FINITELY ADDITIVE MEASURES

BY

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0. **Introduction.** The present paper is concerned with real-valued measures which enjoy the property of finite additivity but not necessarily the property of countable additivity. Our interest in such measures arose from two sources. First, the junior author has been concerned with the space of all finitely additive complex measures on a certain family of sets, which under certain conditions can be made into an algebra over the complex numbers. Second, both of us were informed by S. Kakutani of a result similar to our Theorem 1.23, the proof given by Kakutani being different from ours. The problem of characterizing finitely additive measures in some specific way occurred to us as being quite natural, and this problem we have succeeded in solving under fairly general conditions (Theorem 1.22).

The body of the paper is divided into four sections. In §1, we consider finitely additive measures in a reasonably general context, obtaining first a characterization of such measures in terms of a countably additive part and a purely finitely additive part. Purely finitely additive measures are then characterized explicitly. In §2, we extend the theorem of Fichtenholz and Kantorovič [4](1) and thereby characterize the general bounded linear functional on the Banach space of bounded measurable functions on a general measurable space. In §3, we consider a number of phenomena which appear in the special case of the real number system. Here we exhibit a number of finitely additive measures which have undeniably curious properties. In §4, we describe connections between our finitely additive measures and certain countably additive Borel measures defined on a special class of compact Hausdorff spaces. We are indebted to Professor S. Kakutani for comments on and improvements in the results obtained.

Throughout the present paper, the symbol R designates the real number system, and points are denoted by lower-case Latin letters, sets by capital Latin letters, families of sets by capital script letters. Sets of functions are denoted by capital German letters. For any set X and any $A \subset X$, the characteristic function of A is denoted by χ_A.

1. **General finitely additive measures.**

1.1 Definition. Let X be an abstract set, and let $\mathcal{M}$ be a family of subsets of X closed under the formation of finite unions and of complements. Let $\overline{\mathcal{M}}$ be the smallest family of sets containing $\mathcal{M}$ and closed under the formation of countable unions and of complements.

Presented to the Society, April 28, 1951; received by the editors April 9, 1951.

(1) Numbers in brackets refer to the references at the end of the paper.

1.2 DEFINITION. Let ϕ be a single-valued function defined on $\mathcal{M}$ such that

1.2.1 $\quad -\infty < \phi(A) < +\infty \qquad$ for all $A \in \mathcal{M}$,

1.2.2 $\quad \phi(0) = 0$,

1.2.3 $\quad \sup_{A \in \mathcal{M}} |\phi(A)| < +\infty$,

1.2.4 $\quad \phi(A \cup B) = \phi(A) + \phi(B) \qquad$ for all $A, B \in \mathcal{M}$ such that $A \cap B = 0$.

Then ϕ is said to be a finitely additive measure on $\mathcal{M}$. The set of all such measures for fixed X and $\mathcal{M}$ is denoted by the symbol $\Phi(X, \mathcal{M})$.

1.3 NOTE. We restrict all measures considered to be only finite, partly for simplicity of proofs and partly because the applications envisaged deal exclusively with finite measures. A number of the results of §1 can be proved for measures assuming infinite values, but the proofs involve a number of complications.

1.4 NOTE. Throughout the present section, we shall be concerned only with a single $\Phi(X, \mathcal{M})$ and hence shall use the symbol Φ to denote $\Phi(X, \mathcal{M})$.

1.5 NOTE. One cannot prove 1.2.3 from 1.2.1, 1.2.2, and 1.2.4. Let $X = \{1, 2, 3, \cdots, n, \cdots\}$, let $\sum_{n=1}^{\infty} a_n$ be a convergent but not absolutely convergent series, and let $\mathcal{M}$ consist of finite sets and their complements. Then if $\phi(A) = \sum_{n \in A} a_n$ for all $A \in \mathcal{M}$, one sees immediately that 1.2.3 is violated while 1.2.1, 1.2.2, and 1.2.4 are satisfied.

1.6 DEFINITION. Let ψ be an element of Φ such that for every sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = 0$, we have $\lim_{n \to \infty} \psi(A_n) = 0$. Then ψ is said to be countably additive.

1.7 THEOREM. *A measure γ of Φ is countably additive if and only if, for every sequence $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ of pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$, the equality $\gamma(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \gamma(E_n)$ obtains.*

The proof is simple and is omitted.

1.8 THEOREM. *Let ψ be any countably additive measure. Then ψ admits a unique extension $\bar{\psi}$ over the family $\overline{\mathcal{M}}$ which is countably additive on $\overline{\mathcal{M}}$.*

This result is proved in [6, p. 54, Theorem A].

We now consider general properties of the set of measures Φ.

1.8a NOTE. As G. Birkhoff has shown (see [2, p. 185]), any measure in Φ can be extended over $\overline{\mathcal{M}}$ so as to remain finitely additive. However, uniqueness is often not obtainable. For example, let N_0 be a countably infinite set, and let $\mathcal{J}$ be all subsets of N_0 which are finite or have finite complements. $\overline{\mathcal{J}}$ is then the algebra of all subsets of N_0. The measure π such that $\pi(A)$ is 0 or 1 as A or A' is finite is obviously finitely additive. Its extensions over $\overline{\mathcal{J}}$ can be identified with the finite Borel measures on the space $\beta N_0 \cap N_0'$; and

there are $2^{2^{\aleph_0}}$ such measures. For any completely regular space X, the space βX is a compact Hausdorff space such that X is a dense subspace of βX and every bounded continuous real-valued function on X has a continuous extension over βX. (See [8, pp. 64–67] and [16, p. 59].)

1.9 THEOREM. *For arbitrary $\phi\in\Phi$ and $\alpha\in R$, let the set-function $\alpha\phi$ be defined by the relation $(\alpha\phi)(E)=\alpha\cdot\phi(E)$ for all $E\in\mathfrak{M}$. For arbitrary ϕ, $\gamma\in\Phi$, let the set-function $\phi+\gamma$ be defined by the relation $(\phi+\gamma)(E)=\phi(E)+\gamma(E)$ for all $E\in\mathfrak{M}$. Under these definitions, Φ is a linear space.*

This result being very simple, we omit the proof.

1.10 DEFINITION. For an arbitrary $\phi\in\Phi$, we write $\phi\geqq 0$ if the inequality $\phi(E)\geqq 0$ is valid for all $E\in\mathfrak{M}$. For arbitrary ϕ and $\gamma\in\Phi$, we write $\phi\leqq\gamma$ if $\gamma-\phi\geqq 0$.

1.11 THEOREM. *Under the partial ordering* 1.10, *the space Φ is a lattice. For arbitrary ϕ and $\gamma\in\Phi$, the measure $\phi\wedge\gamma$ is defined by the relation*

$$(\phi\wedge\gamma)(E) = \inf_{T\subset E,\, T\in\mathfrak{M}} (\phi(T)+\gamma(E\cap T')), \tag{1.11.1}$$

for all $E\in\mathfrak{M}$. The measure $\phi\vee\gamma$ is defined by the relation

$$\phi\vee\gamma = -((-\phi)\wedge(-\gamma)). \tag{1.11.2}$$

For a proof of this result, see [3, p. 319].

1.12 THEOREM. *Let ϕ be an arbitrary element of Φ. Writing $\phi\vee 0$ as ϕ_+ and $(-\phi)\vee 0$ as ϕ_-, we have the relations $\phi=\phi_+-\phi_-$ and $\phi_+\wedge\phi_-=0$.*

This result is obvious from 1.11.

Theorem 1.12 permits us in many cases to confine our attention to nonnegative measures $\phi\in\Phi$. We now define a class of finitely additive measures which are, so to say, as unlike countably additive measures as possible.

1.13 DEFINITION. Let ϕ be a measure in Φ such that $0\leqq\phi$. If every countably additive measure ψ such that $0\leqq\psi\leqq\phi$ is identically zero, then ϕ is said to be purely finitely additive. If $\phi\in\Phi$ and both ϕ_+ and ϕ_- are purely finitely additive, then ϕ is said to be purely finitely additive.

We now describe a number of properties possessed by countably additive and purely finitely additive measures.

1.14 THEOREM. *Let ψ_1 and ψ_2 be countably additive. Then $\psi_1+\psi_2$, $\psi_1\wedge\psi_2$, and $\psi_1\vee\psi_2$ are also countably additive. If ψ is countably additive and $\alpha\in R$, then $\alpha\psi$ is countably additive.*

The assertions made about $\psi_1+\psi_2$ and $\alpha\psi$ are immediate consequences of 1.9 and 1.6. Consider now $\psi_1\wedge\psi_2$. Let $\{A_n\}_{n=1}^{\infty}$ be any decreasing sequence of elements of $\mathfrak{M}$ such that $\bigcap_{n=1}^{\infty} A_n=0$. Since $(\psi_1\wedge\psi_2)(A_n)\leqq\min(\psi_1(A_n), \psi_2(A_n))$, it follows from 1.6 that $\liminf_{n\to\infty}(\psi_1\wedge\psi_2)(A_n)\leqq 0$. Assume that

$\liminf_{n\to\infty} (\psi_1\wedge\psi_2)(A_n)<0$. Then there exists a real number $t<0$ and a sequence of positive integers $n_1<n_2<\cdots<n_k<\cdots$ such that $(\psi_1\wedge\psi_2)(A_{n_k})<t$ $(k=1, 2, 3, \cdots)$. By 1.11.1, there exist sets T_{n_k} such that $T_{n_k}\in\mathfrak{M}$, $T_{n_k}\subset A_{n_k}$, and $\psi_1(T_{n_k})+\psi_2(A_{n_k}\cap T'_{n_k})<t$ $(k=1, 2, 3, \cdots)$. For all k, then, $\psi_1(T_{n_k})<t/2$ or $\psi_2(A_{n_k}\cap T'_{n_k})<t/2$. This obviously implies that an infinite subsequence of at least one of the sequences $\{\psi_1(T_{n_k})\}_{k=1}^{\infty}$ and $\{\psi_2(A_{n_k}\cap T'_{n_k})\}_{n=1}^{\infty}$ consists entirely of numbers less than $t/2$. Since $\bigcap_{k=1}^{\infty}T_{n_k}=\bigcap_{k=1}^{\infty}(A_{n_k}\cap T'_{n_k})=0$, this contradicts the hypothesis that ψ_1 and ψ_2 are countably additive, and establishes the fact that $\liminf_{n\to\infty} (\psi_1\wedge\psi_2)(A_n)\geqq 0$. We may now infer that $\lim_{n\to\infty} (\psi_1\wedge\psi_2)(A_n)=0$ and that $\psi_1\wedge\psi_2$ is countably additive. The fact that $\psi_1\vee\psi_2$ is countably additive now follows from 1.11.2 and the fact that $-\psi$ is countably additive if ψ is.

1.15 THEOREM. *Let $\phi\in\Phi$ have the property that for certain countably additive measures ψ_1, ψ_2, the inequality $\psi_1\leqq\phi\leqq\psi_2$ obtains. Then ϕ is countably additive.*

Considering $\phi-\psi_1$ and noting that if $\phi-\psi_1$ is countably additive, then so is ϕ, we reduce the present theorem to the case $\psi_1=0$. Then, if $\{A_n\}_{n=1}^{\infty}$ is any decreasing sequence of sets in $\mathfrak{M}$ such that $\bigcap_{n=1}^{\infty} A_n=0$, the inequalities $0\leqq\phi(A_n)\leqq\psi_2(A_n)$ are evident; by 1.6, $\lim_{n\to\infty} \psi_2(A_n)=0$, and hence $\lim_{n\to\infty} \phi(A_n)=0$. Again by 1.6, it follows that ϕ is countably additive.

1.16 THEOREM. *Let ϕ be a non-negative measure in Φ. Then ϕ is purely finitely additive if and only if $\phi\wedge\psi=0$ for all non-negative countably additive measures ψ.*

Suppose that ϕ is purely finitely additive. Then, if $\gamma\in\Phi$ and $0\leqq\gamma\leqq\psi$, γ is countably additive, by 1.15, and if also $0\leqq\gamma\leqq\phi$, 1.13 shows that $\gamma=0$. Hence $\phi\wedge\psi=0$. Conversely, if $\phi\wedge\psi=0$ for all countably additive $\psi\geqq 0$, it follows at once that ϕ is purely finitely additive.

1.17 THEOREM. *Let π_1 and π_2 be purely finitely additive. Then $\pi_1+\pi_2$, $\pi_1\vee\pi_2$, and $\pi_1\wedge\pi_2$ are also purely finitely additive. If π is purely finitely additive and $\alpha\in R$, then $\alpha\pi$ is purely finitely additive.*

Consider first the case π_1, $\pi_2\geqq 0$. To show that $\pi_1+\pi_2$ is purely finitely additive, it suffices, in view of 1.16, to prove $(\pi_1+\pi_2)\wedge\psi=0$ for all countably additive $\psi\geqq 0$. Thus for all $E\in\mathfrak{M}$, we must compute the number

$$(\pi_1+\pi_2)\wedge\psi(E)=\inf_{A\subset E,\, A\in\mathfrak{M}} [\pi_1(A)+\pi_2(A)+\psi(E\cap A')].$$

Let ϵ_1, ϵ_2, ϵ_1', and ϵ_2' be arbitrary positive real numbers. Then, since $\pi_1\wedge\psi=\pi_2\wedge\psi=0$, there exist sets A_1, $A_2\subset E$ $(A_1, A_2\in\mathfrak{M})$ such that $\pi_1(A_1)<\epsilon_1$, $\pi_2(A_2)<\epsilon_2$, $\psi(A_1')<\epsilon_1'$, $\psi(A_2')<\epsilon_2'$. Then $\pi_1(A_1\cap A_2)+\pi_2(A_1\cap A_2)+\psi((A_1'\cup A_2')\cap E)\leqq\epsilon_1+\epsilon_2+\epsilon_1'+\epsilon_2'$; as this sum may be made arbitrarily small, it follows that $(\pi_1+\pi_2)\wedge\psi=0$, and that $\pi_1+\pi_2$ is purely finitely addi-

tive. The case π_1, π_2 arbitrary follows immediately. If π_1, π_2 are purely finitely additive, then $(\pi_1 \vee 0)$ and $(\pi_2 \vee 0)$ are also. Since $(\pi_1+\pi_2)\vee 0 \leqq \pi_1 \vee 0 + \pi_2 \vee 0$, and since we have just proved that $\pi_1 \vee 0 + \pi_2 \vee 0$ is purely finitely additive, we infer that $(\pi_1+\pi_2)\vee 0$ is also. By an identical argument, we show that $-((\pi_1+\pi_2)\wedge 0) = (-\pi_1-\pi_2)\vee 0$ is purely finitely additive; hence $\pi_1+\pi_2$ is also. If $\pi_1, \pi_2 \geqq 0$, then $0 \leqq \pi_1 \wedge \pi_2 \leqq \pi_1 \vee \pi_2 \leqq \pi_1+\pi_2$. These inequalities make it obvious that $\pi_1 \wedge \pi_2$ and $\pi_1 \vee \pi_2$ are purely finitely additive. For arbitrary π_1, π_2, we note that $(\pi_1 \vee \pi_2)\vee 0 = (\pi_1 \vee 0) \vee (\pi_2 \vee 0)$ and $(\pi_1 \vee \pi_2)\wedge 0 = (\pi_1 \wedge 0) \vee (\pi_2 \wedge 0)$. This reduces the case of arbitrary π_1, π_2 to non-negative π_1, π_2. The final statement of the theorem is established by computing

$$\inf_{A \subset E,\, A \in \mathfrak{M}} [\alpha\pi(A) + \psi(E \cap A')],$$

which is obviously 0 for $\pi \geqq 0$, if ψ is countably additive, and in general by writing $\alpha\pi$ as $\alpha\pi_+ - \alpha\pi_-$.

1.18 THEOREM. *A non-negative measure $\phi \in \Phi$ is purely finitely additive if and only if, for every non-negative countably additive ψ, every $A \in \mathfrak{M}$, and every $\alpha, \beta > 0$, there exists a set T such that $T \subset A$, $T \in \mathfrak{M}$, $\psi(T) < \alpha$, and $\phi(A \cap T') < \beta$.*

This observation follows immediately from 1.16 and 1.11.1.

We now obtain another characterization of purely finitely additive measures in a restricted case.

1.19 THEOREM. *Suppose that $\mathfrak{M} = \overline{\mathfrak{M}}$. Let π be any non-negative purely finitely additive measure and let ψ be any non-negative countably additive measure. Then for every positive real number ϵ, there exists a set $A \in \mathfrak{M}$ such that $\pi(A') = 0$ and $\psi(A) < \epsilon$.*

For A, α, β, and T as in 1.18, we have, taking ϕ of 1.18 to be the measure π now under consideration, $\pi(T) + \pi(T' \cap A) = \pi(A)$; hence $\pi(T) = \pi(A) - \pi(T' \cap A) > \pi(A) - \beta$. Now let ϵ be any positive real number and let $\{\epsilon_n\}_{n=1}^{\infty}$ be any sequence of positive real numbers such that $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. Next, consider the set X itself and let D_1 be a set in $\mathfrak{M}$ such that $\pi(D_1) < \pi(X)/2$ and $\psi(D_1') < \epsilon_1$. Let D_1' be denoted by the symbol A_1. Then $\pi(A_1) > \pi(X)/2$ and $\psi(A_1) < \epsilon_1$. Now consider the set A_1'. By 1.18, there exists a subset D_2 of A_1' such that $D_2 \in \mathfrak{M}$, $\pi(D_2) < \pi(A_1')/2$, and $\psi(D_2' \cap A_1') < \epsilon_2$. Plainly, $\pi(D_2' \cap A_1') > \pi(A_1')/2$. Write $D_1' \cap A_1'$ as A_2. We then have $\pi(A_2) \geqq \pi(A_1')/2$ and $\psi(A_2) < \epsilon_2$. It is also clear that $\pi(A_1 \cup A_2) = \pi(A_1) + \pi(A_2) > \pi(A_1) + \pi(A_1')/2 = \pi(A_1) + [\pi(X) - \pi(A_1)]/2 = \pi(X)/2 + \pi(A_1)/2 > \pi(X)/2 + \pi(X)/4$. Suppose now that disjoint sets $A_1, \cdots, A_n$ in $\mathfrak{M}$ have been found such that $\pi(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \pi(A_i) \geqq \sum_{i=1}^{n} 2^{-i}\pi(X) = (1-2^{-n})\pi(X)$ and $\psi(A_1 \cup \cdots \cup A_n) < \epsilon_1 + \cdots + \epsilon_n$. There exists a set $D_{n+1} \in \mathfrak{M}$ such that $D_{n+1} \subset (A_1 \cup \cdots \cup A_n)'$, such that $\pi(D_{n+1}) < \pi((A_1 \cup \cdots \cup A_n)')/2$ and $\psi(D_{n+1}' \cap (A_1 \cup \cdots \cup A_n)') < \epsilon_{n+1}$. Writing $D_{n+1}' \cap (A_1 \cup \cdots \cup A_n)'$ as A_{n+1},

we immediately see that $\pi(A_{n+1})>\pi((A_1\cup\cdots\cup A_n)')/2$ and $\psi(A_{n+1})<\epsilon_{n+1}$. We next observe that $\pi(A_1\cup\cdots\cup A_{n+1})=\pi(A_1\cup\cdots\cup A_n)+\pi(A_{n+1})>\pi(A_1\cup\cdots\cup A_n)+\pi(A_1\cup\cdots\cup A_n)'/2\geqq(1-2^{-n})\pi(X)+(1-2^{-n})\pi(X)/2=(1-2^{-(n+1)})\pi(X)$. It is also obvious that $\psi(A_1\cup\cdots\cup A_{n+1})<\sum_{i=1}^{n+1}\epsilon_i$. By finite induction, therefore, we may produce a countable sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\overline{\mathcal{M}}$ enjoying, for all n, the properties just described. Now write $\bigcup_{n=1}^{\infty}A_n$ as A. We have $\pi(A)>(1-2^{-n})\pi(X)$, for $n=1, 2, 3, \cdots$, and hence $\pi(A)=\pi(X)$, $\pi(A')=0$. Since ψ is countably additive, we have $\psi(A)=\sum_{n=1}^{\infty}\psi(A_n)<\sum_{n=1}^{\infty}\epsilon_n<\epsilon$.

1.20 NOTE. Theorem 1.19 may fail if $\mathcal{M}\neq\overline{\mathcal{M}}$. To see this, consider the following example. Let X be the half-open interval $[0, 1)$ and let $\mathcal{M}$ be the smallest algebra of sets which contains all intervals $[\alpha, \beta)=E[t;\ t\in R, \alpha\leqq t<\beta]$. Let ω be Lebesgue's singular measure constructed on the Cantor set, which we denote by the symbol C (see [14, pp. 100–101]), and let λ be ordinary Lebesgue measure. One may of course define the measure ω to be countably additive on a class of sets much larger than $\mathcal{M}$; certainly ω, with its domain restricted to $\mathcal{M}$, is a countably additive measure assuming only values between 0 and 1.

Now, let t be any number such that $0\leqq t\leqq 1$. We shall prove later (Theorem 4.1) that there exists a purely finitely additive measure ζ_t on the σ-algebra of all λ-measurable subsets of $[0, 1]$ such that ζ_t assumes only the values 0 and 1, $\zeta_t(N)=0$ if $\lambda(N)=0$, and $\zeta_t(I)=1$ if I is any open interval containing t. Let $\{t_1, \cdots, t_n, \cdots\}$ be any dense subset of C'. (The complement is relative to $[0, 1)$.) Let $\xi=\sum_{n=1}^{\infty}2^{-n}\zeta_{t_n}$. It is easy to see that ξ is a purely finitely additive measure on the σ-algebra of all λ-measurable sets and that ξ remains purely finitely additive when its domain is restricted to $\mathcal{M}$. Now consider any set A in $\mathcal{M}$ such that $\xi(A)=1$. A clearly must contain every point t_n. Thus $A^-\supset\{t_1, \cdots, t_n, \cdots\}^-=C'^-=[0, 1)$. Since $A=\bigcup_{i=1}^{k}[\alpha_i, \beta_i)$, $A^-=\bigcup_{i=1}^{k}[\alpha_i, \beta_i]$, and this implies that A' is a finite set. However, the only finite set in $\mathcal{M}$ is void. Therefore $A=[0, 1)$ and $\omega(A)=1$. This example shows that 1.19 may fail if $\mathcal{M}\neq\overline{\mathcal{M}}$. We note also a generalized form of 1.19.

1.21 THEOREM. *Suppose that $\mathcal{M}=\overline{\mathcal{M}}$. Let ϕ and γ be elements of Φ such that $\phi\wedge\gamma=0$. Let α and β be arbitrary positive numbers. Then there exists a set $A\in\mathcal{M}$ such that $\phi(A)<\alpha$ and $\gamma(A')<\beta$. If ϕ is countably additive, then $\gamma(A')$ can be made zero. If ϕ and γ are both countably additive, then $\phi(A)$ and $\gamma(A')$ can both be made zero.*

The first two assertions are essentially proved in 1.11 and 1.19. The third can be obtained in much the same way as 1.19, and we omit the proof.

The hypothesis that $\mathcal{M}=\overline{\mathcal{M}}$ is essential in the third assertion of 1.21. Using the same X, $\mathcal{M}$, and ω as in 1.20, consider the measure λ. Both λ and ω are countably additive on $\mathcal{M}$, and plainly $\lambda\wedge\omega=0$. However, one can clearly find no set $A\in\mathcal{M}$ such that $\omega(A)=\lambda(A')=0$.

1.22 THEOREM. *Let $\mathfrak{M}=\overline{\mathfrak{M}}$. Then if π is purely finitely additive and not less than* 0 *and ψ is countably additive and not less than* 0, *there exists a decreasing sequence $B_1, B_2, \cdots, B_n, \cdots$ of elements of $\mathfrak{M}$ such that $\lim_{n\to\infty}\psi(B_n)=0$ and $\pi(B_n)=\pi(X)$ $(n=1, 2, 3, \cdots)$. Conversely, if $\phi\in\Phi$ and the above conditions hold for all countably additive ψ, then ϕ is purely finitely additive.*

According to 1.18, there exists a set $C_n\in\mathfrak{M}$ such that $\pi(C_n)=\pi(X)$ and $\psi(C_n)<1/n$ $(n=1, 2, 3, \cdots)$. Putting $B_n=\bigcap_{i=1}^n C_i$, we find $\psi(B_n)\leqq\psi(C_n)<1/n$ and $\pi(B_n')\leqq\pi(C_1')+\cdots+\pi(C_n')=0$. Hence $\pi(B_n)=\pi(X)$. The converse is virtually obvious; if ϕ satisfies the conditions enunciated, and if ψ is countably additive, then $\phi\wedge\psi=0$; and by 1.16, ϕ is purely finitely additive.

Every $\phi\in\Phi$ can be written uniquely as the sum of a countably additive part and a purely finitely additive part. We consider several cases.

1.23 THEOREM. *Let ϕ be a measure in Φ such that $0\leqq\phi$. Then there exist measures ϕ_c and ϕ_p in Φ such that $0\leqq\phi_c$, $0\leqq\phi_p$, ϕ_c is countably additive, ϕ_p is purely finitely additive, and $\phi=\phi_c+\phi_p$.*

Let Γ be the set of all countably additive measures γ in Φ such that

1.24.1 $$0\leqq\gamma\leqq\phi.$$

Let the real number α be defined by the relation

1.24.2 $$\alpha=\sup_{\gamma\in\Gamma}\gamma(X).$$

It is obvious that $0\leqq\alpha\leqq\phi(X)<+\infty$. Select any sequence $\{\gamma_n\}_{n=1}^\infty$ of elements of Γ such that $\lim_{n\to\infty}\gamma_n(X)=\alpha$. Let $\tilde{\gamma}_n$ be defined as $\gamma_1\vee\gamma_2\vee\cdots\vee\gamma_n$, for $n=1, 2, 3, \cdots$. As proved in 1.14, $\tilde{\gamma}_n$ is countably additive. Using 1.8, we extend all of the measures $\tilde{\gamma}_n$ over the family of sets $\overline{\mathfrak{M}}$, on which they are countably additive; we retain the symbol $\tilde{\gamma}_n$ to denote this extended measure. It is clear that $\tilde{\gamma}_1\leqq\tilde{\gamma}_2\leqq\cdots\leqq\tilde{\gamma}_n\leqq\cdots$ and that accordingly, for all $E\in\overline{\mathfrak{M}}$, the number $\lim_{n\to\infty}\tilde{\gamma}_n(E)$ exists and is finite. Let $\phi_c(E)=\lim_{n\to\infty}\tilde{\gamma}_n(E)$, for all $E\in\mathfrak{M}$. It follows from a theorem of Nikodým [13], often referred to as the Vitali-Hahn-Saks theorem, that ϕ_c, with its domain restricted to $\mathfrak{M}$, is a countably additive measure in Φ. We now define ϕ_p as $\phi-\phi_c$. It is obvious that ϕ_p is a non-negative measure in Φ, and we need only show that ϕ_p is purely finitely additive. Suppose that $\psi\in\Phi$ satisfies the inequalities $0\leqq\psi\leqq\phi-\phi_c$ and is countably additive. Then $\phi_c\leqq\psi+\phi_c\leqq\phi$. If $\psi\neq0$, then $\psi(X)>0$, and accordingly $(\psi+\phi_c)(X)>\phi_c(X)=\alpha$. This is an obvious contradiction, and ϕ_p is therefore purely finitely additive.

1.24 THEOREM. *Let ϕ be any measure in Φ. Then ϕ can be uniquely written as the sum of a countably additive measure ϕ_c and a purely finitely additive measure ϕ_p.*

Write ϕ as $\phi_+-\phi_-$. By 1.23, $\phi_+=\phi_c^++\phi_p^+$ and $\phi_-=\phi_c^-+\phi_p^-$. Then ϕ

$=(\phi_c^+-\phi_c^-)+(\phi_p^+-\phi_p^-)$. By 1.14, $\phi_c^- - \phi_c^-$ is countably additive, and by 1.17, $\phi_p^+-\phi_p^-$ is purely finitely additive. To prove uniqueness, suppose that π_i are purely finitely additive, ψ_i are countably additive ($i=1, 2$), and that $\pi_1+\psi_1 =\pi_2+\psi_2$. Then $\psi_1-\psi_2=\pi_2-\pi_1$, and by 1.17, $(\pi_2-\pi_1)\vee 0$ is purely finitely additive. Thus $(\pi_2-\pi_1)\vee 0=(\psi_1-\psi_2)\vee 0$, and as $(\psi_1-\psi_2)\vee 0$ is countably additive, we have $(\psi_1-\psi_2)\vee 0=0$. Similarly, $(\psi_2-\psi_1)\vee 0=0$, and it follows that $\psi_1=\psi_2$. This implies naturally that $\pi_1=\pi_2$ also.

1.25 NOTE. A result related to 1.24 has been announced by Woodbury [18].

2. The space $\mathfrak{L}_\infty$ and its conjugate space.

2.1 DEFINITION. Let T be any set, and let $\mathfrak{M}$ be a family of subsets of X closed under the formation of countable unions and of complements. Let $\mathfrak{N}$ be a proper subfamily of $\mathfrak{M}$ closed under the formation of countable unions and having the additional property that $N\in\mathfrak{N}$, $A\subset T$, and $A\subset N$ imply $A\in\mathfrak{N}$. For an $\mathfrak{M}$-measurable real-valued function $x(t)$ on T, let $\|x\|_\infty$, the ess sup of $|x|$, be the infimum of the set of numbers α such that $E[t;\ |x(t)|<\alpha]\in\mathfrak{N}$, if this set is nonvoid. Otherwise let $\|x\|_\infty=+\infty$.

2.2 THEOREM. *Let $\mathfrak{L}_\infty(T, \mathfrak{M}, \mathfrak{N})$ be the space obtained from the space of all measurable real-valued functions on T with finite norm $\|\cdot\|_\infty$ by identifying two functions x and x' if $\|x-x'\|_\infty=0$. With the usual definitions of sum, multiplication by real numbers, and norm, $\mathfrak{L}_\infty(T, \mathfrak{M}, \mathfrak{N})$ is a real Banach space.*

This result is well known and is stated merely for completeness. We write $\mathfrak{L}_\infty$ for $\mathfrak{L}_\infty(T, \mathfrak{M}, \mathfrak{N})$ where no ambiguity is possible.

We now generalize a theorem of Fichtenholz and Kantorovič and Hildebrandt [4] and [9], obtaining the general bounded linear functional [1, p. 27], on $\mathfrak{L}_\infty$. The set of all such functionals is denoted by $\mathfrak{L}_\infty^*$.

2.3 THEOREM. *Let F be a real-valued functional defined on $\mathfrak{L}_\infty$ such that*

2.3.1 $\quad F(x+y)=F(x)+F(y) \qquad$ *for all $x, y\in\mathfrak{L}_\infty$,*

2.3.2 $\quad F(\alpha x)=\alpha F(x) \qquad$ *for all $\alpha\in R$ and $x\in\mathfrak{L}_\infty$,*

2.3.3 $\quad |F(x)|\leqq A\|x\|_\infty, \qquad$ *for some $A\geqq 0$ and all $x\in\mathfrak{L}_\infty$.*

Then there exists a finitely additive measure ϕ on $\mathfrak{M}$, in the sense of 1.2, *such that*

2.3.4 $$F(x)=\int_T x(t)d\phi(t)$$

for all $x\in\mathfrak{L}_\infty$, and such that $\phi(N)=0$ if *$N\in\mathfrak{N}$. Conversely, if ϕ is a finitely additive measure on $\mathfrak{M}$, in the sense of* 1.2, *such that $\phi(N)=0$ for all $N\in\mathfrak{N}$, then* 2.3.4 *defines a bounded linear functional on $\mathfrak{L}_\infty$. For F and ϕ as in* 2.3.4, *we have $\|F\|=\phi_+(T)+\phi_-(T)=|\phi|(T)$.*

We first note that the integral in 2.3.4 is to be interpreted just as if ϕ

were a countably additive measure and 2.3.4 were a Lebesgue integral. The standard proofs for existence, additivity, homogeneity, and positivity (for $\phi \geqq 0$) can be carried out almost exactly as they are, for example, in [17, pp. 332–337]. The only change required is to replace the lower and upper bounds of $x(t)$ respectively by ess inf $x(t)$ = supremum of real numbers α such that $E[t;\ t \in T,\ x(t) < \alpha] \in \mathcal{N}$, and ess sup $x(t)$ = infimum of real numbers β such that $E[t; t \in T, x(t) > \beta] \in \mathcal{N}$.

Starting, then, with a bounded linear functional on $\mathfrak{L}_\infty$, we consider a general set $E \in \mathfrak{M}$ and the characteristic function χ_E. Plainly, $\chi_E \in \mathfrak{L}_\infty$, and $\|\chi_E\|_\infty = 0$ or 1 as $E \in \mathcal{N}$ or $E \notin \mathcal{N}$. In either case, we apply the functional F to χ_E, obtaining

$$F(\chi_E) = \phi(E). \tag{2.3.5}$$

It is obvious that $\phi(E)$ is a real number for all $E \in \mathfrak{M}$. From 2.3.1, it is clear that ϕ is finitely additive on $\mathfrak{M}$, and from 2.3.3 that $|\phi(E)| \leqq A$ for all $E \in \mathfrak{M}$. Now let $x(t)$ be an arbitrary element of $\mathfrak{L}_\infty$, let a = ess inf $x(t)$, and let b = ess sup $x(t)$. Let $A_i = E[t;\ t \in T,\ n^{-1}[(n-i)a + ib] \leqq x(t) < n^{-1}[(n-i-1)a + (i+1)b]]$ $(i = 0, 1, 2, \cdots, n-2)$ and let $A_{n-1} = E[t; t \in T, n^{-1}[(n-1)b + a] \leqq x(t) \leqq b]$. Let $P_n = \sum_{i=0}^{n-1} n^{-1}[(n-i)a + ib]\chi_{A_i}$. It is obvious that $\|P_n - x\|_\infty \leqq n^{-1}(b-a)$. Hence $F(x) = \lim_{n\to\infty} F(P_n) = \lim_{n\to\infty} \sum_{i=0}^{n-1} n^{-1}[(n-i)a + ib]\phi(A_i) = \int_T x(t)d\phi(t)$. Thus F has the integral representation 2.3.4 as asserted. The converse, of course, is simple: any finitely additive ϕ of the kind described produces a functional satisfying 2.3.1–2.3.3. We now consider the norm $\|F\| = \sup_{\|x\|_\infty \leqq 1} |F(x)|$. Writing the finitely additive ϕ which corresponds to F as $\phi_+ - \phi_-$ and noting that $\phi_+ \wedge \phi_- = 0$, we find for every $\epsilon > 0$ a subset A of T such that $\phi_+(A') < \epsilon$, $\phi_-(A) < \epsilon$ (see Theorem 1.21). Let $g = \chi_A - \chi_{A'}$. Plainly $g \in \mathfrak{L}_\infty$, and $\|g\|_\infty \leqq 1$. (If $\phi \neq 0$, $\|g\|_\infty = 1$.) Consider

$$\begin{aligned}
F(g) &= \int_T g(t)d\phi_+(t) - \int_T g(t)d\phi_-(t) \\
&= \int_A g(t)d\phi_+(t) + \int_{A'} g(t)d\phi_+(t) - \int_A g(t)d\phi_-(t) - \int_{A'} g(t)d\phi_-(t) \\
&= \phi_+(A) - \phi_+(A') - \phi_-(A) + \phi_-(A') > \phi_+(A) + \phi_-(A') - 2\epsilon \\
&> \phi_+(A) + \phi_-(A) + \phi_+(A') + \phi_-(A') - 4\epsilon = |\phi|(A) + |\phi|(A') - 4\epsilon \\
&= |\phi(T)| - 4\epsilon.
\end{aligned}$$

From this reckoning, it follows that $\|F\| \geqq |\phi|(T)$. On the other hand, let $x(t) \in \mathfrak{L}_\infty$ and let $\|x\|_\infty$ be $\leqq 1$. Then $E[t; t \in T, |x(t)| > 1] \in \mathcal{N}$, and $|\int_T x(t)d\phi(t)| = |\int_T \max[\min(x(t), 1), -1]d\phi(t)|$. It is thus no restriction to suppose that $|x(t)| \leqq 1$ for all $t \in T$. Then

$$\left|\int_T x(t)d\phi(t)\right| = \left|\int_T x(t)d\phi_+(t) - \int_T x(t)d\phi_-(t)\right|$$

$$\leqq \left| \int_T x(t)d\phi_+(t) \right| + \left| \int_T x(t)d\phi_-(t) \right|$$
$$\leqq \phi_+(T) + \phi_-(T) = |\phi|(T).$$

We have therefore proved that $\|F\| = |\phi|(T)$.

Provided with the information that finitely additive measures on $\mathcal{M}$ are identifiable with linear functions on $\mathfrak{L}_\infty$, in accordance with 2.2, we are able to prove the existence of large numbers of purely finitely additive measures on $\mathcal{M}$. It may very well be the case, of course, that no countably additive measures other than 0 exist on $\mathcal{M}$. (For a discussion of this matter, and further references, see [2, pp. 185–187].) In this case, every finitely additive measure on $\mathcal{M}$ is purely finitely additive.

The Hahn-Banach theorem of course shows that there exist finitely additive measures not equal to 0 on $\mathcal{M}$. Thus:

2.4 THEOREM. *Let $x(t)$ be any element of $\mathfrak{L}_\infty$ such that $\|x\|_\infty > 0$ and let α be any real number. Then there exists a finitely additive measure ϕ on $\mathcal{M}$ such that $\int_T x(t)d\phi(t) = \alpha$.*

Using a result somewhat more refined than the Hahn-Banach theorem, we can prove somewhat more.

2.5 THEOREM. *Let x be an element of $\mathfrak{L}_\infty$ such that $\|x\|_\infty > 0$ and* ess inf $x(t) \geqq 0$. *Let α be any non-negative real number. Then there exists a finitely additive measure ϕ on $\mathcal{M}$ such that $\phi \geqq 0$ and $\int_T x(t)d\phi(t) = \alpha$.*

A result of M. G. Kreĭn (see [12, Theorem 1.1]) is applied here. The conditions imposed on x imply that there exists a positive real number, say δ, such that $E_0 = E[t,\ t \in T,\ x(t) > \delta] \notin \mathcal{N}$. The space $\mathfrak{S}$ of all $\mathcal{M}$-measurable real-valued functions of the form $y \cdot \chi_{E_0}$, where $y \in \mathfrak{L}_\infty$, obviously forms a Banach space, under the algebraic operations and norm of $\mathfrak{L}_\infty$. Let $\mathfrak{P}$ denote the set of all $z \in \mathfrak{S}$ such that ess inf $z(t) \geqq 0$. It is clear that $x\chi_{E_0}$ is an interior element of the set $\mathfrak{P}$, and that $\mathfrak{P}$ is a linear semigroup in the sense of Kreĭn. The linear subspace $\{ax\chi_{E_0}\}_{a \in R}$ of $\mathfrak{S}$ has the obvious linear functional

$$f(a \cdot x \cdot \chi_{E_0}) = a \cdot \alpha.$$

According to Kreĭn's theorem, f admits an extension F over all of $\mathfrak{S}$ such that $F(p) \geqq 0$ for all $p \in \mathfrak{P}$. We apply 2.3 to the space $\mathfrak{S}$ and the functional F in an obvious way, and find that $F(y) = \int_{E_0} y(t)d\phi(t)$ for a finitely additive ϕ defined on all $\mathcal{M}$-measurable subsets of E_0. For $A \in \mathcal{M}$ and $A \subset E_0$, we have $0 \leqq F(\chi_A) = \phi(A)$, and hence ϕ is non-negative. For arbitrary $A \in \mathcal{M}$, let $\phi(A) = \phi(A \cap E_0)$. This ϕ clearly satisfies all conditions of the present theorem.

In our further investigations (§3), we require the following result.

2.6 THEOREM. *Let ϕ be any measure in $\mathfrak{L}_\infty^*(T, \mathcal{M}, \mathcal{N})$. Then ϕ is purely*

finitely additive as a measure on $\mathcal{M}$ *if and only if* $\phi_+ \wedge \psi = \phi_- \wedge \psi = 0$ *for all countably additive measures* ψ *in* $\mathfrak{L}_\infty^*(T, \mathcal{M}, \mathcal{N})$ *which are not less than* 0.

The sufficiency being obvious, consider a measure ϕ satisfying the condition stated. Then, plainly, any finitely additive measure γ such that $0 \leqq \gamma \leqq \phi_+$ must vanish for all sets in $\mathcal{N}$; and the conditions stated then imply that ϕ_+ is purely finitely additive in the sense of 1.13. ϕ_- is treated similarly.

3. **The real line.** We here consider finitely additive measures on R, applying results of §§1 and 2 to this special case. All of the results of the present section can be extended by obvious devices to n-dimensional Euclidean space R^n; others admit extensions to more or less arbitrary topological spaces. To avoid technical complications, we confine ourselves to R. Throughout the present section, we denote the family of Lebesgue measurable sets in R by $\mathcal{L}$, the family of sets of Lebesgue measure zero by $\mathcal{Z}$, and by λ a fixed countably additive measure on $\mathcal{L}$ such that $0 \leqq \lambda(E) \leqq 1$ for all $E \in \mathcal{L}$, $\lambda(R) = 1$, and such that $\lambda(E) = 0$ if and only if $E \in \mathcal{Z}$. (For example, $\lambda(E)$ could be taken as $\pi^{-1/2} \int_E e^{-t^2} dt$.) In the present section, we denote the space $\mathfrak{L}_\infty(R, \mathcal{L}, \mathcal{Z})$ simply by $\mathfrak{L}_\infty(R)$. We refer to $\mathfrak{L}_\infty^*(R)$ either as a space of functionals or of finitely additive measures.

We first characterize the measures in $\mathfrak{L}_\infty^*(R)$ which are purely finitely additive.

3.1 THEOREM. *Let* ϕ *be any measure in* $\mathfrak{L}_\infty^*(R)$. *Then* ϕ *is purely finitely additive if and only if there exists a monotone decreasing sequence* $A_1, A_2, \cdots, A_n, \cdots$ *of sets in* $\mathcal{L}$ *such that* $\lim_{n\to\infty} \lambda(A_n) = 0$ *and such that* $|\phi|(A_n') = 0$ $(n = 1, 2, 3, \cdots)$.

The necessity follows from 1.22, since $\mathcal{L}$ is certainly a σ-algebra, and since λ is countably additive. Sufficiency is proved as follows. Let ψ be any finite-valued countably additive measure on $\mathcal{L}$ (not necessarily absolutely continuous with respect to λ) such that $|\phi| \geqq \psi \geqq 0$. Then $\psi(N) = 0$ necessarily for $N \in \mathcal{Z}$, and ψ is absolutely continuous with respect to λ. For $P \in \mathcal{L}$, $\psi(P) = \psi(A_1' \cap P) + \sum_{n=1}^\infty \psi(P \cap A_n \cap A_{n+1}') + \psi(P \cap \bigcap_{n=1}^\infty A_n) = 0$, since $\psi \leqq |\phi|$. Therefore $|\phi|$ is purely finitely additive, and as $\phi_+, \phi_- \leqq |\phi|$, ϕ is purely finitely additive.

We now explore the various peculiar properties that finitely additive measures in $\mathfrak{L}_\infty^*(R)$ may exhibit.

3.2 THEOREM. *There exist nonzero measures* ϕ *in* $\mathfrak{L}_\infty^*(R)$ *such that* $\int_{-\infty}^\infty x(t) d\phi(t) = 0$ *for every* $x \in \mathfrak{L}_\infty$ *which vanishes outside of some bounded set. Such measures are necessarily purely finitely additive. The relations* $\phi_+ = 0$, $\phi_- = 0$, *and* $\phi_+ \neq 0 \neq \phi_-$ *are all possible.*

Let $\mathfrak{B}$ be the set of all $x \in \mathfrak{L}_\infty(R)$ such that for some $a > 0$ (dependent on x), $\|x(1 - \chi_{[-a,a]})\|_\infty = 0$. It is clear that $\mathfrak{B}$ is a linear subspace of $\mathfrak{L}_\infty$. Let $\tilde{\mathfrak{B}}$ be the subspace of all $x + \alpha \cdot 1$, for $x \in \mathfrak{B}$ and $\alpha \in R$. Let $f(x + \alpha \cdot 1) = \alpha$. By Kreĭn's

extension theorem (see 2.4), we know that f admits a positive extension F over all of $\mathfrak{L}_\infty$. F is bounded because positive. The measure corresponding to F by 2.3, which we write as ϕ, plainly has the property that $\phi(E)=F(\chi_E)=f(\chi_E)=0$ for all bounded sets E.

We may replace $\mathfrak{B}$ in the construction above by the subspace $\mathfrak{B}_{-\infty}$ of all functions $x\in\mathfrak{L}_\infty$ such that for some $t\in R$ (dependent upon x), $\|x(1-\chi_{(-\infty,t]})\|_\infty=0$ or by the subspace $\mathfrak{B}_{+\infty}$ of all $x\in\mathfrak{L}_\infty$ such that for some $t\in R$, $\|x(1-\chi_{[t,+\infty)})\|_\infty=0$. Doing so, we obtain positive measures $\phi_{+\infty}$ and $\phi_{-\infty}$ which vanish for all sets bounded above and below, respectively. Such measures are said to be confined to $+\infty$ or to $-\infty$, respectively. By taking appropriate linear combinations of these measures, we can produce all measures specified in the present theorem.

Finally, it is easy to see that any measure $\phi\in\mathfrak{L}_\infty^*(R)$ which vanishes for all bounded sets must be purely finitely additive. We note that ϕ_+ and ϕ_- must both vanish for all bounded sets if ϕ does so. Using A_n as $(-\infty, -n)\cup(n, +\infty)$ $(n=1, 2, 3, \cdots)$, we apply 3.1 and see that ϕ must be purely finitely additive.

Fubini's theorem completely loses its validity for finitely additive measures.

3.3 THEOREM. *Let α and β be any two real numbers. There exists $y\in\mathfrak{L}_\infty(R)$ and positive measures ϕ_1 and $\phi_2\in\mathfrak{L}_\infty^*$ such that*

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} y(t+u)d\phi_1(t)\right]d\phi_2(u) = \alpha \tag{3.3.1}$$

and

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} y(t+u)d\phi_2(u)\right]d\phi_1(t) = \beta. \tag{3.3.2}$$

From 3.2, it is plain that there are positive measures ϕ_1, $\phi_2\in\mathfrak{L}_\infty^*$ such that $\phi_1(R)=\phi_2(R)=1$, $\phi_1((a, +\infty))=1$, and $\phi_2((-\infty, a))=1$ for all $a\in R$. Let $y(t)$ be defined as $\beta\chi_{(-\infty,0)}+\alpha\chi_{[0,+\infty)}$. For all $u\in R$, we have $y(t+u)=\alpha$ for $t\geqq -u$. Hence $\int_{-\infty}^{+\infty}y(t+u)d\phi_1(t)=\alpha$ for all $u\in R$. The equality 3.3.1 follows immediately, and 3.3.2 is established by an identical argument.

3.4 THEOREM. *There exists a nonzero finitely additive measure ζ such that*

$$\int_{-\infty}^{+\infty} c(t)d\zeta(t) = 0 \tag{3.4.1}$$

for all bounded continuous functions c. Any measure satisfying 3.4.1 *cannot be positive, however, and in fact $\zeta_+(R)=\zeta_-(R)$. Furthermore, any measure ζ satisfying* 3.4.1 *must be purely finitely additive.*

We prove this result by using the Hahn-Banach theorem in its general

form (see [1, pp. 27–28]). For an arbitrary but fixed $a \in R$, and $x \in \mathfrak{L}_\infty(R)$, write ess lim $\sup_{t \downarrow a} x(t)$ for the number $\inf_{\epsilon>0}$ ess $\sup_{a<t<a+\epsilon} x(t)$, and write ess lim $\inf_{t \downarrow a} x(t)$ for $\sup_{\epsilon>0}$ ess $\inf_{a<t<a+\epsilon} x(t)$. The numbers ess lim $\sup_{t \uparrow a} x(t)$ and ess lim $\inf_{t \uparrow a} x(t)$ are defined dually. Let $p(x) =$ ess lim $\sup_{t \downarrow a} x(t)$ $-$ess lim $\inf_{t \uparrow a} x(t)$, for all $x \in \mathfrak{L}_\infty(R)$. It is easy to verify that $p(x+y) \leqq p(x) + p(y)$ for all $x, y \in \mathfrak{L}_\infty(R)$ and that $p(\alpha x) = \alpha p(x)$ for $\alpha \geqq 0$. Consider now the subspace $\mathfrak{C}(R)$ of $\mathfrak{L}_\infty(R)$ consisting of all bounded continuous functions. It is clear that $p(c) = 0$ for all $c \in \mathfrak{C}$; thus the linear functional $Z(c) = 0$ has the property that $Z(c) \leqq p(c)$ for $c \in \mathfrak{C}$. By the Hahn-Banach theorem, Z admits a linear extension Z over $\mathfrak{L}_\infty$ such that $-p(-x) \leqq Z(x) \leqq p(x)$ for all $x \in \mathfrak{L}_\infty$. Since $|p(x)| \leqq 2 \cdot \|x\|_\infty$ for all $x \in \mathfrak{L}_\infty$, it follows that $Z \in \mathfrak{L}_\infty^*(R)$, and that $Z(x) = p(x)$ whenever $p(x) = -p(-x)$. For example $p(\chi_{[a,+\infty)}) = 1$, $p(-\chi_{[a,+\infty)}) = -1$, so that $Z(\chi_{[a,+\infty)}) = 1$. If $c(t) \in \mathfrak{C}(R)$, then it is obvious that $p(c) = 0$. Therefore $Z(c) = 0$. If we write ζ for the finitely additive measure associated with Z by 2.3, we obviously have $\zeta((a, b)) = \zeta((a, +\infty)) = 1$ for all $b > a$, $\zeta((b, a)) = ((-\infty, a)) = -1$ for all $b < a$, and $\zeta((b, c)) = 0$ for $b < a < c$.

It is obvious that $\zeta(R) = \int_{-\infty}^{\infty} 1 d\zeta(t) = 0$ for any ζ satisfying 3.4.1, and hence $\zeta_+(R) = \zeta_-(R)$. The fact that any ζ satisfying 3.4.1 is of necessity purely finitely additive may be established by a straightforward argument; we omit the proof.

We obtain even more bizarre measures by selecting different $p(x)$. For example:

3.5 THEOREM. *Let $\alpha \in R$. There exists a finitely additive measure ϕ such that*

$$\int_{-\infty}^{+\infty} x(t) d\phi(t) = 0 \tag{3.5.1}$$

for all $x \in \mathfrak{L}_\infty$ such that ess $\lim_{t \downarrow a} x(t)$ *exists. Such a measure ϕ cannot be positive and must be purely finitely additive.*

This is obtained by taking $p(x) =$ ess lim $\sup_{t \downarrow a} x(t) -$ess lim $\inf_{t \downarrow a} x(t)$. The proof of 3.4 is then repeated *verbatim*.

For purposes of later application, we prove the existence of a measure in $\mathfrak{L}_\infty^*(R)$ having a certain unit property.

3.6 THEOREM. *Let a be any real number. There exists a measure $\epsilon_a \in \mathfrak{L}_\infty^*(R)$ such that*

$$\int_{-\infty}^{+\infty} x(t+u) d\epsilon_a(u) = x(t+a) \tag{3.6.1}$$

for all $x \in \mathfrak{L}_\infty(R)$.

We again use a transfinite device, which appears, indeed, to be an essential adjunct of the present topic. Here we consider the nonlinear functional

3.6.2 $$d(x) = \operatorname{ess\,lim\,sup}_{t\downarrow 0} \frac{1}{t}\int_a^{a+t} x(u)du,$$

where the integral is the ordinary Lebesgue integral and x is a generic element of $\mathfrak{L}_\infty(R)$. It is easy to see that $d(tx)=td(x)$ for all $t\geqq 0$ and that $p(x+y)\leqq p(x)+p(y)$. Furthermore,

3.6.3 $$-d(-x) = \operatorname{ess\,lim\,inf}_{t\downarrow 0} \frac{1}{t}\int_a^{a+t} x(u)du.$$

Finally, $|d(x)|\leqq \|x\|_\infty$. Let $\mathfrak{F}$ be the linear subspace of $\mathfrak{L}_\infty(R)$ consisting of all x such that $d(x)=-d(-x)$. On $\mathfrak{F}$, d is a bounded linear functional. By the Hahn-Banach theorem, d admits an extension D over $\mathfrak{L}_\infty(R)$ such that $-d(-x)\leqq D(x)\leqq d(x)$ for all $x\in\mathfrak{L}_\infty(R)$. Consider any function $x\in\mathfrak{L}_\infty(R)$, and let v be any fixed real number. Then $D(x(v+u))=$

3.6.4 $$\operatorname{ess\,lim}_{t\downarrow 0} \frac{1}{t}\int_a^{a+t} x(v+u)du$$

if this limit exists. 3.6.4 may be re-written as

3.6.5 $$\operatorname{ess\,lim}_{t\downarrow 0} \frac{1}{t}\int_{a+v}^{a+v+t} x(u)du;$$

a celebrated theorem of Lebesgue (see for example [17, p. 362]) asserts that 3.6.5 exists for almost all $a+v$ and equals $x(a+v)$. Thus, a being fixed, the limit 3.6.4 exists for almost all v and equals $x(a+v)$. Therefore, in the sense of $\mathfrak{L}_\infty$, $D(x(v+u))=x(a+v)$. If we write the measure associated with D as ϵ_a, the relation 3.6.1 follows at once.

4. Representation theory for $\mathfrak{L}_\infty$. We now return to the case of a general $\mathfrak{L}_\infty(X, \mathcal{M}, \mathcal{N})$ as treated in §2, finding that the finitely additive measures considered in 2.3 can be regarded as countably additive Borel measures defined on a certain compact Hausdorff space. The countably additive and purely finitely additive parts of each of the former measures are thereupon characterized in terms of their countably additive counterparts.

4.1 THEOREM. *Let $\mathcal{E}_0$ be any subfamily of $\mathcal{M}$ such that for every finite family $E_1, E_2, \cdots, E_m$ of sets in $\mathcal{E}_0$, the set $\bigcap_{i=1}^n E_i\notin\mathcal{N}$. Then there exists a measure ω in $\mathfrak{L}_\infty^*$ such that:*

4.1.1 *ω assumes only the values* 0 *and* 1;

4.1.2 *$\omega(E)=1$ for all $E\in\mathcal{E}_0$.*

Let $\boldsymbol{P}$ be the collection of all subfamilies $\mathcal{A}$ of $\mathcal{M}$ such that:

4.1.3 $\mathcal{A}\cap\mathcal{N}=0$;

4.1.4 $\mathcal{E}_0\subset\mathcal{A}$;

4.1.5 $A, B\in\mathcal{A}$ imply that $A\cap B\in\mathcal{A}$;

4.1.6 $A \in \mathcal{A}$ and $C \in \mathcal{M}$ and $C \supset A$ imply $C \in \mathcal{A}$.

There is obviously some family in $\boldsymbol{P}$: namely, the family of all $A \in \mathcal{M}$ such that $A \supset E$ for some $E \in \mathcal{E}_0$. It is obvious that if $\boldsymbol{Q}$ is a subcollection of $\boldsymbol{P}$ completely ordered by set-inclusion, then $\bigcup_{\mathcal{A} \in \boldsymbol{Q}} \mathcal{A} \in \boldsymbol{P}$. Hence, by Zorn's Lemma, $\boldsymbol{P}$ contains a maximal element $\mathcal{A}_0$. We note that if B_1 and B_2 are elements of $\mathcal{M}$ such that $B_1 \cap B_2 \in \mathcal{N}$, then at most one of B_1 and B_2 is in $\mathcal{A}_0$. This follows at once from 4.1.5 and 4.1.3. Next, let B be any set in $\mathcal{M}$. If $B \notin \mathcal{A}_0$, a simple argument based on the maximality of $\mathcal{A}_0$ shows that $B \cap A_0 \in \mathcal{N}$ for some $A_0 \in \mathcal{A}_0$. One sees similarly that $N \in \mathcal{N}$ implies $N' \in \mathcal{A}_0$. Hence $B' \cup A_0' \in \mathcal{A}_0$; this implies that $B' \cap A_0 \in \mathcal{A}_0$ and hence that $B' \in \mathcal{A}_0$. Therefore, exactly one of the sets B and B' is in $\mathcal{A}_0$. Now, we write

4.1.7
$$\omega(B) = \begin{cases} 1 & \text{if } B \in \mathcal{A}_0, \\ 0 & \text{if } B \notin \mathcal{A}_0, \end{cases}$$

and obtain in this fashion a measure satisfying 4.1.1 and 4.1.2.

4.2. THEOREM. *Let Ω be the space of all measures in $\mathfrak{L}_\infty^*$ which assume only the values* 0 *and* 1. *For $E \in \mathcal{M}$, let Δ_E consist of all $\omega \in \Omega$ such that $\omega(E) = 1$. Let Δ_E be assigned as a neighborhood of every $\omega \in \Delta_E$. Under this definition of neighborhoods, Ω is a compact Hausdorff space.*

The usual neighborhood axioms are obviously satisfied; we note that $\Delta_{E_1} \cap \Delta_{E_2} = \Delta_{E_1 \cap E_2}$. If $\omega_1 \neq \omega_2$, then there exists an $E \in \mathcal{M}$ such that $\omega_1(E) = 1$ and $\omega_2(E) = 0$. It is plain that $\omega_1 \in \Delta_E$, $\omega_2 \in \Delta_{E'}$, and that $\Delta_E \cap \Delta_{E'} = 0$. Hence Ω is a Hausdorff space. Compactness is easily established. Let $\{\Delta_E\}_{E \in \mathcal{A}}$ be any covering of Ω by sets Δ_E. We note the equality $\Delta_{E'} = (\Delta_E)'$. Thus $\{\Delta_{E'}\}$ is a family of closed subsets of Ω with total intersection void. If $\{\Delta_E\}$ admits no finite subcovering, then $\{\Delta_{E'}\}$ admits no finite subfamily with total intersection void. That is, for every $E_1, \cdots, E_n \in \mathcal{A}$, there exists a measure $\omega \in \Omega$ such that $\omega(E_i') = 1$ $(i = 1, \cdots, m)$. Hence $\bigcap_{i=1}^n E_i' \notin \mathcal{N}$, and it follows that there exists some $\omega_0 \in \Omega$ such that $\omega_0(E') = 1$ for all $E \in \mathcal{A}$ (see 4.1). This evident contradiction proves the present theorem.

4.3 THEOREM. *Let $x \in \mathfrak{L}_\infty$. Then, writing*

4.3.1
$$\bar{x}(\omega) = \int_T x(t)\,d\omega(t),$$

we represent x as a real-valued function on Ω. Every such function $\bar{x}$ is continuous on Ω; and conversely, every continuous real function on Ω is obtainable as a function $\bar{x}$ for some $x \in \mathfrak{L}_\infty$. The correspondence $x \to \bar{x}$ has the properties that

4.3.2
$$\overline{(\alpha x + \beta y)} = \alpha \bar{x} + \beta \bar{y},$$

4.3.3
$$\overline{xy} = \bar{x}\bar{y},$$

4.3.4 $$\|x\|_\infty = \sup_{\omega\in\Omega} |\bar{x}(\omega)|.$$

Hence $\mathfrak{L}_\infty$ can be identified with the algebra $\mathfrak{C}(\Omega, R)$ of all real-valued continuous functions on Ω.

We first show that $\bar{x}$, defined by 4.3.1, is continuous on Ω. Let ω_0 be a fixed element of Ω. In the topology induced by the norm, the map $x\to\int_T x(t)d\omega_0(t)$ is obviously continuous. Let $\alpha=\text{ess inf}_{t\in T}\ x(t)$, $\beta=\text{ess sup}_{t\in T}\ x(t)$, and let A_i be defined as $E[t;\ t\in T,\ 2^{-n}(i\beta+(2^n-i)\alpha)\leqq x(t)<2^{-n}((i+1)\beta+(2^n-i-1)\alpha)]$ for $i=0, 1, 2, \cdots, 2^n-2$, and let A_{2^n-1} be $E[t;t\in T,\beta-2^{-n}(\beta-\alpha)\leqq x(t)\leqq\beta]$. Let $s_n(t)=\sum_{i=0}^{2^n-1} 2^{-n}(i\beta+(2^n-i)\alpha)\chi_{A_i}$. Plainly $\|x-s_n\|_\infty\leqq(\beta-\alpha)2^{-n}$, and since $\|\omega_0\|=1$, considered as a linear functional, it follows that

$$\left|\int_T x(t)d\omega_0(t) - \int_T s_n(t)d\omega_0(t)\right| \leqq (\beta-\alpha)2^{-n}.$$

However, $\omega_0(A_i)=1$ for exactly one value of i. For every positive integer n, let B_n be the set A_i such that $\omega_0(A_i)=1$, and let C_n be the number $2^{-n}(i\beta+(2^n-i)\alpha)$. It is clear that $B_1\supset B_2\supset\cdots\supset B_n\supset\cdots$, that $C_n\to$ some limit C, and that $C=\int_T x(t)d\omega_0(t)$. Now, if ω is any element of Ω which is in Δ_{B_n}, we have $\omega(B_n)=1$, and $\int_T s_n(t)d\omega(t)=C_n$, and hence $|\bar{x}(\omega)-\bar{x}(\omega_0)|\leqq(\beta-\alpha)2^{-n}$. This shows that $\bar{x}$ is continuous.

Property 4.3.2 is obvious. To prove 4.3.3, we must prove

4.3.5 $$\int_T x(t)y(t)d\omega(t) = \int_T x(t)d\omega(t)\int_T y(t)d\omega(t).$$

For any sets E, $F\in\mathfrak{M}$, we have

$$\int_T \chi_E(t)\chi_F(t)d\omega(t) = \int_T \chi_{E\cap F}(t)d\omega(t) = \omega(E\cap F) = \omega(E)\cdot\omega(F)$$
$$= \int_T \chi_E(t)d\omega(t)\cdot\int_T \chi_F(t)d\omega(t).$$

For any function

4.3.6 $$x = \sum_{i=1}^{n} \alpha_i\chi_{E_i} \qquad (\alpha_i\in R,\ E_i\in\mathfrak{M}),$$

we have

$$\int_T x^2(t)d\omega(t) = \sum_{i,j=1}^{n} \alpha_i\alpha_j\omega(E_i)\omega(E_j) = \left(\int_T x(t)d\omega(t)\right)^2.$$

Since functions of the form 4.3.6 are dense in $\mathfrak{L}_\infty$, we verify 4.3.5 by con-

tinuity for the case $x=y$. Therefore, for arbitrary x and y,

$$\int_T [x(t)+y(t)]^2 d\omega(t) = \left[\int_T x(t)d\omega(t)\right]^2 + 2\int_T x(t)y(t)d\omega(t) + \left[\int_T y(t)d\omega(t)\right]^2; \tag{4.3.7}$$

also

$$\begin{aligned}\int_T [x(t)+y(t)]^2 d\omega(t) &= \left[\int_T [x(t)+y(t)]d\omega(t)\right]^2 \\ &= \left[\int_T x(t)d\omega(t)\right]^2 + 2\int_T x(t)d\omega(t)\int_T y(t)d\omega(t) \\ &\quad + \left[\int_T y(t)d\omega(t)\right]^2.\end{aligned} \tag{4.3.8}$$

Comparing 4.3.7 and 4.3.8, we have 4.3.5.

We now prove 4.3.4. Let x be any nonzero element of $\mathfrak{L}_\infty$. Write $\alpha = \text{ess inf}_{t\in T}\, x(t)$, $\beta = \text{ess sup}_{t\in T}\, x(t)$, and define a as β if $|\beta| \geqq |\alpha|$ and as α if $|\beta| < |\alpha|$. Then for every $\epsilon > 0$, the set $D_\epsilon = E[t, t\in T, |x(t)-a| < \epsilon]$ is not in $\mathfrak{N}$. By 4.1, there exists a measure $\omega \in \Omega$ such that $\omega(D_\epsilon)=1$, and for such an ω, it is clear that $|\int_T x(t)d\omega(t) - a| < \epsilon$. 4.3.4 follows at once, since $\|x\|_\infty = \max(|\alpha|, |\beta|)$.

To show that every $f \in \mathfrak{C}(\Omega, R)$ is representable as some $\bar{x}$, we use the Stone-Weierstrass approximation theorem [7]. For $\omega_1 \neq \omega_2$, consider any set $E \in \mathfrak{M}$ such that $\omega_1(E)=0$, $\omega_2(E)=1$. Then, setting $x(t)=\chi_E(t)$, we have $\bar{x}(\omega_1)=0$, $\bar{x}(\omega_2)=1$. The collection of functions $\bar{x}$ on Ω is therefore a dense subring of $\mathfrak{C}(\Omega, R)$, and being uniformly closed, in view of 4.3.4, it coincides with $\mathfrak{C}(\Omega, R)$. This observation completes the present proof.

4.4 THEOREM. *Let F be any functional in $\mathfrak{L}_\infty^*$ such that $F(xy)=F(x)F(y)$ for all $x, y \in \mathfrak{L}_\infty$. Then the measure corresponding to F under 2.3 is a measure ω in Ω.*

Any multiplicative linear functional on $\mathfrak{C}(\Omega, R)$ has the form $f \to f(\omega_0)$ for some $\omega_0 \in \Omega$, by a well known theorem. Since $\mathfrak{L}_\infty$ is identifiable with $\mathfrak{C}(\Omega, R)$, as 4.3 shows, we infer that $F(x)$ has the form $x \to \int_T x(t)d\omega_0(t)$ for the same ω_0. (This fact is also apparent from examining $F(\chi_E)$ for $E \in \mathfrak{M}$.)

4.5 Having established the representation theorem 4.3, we now apply a theorem of Kakutani [10] to characterize all bounded linear functionals f on $\mathfrak{C}(\Omega, R) = \mathfrak{L}_\infty$. The theorem in question states that for every such f, there exists a countably additive, finite-valued, regular, Borel measure $\bar{\phi}$ on Ω such that

4.5.1 $$f(\bar{x}) = \int_\Omega \bar{x}(\omega)d\bar{\phi}(\omega).$$

The set of all such measures is denoted by $\mathfrak{C}^*(\Omega)$. The converse assertion, that every such $\bar{\phi}$ produces a bounded linear functional on $\mathfrak{C}(\Omega, R)$, is obvious. Thus the space $\mathfrak{L}_\infty^*$ has two distinct representations: first, as the space of all finitely additive measures on $\mathfrak{M}$; second, as the space of all countably additive, regular, Borel measures on Ω. Therefore we are in possession of a one-to-one mapping carrying all of our finitely additive $\mathfrak{M}$-measures on T into all countably additive regular Borel measures on Ω. We denote the image of $\gamma \in \mathfrak{L}_\infty^*$ by $\bar{\gamma}$. It is obvious that something must happen to change mere finite additivity on T into countable additivity on Ω. We proceed to study this phenomenon. First, we note a simple consequence of the representation 4.5.1.

4.6 THEOREM. *Every measure γ in $\mathfrak{L}_\infty^*$ is the weak limit of linear combinations of* 0–1 *measures, in the sense that for every $\epsilon > 0$ and $x_1, \cdots, x_m \in \mathfrak{L}_\infty$, there exists a measure $\delta = \sum_{i=1}^k \alpha_i \omega_i$ such that*

4.6.1 $$\left| \int_T x_j(t)d\gamma(t) - \int_T x_j(t)d\delta(t) \right| < \epsilon, \qquad j = 1, 2, \cdots, m.$$

This is proved by transferring the corresponding theorem for $\mathfrak{C}^*(\Omega)$, proved by Kakutani [10], back to the space $\mathfrak{L}_\infty$.

4.7 Let A be any set in $\mathfrak{M}$. Let us denote by $\overline{A}$ the set Δ_A contained in Ω. Plainly $\overline{A}$ is an open-and-closed set, and it is easy to show that the correspondence $A \to \overline{A}$ maps $\mathfrak{M}$ in a one-to-one fashion into the family $\overline{\mathfrak{M}}$ of all open-and-closed subsets of Ω, with preservation of finite unions, intersections, and complements. Furthermore, it is obvious that $\overline{\chi_A} = \chi_{\overline{A}}$. From these facts we infer, for an arbitrary $\gamma \in \mathfrak{L}_\infty^*$, the equalities

$$\gamma(A) = \int_T \chi_A(t)d\gamma(t) = \int_\Omega \overline{\chi_A}(\omega)d\bar{\gamma}(\omega) = \int_\Omega \chi_{\overline{A}}(\omega)d\bar{\gamma}(\omega) = \bar{\gamma}(\overline{A}).$$

Hence the measures γ and $\bar{\gamma}$ are duplicates of each other on the families of sets $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ respectively. The difference in behavior arises, naturally, from the facts that the correspondence between $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ need not preserve countably infinite set operations, and that $\bar{\gamma}$ is defined on a family of sets which may properly contain $\overline{\mathfrak{M}}$.

4.8 THEOREM. *The measure $\bar{\omega}$ on Ω is the measure such that $\bar{\omega}(\Gamma) = 1$ or* 0 *as $\omega \in \Gamma$ or $\omega \notin \Gamma$, for all Borel sets Γ. Conversely, any Borel measure μ on Ω assuming the values* 0 *and* 1 *and no other values is some $\bar{\omega}$.*

For Γ an open-and-closed set in Ω, that is, $\Gamma = \overline{A}$ for some $A \in \mathfrak{M}$, it is obvious that $\bar{\omega}(\overline{A}) = \omega(A)$ and $\bar{\omega}(\overline{A}) = 1$ or 0 as $\omega \in \Delta_A$ or $\omega \in \Delta_{A'}$. Suppose that Γ is any closed set in Ω, and that $\omega \in \Gamma$. Since $\bar{\omega}$ is regular, there exists

an open set P such that $\Gamma \subset P$ and $\bar{\omega}(P) - \bar{\omega}(\Gamma)$ is arbitrarily small. It is clear that P may be taken as open-and-closed, and therefore $\bar{\omega}(\Gamma) = 1$. If Γ is open and $\omega \in \Gamma$, then there exists an open-and-closed set Σ such that $\omega \in \Sigma \subset \Gamma$ and $\bar{\omega}(\Sigma)$ is arbitrarily close to $\bar{\omega}(\Gamma)$. This implies that $\bar{\omega}(\Gamma) = 1$. Now suppose that Γ is an arbitrary Borel set. If $\omega \in \Gamma$, then, using again the regularity of $\bar{\omega}$, we have $\bar{\omega}(\Gamma) = \sup \bar{\omega}(P)$ taken over all closed $P \subset \Gamma$. This number is clearly 1. If $\omega \in \Gamma'$, then $\bar{\omega}(\Gamma') = 1$ and $\bar{\omega}(\Gamma) = 0$.

To prove the converse, we note that a measure μ of the type described must be regular. Hence $\mu = \bar{\gamma}$ for some $\gamma \in \mathfrak{L}_\infty^*$; γ therefore assumes only the values 0 and 1 and is thus a measure ω. This brings us back to the first assertion of the theorem.

4.9 Definition. A set $A \in \mathcal{M} \cap \mathcal{N}'$ is said to be atomic if $B \in \mathcal{M}$, $B \subset A$, and $B' \cap A \notin \mathcal{N}$ imply $B \in \mathcal{N}$.

4.10 Theorem. *A point $\omega \in \Omega$ is isolated if and only if the family $E[M;\ M \in \mathcal{M},\ \omega(M) = 1]$ contains an atomic element. Any such ω is countably additive.*

This theorem is easy to prove from 4.9 and 4.2; accordingly, we omit the proof. A curious result concerning G_δ's in Ω is the following.

4.11 Theorem. *Let Γ be a non-open closed G_δ in Ω. Then there exists a sequence $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ of sets in $\mathcal{M}$ where $A_n \cap A'_{n+1} \notin \mathcal{N}$ $(n = 1, 2, 3, \cdots)$ such that Γ consists of all $\omega \in \Omega$ with $\omega(A_n) = 1$ for all n. The interior of Γ consists of all ω such that $\omega(\bigcap_{n=1}^\infty A_n) = 1$. The boundary of Γ contains at least $2^{2^{\aleph_0}}$ elements. Conversely, for any sequence $\{A_n\}_{n=1}^\infty$ of sets as described above, the set of all $\omega \in \Omega$ such that $\omega(A_n) = 1$ for all n is a non-open closed G_δ in Ω.*

Since Ω is a normal space, there exists a continuous function $\bar{x}$ such that $\bar{x}(\omega) = 0$ for all $\omega \in \Gamma$ and $\bar{x}(\omega) > 0$ for all $\omega \in \Gamma'$ (this is easily proved for any closed G_δ in any normal space). Let $A_n = E[t;\ t \in T,\ x(t) \leqq n^{-1}]$ $(n = 1, 2, 3, \cdots)$. On account of the properties of the ω-integral set forth in 4.3, we see that $\omega(A_n) = 1$ if $\omega \in \Gamma$ $(n = 1, 2, 3, \cdots)$ and $\omega(A_n) = 0$ for some positive integer n if $\omega \in \Gamma'$. Thus Γ is characterized as the set of all measures in Ω such that $\omega(A_n) = 1$ for all n. If all but a finite number of the A_n are equal, up to sets in $\mathcal{N}$, then Γ is plainly the open-and-closed subset of Ω consisting of all ω such that $\omega(\bigcap_{n=1}^\infty A_n) = 1$. Therefore, we incur no curtailment of generality by supposing that $B_n = A_n \cap A'_{n+1} \notin \mathcal{N}$ $(n = 1, 2, 3, \cdots)$. Next, writing A_0 for $\bigcap_{n=1}^\infty A_n$, we clearly have $\Delta_{A_0} \subset \Gamma'^{-1}$, since Δ_{A_0} is an open-and-closed set contained in Γ. The other elements of Γ are the measures ω such that $\omega(A_0) = 0$ and $\omega(A_n) = 1$ $(n = 1, 2, 3, \cdots)$. An argument based on the Čech-Stone β [8] for the integers shows that there are at least $2^{2^{\aleph_0}}$ distinct measures ω such that $\omega(A_0) = 0$ and $\omega(\bigcup_{k=1}^\infty B_{n_k}) = 1$ for some sequence $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ of positive integers. Let ω be any of these measures and

let C be a set in $\mathcal{M}$ such that $\omega(C)=1$. Then for some positive integer n_0, $C\cap A'_{n_0}\notin\mathcal{N}$. For, if $C\cap A'_n\in\mathcal{N}$ for all n, then $\bigcup_{n=1}^{\infty}(C\cap A'_n)=C\cap\bigcup_{n=1}^{\infty}A'_n=C\cap(\bigcap_{n=1}^{\infty}A_n)'=C\cap A'_0\in\mathcal{N}$, and $\omega(C)=\omega(C\cap A'_0)=1$ would be impossible. There is by 4.1 a measure δ such that $\delta(C\cap A'_{n_0})=1$. It is plain that $\delta\in\Delta_C$, that $\delta(A'_{n_0})=1$, and that $\delta\notin\Gamma$. Hence $\omega\notin\Gamma'^{-1}$. This proves the first part of the present theorem. The converse is now obvious.

4.12 COROLLARY. *The interior of a closed G_δ is closed.*

4.13 COROLLARY. *A point ω of Ω is a G_δ if and only if it is isolated.*

Using 4.11, we obtain a characterization of countably additive measures in $\mathfrak{L}_\infty^*$ as follows. (Of course, there may be no such measures $\neq 0$.)

4.14 THEOREM. *A measure ϕ in $\mathfrak{L}_\infty^*$ is countably additive if and only if its counterpart $\bar\phi$ in $\mathfrak{C}^*(\Omega)$ vanishes for every nowhere-dense closed G_δ in Ω.*

Suppose that ϕ is countably additive. Then, if $A_1\supset A_2\supset\cdots\supset A_n\supset\cdots$ and $\bigcap_{n=1}^{\infty}A_n\in\mathcal{N}$, we have $\lim_{n\to\infty}\phi(A_n)=0$. For Γ any nowhere-dense G_δ in Ω, there exist $A_1, A_2, \cdots$ as above such that $\Gamma=\bigcap_{n=1}^{\infty}\Delta_{A_n}$, as shown in 4.11. Clearly $\bar\phi(\Gamma)=\lim_{n\to\infty}\bar\phi(\Delta_{A_n})=\lim_{n\to\infty}\phi(A_n)=0$. The converse is proved in the same way.

A description of purely finitely additive measures analogous to 4.14 can be obtained in a certain special case.

4.15 THEOREM. *Suppose that $\mathfrak{L}_\infty^*$ contains a countably additive measure λ such that $\lambda(A)=0$ if and only if $A\in\mathcal{N}$. Then, if $\phi\in\mathfrak{L}_\infty^*$ and ϕ is purely finitely additive, there exists a nowhere-dense closed G_δ, Π, in Ω such that $\bar\phi(\mathrm{B})=0$ for every Borel set $\mathrm{B}\subset\Pi'$.*

To obtain this result, we use 1.22. Considering $|\phi|$ and $|\lambda|$ instead of ϕ and λ, we may suppose $\phi\geqq 0$, $\lambda\geqq 0$. Let ϕ be purely finitely additive. Let $\{B_n\}_{n=1}^{\infty}$ be as described in 1.22. Then $\lambda(\bigcap_{n=1}^{\infty}B_n)=\lim_{n\to\infty}\lambda(B_n)=0$, and hence $C=\bigcap_{n=1}^{\infty}B_n\in\mathcal{N}$. This shows that $\phi(B_n\cap C')=\phi(B_n)=\phi(T)$ $(n=1, 2, 3, \cdots)$ and hence $\bar\phi(\Delta_{B_n})=\bar\phi(\Omega)$ $(n=1, 2, 3, \cdots)$. Since $\bar\phi$ is countably additive, $\bar\phi(\bigcap_{n=1}^{\infty}\Delta_{B_n})=\bar\phi(\Omega)$. Since $C\in\mathcal{N}$, $\bigcap_{n=1}^{\infty}\Delta_{B_n}$ is a nowhere-dense closed G_δ (see 4.11) which can be used as Π in the present theorem.

4.16 THEOREM. *For a general $\mathcal{M}$, $\mathcal{N}$, and T, any measure ϕ such that $\bar\phi$ is confined to a closed nowhere dense G_δ is purely finitely additive.*

We omit the proof of this result.

The question arises as to extending 4.15 to the case where $\mathcal{M}$ does not admit a countably additive measure vanishing exactly on $\mathcal{N}$. It may be conjectured, for example, that if $\mathfrak{L}_\infty^*$ contains no countably additive measures at all, then every counterpart measure $\bar\phi$ is confined to a closed nowhere dense G_δ in Ω. We have not succeeded in deciding this.

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