where $(\eta^1, \cdots, \eta^n)$ are the components in $(v)$ of the unit tangent vector at $O$ to the unique geodesic $g$ from $O$ to $P$ whose initial direction and length $r$ are near those of $g_0$. Let $r_0$ be the length of $g_0$ and $(\eta_0) = (\eta^1_0, \cdots, \eta^n_0)$ the components of the unit tangent vector to $g_0$ at $O$.

Let $\eta^p = \eta^p(\lambda^2, \cdots, \lambda^n)$ be a nonsingular parametrization of the unit sphere $\sum \eta^p \eta^p = 1$ in $(\eta)$-space, neighboring $(\eta) = (\eta_0)$, and suppose $\eta^p(\lambda^0, \cdots, \lambda^n) = \eta^n_0$. Then we can use $(r, \lambda)$ as geodesic polar coordinates near $g_0$; that is, over a domain $0 \leq r < r_0 + \epsilon, \lambda^a_0 - \epsilon < \lambda^a < \lambda^a + \epsilon$. Since geodesics spheres about $O$ are orthogonal to the geodesics issuing from $O$, and since $r$ is length from $O$ along these geodesics, the metric tensor in the system $(r, \lambda)$ has the form

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(1) (1) is essentially a classical theorem of S. Bernstein, Math. Zeit. vol. 26 (1927) pp. 151–158. (2) follows from the fact that at every point of a minimal surface the sum of the principal normal curvatures is zero, hence their product is nonpositive.

(2) $r \geq 4$ is sufficient for all purposes in this paper.

(3) The summation convention of tensor analysis is used throughout.

(4) Numbers in brackets refer to the list of references at the end of the paper.
\[ ds^2 = dr^2 + h_{ab}(r, \lambda)d\lambda^a d\lambda^b, \quad \alpha, \beta = 2, \ldots, n. \]

Using (1), we see that
\[ h_{ab}(r, \lambda) = r^2 a_{pq}(\eta(\lambda)) \frac{\partial \eta^p}{\partial \lambda^a} \frac{\partial \eta^q}{\partial \lambda^b}. \]

Differentiating, and using (1), we find
\[ h_{ab}(0, \lambda) = (0), \quad \frac{\partial h_{ab}}{\partial r}(0, \lambda) = 0, \]
\[ \frac{\partial^2 h_{ab}}{\partial r^2}(0, \lambda) = 2 \sum \frac{\partial \eta^p}{\partial \lambda^a} \frac{\partial \eta^p}{\partial \lambda^b}. \]

Since
\[ \sum \eta^p \eta^p = \sum \frac{\partial \eta^p}{\partial \lambda^a} \frac{\partial \eta^p}{\partial \lambda^b} d\lambda^a d\lambda^b \]

it follows that
\[ \frac{\partial^2 h_{ab}}{\partial r^2}(0, \lambda) T^a T^b > 0 \quad \text{if} \ (T) \neq (O). \]

Denoting \((\partial h_{ab}/\partial r)T^a V^b\) by \(H(T, V)\), we get from (3) and (4) and the compactness of spheres the following lemma.

**Lemma 1.** Let \(O\) be a point of a Riemannian manifold \(R_n\). There exists a neighborhood \(N(O)\) such that if \(ds^2 = dr^2 + h_{ab}(r, \lambda)d\lambda^a d\lambda^b\) is the metric in a polar coordinate system based on \(O\), then \(H(T, T) = (\partial h_{ab}/\partial r)T^a T^b\) is positive definite in \(N(O) - O\).

Now let \(R_n\) be a Riemannian manifold of nowhere positive sectional curvature. There exists no pair of conjugate points in \(R_n\). Let \(P_0\) be a point which can be joined to \(O\) by a geodesic \(g_0\), and let \((r, \lambda)\) be a polar coordinate system near \(g_0\) based on \(O\). Let \(\bar{R}_{\alpha\beta\gamma\delta}\) be the Riemann tensor in this coordinate system; then
\[ \bar{R}_{\alpha\beta\gamma\delta} = -\frac{1}{2} \frac{\partial^2 h_{\beta\delta}}{\partial r^2} + \frac{1}{4} h^{\alpha\gamma} \frac{\partial h_{\alpha\beta}}{\partial r} \frac{\partial h_{\gamma\delta}}{\partial r}, \quad \beta, \delta = 2, \ldots, n. \]

Hence the sectional curvature of \(R_n\) with respect to a 2-plane determined by a tangent vector to \(g_0\) at \(P\) and a unit vector \((T)\) orthogonal to \(g_0\) at \(P\) is
\[ \bar{K}(T) = \frac{1}{2} \frac{\partial^2 h_{\beta\delta}}{\partial r^2} T^\beta T^\delta + \frac{1}{4} h^{\alpha\gamma} \left( \frac{\partial h_{\alpha\beta}}{\partial r} T^\beta \right) \left( \frac{\partial h_{\gamma\delta}}{\partial r} T^\delta \right). \]

Then \(H(T, T)\) is positive definite at \(P_0\); for if \(H(T_0, T_0) \leq 0\) at \(P_0\) for some
(T₀) ≠ (O), then by (3) and (4) there must be on g₀ a point at which \( dH(T₀, T₀)/dr = 0 \) and \( H(T₀, T₀) > 0 \), which by (5) contradicts the non-positiveness of \( K(T) \). Hence we have the following lemma.

**Lemma 2.** Let O be a point of a Riemannian manifold \( R_n \) of nowhere positive sectional curvature. If \( P₀ \in R_n \) can be joined to O by a geodesic and if \( ds^2 = dr^2 + h_ab(r, \lambda)d\lambda^ad\lambda^b \) is the metric near \( P₀ \) in a polar coordinate system based on O, then \( H(T, T) \) is positive definite at \( P₀ \).

Little can be said about the size of \( N(O) \) in the case of variable positive curvature. However, in case of constant positive curvature \( \bar{K} \), no geodesic of length less than \( \pi/\sqrt{\bar{K}} \) contains a pair of conjugate points, and also we can choose polar coordinates so that

\[
\sin^2 \frac{\lambda}{\bar{K}^{1/2}} b_{\alpha\beta} = \frac{\sin \lambda}{\bar{K}^{1/2}} b_{\alpha\beta}(\lambda),
\]

where \( b_{\alpha\beta} \) is positive definite. Hence we have the following lemma.

**Lemma 3.** Let O be a point of a Riemannian manifold \( R_n \) of constant positive curvature \( \bar{K} \). If \( P₀ \in R_n \) can be joined to O by a geodesic \( g₀ \) of length less than \( \pi/2(\sqrt{\bar{K}})^{1/2} \) (less than or equal to \( \pi/2(\sqrt{\bar{K}})^{1/2} \)), and if \( ds^2 = dr^2 + h_ab(r, \lambda)d\lambda^ad\lambda^b \) is the metric near \( P₀ \) in any polar coordinate system based on O, then \( H(T, T) \) is positive definite (positive semi-definite) at \( P₀ \).

3. **Curvature of closed hypersurfaces.** Let \( R_{n-1} \) be a hypersurface of \( R_n \), imbedded in \( R_n \) differentiably of class \( C^{r+1} \), with local imbedding equations \( x^p = x^p(u) \) in terms of coordinate systems \( (x) \) on \( R_n \) and \( (u) = (u^1, \ldots, u^{n-1}) \) on \( R_{n-1} \). It is known [2] that the following relation holds between the Riemann tensors on \( R_{n-1} \) and \( R_n \):

\[
R_{ijkl} - R_{k_1k_2}^{i_1i_2} \frac{\partial x^{i_1}}{\partial u^{i_1}} \frac{\partial x^{i_2}}{\partial u^{i_2}} \frac{\partial x^{j}}{\partial u^{j}} \frac{\partial x^{l}}{\partial u^{l}} = b_{ij}b_{kl} - b_{ik}b_{jl}, \quad i, j, k, l = 1, \ldots, n - 1,
\]

where \( b_{ij} \) is the second fundamental form of \( R_{n-1}(\xi) \). Also, \( b_{ij} \) satisfies

\[
\frac{\partial^2 x^p}{\partial u^i \partial u^j} + \Gamma^p_{qr} \frac{\partial x^q}{\partial u^i} \frac{\partial x^r}{\partial u^j} - \Gamma^p_{ij} \frac{\partial x^p}{\partial u^k} = b_{ij} \xi^p
\]

where \( (\xi) \) is the normal to \( R_{n-1} \).

Now suppose \( R_n \) is complete [3; 4], so that every pair of points A, B of \( R_n \) can be joined by a geodesic which is a shortest curve joining A to B; the

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(9) I am indebted to the referee for the suggestion that the Gauss equation (6) be used. My original proof of the theorems to follow was longer, and used specialized coordinate systems on \( R_{n-1} \).
distance between $A$ and $B$ is defined to be the length of a shortest geodesic joining them. Let $O$ be any point of $R_n$, and let $M$ be the minimum point locus [5; 6] with respect to $O$. Then $R_n = C_n \cup M$, where $C_n$ is an open $n$-cell with $M$ as its singular boundary; $M$ is at most $(n - 1)$-dimensional. Every point $P$ of $C_n$ can be joined to $O$ by a unique geodesic lying in $C_n$, and the length of such a geodesic measures the distance $OP$. The distance $OP$ is a function of class $C^{r-2}$ of $P$ for $P \neq 0$ in $C_n$.

Assume next that $R_{n-1}$ is closed.(6) Let $P_0$ be a point of $R_{n-1}$ whose distance $r_0$ in $R_n$ from $O$ is maximum. Suppose $P_0$ lies in $C_n(7)$; we can use the polar coordinates $(r, \lambda)$ of the previous section about the geodesic $OP_0$ in $C_n$, and the length $r$ of geodesics in $C_n$ from $O$ to points $P$ near $P_0$ measures the distance from $O$ to $P$. Then at $P_0$ we have $\partial r/\partial u^i = 0$, so that (7) becomes for $\rho = 1$

\[
\frac{\partial^2 r}{\partial u^i \partial u^j} - \frac{1}{2} \frac{\partial h_{\alpha\beta}}{\partial r} \frac{\partial \lambda^\alpha}{\partial u^i} \frac{\partial \lambda^\beta}{\partial u^j} = b_{ij}.
\]

Now at $P_0$, $\partial^2 r/\partial u^i \partial u^j$ is negative semi-definite, and if $H(T, T)$ is positive definite at $P_0$, then $b_{ij}$ is negative definite there. This implies that $(b_{ik}b_{jl} - b_{il}b_{jk})\eta^i \eta^j \xi^k \xi^l > 0$ for all $(\eta), (\xi)$ neither of which is $(O)$. Let $T^a = (\partial \lambda^a/\partial u^i)\eta^i$, $V^a = (\partial \lambda^a/\partial u^i)\xi^i$. Then by (6) the relative curvature of $R_{n-1}$ at $P_0$ (defined as the difference $K(\eta, \xi) - \overline{K}(T, V)$ between corresponding sectional curvatures of $R_{n-1}$ and $R_n$ at $P_0$) is positive. Hence using Lemmas 1, 2, 3 we have the following theorem.

**Theorem 1.** Every point $O$ of $R_n$ has a neighborhood $N(O)$ such that every closed $R_{n-1}$ in $N(O)$ has a point $P_0$ at which all relative curvatures of $R_{n-1}$ are positive. If $R_n$ is complete and has everywhere nonpositive sectional curvature, and if a closed $R_{n-1}$ in $R_n$ has the property that there exists a point $O$ of $R_n$ such that some point $P_0$ of $R_{n-1}$ furthest from $O$ is not on the minimum point locus with respect to $O$, then at $P_0$ all relative curvatures of $R_{n-1}$ are positive. If $R_n$ is complete and has constant positive curvature $K$, and if a closed $R_{n-1}$ has the property that there exists a point $O$ of $R_n$ such that all points of $R_{n-1}$ have distance less than $\pi/(2K)^{1/2}$ from $O$ and some point $P_0$ of $R_{n-1}$ furthest from $O$ is not on the minimum point locus with respect to $O$, then at $P_0$ all relative curvatures of $R_{n-1}$ are positive.

An example of a closed hypersurface $R_{n-1}$ in an $R_n$ of nonpositive curvature with the striking property that, no matter what point $O$ of $R_n$ is chosen, all points of $R_{n-1}$ furthest from $O$ are on the minimum point locus with respect to $O$ is furnished by the flat 2-torus $R_2$ naturally imbedded in the flat

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(6) Closed = compact.

(7) This assumption and the assumption of completeness of $R_n$ are not needed if $R_{n-1}$ lies in a sufficiently small neighborhood of $O$. 

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3-torus \( \mathbb{R}_3 \), the direct product of the flat 2-torus and the circle; here the relative curvature of \( \mathbb{R}_3 \) is zero. This example is best visualized as the square with opposite sides identified imbedded in the cube with opposite faces identified.

An example to show the need for the assumption about "less than \( \pi/2(K)^{1/2} \)" in the case of constant positive curvature is furnished by the flat 2-torus imbedded in the 3-sphere in \( E_4 \) by means of \( x = \cos \phi, y = \sin \phi, z = \cos \theta, w = \sin \theta \). This flat torus in \( E_4 \) also serves to show that Theorem 1 breaks down if \( \mathbb{R}_{n-1} \) is replaced by \( \mathbb{R}_m \) with \( m < n - 1 \).

If \( \mathbb{R}_n \) is simply-connected and has everywhere nonpositive sectional curvature, it is known [5; 1] that it is homeomorphic to \( E_n \) and the minimum point locus is absent. If \( \mathbb{R}_n \) is simply-connected and has constant positive curvature, it is known to be isometric to a sphere [7], and the minimum point locus with respect to any point \( O \) is simply the antipodal point. Hence we have the following theorem.

**Theorem 2.** Let \( \mathbb{R}_n \) be complete and simply-connected, and have everywhere nonpositive sectional curvature. Let \( O \in \mathbb{R}_n \), and let the hypersurface \( \mathbb{R}_{n-1} \) be closed. Then all relative curvatures of \( \mathbb{R}_{n-1} \) at every point \( P_0 \) furthest from \( O \) are positive. If \( \mathbb{R}_n \) is a sphere, and if \( \mathbb{R}_{n-1} \) lies in an open hemisphere of \( \mathbb{R}_n \), then at every point of \( \mathbb{R}_{n-1} \) furthest from the pole of the hemisphere all relative curvatures of \( \mathbb{R}_{n-1} \) are positive.

Theorems 1 and 2 can be strengthened slightly by everywhere replacing the phrase "relative curvatures of \( \mathbb{R}_{n-1} \) are positive" by "sectional curvatures of \( \mathbb{R}_{n-1} \) are greater than or equal to the corresponding sectional curvatures of the geodesic hypersphere about \( O \) through \( P_0 \), which in turn exceed the corresponding sectional curvatures of \( \mathbb{R}_n \)."

For at \( P_0 \) the sectional curvatures of the geodesic hypersphere in question are

\[
K_0(T, V) = K(T, V) + [H(T, T)H(V, V) - H^2(T, V)]/4
\]

so that by (6) and (8) we have, denoting \( \rho(\eta, \xi) = (\partial^2 r/\partial u^i \partial u^i)\eta^i \xi^i \),

\[
K(\eta, \xi) = K_0(T, V) + \rho(\eta, \eta)\rho(\xi, \xi) - \rho^2(\eta, \xi)
\]

\[
- \frac{1}{2} \left[ \rho(\eta, \eta)H(V, V) + \rho(\xi, \xi)H(T, T) - 2\rho(\eta, \xi)H(T, V) \right]
\]

\[
= K_0(T, V) + \rho(\eta, \eta)\rho(\xi, \xi) - \rho^2(\eta, \xi)
\]

\[
- \frac{1}{2\rho(\eta, \eta)} \left[ H(A, A) + (\rho(\eta, \eta)\rho(\xi, \xi) - \rho^2(\eta, \xi))H(T, T) \right]
\]

where \( A^a = \rho(\eta, \eta) V^a - \rho(\eta, \xi) T^a \). The required results follow from Lemmas 1, 2, 3, and equations (10) and (9).

This implies, for example, that a closed \( \mathbb{R}_{n-1} \) in hyperbolic \( n \)-space \( H_n \) has
a point where all sectional curvatures are positive.

4. Minimal varieties. An \( m \)-dimensional minimal variety in a Riemannian manifold \( R_n \) is by definition an extremal \( m \)-manifold in the calculus of variations problem associated with the multiple integral for \( m \)-dimensional area in \( R_n \). Let \( R_m \) be such a minimal variety in \( R_n \) imbedded differentiably of class \( C^{r+1} \). Let \( x^p = x^p(u^1, \ldots, u^m) \), \( p = 1, \ldots, n \), be the equations of this imbedding in terms of a local coordinate system \( (x) \) in \( R_n \) and a local coordinate system \( (u) \) in \( R_m \).

Now since \( R_m \) is a sub-manifold of \( R_n \), we have at every point \( P \) of \( R_m \) [2]

\[
R_{ijkl} = \frac{\partial x^p}{\partial u^i} \frac{\partial x^q}{\partial u^j} \frac{\partial x^r}{\partial u^k} \frac{\partial x^s}{\partial u^l} + \sum_{\sigma=m+1}^n b_{ik\sigma} b_{jls} - b_{i\sigma} b_{jkl},
\]

where \( b_{ij\sigma} \) is the \( \sigma \)th “second fundamental form” of \( R_m \). If \( (\eta), (\zeta) \) are two unit orthogonal vectors tangent to \( R_m \) at \( P \), we have

\[
K(\eta, \zeta) = K(T, V) + \sum_{\sigma=m+1}^n (b_{ik\sigma} b_{jls} - b_{i\sigma} b_{jkl}) \eta^i \zeta^j \zeta^l.
\]

Keeping \( (\eta) \) fixed, we allow \( (\zeta) \) to range over an \(( m-1 \))-dimensional unit orthogonal frame tangent to \( R_m \) and orthogonal to \( (\eta) \), and sum. We obtain

\[
k(\eta) = k(T) + g^{il} \sum_{\sigma} (b_{ik\sigma} b_{jls} - b_{i\sigma} b_{jkl}) \eta^k \eta^l,
\]

where \( k(\eta) \) is the Ricci curvature of \( R_m \) in the direction \( (\eta) \) and \( k(T) \) is the Ricci curvature in the same direction of the geodesic \( m \)-manifold tangent to \( R_m \) at \( P \). But since \( R_m \) is a minimal variety, we know [2]

\[
g^{il} b_{jls} = 0, \quad \sigma = m + 1, \ldots, n,
\]

so that

\[
k(\eta) = k(T) - g^{il} \sum_{\sigma} A_{is} A_{ls}.
\]

For each \( \sigma \), \( g^{il} A_{is} A_{ls} \geq 0 \), by the positive definiteness of \( g^{il} \). Hence we have the following theorem.

**Theorem 3.** At each point \( P \), the Ricci curvature of a minimal variety \( R_m \) (for arbitrary \( 2 \leq m < n \)) imbedded in a Riemannian manifold \( R_n \) does not exceed the corresponding Ricci curvature of the geodesic \( m \)-manifold tangent to \( R_m \) at \( P \).

This combines with Theorems 1 and 2 to produce the following theorem.

**Theorem 4.** If there exists a closed minimal hypersurface \( R_{m-1} \) in a complete Riemannian manifold \( R_n \) of nonpositive curvature, then all points of \( R_{m-1} \) furthest from any fixed point \( O \) of \( R_n \) are on the minimum point locus with re-
spect to \( O \); hence there exists no closed minimal hypersurface in a complete simply-connected Riemannian manifold of nonpositive curvature, in particular in \( E_n \) or \( H_n \). If there exists a closed minimal hypersurface \( R_{n-1} \) in a complete Riemannian manifold \( R_n \) of constant curvature \( K > 0 \), then every point of \( R_{n-1} \) furthest from any fixed point \( O \) of \( R_n \) is either at least \( \pi/2(K)^{1/2} \) distant from \( O \) or is on the minimum point locus with respect to \( O \); hence there exists no closed minimal hypersurface in an open hemisphere.

The flat 2-torus \( T_2 \) is a closed minimal hypersurface in the flat 3-torus \( T_3 \); the equatorial 2-sphere \( S_2 \) in the 3-sphere \( S_3 \) is a closed minimal hypersurface. Both these examples illustrate the theorem.

References