ON THE OSCILLATION OF SUMS OF RANDOM VARIABLES

BY

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1. Introduction. Let $X$ be a random variable with distribution function $F(x)$ and characteristic function $\phi(z) = \int_{-\infty}^{\infty} e^{itz} dF(x)$. The sequence of partial sums $\{S_n\}$ will be said to be generated by $X$ if $S_n = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n, \ldots$ are independent, identically distributed random variables with distribution function $F(x)$.

Let the abbreviations i.o. and f.o. denote the phrases “infinitely often” and “finitely often,” respectively. The sequence $\{S_n\}$ generated by the random variable $X$ is said to oscillate if

$$P\{S_n > 0 \text{ i.o.}\} = P\{S_n \leq 0 \text{ i.o.}\} = 1. \quad (1.1)$$

A sufficient condition for oscillation of the sequence $S_n$, convenient for the application of results concerning partial limit laws of normed sums of independent and identically distributed random variables, will be obtained. When $E(|X|) < \infty$, the problem considered will be shown to be equivalent to a problem dealt with by K. L. Chung and W. H. J. Fuchs [1] (2).

The necessary and sufficient condition for oscillation of the sequence $\{S_n\}$ generated by $X$ will be obtained. The necessary and sufficient condition is used to obtain a class of random variables each of which generates sums $S_n$ which satisfy

$$P\left\{\lim_{n \to \infty} |S_n| = \infty\right\} = P\left\{\lim_{n \to \infty} \inf S_n = -\infty\right\}$$

$$= P\left\{\lim_{n \to \infty} \sup S_n = \infty\right\} = 1 \quad (1.2)$$

and

$$\lim_{n \to \infty} P\{S_n > 0\} = 0 \quad (1.3)$$

simultaneously.

2. Preliminary results.

Lemma 2.1. If $\lim \sup_{n \to \infty} P\{S_n > 0\} > 0 \quad (\lim \sup_{n \to \infty} P\{S_n < 0\} > 0)$, then $P\{S_n > 0 \text{ i.o.}\} = 1 \quad (P\{S_n < 0 \text{ i.o.}\} = 1).$
Now \( \{ S_n < 0 \text{ i.o.} \} = \prod_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{ S_n > 0 \} \). But \( P \{ \bigcup_{n=m}^{\infty} \{ S_n > 0 \} \} \geq \limsup_{n \to \infty} P \{ S_n > 0 \} > 0 \). Since \( \bigcup_{n=m}^{\infty} \{ S_n > 0 \} \) is a monotone decreasing sequence of events,

\[
P \{ S_n > 0 \text{ i.o.} \} = \lim_{m \to \infty} P \left\{ \bigcup_{n=m}^{\infty} \{ S_n > 0 \} \right\} > 0.
\]

P. Lévy has shown (see [4, p. 147]) that if \( A_1, A_2, \cdots \) are given constants,

\[
P \{ S_n > A_n \text{ i.o.} \}
\]

can be only either zero or one. Hence

\[
P \{ S_n > 0 \text{ i.o.} \} = 1.
\]

One shows that \( \limsup_{n \to \infty} P \{ S_n < 0 \} > 0 \) implies \( P \{ S_n < 0 \text{ i.o.} \} = 1 \) in like manner.

K. L. Chung and W. H. J. Fuchs [1] have studied recurrent values of partial sums \( S_n \) generated by a random variable \( X \). The value \( b \) is said to be recurrent if

\[
P \{ |S_n - b| < \epsilon \text{ i.o.} \} = 1
\]

for every \( \epsilon > 0 \). They have shown that the set of recurrent values is a closed additive group. In particular, 0 is the one and only recurrent value if and only if \( X = 0 \) with probability 1.

**Lemma 2.2.** If the partial sums \( S_n \) generated by \( X \) have recurrent values and if \( X \neq 0 \) with positive probability, the sequence \( \{ S_n \} \) oscillates.

Zero cannot be the only recurrent value since \( X \neq 0 \) with positive probability. There must then be two values \( a > 0, b < 0 \) which are recurrent values, that is, given any \( \epsilon > 0 \)

\[
P \{ |S_n - b| < \epsilon \text{ i.o.} \} = P \{ |S_n - a| < \epsilon \text{ i.o.} \} = 1.
\]

Let \( \epsilon < \min (a, |b|) \). Then

\[
P \{ S_n > 0 \text{ i.o.} \} = P \{ |S_n - a| < \epsilon \text{ i.o.} \} = 1,
\]

\[
P \{ S_n < 0 \text{ i.o.} \} = P \{ |S_n - b| < \epsilon \text{ i.o.} \} = 1.
\]

**Theorem 2.1.** If \( E(|X|) < \infty \) and \( X \neq 0 \) with positive probability, the problem of oscillation and the Chung-Fuchs problem are equivalent. Under these conditions, oscillation takes place if and only if \( E(X) = 0 \).

If \( X \neq 0 \) with positive probability, Lemma 2.2 implies that the Chung-Fuchs problem is included in the problem of oscillation. Chung and Fuchs [1] have shown that if \( E(|X|) < \infty \) and \( E(X) = 0 \), there are recurrent values and hence there is oscillation. But if \( E(X) = m \neq 0 \), the strong law of large numbers tells us that
and hence (1.1) cannot be true. The equivalence of the two problems when $E(|X|) < \infty$ is demonstrated.

If $f(z)$ is a complex-valued function, let $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ denote the real and imaginary parts of $f(z)$, respectively.

Let

\[
\log_2 z = \log \{-\log z\} \quad \text{and} \quad \log_n z = \log \{\log_{n-1} z\}, \quad n > 2.
\]

Lemma 2.3. The sequence $\{S_n\}$ generated by $X$ has no recurrent values if the characteristic function $\phi(z)$ of $X$ is such that

\[
\liminf_{z \to 0+} \frac{1 - \operatorname{Re} \phi(z)}{z \log z \cdots \log_{n-1} z \log_{n+1} z} > k > 0
\]

for some $\epsilon > 0$ and an integer $n \geq 1$.

A necessary and sufficient condition that there be recurrent values [1] is that

\[
\lim_{\rho \to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho \phi(z)} \, dz = \infty, \quad \delta > 0.
\]

But

\[
\lim_{\rho \to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho \phi(z)} \, dz = \lim_{\rho \to 1-} \int_{-\delta}^{\delta} \frac{1 - \rho \operatorname{Re} \phi(z)}{(1 - \rho \operatorname{Re} \phi(z))^2 + (\rho \operatorname{Im} \phi(z))^2} \, dz \\
\leq \lim_{\rho \to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho \operatorname{Re} \phi(z)} \, dz \\
\leq \int_{-\delta}^{\delta} \frac{1}{1 - \operatorname{Re} \phi(z)} \, dz.
\]

But this last integral converges if (2.1) is satisfied.

Lemma 2.4. A necessary and sufficient condition that $S_n$ oscillate is that

\[
\sum_{i=1}^{\infty} P\{S_i > 0, S_{i+1} \leq 0\} = \infty
\]

and

\[
\sum_{i=1}^{\infty} P\{S_i \leq 0, S_{i+1} > 0\} = \infty.
\]

The necessity is trivial since the theorem of Borel-Cantelli tells us that
\[
\sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0\} < \infty
\]
or
\[
\sum_{j=1}^{\infty} P\{S_j \leq 0, S_{j+1} > 0\} < \infty
\]
implies that there is no oscillation with probability 1.

Now
\[
P\{S_n > 0 \text{ f.o.}\} = P\{S_n \leq 0; n = 1, 2, \ldots \}
\]
\[
+ \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+n} \leq 0; n = 1, 2, \ldots \}
\]
\[
\geq P\{S_n \leq 0; n = 1, 2, \ldots \}
\]
\[
+ \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0, S_{j+n} - S_{j+1} \leq 0; n = 1, 2, \ldots \}
\]
\[
= P\{S_n \leq 0; n = 1, 2, \ldots \}
\]
\[
+ \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0\} P\{S_n \leq 0; n = 1, 2, \ldots \}
\]
\[
= P\{S_n \leq 0; n = 1, 2, \ldots \} \left[ 1 + \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0\} \right].
\]
The divergence of \( \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0\} \) implies that \( P\{S_n \leq 0; n = 1, 2, \ldots \} = 0 \). But
\[
P\{S_n > 0 \text{ f.o.}\} \leq \lim_{m \to \infty} P\{S_n \leq 0; n = 1, 2, \ldots \} \left[ 1 + \sum_{j=1}^{m} P\{S_j > 0\} \right].
\]
Hence the divergence of \( \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leq 0\} \) implies that
\[
P\{S_n > 0 \text{ f.o.}\} = 0.
\]

An analogous argument indicates that
\[
\sum_{j=1}^{\infty} P\{S_j \leq 0, S_{j+1} > 0\} = \infty \quad \text{implies that} \quad P\{S_n \leq 0 \text{ f.o.}\} = 0.
\]

3. **Application of the sufficient condition.** One makes use of Lemma 2.1 and the study of the infinitely divisible laws as partial limit laws of normed sums of independent random variables to characterize a class of random variables each of which generates partial sums \( S_n \) that oscillate. Random variables with no finite first moment are of interest since Chung and Fuchs have completely solved the problem for random variables having a first moment.
Theorem 3.1. Let the sequence $S_n$ be generated by a random variable $X$ with distribution function $F(x)$. If there are monotone sequences $n_k \to \infty$ and $a_k \to \infty$ of positive integers and positive numbers respectively such that

(i) \[ \lim_{k \to \infty} \frac{n_k}{a_k} \left[ \int_{|x| < r_a_k} x dF(x) + \int_{|x| > r_a_k} \frac{x}{1 + (x/a_k)^2} dF(x) \right] = m, \]

(ii) \[ \lim_{k \to \infty} \frac{n_k}{a_k^2} \left[ \int_{|x| < r_a_k} x^2 dF(x) - \left( \int_{|x| < r_a_k} x dF(x) \right)^2 \right] = \sigma^2, \]

(iii) \[ \lim_{k \to \infty} n_k (1 - F(xa_k)) = -\Omega(x), \quad x > 0, \]

\[ \lim_{k \to \infty} n_k F(-xa_k) = \Omega(-x), \quad x > 0, \]

where $\Omega(x)$ is finite for $|x| > 0$ and $\Omega(-\infty) = \Omega(\infty) = 0$,

(iv) $\Omega(x)$ increases somewhere in both the ranges $x > 0$ and $x < 0$,

then the sequence $\{S_n\}$ oscillates.

W. Doeblin [2] has shown that if conditions (i), (ii), (iii) are satisfied, the normed subsequence $S_{n_k}/a_k$ has a limiting distribution function as $k \to \infty$ whose characteristic function is

\[ \exp \left\{ i m \nu - \frac{\sigma^2}{2} \nu + \int_{-\infty}^{\infty} \left( e^{i \nu x} - 1 - \frac{i \nu x}{1 + x^2} \right) d\Omega(x) \right\}. \]

Note that $\Omega(x)$ is nondecreasing in the ranges $x > 0$ and $x < 0$. If $\Omega(x)$ increases somewhere in both the ranges $x > 0$ and $x < 0$, the limiting distribution function attributes positive probability to both positive and negative values, that is,

\[ \lim_{k \to \infty} P \left\{ \frac{S_{n_k}}{a_k} > 0 \right\} > 0, \quad \lim_{k \to \infty} P \left\{ \frac{S_{n_k}}{a_k} < 0 \right\} > 0. \]

This follows from the fact that the characteristic function

\[ \exp \left\{ \int_{-\infty}^{\infty} \left( e^{i \nu x} - 1 - \frac{i \nu x}{1 + x^2} \right) d\Omega(x) \right\} \]

is the limit of a sequence each element of which is the characteristic function of a weighted sum of properly centered independent and non-identically distributed Poisson random variables (see [4, pp. 173–180]).

But then Lemma 2.1 implies that the sequence $\{S_n\}$ oscillates.

4. Application of the necessary and sufficient condition. The object of this section is to obtain Theorems 4.2 and 4.3 which give a characterization of a class of random variables satisfying (1.2) and (1.3) simultaneously.
Lemma 4.1. Let \( F(x) \) be a distribution function with \( \int |x|^{-\alpha} dF(x) < \infty \) for some \( \alpha, 0 < \alpha < 1 \). Given \( Y > 0 \), let

\[
F(x; Y) =
\begin{cases}
0 & \text{if } x \leq -Y, \\
F(x) - F(-Y) & \text{if } |x| < Y, \\
F(Y) - F(-Y) & \text{if } x \geq Y,
\end{cases}
\]

and \( \phi(z; Y) \) be the Fourier-Stieltjes transform of \( F(x; Y) \). Then \( \text{Im} \phi(z; Y) = o(\frac{1}{|z|^\alpha}) \) at \( z = 0 \) uniformly for all \( Y > 0 \).

Now \( |\text{Im} \phi(z,Y)| = \int_{|z|<Y} \sin \alpha \text{d}F(x) \leq \int_{|z|<Y} |x\alpha| \sin \alpha \text{d}F(x) = \frac{1}{|z|^\alpha}o(1) \) uniformly for all \( Y > 0 \) by the Lebesgue theorem on dominated convergence.

Let \( X \) be a lattice random variable, i.e., there is a largest \( h > 0 \) such that

\[
P\{X = kh\} = p_k \geq 0, \quad k = 0, \pm 1, \pm 2, \ldots,
\]

and

\[
\sum_{k=-\infty}^{\infty} p_k = 1.
\]

Let

\[
\phi_1(z) = \sum_{j=-\infty}^{-1} p_je^{ijhz}, \quad \phi_2(z) = \sum_{j=1}^{\infty} p_je^{ijhz}.
\]

Theorem 4.1. Let \( X \) be a lattice random variable with \( E(|X|^\alpha) < \infty \) for some \( \alpha, 0 < \alpha < 1 \), and let the partial sums generated by \( X \) have no recurrent values. A necessary and sufficient condition that the partial sums generated by \( X \) oscillate is that

\[
\lim_{\rho \to 1-} \left[ \frac{h(1-\rho)}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\text{Im} \phi_2(z)}{|1 - \rho \phi(z)|^2} \text{d}z 
\right. + \frac{\rho h}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 + \cos zh} \frac{\{\phi_1(0) - \text{Re} \phi_1(z)\} \text{Im} \phi_2(z)}{|1 - \rho \phi(z)|^2} - \frac{\{\phi_2(0) - \text{Re} \phi_2(z)\} \text{Im} \phi_1(z)}{|1 - \rho \phi(z)|^2} \text{d}z 
\]

\[
(4.1)
\]

\[
= \lim_{\rho \to 1-} \left[ \frac{-h(1-\rho)}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\text{Im} \phi_1(z)}{|1 - \rho \phi(z)|^2} \text{d}z 
\right. + \frac{\rho h}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 + \cos zh} \frac{\{\phi_1(0) - \text{Re} \phi_1(z)\} \text{Im} \phi_2(z)}{|1 - \rho \phi(z)|^2} - \frac{\{\phi_2(0) - \text{Re} \phi_2(z)\} \text{Im} \phi_1(z)}{|1 - \rho \phi(z)|^2} \text{d}z 
\]

\[
(4.2)
\]

\[
= \infty.
\]
Lemma 2.4 implies that

\begin{align}
\lim_{p \to 1^-} \sum_{j=1}^{\infty} p^j P\{S_j > 0, S_{j+1} \leq 0\} &= \infty, \\
\lim_{p \to 1^-} \sum_{j=1}^{\infty} p^j P\{S_j \leq 0, S_{j+1} > 0\} &= \infty
\end{align}

is a necessary and sufficient condition for oscillation.

First consider (4.3).

\begin{align}
\sum_{j=1}^{\infty} p^j P\{S_j > 0, S_{j+1} \leq 0\} &= \sum_{j=1}^{\infty} p^j \sum_{k=1}^{\infty} P\{S_j = kh\} \sum_{n=-\infty}^{-k} P\{X = nh\}.
\end{align}

But \( P\{S_j = kh\} = (h/2\pi) \int_{-\pi/h}^{\pi/h} \phi(z)e^{-ikhz}dz \) where \( \phi(z) \) is the characteristic function of \( X \). Expression (4.5) can then be rewritten as

\begin{align}
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \phi(z)e^{-ikhz}dz \sum_{n=-\infty}^{-k} p_n
\end{align}

\begin{align}
&= \sum_{k=1}^{\infty} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ikhz} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz \sum_{j=-\infty}^{-k} p_j \\
&= \sum_{j=-\infty}^{-1} \sum_{k=1}^{\infty} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ikhz} - e^{i(j-1)hz} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz p_j \\
&= \sum_{j=-\infty}^{-1} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ihz} \frac{\phi_1(0) - \phi_1(z)}{1 - e^{-ihz}} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz \\
&= \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ihz} \left\{ \phi_1(0) - \phi_1(z) \right\} \frac{\rho \phi(z)}{1 - e^{-ihz}} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz
\end{align}

where the last interchange of summation and integration follows by applying Lemma 4.1. Now expression (4.6) can be rewritten as

\begin{align}
- \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \left\{ \phi_1(0) - \phi_1(z) \right\} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz \\
+ \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \phi_1(0) - \phi_1(z) \frac{\rho \phi(z)}{1 - e^{-ihz}} \frac{\rho \phi(z)}{1 - \rho \phi(z)} dz.
\end{align}

But expression (4.7) equals

\begin{align}
- P\{X < 0\} \sum_{n=1}^{\infty} \rho^n P\{S_n = 0\} + \sum_{j=-1}^{-\infty} p_j \sum_{n=1}^{\infty} \rho^n P\{S_n = -jh\}
\end{align}

which is bounded by
\[
\sum_{j=1}^{\infty} P\{S_j = 0\} < \infty
\]

in absolute value. The assumption that the partial sums have no recurrent values implies that \(\sum_{j=1}^{\infty} P\{S_j = 0\} < \infty\) [1]. Hence we need consider only (4.8) which can be written as

\[
\begin{align*}
(4.9) & \quad - \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - e^{-ihz}} \, dz \\
(4.10) & \quad + \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - e^{-ihz}} \frac{1}{1 - \rho \phi(z)} \, dz.
\end{align*}
\]

But (4.9) equals

\[- P\{X < 0\}\]

so that only (4.10) need be considered. Expression (4.10) equals

\[
\begin{align*}
(4.11) & \quad \frac{h}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - \rho \phi(z)} \, dz \\
(4.12) & \quad - \frac{hi}{4\pi} \int_{-\pi/h}^{\pi/h} \sin zh \frac{\phi_1(0) - \phi_1(z)}{1 - \cos zh \left| 1 - \rho \phi(z) \right|^2} \, dz.
\end{align*}
\]

But (4.11) is bounded in absolute value by

\[
1 + \sum_{j=1}^{\infty} P\{S_j = 0\} < \infty
\]

so that only (4.12) need be considered. Expression (4.12) can be rewritten as

\[
\begin{align*}
(4.13) & \quad - \frac{hi}{4\pi} \int_{-\pi/h}^{\pi/h} \sin zh \frac{\phi_1(0) - \phi_1(z)}{1 - \cos zh \left| 1 - \rho \phi(z) \right|^2} (1 - \rho \phi(z)) \, dz \\
& \quad + \frac{\rho h}{4\pi} \int_{-\pi/h}^{\pi/h} \sin zh \left( \left\{ \phi_1(0) - \text{Re} \phi_2(z) \right\} \left| 1 - \rho \phi(z) \right|^2 - \left\{ \phi_2(0) - \text{Re} \phi_2(z) \right\} \left| 1 - \rho \phi(z) \right|^2 \right) \, dz.
\end{align*}
\]
Hence equation (4.3) is true if and only if the limit of (4.13) as \( p \to 1^- \) is infinite. One derives condition (4.1) in a completely analogous manner from equation (4.4).

Given the distribution function \( F(x) \), consider the two auxiliary distribution functions

\[
F_2(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
\frac{F(x)}{1 - F(0+)} & \text{if } x > 0
\end{cases}
\]

and

\[
F_1(x) = \begin{cases} 
\frac{F(x)}{F(0-)} & \text{if } x < 0, \\
1 & \text{if } x \geq 0.
\end{cases}
\]

Let \( \phi(z), \Phi_1(z), \Phi_2(z) \) be the characteristic functions of \( F(x), F_1(x), \) and \( F_2(x), \) respectively.

Lemma 4.2. Let \( F(x) \) be such that

\[
1 - F_2(x) = x^{-\alpha}h_2(x), \quad F_1(x) = x^{-\beta}h_1(x),
\]

where \( 0 < \alpha, \beta < 1 \) and

\[
\lim_{z \to \infty} \frac{h_2(cz)}{h_2(x)} = \lim_{z \to \infty} \frac{h_1(cz)}{h_1(x)} = 1
\]

for every constant \( c > 0. \) Then

\[
\frac{1 - \text{Re } \Phi_2(z)}{\Gamma(1 - \alpha) \cos (\pi \alpha/2)}, \quad \frac{-\text{Im } \Phi_2(z)}{\sin (\pi \alpha/2) \Gamma(1 - \alpha)} \sim 1 - F_2 \left( \frac{1}{z} \right) \quad \text{as } z \to 0^+
\]

and

\[
\frac{1 - \text{Re } \Phi_1(z)}{\Gamma(1 - \alpha) \cos (\pi \alpha/2)}, \quad \frac{-\text{Im } \Phi_1(z)}{\sin (\pi \alpha/2) \Gamma(1 - \alpha)} \sim F_1 \left( -\frac{1}{z} \right) \quad \text{as } z \to 0^+.
\]

Let \( \{a_k\}, \{b_k\} \) be positive sequences such that

\[
1 - F_2(a_k) \sim 1/k, \quad F_1(-b_k) \sim 1/k
\]

as \( k \to \infty. \) Let \( Y_1, Y_2, \ldots \) be independent random variables with common distribution function \( F_2(x), \) while \( Z_1, Z_2, \ldots, \) are independent random variables with common distribution function \( F_1(x). \) Let \( S_n \) and \( T_n \) be their corresponding partial sums. Then [2, 3]

\[
P\{S_n \leq xa_n\} \to G_\alpha(x), \quad P\{T_n \leq xb_n\} \to H_\beta(x)
\]
where \( G_\alpha(x), H_\beta(x) \) have the characteristic functions \( \gamma_\alpha(z), \delta_\beta(z) \) and
\[
\delta_\beta(z) = \varphi_\beta(z), \gamma_\alpha(z) = \exp - \left\{ z \left( \cos \frac{\pi \alpha}{2} - i \sin \frac{\pi \alpha}{2} \sgn z \right) \Gamma(1 - \alpha) \right\}.
\]
Then
\[
n(1 - \Re \Phi_2(z/a_n)) \to z^\alpha \cos \frac{\pi \alpha}{2} \Gamma(1 - \alpha),
n \Im \Phi_2(z/a_n) \to z^\alpha \sin \frac{\pi \alpha}{2} \Gamma(1 - \alpha)
\]
as \( n \to \infty \), \( z > 0 \). But the choice of \( \{a_k\} \) and the properties of \( h_2(x) \) imply that
\[
n(1 - F_2(a_n x)) \to \frac{1}{x^\alpha}
\]
as \( n \to \infty \). This implies that
\[
\frac{1 - \Re \Phi_2(z)}{\cos (\pi \alpha/2) \Gamma(1 - \alpha)}, \quad \frac{\Im \Phi_2(z)}{\sin (\pi \alpha/2) \Gamma(1 - \alpha)} \sim 1 - F_2 \left( \frac{1}{z} \right)
\]
as \( z \to 0^+ \). The analogous result for \( 1 - \Re \Phi_1(z), \Im \Phi_1(z) \) follows in like manner.

**Theorem 4.2.** Let \( F(x) \) be such that
\[
1 - F_2(x) = x^{-\alpha} h_2(x), \quad F_1(x) = x^{-\beta} h_1(x),
\]
\( 0 < \alpha, \beta < 1 \), where
\[
\lim_{x \to \infty} \frac{h_2(cx)}{h_2(x)} = \lim_{x \to \infty} \frac{h_1(cx)}{h_1(x)} = 1
\]
for every constant \( c > 0 \). There is no oscillation of the partial sums generated by a random variable \( X \) with distribution function \( F(x) \) if \( \alpha \neq \beta \). Assume \( \alpha = \beta \). Then a sufficient condition for the partial sums generated to oscillate is that there be a positive integer \( n \) and constants \( k_1, k_2 > 0 \) such that
\[
(A) \quad \frac{-k_2}{\log z \cdots \log_n z} < \frac{h_1(-1/z)}{h_2(1/z)} < -k_1 \log z \cdots \log_n z
\]
as \( z \to 0^+ \). On the other hand, if there is a positive integer \( n \) and there are constants \( \epsilon, k > 0 \) such that
\[
(B) \quad \frac{h_1(-1/z)}{h_2(1/z)} > -k \log z \cdots \log_{n-1} z \log_{n+\epsilon} z
\]
or

\[ \frac{h_2(1/z)}{h_1(-1/z)} > -k \log z \cdots \log_{n-1} z \log^{1+n} z \]

as \( z \to 0^+ \), there is no oscillation.

Let \( X \) first be a lattice random variable. If \( X \) has a distribution function of the form specified above, Lemmas 4.2 and 2.3 imply that there are no recurrent values. In investigating expressions (4.1) and (4.2) as \( \rho \to 1^- \), it is clear that one need only consider the indicated integrations over a neighborhood \((-\epsilon, \epsilon)\), \( \epsilon > 0 \), of zero. Lemma 4.2 implies that

\[ \phi_2(0) - \text{Re} \phi_2(z) \sim z^\alpha h_2 \left( \frac{1}{z} \right) \Gamma(1-\alpha)(1-F(0+)) \]

and

\[ \phi_1(0) - \text{Re} \phi_1(z) \sim z^\beta h_1 \left( -\frac{1}{z} \right) \Gamma(1-\alpha)F(0-) \]

as \( z \to 0^+ \). Now

\[ 0 \leq -\frac{h(1-\rho)}{4\pi} \int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh} \frac{\text{Im} \phi_1(z)}{|1-\rho \phi(z)|^2} \, dz \]

\[ \leq -\frac{h}{4\pi} \int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh} \frac{\text{Im} \phi_1(z)}{1-\text{Re} \phi(z)} \, dz. \]

The non-negativity of the integrand and Fatou's lemma imply that

\[ \liminf_{\rho \to 1^-} \int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh} \left( \frac{\phi_1(0) - \text{Re} \phi_1(z)}{|1-\rho \phi(z)|^2} \, \text{Im} \phi_2(z) - \frac{\phi_2(0) - \text{Re} \phi_2(z)}{|1-\rho \phi(z)|^2} \, \text{Im} \phi_1(z) \right) \, dz \]

\[ \leq \int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh} \left( \frac{\phi_1(0) - \text{Re} \phi_1(z)}{|1-\phi(z)|^2} \, \text{Im} \phi_2(z) - \frac{\phi_2(0) - \text{Re} \phi_2(z)}{|1-\phi(z)|^2} \, \text{Im} \phi_1(z) \right) \, dz. \]

However

\[ \frac{1}{|1-\rho \phi(z)|^2} \leq \frac{1}{(1-\text{Re} \phi(z))^2}, \quad 0 < \rho \leq 1. \]

But \( |1-\phi(z)| \) and \( 1-\text{Re} \phi(z) \) have the same behavior at zero as can be seen
by making use of (4.14) and (4.15). Hence

\[ \int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1 - \cos zh} \frac{\{\phi_1(0) - \text{Re} \phi_1(z)\} \text{Im} \phi_1(z) - \{\phi_2(0) - \text{Re} \phi_2(z)\} \text{Im} \phi_1(z)}{|1 - \rho \phi(z)|^2} \, dz \]

diverges as \( \rho \to 1^- \) if and only if

\[ (4.17) \int_{0}^{\epsilon} \frac{1}{z} \frac{z^{\alpha+\beta} h_2(1/z) h_1(-1/z)}{z^{2\alpha} h_1^2(-1/z) + z^{2\beta} h_2(1/z)} \, dz \]

diverges.

Let \( \beta > \alpha \). By making use of (4.14) and (4.15), we see that (4.16) is bounded by a multiple of

\[ z^\beta \int_{0}^{\epsilon} \frac{1}{z} h_1(-1/z) h_2(1/z) \, dz \]

which converges since

\[ \frac{h_1(-1/z)}{h_2(1/z)} = o(z^\gamma) \]

as \( z \to 0^+ \) for every \( \gamma < 0 \). Expression (4.17) converges for the same reason. Hence, (4.2) converges and there is no oscillation. If \( \alpha > \beta \), a similar argument shows that (4.1) converges.

Let \( \alpha = \beta \). Assume condition (C). Then both (4.16) and (4.17) are bounded by a multiple of

\[ -\int_{0}^{\epsilon} \frac{dz}{z \log z \cdots \log_{n-1} z \log_{n+\epsilon} z} \]

which converges. Expression (4.2) converges and there is no oscillation. If condition (B) were valid, a similar argument would show that (4.1) converges.

Let \( \alpha = \beta \) and assume condition (A). Expression (4.16) is bounded below by zero and (4.17) is greater than

\[ \int_{0}^{\epsilon} \frac{dz}{z \log z \cdots \log_{n} z} \]

which diverges. Hence (4.2) diverges. A similar argument shows that (4.1) diverges. It then follows that the partial sums generated oscillate.

Now let \( X \) be a random variable whose distribution function satisfies the assumptions of Theorem 4.2. Let \( C(x) \) denote the greatest integer less than or equal to \( x \). Consider the two auxiliary lattice random variables

\[ X_- = hC(X/h), \quad X_+ = X_- + h, \]
where $h$ is such that $P\{X_-=h\}, P\{X_+=h\} > 0$. Now

(4.18) \[ X_- \leq X \leq X_+. \]

$X_-, X, X_+$ have distribution functions with the same behavior at $-\infty$ and $+\infty$. Therefore $X_-, X, X_+$ have characteristic functions with the same behavior as $z \to 0$. In view of what has been proved, $X_-$ and $X_+$ generate partial sums which either oscillate together or are both positive finitely often with probability one or are both negative finitely often with probability one. Hence, the theorem applies to $X$ in view of (4.18).

**Theorem 4.3.** Let $F(x)$ be such that

\[ 1 - F_2(x) = x^{-\alpha}h_2(x), \quad F_1(x) = x^{-\alpha}h_1(x), \quad 0 < \alpha < 1, \]

where

\[ \lim_{x \to \infty} \frac{h_2(cx)}{h_2(x)} = \lim_{x \to \infty} \frac{h_1(cx)}{h_1(x)} = 1 \]

for every constant $c > 0$. Assume condition (A) and

\[ \lim_{x \to \infty} \frac{h_2(x)}{h_2(-x)} = 0, \quad \left( \lim_{x \to \infty} \frac{h_1(-x)}{h_2(x)} = 0 \right). \]

Then

(4.19) \[ P\left\{ \lim_{k \to \infty} |S_k| = \infty \right\} = P\left\{ \lim \inf_{k \to \infty} S_k = -\infty \right\} = P\left\{ \lim \sup_{k \to \infty} S_k = \infty \right\} = 1, \]

(4.20) \[ \lim_{k \to \infty} P\{S_k > 0\} = 0, \quad \left( \lim_{k \to \infty} P\{S_k < 0\} = 0 \right) \]

simultaneously.

If condition (A) is valid, the partial sums generated oscillate but have no recurrent values. Whenever there is oscillation and there are no recurrent values (4.19) is true. For then

\[ \sum_{k=1}^{\infty} P\{ |S_k| < m \} < \infty \]

for all $m > 0$ [1] and hence by the theorem of Borel-Cantelli

\[ P\left\{ \lim_{k \to \infty} |S_k| = \infty \right\} = 1. \]
The fact that there is oscillation implies that
\[ P \left\{ \lim \inf_{k \to \infty} S_k = -\infty \right\} = P \left\{ \lim \sup_{k \to \infty} S_k = \infty \right\} = 1. \]

Now assume
\[ \lim_{x \to \infty} \frac{h_2(x)}{h_1(-x)} = 0. \]

Let \( \{b_k\} \) be such that
\[ F_1(-b_k) \sim 1/k \]
as \( k \to \infty \). Then
\[ 1 - F_2(b_k) = o(1/k) \]
as \( k \to \infty \) and
\[ P \{ S_k < x b_k \} \to H_\alpha(x) \]
as \( n \to \infty \) by [2]. But \( H_\alpha(x) \) is the distribution function of a random variable that is nonpositive. Hence
\[ \lim_{k \to \infty} P \{ S_k > 0 \} = 0. \]

A similar argument shows that
\[ \lim_{x \to \infty} \frac{h_1(-x)}{h_2(x)} = 0 \]
implies that
\[ \lim_{k \to \infty} P \{ S_k < 0 \} = 0. \]

**References**