THE AUTOMORPHISM GROUP OF A LIE GROUP

BY

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Introduction. The group $A(G)$ of all continuous and open automorphisms of a locally compact topological group $G$ may be regarded as a topological group, the topology being defined in the usual fashion from the compact and the open subsets of $G$ (see §1). In general, this topological structure of $A(G)$ is somewhat pathological. For instance, if $G$ is the discretely topologized additive group of an infinite-dimensional vector space over an arbitrary field, then $A(G)$ already fails to be locally compact.

On the other hand, if $G$ is a connected Lie group, we shall show without any difficulty that the compact-open topology of $A(G)$ coincides with the topology obtained by identifying $A(G)$ with a closed subgroup of the linear group of automorphisms of the Lie algebra of $G$, as was done by Chevalley (in [1]) in order to make $A(G)$ into a Lie group. We shall then deduce that $A(G)$ is a Lie group whenever the group of its components, $G/G_0$, is finitely generated($^*$), where $G_0$ denotes the component of the identity element in $G$.

The other questions with which we shall be concerned are the following: Let $I(G_0)$ denote the group of the inner automorphisms of $G_0$, and let $E(G_0, G)$ denote the natural image in $A(G_0)$ of $A(G)$. Regard $I(G_0)$ and $E(G_0, G)$ as subgroups of $A(G_0)$. Are these subgroups closed in $A(G_0)$? Is $E(G_0, G)$ topologically, as well as group-theoretically, isomorphic with the corresponding factor group of $A(G)$?

We shall show, under the assumption that $G/G_0$ is finitely generated, that these questions are related as follows: The natural continuous homomorphism of $A(G)$ onto $E(G_0, G)$ is open if and only if $E(G_0, G)$ is closed in $A(G_0)$. Under the stronger assumption that $G/G_0$ is finite, a sufficient condition for $E(G_0, G)$ to be closed in $A(G_0)$ is that $I(G_0)$ be closed in $A(G_0)$. Finally, in order to throw some light on the difficulties which are involved here, we shall give a simple example in which $I(G_0)$ and $E(G_0, G)$ are not closed in $A(G_0)$. In this example, $G$ has only two components and $G_0$ is homeomorphic with Euclidean 5-space.

1. Topological preparation. We shall describe the topology of a group $G$ in terms of a fundamental system $\mathcal{V}$ of neighborhoods $V$ of the identity element. A system $\mathcal{V}$ of subsets of $G$ will define a Hausdorff topology consistent with the group operations if and only if it satisfies the following conditions($^\dagger$):

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($^*$) I am indebted to the referee for the remark that my original requirement, "$G/G_0$ finite", can be relaxed to the present one.

($^\dagger$) We are taking these from §2 of [4].
I. The intersection of all $V \in \mathcal{B}$ is the set consisting of the identity element of $G$ only.

II. If $V_1$ and $V_2$ are sets belonging to $\mathcal{B}$, there is a $V \in \mathcal{B}$ such that $V \subseteq V_1 \cap V_2$.

III. For every $V \in \mathcal{B}$ there is a $W \in \mathcal{B}$ such that $W^{-1}W \subseteq V$.

IV. For every $g \in G$ and $V \in \mathcal{B}$ there is a $W \in \mathcal{B}$ such that $W \subseteq g^{-1}Vg$.

The neighborhoods of the identity element are then all the sets containing a set belonging to $\mathcal{B}$.

If $C$ is any compact subset of $G$ and $V \in \mathcal{B}$, we denote by $N(C, V)$ the set of all $a \in A(G)$ for which $a(x)x^{-1} \in V$ and $a^{-1}(x)x^{-1} \in V$, whenever $x \in C$.

We claim that if $G$ is locally compact, the system of these $N(C, V)$ satisfies conditions I–IV above. In fact, I holds quite evidently. If $V \subseteq V_1 \cap V_2$, we clearly have $N(C \cup C_2, V) \subseteq N(C_1, V_1) \cap N(C_2, V_2)$, so that II is satisfied.

In order to verify III we proceed as follows: Since $G$ is locally compact, given $V \in \mathcal{B}$, there is a compact set $C_0$ and a $V_0 \in \mathcal{B}$ such that $V_0 \subseteq C_0$ and $V_0V_0V_0 \subseteq V$. From the identity $(\beta^{-1}(\alpha)(c))c^{-1} = [\beta^{-1}(\alpha(c)c^{-1})(\alpha(c)c^{-1})^{-1}] \cdot [\alpha(c)c^{-1}]^{-1}$, we can see immediately that we have then $N(C \cup C_0, V_0)^{-1}N(C \cup C_0, V_0) \subseteq N(C, V)$, which shows that III is satisfied.

Finally, we have, with $a \in A(G)$,

$$\alpha^{-1}N(\alpha(C), \alpha(V)) \alpha \subseteq N(C, V),$$

whence IV holds.

From now on, if $G$ is any locally compact group, $A(G)$ will denote the group of all continuous and open automorphisms of $G$, with the topology defined by the $N(C, V)$, where $C$ ranges over the compact subsets of $G$ and $V$ over the set of neighborhoods of the identity element in $G$.

Next we shall prove two elementary results which we shall need later on.

**Lemma 1.** Let $G$ be a topological group, $U$ a neighborhood of the identity element, $C$ a compact subset of $G$. Then the intersection of all the sets $c^{-1}Uc$, with $c \in C$, is a neighborhood of the identity element.

**Proof.** We can find $V \in \mathcal{B}$ such that $VVV^{-1} \subseteq U$. Then we have $\gamma^{-1}U \gamma \supseteq V$, for every $\gamma \in V$. Since $C$ is compact, there are elements $c_1, \ldots, c_n$ in $C$ such that $C$ is contained in the union of the $n$ sets $V_c$. Now if $c$ is any element in $C$, we write $c = yc_i$, with $y \in V$. Then $c^{-1}Uc = c_i^{-1}y^{-1}Uyc_i \supseteq c_i^{-1}Vc_i$. Hence the intersection of all the sets $c^{-1}Uc$ contains the finite intersection of the $c_i^{-1}Vc_i$ and is therefore a neighborhood of the identity element.

**Lemma 2.** Suppose that $G$ is connected and locally compact. Let $C$ be a compact subset of $G$, $V$ a neighborhood of the identity, and $S$ a compact neighborhood of the identity. Then there exists a neighborhood $W$ of the identity such that $N(S, W) \subseteq N(C, V)$.

**Proof.** Since $G$ is connected and $S$ a neighborhood of the identity, we have $G = \bigcup_{n=1}^{\infty} S^n$. Since $C$ is compact and since $S^n \subseteq S^{n+1}$, it follows that $C \subseteq S^m$, for some $m$. Now choose a neighborhood $T$ of the identity such that $T^m \subseteq V$,
and put $W = \cap_{k \in \mathbb{N}} x^{-1} T x$. By Lemma 1, $W$ is a neighborhood of the identity, for $S^n$ is compact. Now let $\alpha \in A(S, W)$ and $e \in C$. We have $e = x_1, \ldots, x_n$, with $x_i \in S$. Put $c_k = x_1 \cdot \ldots x_k$, and suppose we have already shown that $\alpha(c_k)c_k^{-1} \in T^k$. Then we have $\alpha(c_{k+1})c_{k+1}^{-1} = (\alpha(c_k)c_k^{-1})c_k(\alpha(x_{k+1})x_{k+1}^{-1})c_{k+1}^{-1} \in T^kCkWc_{k+1}^{-1} \subseteq T^{k+1}$. Hence we get $\alpha(c)c^{-1} \in T^n \subseteq V$, which clearly suffices to establish our lemma.

2. Automorphism groups. Let $G$ be a connected Lie group. An automorphism $\alpha \in A(G)$ induces an automorphism $\tilde{\alpha}$ of the Lie algebra $\mathfrak{g}$ of $G$. We denote by $A(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, with the topology induced by that of the full linear group of which $A(\mathfrak{g})$ is clearly a closed subgroup. It is shown in [1] that the mapping $\alpha \rightarrow \tilde{\alpha}$ is a group isomorphism of $A(G)$ onto a closed subgroup of $A(\mathfrak{g})$. (If $G$ is simply-connected the image of $A(G)$ coincides with $A(\mathfrak{g})$.) We shall prove the following result:

**Theorem 1.** Let $G$ be a connected Lie group. Then the group isomorphism $\alpha \rightarrow \tilde{\alpha}$ of $A(G)$ onto the corresponding closed subgroup of $A(\mathfrak{g})$ is also a homeomorphism.

**Proof.** We denote by $e$ the “exponential mapping” of $\mathfrak{g}$ into $G$. We have then, for $\alpha \in A(G)$, $e\alpha = a\alpha$, and $e$ gives an analytic isomorphism between a neighborhood of $0$ in $\mathfrak{g}$ and a neighborhood of the identity in $G$. Let $z_1, \ldots, z_n$ be a linear basis for $\mathfrak{g}$ such that the corresponding solid sphere $S_2$ of radius $2$, in the Euclidean metric defined by our basis, around $0$ in $\mathfrak{g}$, is mapped by $e$ 1-1 and analytically onto the canonical sphere $Z_2 = e(S_2)$ around the identity element in $G$. For any positive real number $p$, $S_p$ will denote the closed solid sphere of radius $p$ around $0$ in $\mathfrak{g}$, and we set $Z_p = e(S_p)$.

Now if $N$ is any neighborhood of the identity in $A(\mathfrak{g})$, there is a real number $s$ such that $0 < s < 1$ and such that every $\tau \in A(\mathfrak{g})$ satisfying $\tau(z_i) - z_i \in B_s \subseteq N$. It follows from the elementary properties of the exponential mapping $e$ that there is a real number $q > 1$ and a real number $r$, $s > r > 0$, such that, for all $a, b \in B_r$, we have $e(a)e(b) = e(a + b + c)$, with $|c| < q \cdot |a| + |b|$, where $|u|$ denotes the distance of $u$ from $0$ in $\mathfrak{g}$. Let $\alpha \in N(Z_r, Z_{r/2^q})$. Then, for $0 \leq t \leq r$, $e(\alpha(tz_i))e(tz_i)^{-1} = e(u_i(t))$, where $u_i(t) \subseteq B_{t^2r^2}$. Hence $e(t\alpha(z_i)) = e(u_i(t))e(tz_i) = e(u_i(t) + tz_i + v_i(t))$, with $|v_i(t)| < q |z_i| t \leq r^2/2$. In particular, this shows that $t\alpha(z_i)$ remains in $B_{r}$ as $t$ varies from 0 to $r$, and that we must have $r(\alpha(z) - z_i) = u_i(r) + v_i(r)$. Therefore, $|\alpha(z_i) - z_i| \leq r/2 + r^2/2 < r < s$, whence $\alpha \in N$. Thus we have shown that $N(Z_r, Z_{r/2^q}) \subseteq N$, and this implies that the mapping $\alpha \rightarrow \tilde{\alpha}$ is continuous.

On the other hand, given any neighborhood $V$ of the identity in $G$, we can find a real number $r > 0$ such that $e(u+z) \subseteq V(z)$ whenever $|z| \leq 1$ and $|u| \leq r$, because $e$ is uniformly continuous on $B_r$. Hence if $\alpha$ is such that $\alpha(z) - z \in B_r$ for all $z \in B_1$, then $\alpha(x)x^{-1} \subseteq V$ for all $x \in Z_1$. Since the $N(Z_1, V)$
constitute a fundamental system of neighborhoods of the identity in $A(G)$, by Lemma 2, and since the $\alpha$ satisfying the above conditions make up a neighborhood of the identity in the image of $A(G)$ in $A(\mathbb{G})$, we conclude that the mapping $\alpha \mapsto \alpha$ is also continuous. This completes the proof of Theorem 1.

**Theorem 2.** Let $G$ be a Lie group, and let $G_0$ denote the component of the identity in $G$. Suppose that the group of components, $G/G_0$, is finitely generated. Then $A(G)$ is a Lie group and has at most countably many components.

**Proof.** Our assumption on $G/G_0$ means that there is a finite set $g_1, \cdots, g_m$ of elements of $G$ such that every component $P$ of $G$ is of the form $P = pG_0$, where $p$ is a product of $g_i$'s (with repetitions allowed). Now let $B$ denote the subgroup of $A(G)$ which consists of all those automorphisms that map each component of $G$ onto itself. It is clear from the form $pG_0$ of a component that we have $N((g_1, \cdots, g_m), G_0) \subseteq B$, whence we see that $B$ is open in $A(G)$. (Note that $G_0$ is open in $G$.) Furthermore, since $A(G)/B$ is isomorphic with a subgroup of the automorphism group $A(G/G_0)$ of the finitely generated group $G/G_0$, it follows that $A(G)/B$ has at most countably many elements. Hence it will suffice to prove that $B$ is a Lie group with at most countably many components.

Let $H$ be the semi-direct product $(G_0^n \times A(G_0))_A$ whose elements are the $m+1$-tuples $(c_1, \cdots, c_m, \alpha)$, with $c_i \in G_0$ and $\alpha \in A(G_0)$, and where products are defined by the formula $(c_1, \cdots, c_m, \alpha) (d_1, \cdots, d_m, \beta) = (c_1\alpha(d_1), \cdots, c_m\alpha(d_m), \alpha\beta)$. If we topologize $H$ by the natural product topology it is evident, since $A(G_0)$ is a Lie group, that $H$ is a Lie group. Furthermore, since $A(G_0)$ may, according to Theorem 1, be identified with a subgroup of the full linear group, its topology satisfies the second axiom of countability, and the same holds therefore for $H$.

Now we define a mapping $\phi$ of $B$ into $H$ by setting $\phi(b) = (g_1^{-1}b(g_1), \cdots, g_m^{-1}b(g_m), \beta)$, where $\beta$ is the restriction of $b$ to $G_0$. It is immediately seen that $\phi$ is a continuous isomorphism of $B$ into $H$. We show next that $\phi^{-1}$ maps $\phi(B)$ continuously onto $B$. For this it suffices to show that, if $V$ is any neighborhood of the identity in $G$ and $C$ any compact subset of $G$, there is a neighborhood $M$ of the identity in $H$ for which $\phi^{-1}(M \cap \phi(B)) \subseteq N(C, V)$. Now, since $C$ is compact, there is a finite set $p_1, \cdots, p_k$ of products of the $g_i$'s such that $C \subseteq \bigcup_{j=1}^k p_jG_0$. Let $S = \bigcup_{j=1}^k (p_j^{-1}C) \cap G_0$, and let $W$ be a neighborhood of the identity in $G_0$. Note that $S$ is compact, so that the corresponding subset $N_0(S, W)$ of $A(G_0)$ is a neighborhood of the identity in $A(G_0)$. Put $U = W \times \cdots \times W \times N_0(S, W)$. Then $U$ is a neighborhood of the identity in $H$, and so is $M = U \cup U^{-1}$. Since every element of $C$ can be written in the form $c = p_is$, with $s \in S$, it is not hard to see that, with a suitable choice of $W$, the neighborhood $M$ will satisfy our above requirement. Hence $\phi$ is a homeomorphism of $B$ onto $\phi(B)$. 

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Now for each component $P$ of $G$ select an explicit product $p$ of $g_i$'s such that $P = pG$. Let $\eta = (c_1, \cdots, c_m, \alpha)$ be an arbitrary element $H$. Let $p'$ denote the element of $G$ which is obtained from $p$ by replacing each $g_i$ by $g_i\alpha_i$. For $x \in G_0$, define $\tilde{\eta}(px) = p'\alpha(x)$. Then $\tilde{\eta}$ is evidently a homeomorphism of $G$ onto itself. It follows that $\eta$ will belong to $\phi(B)$ if and only if $\tilde{\eta}(uv) = \tilde{\eta}(u)\tilde{\eta}(v)$, for all $u, v \in G$. This shows that $\phi(B)$ is closed in $H$. Hence $\phi(B)$ is a Lie group. Since $H$ satisfies the second axiom of countability, so does $\phi(B)$. Since the components of a Lie group are open sets, it follows that the number of components of $B$ is at most countable. Hence also $B$ is a Lie group with at most countably many components, and our proof is complete.

3. The restriction homomorphism. Let $G$ be a Lie group, $G_0$ the component of the identity element in $G$. We assume that $G/G_0$ is finitely generated, or—which is equivalent—that $G$ is generated by a compact subset. The restriction of automorphisms to $G_0$ evidently gives a continuous homomorphism, $\rho$ say, of $A(G)$ onto a subgroup $E(G_0, G)$ of $A(G_0)$. Let $R$ denote the kernel of $\rho$. It is natural to inquire under what conditions $A(G)/R$ is isomorphic, as a topological group, with $E(G_0, G)$, or, equivalently, under what conditions $\rho$ is open. A superficial answer is given by the following theorem:

**Theorem 3.** The restriction homomorphism $\rho$ of $A(G)$ onto $E(G_0, G)$ is open if and only if $E(G_0, G)$ is closed in $A(G_0)$.

**Proof.** Suppose first that $\rho$ is open. Then $E(G_0, G)$ is homeomorphic with $A(G)/R$ and hence is locally compact. It follows from this, as is well known, that $E(G_0, G)$ is closed in $A(G_0)$.(5) Conversely, if $E(G_0, G)$ is closed in $A(G_0)$, then it is locally compact. Furthermore, as a subspace of $A(G_0)$, it satisfies the second axiom of countability. By Theorem 2, $A(G)$ is locally compact and satisfies the second axiom of countability. By a well known result(6), the continuous homomorphism $\rho$ must therefore be open. This completes the proof.

We shall now give an example in which $E(G_0, G)$ is not closed in $A(G_0)$:

Let $C$ denote the additive group of the complex numbers, $R$ the additive group of the real numbers, both with the ordinary topology. We form the semi-direct product $G_0 = (C \times C \times R)_h$ with the multiplication

$$(c_1, c_2, r)(c_1', c_2', r') = (c_1 + e^{2\pi i h}r_1, c_2 + e^{2\pi i h}r_2, r + r'),$$

where $h$ is a fixed irrational real number. Evidently, $G_0$ is a Lie group, and its underlying space is Euclidean 5-space.

Next we construct a semi-direct product $G$ of $G_0$ by a group of order 2, with a generator $g$ such that $g^2$ is the identity element $(0, 0, 0)$ of $G_0$ and $G$,

(5) See §3 of [4].

(6) Theorem 13, chap. III, in [3].
and \( g(c_1, c_2, r) g^{-1} = \gamma(c_1, c_2, r) = (\overline{c_1}, \overline{c_2}, -r) \), where \( \overline{c} \) denotes the complex conjugate of \( c \).

With \( s, t \) arbitrary real numbers, let \( \alpha = \alpha_{s,t} \) be the automorphism of \( G_0 \) which is defined by setting \( \alpha(c_1, c_2, r) = (e^{2\pi i c_1}, e^{2\pi i c_2}, r) \). We shall determine the extensions of \( \alpha \) to \( G \). Let \( \tilde{\alpha} \in A(G) \) be such that \( \rho(\tilde{\alpha}) = \alpha \). We must have \( \tilde{\alpha}(g) = zg \), with \( z \in G_0 \). If we write down the equations which express the fact that \( \tilde{\alpha} \) is a homomorphism, we find that we must have: (1) \( \gamma(z)z = (0, 0, 0) \), and (2) \( z\alpha\gamma(x) = \gamma\alpha(x)z \), for all \( x \in G_0 \).

Conversely, every element \( z \in G_0 \) which satisfies these conditions defines an extension \( \tilde{\alpha} \in A(G) \) of \( \alpha \).

Write \( z = (c_1, c_2, r) \). If we write down conditions (2) with \( x = (0, 0, u) \), we find that we must have \( c_1 = 0 = c_2 \). Then condition (1) holds with arbitrary \( r \in R \). If we rewrite conditions (2) with \( z = (0, 0, r) \) and all \( x \in G_0 \), we find that they are equivalent to the condition that \( r + 2s \) and \( hr + 2t \) be integers. Now, since \( h \) is irrational, we can find a sequence of integers \( k_n \) such that the congruence class mod 1 of \( hk_n/2 \) approaches the congruence class of \( 1/3 \) as \( n \) becomes large. Put \( s_n = 1/2hn \) and \( t_n = 1/2n + hk_n/2 \). Let \( \alpha_n = \alpha_{s_n, t_n} \). If \( r_n = -k_n - 1/2n \), then it satisfies our above conditions on \( r \), whence we conclude that each \( \alpha_n \) belongs to \( E(G_0, G) \). On the other hand, \( \alpha_n \) evidently approaches the automorphism \( \alpha_{0,1/3} \) in \( A(G_0) \), and we claim that \( \alpha_{0,1/3} \in E(G_0, G) \). In fact, the conditions for the number \( r \) needed for extending \( \alpha_{0,1/3} \) become \( r = m \), and \( hr = n - 2/3 \), where \( m \) and \( n \) are integers. But these conditions are incompatible, because \( h \) is irrational. Hence \( E(G_0, G) \) is not closed in \( A(G_0) \).

It will be apparent from the next theorem that in the above example the group \( I(G_0) \) of the inner automorphisms of \( G_0 \) is not closed in \( A(G_0) \); a fact which could also be shown quite directly by considering the above automorphisms \( \alpha_{s,t} \).

**Theorem 4.** If \( G/G_0 \) is finite and \( I(G_0) \) is closed in \( A(G_0) \), then the restriction homomorphism \( \rho \) of \( A(G) \) onto \( E(G_0, G) \) is open.

**Proof.** Let \( B \) denote the subgroup of \( A(G) \) whose elements map each component \( g_i G_0, i = 1, \ldots, m, \) of \( G \) onto itself. We claim that it suffices to show that the restriction of \( \rho \) to \( B \) is an open homomorphism of \( B \) onto \( \rho(B) \). In fact, if this has been proved, we may conclude that \( \rho(B) \) is homeomorphic with \( B/R \cap B \), where \( R \) is the kernel of \( \rho \), and hence that \( \rho(B) \) is locally compact. This implies that \( \rho(B) \) is closed in \( E(G_0, G) \). Since \( A(G)/B \) is finite, so is \( E(G_0, G)/\rho(B) \). Hence the complement of \( \rho(B) \) in \( E(G_0, G) \) is the union of a finite number of cosets of \( \rho(B) \) and hence is closed. Hence \( \rho(B) \) is open in \( E(G_0, G) \). It follows that \( \rho \) is an open homomorphism of \( A(G) \) onto \( E(G_0, G) \), which proves our claim.

Now let us observe that our assumption on \( I(G_0) \) implies that the natural homomorphism of \( G_0 \) onto \( I(G_0) \) is open, as well as continuous. Indeed, the continuity is independent of our assumption; for, given a compact subset...
C of $G_0$ and a neighborhood $V$ of the identity in $G_0$, it is clear that for each $c \in C$ we can find a neighborhood $V_c$ of the identity such that $uxu^{-1}x^{-1} \in V$ for all $u \in V_x$ and $x \in cV_x$. Since $C$ is compact, there is a finite subset $c_1, \ldots, c_q$ of $C$ such that $C \subseteq \bigcup_{i=1}^q c_i V_{c_i}$. Then, if $W = \bigcap_{i=1}^q V_{c_i}$, we have $uxu^{-1}x^{-1} \in V$ for all $u \in W$ and $x \in C$, which proves that the natural homomorphism of $G_0$ onto $I(G_0)$ is always continuous. Now if $I(G_0)$ is closed in $A(G_0)$, then it is a connected Lie group. Since $G_0$ is also a connected Lie group, the continuous natural homomorphism of $G_0$ onto $I(G_0)$ must automatically be open.

It will be convenient to identify $B$ with the closed subgroup $\phi(B)$ of the group $H$ which we introduced in the proof of Theorem 2. It is easy to check that $\phi(R \cap B)$ is precisely the set of elements $(z_1, \ldots, z_m, 1) \in H$, where 1 stands for the identity automorphism of $G_0$, and where the elements $z_i$ belong to the center, $Z_0$, say, of $G_0$ and satisfy the relations $g_i^{-1}z_ig_ig_j^{-1}z_j = z_{k(i,j)}$, the index $k(i,j)$ being determined by the relation $g_ig_j^{-1}g_{i(j)}g_{i(j)}^{-1}G_{i(j)}G_{i(j)}^{-1}$. The index $k(i,j)$ being determined by the relation $g_ig_j^{-1}g_{i(j)}g_{i(j)}^{-1}G_{i(j)}G_{i(j)}^{-1}$.

What we still have to prove therefore amounts to the following: Given a neighborhood $U$ of the identity in $G_0$, there is a neighborhood $M$ of the identity in $A(G_0)$, such that, for every element $(c_1, \ldots, c_m, \alpha) \in \phi(B)$ with $\alpha \in M$, we can find elements $z_i$ as above and such that $z_ic_i \in U$.

Since $Z_0$ is a Lie group, there is a neighborhood $D$ of the identity in $G_0$ which has the following properties:

1. For every $z \in D \cap Z_0$, there is an element $t \in D \cap Z_0$ such that $t^m = z$.
2. If $s$, $t$ belong to $(g_j^{-1}Dg_jDD^{-1}) \cap Z_0$ and $s^m = t^m$, then $s = t$.
3. $DD \subseteq U$.

Let us choose a neighborhood $S$ of the identity in $G_0$ such that $S^{2m} \subseteq D$. Let $\gamma_i$ denote the inner automorphism $u \rightarrow g_i^{-1}ug_i$, and choose a neighborhood $V$ of the identity in $G_0$ such that $V = V^{-1}$ and $V\gamma_i\gamma_j^{-1}(VV) \subseteq S$, for all $i, j, k$.

Now observe that if $\phi(b) = (c_1, \ldots, c_m, \alpha)$, then $\gamma_i\alpha\gamma_i^{-1}\alpha^{-1}(x) = c_\alpha x\alpha^{-1}$, for every $x \in G_0$. Since the homomorphism of $G_0$ onto $I(G_0)$ is open, we can find a neighborhood $M'$ of the identity in $A(G_0)$ such that every automorphism belonging to $I(G_0) \cap M'$ is effected by an element belonging to $V$. Choose a neighborhood $M$ of the identity in $A(G_0)$ such that, for every $\alpha \in M$, we have $\gamma_i\alpha\gamma_i^{-1}\alpha^{-1} \in M'$, for each $i$, and also $\alpha(a(i, j))a(i, j)^{-1} \in V$, for each pair $(i, j)$, where $a(i, j) = g_i^{-1}g_{i(j)}^{-1}g_{i(j)}g_i$.

Now let $\alpha \in M$ and $(c_1, \ldots, c_m, \alpha) = \phi(b)$. Then the inner automorphism of $G_0$ which is effected by $c_i$ is also effected by an element $v_i \in V$, whence $c_i = v_i c_i$, with $t_i \in Z_0$. We have $\gamma_j(c_j) = g_j^{-1}g_j^{-1}b(g_jg_i) = g_j^{-1}g_j^{-1}g_{i(j)}g_{i(j)}^{-1}a(a(i, j))$

(?) For the argument which now follows I am indebted to T. Nakayama. The idea of this proof is that if $\alpha$ is "small" enough the $c_i$ can be replaced by "small" elements. This possibility is due to the fact that a factor set, defined on $G/G_0$, and with sufficiently small values in $Z_0$, must be a transformation set, because unique divisibility holds near the identity in $Z_0$. Such a device has been used by Iwasawa on p. 510 of [2] in dealing with the automorphisms of a compact group.
\[ a(i, j)^{-1} c_{k(i, j)} a(i, j), \text{ whence } \gamma_j(t_i) t_i^{-1} = a(i, j)^{-1} v_{k(i, j)} a(i, j) v_j^{-1} \gamma_j(v_i^{-1}) \]
\[ = \gamma_j(t_i) v_j^{-1} a(i, j) a(i, j) v_j^{-1} \gamma_j(v_i^{-1}) \in S^2. \]

Hence \( \prod_{i=1}^m \gamma_j(t_i) t_i^{-1} \in S^m \subseteq D, \) i.e., \( \gamma_j(t)^{-1} \in D \cap Z_0, \) where \( t = t_1 \cdots t_m. \) By property (1) of \( D, \) there are elements \( u_j \in D \cap Z_0 \) such that \( u_j^m = \gamma_j(t)^{-1} \), and then we have \( (\gamma_j(u_i) u_j u_i^{-1})^m = (\gamma_j(t_i) t_i^{-1})^m. \) By property (2) of \( D, \) it follows that \( \gamma_j(u_i) u_j u_i^{-1} = \gamma_j(t_i) t_i^{-1}. \) Set \( z_i = u_i a_i^{-1}. \) Then \( g_j^{-1} z_i g_j z_i = z_{k(i, j)}, \) and \( z_i c_i = u_i a_i \in DD \subseteq U. \) This is what we had to prove in order to establish Theorem 4.

**Corollary.** If \( G_0/Z_0 \) is compact, or if \( G_0 \) is semisimple, then the restriction homomorphism of \( A(G) \) onto \( E(G_0, G) \) is open.

**Proof.** If \( G_0/Z_0 \) is compact, so is \( I(G_0), \) and hence \( I(G_0) \) is closed in \( A(G_0). \) If \( G_0 \) is semisimple, then \( I(G_0) \) coincides with the component of the identity in \( A(G_0), \) and hence is closed in \( A(G_0). \)

**Bibliography**