THE AUTOMORPHISM GROUP OF A LIE GROUP

BY

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Introduction. The group $A(G)$ of all continuous and open automorphisms of a locally compact topological group $G$ may be regarded as a topological group, the topology being defined in the usual fashion from the compact and the open subsets of $G$ (see §1). In general, this topological structure of $A(G)$ is somewhat pathological. For instance, if $G$ is the discretely topologized additive group of an infinite-dimensional vector space over an arbitrary field, then $A(G)$ already fails to be locally compact.

On the other hand, if $G$ is a connected Lie group, we shall show without any difficulty that the compact-open topology of $A(G)$ coincides with the topology obtained by identifying $A(G)$ with a closed subgroup of the linear group of automorphisms of the Lie algebra of $G$, as was done by Chevalley (in [1]) in order to make $A(G)$ into a Lie group. We shall then deduce that $A(G)$ is a Lie group whenever the group of its components, $G/G_0$, is finitely generated\(^1\), where $G_0$ denotes the component of the identity element in $G$.

The other questions with which we shall be concerned are the following: Let $I(G_0)$ denote the group of the inner automorphisms of $G_0$, and let $E(G_0, G)$ denote the natural image in $A(G_0)$ of $A(G)$. Regard $I(G_0)$ and $E(G_0, G)$ as subgroups of $A(G_0)$. Are these subgroups closed in $A(G_0)$? Is $E(G_0, G)$ topologically, as well as group-theoretically, isomorphic with the corresponding factor group of $A(G)$?

We shall show, under the assumption that $G/G_0$ is finitely generated, that these questions are related as follows: The natural continuous homomorphism of $A(G)$ onto $E(G_0, G)$ is open if and only if $E(G_0, G)$ is closed in $A(G_0)$. Under the stronger assumption that $G/G_0$ is finite, a sufficient condition for $E(G_0, G)$ to be closed in $A(G_0)$ is that $I(G_0)$ be closed in $A(G_0)$. Finally, in order to throw some light on the difficulties which are involved here, we shall give a simple example in which $I(G_0)$ and $E(G_0, G)$ are not closed in $A(G_0)$. In this example, $G$ has only two components and $G_0$ is homeomorphic with Euclidean 5-space.

1. Topological preparation. We shall describe the topology of a group $G$ in terms of a fundamental system $\mathfrak{B}$ of neighborhoods $V$ of the identity element. A system $\mathfrak{B}$ of subsets of $G$ will define a Hausdorff topology consistent with the group operations if and only if it satisfies the following conditions\(^2\):

\(^{1}\) I am indebted to the referee for the remark that my original requirement, "$G/G_0$ finite", can be relaxed to the present one.

\(^{2}\) We are taking these from §2 of [4].
I. The intersection of all $V \in \mathcal{B}$ is the set consisting of the identity element of $G$ only.

II. If $V_1$ and $V_2$ are sets belonging to $\mathcal{B}$, there is a $V \in \mathcal{B}$ such that $V \subseteq V_1 \cap V_2$.

III. For every $V \in \mathcal{B}$ there is a $W \in \mathcal{B}$ such that $W^{-1}W \subseteq V$.

IV. For every $g \in G$ and $V \in \mathcal{B}$ there is a $W \in \mathcal{B}$ such that $W \subseteq gVg^{-1}$.

The neighborhoods of the identity element are then all the sets containing a set belonging to $\mathcal{B}$.

If $C$ is any compact subset of $G$ and $V \in \mathcal{B}$, we denote by $N(C, V)$ the set of all $\alpha \in \mathcal{A}(G)$ for which $\alpha(x)x^{-1} \in V$ and $\alpha^{-1}(x)x^{-1} \in V$, whenever $x \in C$.

We claim that if $G$ is locally compact, the system of these $N(C, V)$ satisfies conditions I–IV above. In fact, I holds quite evidently. If $V \subseteq V_1 \cap V_2$, we clearly have $N(C_1 \cup C_2, V) \subseteq N(C_1, V_1) \cap N(C_2, V_2)$, so that II is satisfied.

In order to verify III we proceed as follows: Since $G$ is locally compact, given $V \in \mathcal{B}$, there is a compact set $C_0$ and a $V_0 \in \mathcal{B}$ such that $V_0 \subseteq C_0$ and $V_0 \subseteq V$. From the identity $(\beta^{-1}\alpha)(c)c^{-1} = [\beta^{-1}(\alpha(c)c^{-1})(\alpha(c)c^{-1})^{-1}] \cdot [\alpha(c)c^{-1}]$ we can see immediately that we have then $N(C \cup C_0, V_0)^{-1} N(C \cup C_0, V_0) \subseteq N(C, V)$, which shows that III is satisfied. Finally, we have, with $\alpha \in \mathcal{A}(G)$, $\alpha^{-1}N(\alpha(C), \alpha(V))\alpha \subseteq N(C, V)$, whence IV holds.

From now on, if $G$ is any locally compact group, $\mathcal{A}(G)$ will denote the group of all continuous and open automorphisms of $G$, with the topology defined by the $N(C, V)$, where $C$ ranges over the compact subsets of $G$ and $V$ over the set of neighborhoods of the identity element in $G$.

Next we shall prove two elementary results which we shall need later on.

**Lemma 1.** Let $G$ be a topological group, $U$ a neighborhood of the identity element, $C$ a compact subset of $G$. Then the intersection of all the sets $c^{-1}Uc$, with $c \in C$, is a neighborhood of the identity element.

**Proof.** We can find $V \in \mathcal{B}$ such that $VVV^{-1} \subseteq U$. Then we have $y^{-1}Uy \supseteq V$, for every $y \in V$. Since $C$ is compact, there are elements $c_1, \ldots, c_n$ in $C$ such that $C$ is contained in the union of the $n$ sets $Vc_i$. If $c$ is any element in $C$, we write $c = yc_i$, with $y \in V$. Then $c^{-1}Uc = c_i^{-1}y^{-1}Uyc_i \supseteq c_i^{-1}Vc_i$. Hence the intersection of all the sets $c^{-1}Uc$ contains the finite intersection of the $c_i^{-1}Vc_i$ and is therefore a neighborhood of the identity element.

**Lemma 2.** Suppose that $G$ is connected and locally compact. Let $C$ be a compact subset of $G$, $V$ a neighborhood of the identity, and $S$ a compact neighborhood of the identity. Then there exists a neighborhood $W$ of the identity such that $N(S, W) \subseteq N(C, V)$.

**Proof.** Since $G$ is connected and $S$ a neighborhood of the identity, we have $G = \bigcup_{n=1}^{\infty} S^n$. Since $C$ is compact and since $S^n \subseteq S^{n+1}$, it follows that $C \subseteq S^m$, for some $m$. Now choose a neighborhood $T$ of the identity such that $T^n \subseteq V$,
and put $W = \cap_{x \in S} x^{-1}Tx$. By Lemma 1, $W$ is a neighborhood of the identity, for $S^n$ is compact (3). Now let $a \in N(S, W)$ and $c \in C$. We have $c = x_1 \cdots x_n$, with $x_i \in S$. Put $c_k = x_1 \cdots x_k$, and suppose we have already shown that $\alpha(c_k)c_k^{-1} \in T^k$. Then we have $\alpha(c_{k+1})c_{k+1}^{-1} = (\alpha(c_k)c_k^{-1})c_k(\alpha(x_{k+1})x_{k+1}^{-1})c_k^{-1} \in T^k c_k W c_k^{-1} \subseteq T^{k+1}$. Hence we get $\alpha(c)c^{-1} \subseteq T^m \subseteq V$, which clearly suffices to establish our lemma.

2. Automorphism groups. Let $G$ be a connected Lie group. An automorphism $\alpha \in A(G)$ induces an automorphism $\hat{\alpha}$ of the Lie algebra $\mathfrak{g}$ of $G$. We denote by $A(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, with the topology induced by that of the full linear group of which $A(\mathfrak{g})$ is clearly a closed subgroup. It is shown in [1] that the mapping $\alpha \rightarrow \hat{\alpha}$ is a group isomorphism of $A(G)$ onto a closed subgroup of $A(\mathfrak{g})$. (If $G$ is simply-connected the image of $A(G)$ coincides with $A(\mathfrak{g})$.) We shall prove the following result:

Theorem 1. Let $G$ be a connected Lie group. Then the group isomorphism $\alpha \rightarrow \hat{\alpha}$ of $A(G)$ onto the corresponding closed subgroup of $A(\mathfrak{g})$ is also a homeomorphism.

Proof. We denote by $e$ the “exponential mapping” of $\mathfrak{g}$ into $G$ (4). We have then, for $\alpha \in A(G)$, $e\alpha = \alpha e$, and $e$ gives an analytic isomorphism between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of the identity in $G$. Let $z_1, \cdots, z_n$ be a linear basis for $\mathfrak{g}$ such that the corresponding solid sphere $S_2$ of radius 2, in the Euclidean metric defined by our basis, around 0 in $\mathfrak{g}$, is mapped by $e$-1 and analytically onto the canonical sphere $Z_2 = e(S_2)$ around the identity element in $G$. For any positive real number $p$, $S_p$ will denote the closed solid sphere of radius $p$ around 0 in $\mathfrak{g}$, and we set $Z_p = e(S_p)$.

Now if $N$ is any neighborhood of the identity in $A(\mathfrak{g})$, there is a real number $s$ such that $0 < s < 1$ and such that every $\tau \in A(\mathfrak{g})$ satisfying $\tau(z_i) - z_i \in S_2$ belongs to $N$. It follows from the elementary properties of the exponential mapping $e$ that there is a real number $q > 1$ and a real number $r$, $s > r > 0$, such that, for all $a, b \in S_r$, we have $e(a)e(b) = e(a + b + c)$, with $|c| < q |a| |b|$, where $|u|$ denotes the distance of $u$ from 0 in $\mathfrak{g}$. Let $\alpha \in N(Z_r, Z_r^2)$. Then, for $0 \leq t \leq r$, $e(t\alpha(z_i))e(tz_i)^{-1} = e(u_i(t))$, where $u_i(t) \in S_r^2 \subseteq Z_r^2$. Hence $e(t\alpha(z_i)) = e(u_i(t))e(tz_i) = e(u_i(t) + tz_i + v_i(t))$, with $|v_i(t)| < q |u_i(t)| t \leq r^2/2$. In particular, this shows that $t\alpha(z_i)$ remains in $S_r$ as $t$ varies from 0 to $r$, and that we must have $r(\alpha(z_i) - z_i) = u_i(r) + v_i(r)$. Therefore, $|\alpha(z_i) - z_i| \leq r/2q + r^2/2 < r < s$, whence $\alpha \in N$. Thus we have shown that $N(Z_r, Z_r^2) \subseteq N$, and this implies that the mapping $\alpha \rightarrow \hat{\alpha}$ is continuous.

On the other hand, given any neighborhood $V$ of the identity in $G$, we can find a real number $r > 0$ such that $e(u+z) \in Ve(z)$ whenever $|z| \leq 1$ and $|u| \leq r$, because $e$ is uniformly continuous on $S_r$. Hence if $\hat{\alpha}$ is such that $\alpha(z) - z \in S_r$, for all $z \in Z_1$, then $\alpha(x)x^{-1} \in V$ for all $x \in Z_1$. Since the $N(Z_1, V)$

(3) If $A$ and $B$ are compact, so is $AB$; see §3 of [4].
(4) See §VIII, Chap. IV, of [1].
constitute a fundamental system of neighborhoods of the identity in $A(G)$, by Lemma 2, and since the $\alpha$ satisfying the above conditions make up a neighborhood of the identity in the image of $A(G)$ in $A(\mathfrak{g})$, we conclude that the mapping $\alpha \mapsto \alpha$ is also continuous. This completes the proof of Theorem 1.

**Theorem 2.** Let $G$ be a Lie group, and let $G_0$ denote the component of the identity in $G$. Suppose that the group of components, $G/G_0$, is finitely generated. Then $A(G)$ is a Lie group and has at most countably many components.

**Proof.** Our assumption on $G/G_0$ means that there is a finite set $g_1, \ldots, g_m$ of elements of $G$ such that every component $P$ of $G$ is of the form $P = pG_0$, where $p$ is a product of $g_i$'s (with repetitions allowed). Now let $B$ denote the subgroup of $A(G)$ which consists of all those automorphisms that map each component of $G$ onto itself. It is clear from the form $pG_0$ of a component that we have $N((g_1, \ldots, g_m), G_0) \subseteq B$, whence we see that $B$ is open in $A(G)$. (Note that $G_0$ is open in $G$.) Furthermore, since $A(G)/B$ is isomorphic with a subgroup of the automorphism group $A(G/G_0)$ of the finitely generated group $G/G_0$, it follows that $A(G)/B$ has at most countably many elements. Hence it will suffice to prove that $B$ is a Lie group with at most countably many components.

Let $H$ be the semi-direct product $(G_0^n \times A(G_0))_{\alpha}$ whose elements are the $m+1$-tuples $(c_1, \ldots, c_m, \alpha)$, with $c_i \in G_0$ and $\alpha \in A(G_0)$, and where products are defined by the formula $(c_1, \ldots, c_m, \alpha)(d_1, \ldots, d_m, \beta) = (c_1 \alpha(d_1), \ldots, c_m \alpha(d_m), \alpha \beta)$. If we topologize $H$ by the natural product topology it is evident, since $A(G_0)$ is a Lie group, that $H$ is a Lie group. Furthermore, since $A(G_0)$ may, according to Theorem 1, be identified with a subgroup of the full linear group, its topology satisfies the second axiom of countability, and the same holds therefore for $H$.

Now we define a mapping $\phi$ of $B$ into $H$ by setting $\phi(b) = (g_1^{-1}b(g_1), \ldots, g_m^{-1}b(g_m), \beta)$, where $\beta$ is the restriction of $b$ to $G_0$. It is immediately seen that $\phi$ is a continuous isomorphism of $B$ into $H$. We show next that $\phi^{-1}$ maps $\phi(B)$ continuously onto $B$. For this it suffices to show that, if $V$ is any neighborhood of the identity in $G$ and $C$ any compact subset of $G$, there is a neighborhood $M$ of the identity in $H$ for which $\phi^{-1}(M \cap \phi(B)) \subseteq N(C, V)$. Now, since $C$ is compact, there is a finite set $p_1, \ldots, p_k$ of products of the $g_i$'s such that $C \subseteq \bigcup_{j=1}^{k} p_j G_0$. Let $S = \bigcup_{j=1}^{k} (p_j^{-1}C) \cap G_0$, and let $W$ be a neighborhood of the identity in $G_0$. Note that $S$ is compact, so that the corresponding subset $N_0(S, W)$ of $A(G_0)$ is a neighborhood of the identity in $A(G_0)$. Put $U = W \times \cdots \times W \times N_0(S, W)$. Then $U$ is a neighborhood of the identity in $H$, and so is $M = U \cap U^{-1}$. Since every element of $C$ can be written in the form $c = p_s s$, with $s \in S$, it is not hard to see that, with a suitable choice of $W$, the neighborhood $M$ will satisfy our above requirement. Hence $\phi$ is a homeomorphism of $B$ onto $\phi(B)$. 

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Now for each component $P$ of $G$ select an explicit product $p$ of $g_i$'s such that $P = \rho P$. Let $\eta = (c_1, \ldots, c_m, \alpha)$ be an arbitrary element $H$. Let $\rho'$ denote the element of $G$ which is obtained from $p$ by replacing each $g_i$ by $g_i'c_i$. For $x \in G_0$, define $\tilde{\eta}(px) = \rho'\alpha(x)$. Then $\tilde{\eta}$ is evidently a homeomorphism of $G$ onto itself. It follows that $\eta$ will belong to $\phi(B)$ if and only if $\tilde{\eta}(uv) = \tilde{\eta}(u)\tilde{\eta}(v)$, for all $u, v \in G$. This shows that $\phi(B)$ is closed in $H$. Hence $\phi(B)$ is a Lie group. Since $H$ satisfies the second axiom of countability, so does $\phi(B)$. Since the components of a Lie group are open sets, it follows that the number of components of $B$ is at most countable. Hence also $B$ is a Lie group with at most countably many components, and our proof is complete.

3. The restriction homomorphism. Let $G$ be a Lie group, $G_0$ the component of the identity element in $G$. We assume that $G/G_0$ is finitely generated, or—which is equivalent—that $G$ is generated by a compact subset. The restriction of automorphisms to $G_0$ evidently gives a continuous homomorphism, $\rho$ say, of $\mathcal{A}(G)$ onto a subgroup $\mathcal{E}(G_0, G)$ of $\mathcal{A}(G_0)$. Let $R$ denote the kernel of $\rho$. It is natural to inquire under what conditions $\mathcal{A}(G)/R$ is isomorphic, as a topological group, with $\mathcal{E}(G_0, G)$, or, equivalently, under what conditions $\rho$ is open. A superficial answer is given by the following theorem:

**Theorem 3.** The restriction homomorphism $\rho$ of $\mathcal{A}(G)$ onto $\mathcal{E}(G_0, G)$ is open if and only if $\mathcal{E}(G_0, G)$ is closed in $\mathcal{A}(G_0)$.

**Proof.** Suppose first that $\rho$ is open. Then $\mathcal{E}(G_0, G)$ is homeomorphic with $\mathcal{A}(G)/R$ and hence is locally compact. It follows from this, as is well known, that $E(G_0, G)$ is closed in $A(G_0)$. Conversely, if $E(G_0, G)$ is closed in $A(G_0)$, then it is locally compact. Furthermore, as a subspace of $A(G_0)$, it satisfies the second axiom of countability. By Theorem 2, $A(G)$ is locally compact and satisfies the second axiom of countability. By a well known result, the continuous homomorphism $\rho$ must therefore be open. This completes the proof.

We shall now give an example in which $E(G_0, G)$ is not closed in $A(G_0)$: Let $C$ denote the additive group of the complex numbers, $R$ the additive group of the real numbers, both with the ordinary topology. We form the semi-direct product $G_0 = (C \times C \times R)_h$ with the multiplication

$$(c_1, c_2, r)(c'_1, c'_2, r') = (c_1 + e^{2\pi ihr}c'_1, c_2 + e^{2\pi ihc_2}, r + r'),$$

where $h$ is a fixed irrational real number. Evidently, $G_0$ is a Lie group, and its underlying space is Euclidean 5-space.

Next we construct a semi-direct product $G$ of $G_0$ by a group of order 2, with a generator $g$ such that $g^2$ is the identity element $(0, 0, 0)$ of $G_0$ and $G$.

(\text{\textsuperscript{1}}) See §3 of [4].

(\text{\textsuperscript{2}}) Theorem 13, chap. III, in [3].
and $g(c_1, c_2, r)g^{-1} = \gamma(c_1, c_2, r) = (\overline{c}_1, \overline{c}_2, -r)$, where $\overline{c}$ denotes the complex conjugate of $c$.

With $s$, $t$ arbitrary real numbers, let $\alpha = \alpha_{s,t}$ be the automorphism of $G_0$ which is defined by setting $\alpha(c_1, c_2, r) = (e^{2\pi i s}c_1, e^{2\pi i t}c_2, r)$. We shall determine the extensions of $\alpha$ to $G$. Let $\tilde{\alpha} \in A(G)$ be such that $\rho(\tilde{\alpha}) = \alpha$. We must have $\tilde{\alpha}(\gamma) = g\gamma$, with $\gamma \in G_0$. If we write down the equations which express the fact that $\tilde{\alpha}$ is a homomorphism, we find that we must have: (1) $\gamma(z)z = (0, 0, 0)$, and (2) $\gamma \alpha(x) = \gamma \alpha(x)\gamma$, for all $x \in G_0$.

Conversely, every element $\gamma \in G_0$ which satisfies these conditions defines an extension $\tilde{\alpha} \in A(G)$ of $\alpha$.

Write $\gamma = (c_1, c_2, r)$. If we write down conditions (2) with $x = (0, 0, u)$, we find that we must have $c_1 = 0 = c_2$. Then condition (1) holds with arbitrary $r \in \mathbb{R}$. If we rewrite conditions (2) with $\gamma = (0, 0, r)$ and all $x \in G_0$, we find that they are equivalent to the condition that $r + 2s$ and $hr + 2t$ be integers. Now, since $h$ is irrational, we can find a sequence of integers $k_n$ such that the congruence class mod 1 of $hk_n/2$ approaches the congruence class of $1/3$ as $n$ becomes large. Put $s_n = 1/2hn$ and $r_n = 1/2n + hk_n/2$. Let $\alpha_n = \alpha_{s_n, r_n}$. If $r_n = -k_n - 1/hn$, then it satisfies our above conditions on $r$, whence we conclude that each $\alpha_n$ belongs to $E(G_0, G)$. On the other hand, $\alpha_n$ evidently approaches the automorphism $\alpha_{0, 1/3}$ in $A(G_0)$, and we claim that $\alpha_{0, 1/3} \in E(G_0, G)$. In fact, the conditions for the number $r$ needed for extending $\alpha_{0, 1/3}$ become $r = m$, and $hr = n - 2/3$, where $m$ and $n$ are integers. But these conditions are incompatible, because $h$ is irrational. Hence $E(G_0, G)$ is not closed in $A(G_0)$.

It will be apparent from the next theorem that in the above example the group $I(G_0)$ of the inner automorphisms of $G_0$ is not closed in $A(G_0)$; a fact which could also be shown quite directly by considering the above automorphisms $\alpha_{s,t}$.

**Theorem 4.** If $G/G_0$ is finite and $I(G_0)$ is closed in $A(G_0)$, then the restriction homomorphism $\rho$ of $A(G)$ onto $E(G_0, G)$ is open.

**Proof.** Let $B$ denote the subgroup of $A(G)$ whose elements map each component $g_i G_0$, $i = 1, \ldots, m$, of $G$ onto itself. We claim that it suffices to show that the restriction of $\rho$ to $B$ is an open homomorphism of $B$ onto $\rho(B)$. In fact, if this has been proved, we may conclude that $\rho(B)$ is homeomorphic with $B/R \cap B$, where $R$ is the kernel of $\rho$, and hence that $\rho(B)$ is locally compact. This implies that $\rho(B)$ is closed in $E(G_0, G)$. Since $A(G)/B$ is finite, so is $E(G_0, G)/\rho(B)$. Hence the complement of $\rho(B)$ in $E(G_0, G)$ is the union of a finite number of cosets of $\rho(B)$ and hence is closed. Hence $\rho(B)$ is open in $E(G_0, G)$. It follows that $\rho$ is an open homomorphism of $A(G)$ onto $E(G_0, G)$, which proves our claim.

Now let us observe that our assumption on $I(G_0)$ implies that the natural homomorphism of $G_0$ onto $I(G_0)$ is open, as well as continuous. Indeed, the continuity is independent of our assumption; for, given a compact subset
C of $G_0$ and a neighborhood $V$ of the identity in $G_0$, it is clear that for each $c \in C$ we can find a neighborhood $V_c$ of the identity such that $uxu^{-1}x^{-1} \in V$ for all $u \in V_c$ and $x \in cV_c$. Since $C$ is compact, there is a finite subset $c_1, \ldots, c_q$ of $C$ such that $C \subseteq \bigcup_{i=1}^{q} c_i V_c$. Then, if $W = \cap_{i=1}^{q} V_{c_i}$, we have $uxu^{-1}x^{-1} \in V$ for all $u \in W$ and $x \in C$, which proves that the natural homomorphism of $G_0$ onto $I(G_0)$ is always continuous. Now if $I(G_0)$ is closed in $A(G_0)$, then it is a connected Lie group. Since $G_0$ is also a connected Lie group, the continuous natural homomorphism of $G_0$ onto $I(G_0)$ must automatically be open.

It will be convenient to identify $B$ with the closed subgroup $\phi(B)$ of the group $H$ which we introduced in the proof of Theorem 2. It is easy to check that $\phi(R \cap B)$ is precisely the set of elements $(z_1, \ldots, z_m, 1) \in H$, where 1 stands for the identity automorphism of $G_0$, and where the elements $z_i$ belong to the center, $Z_0$, say, of $G_0$ and satisfy the relations $g_j^{-1}z_ikg_jz_j = z_{k(i, j)}$, the index $k(i, j)$ being determined by the relation $g_jg_i = g_{k(i, j)}g_0$.

What we still have to prove therefore amounts to the following: Given a neighborhood $U$ of the identity in $G_0$, there is a neighborhood $M$ of the identity in $I(G_0)$, such that, for every element $(c_1, \ldots, c_m, \alpha) \in \phi(B)$ with $\alpha \in M$, we can find elements $z_i$ as above and such that $z_i c_i \subseteq U$. Since $Z_0$ is a Lie group, there is a neighborhood $D$ of the identity in $G_0$ which has the following properties:

1. For every $z \in D \cap Z_0$, there is an element $t \in D \cap Z_0$ such that $t^m = z$.
2. If $s$, $t$ belong to $(g_j^{-1}Dg_jDD^{-1}) \cap Z_0$ and $s^m = t^m$, then $s = t$.
3. $DD \subseteq U$.

Let us choose a neighborhood $S$ of the identity in $G_0$ such that $S^2 \subseteq D$. Let $\gamma_i$ denote the inner automorphism $u \rightarrow g_i^{-1}ug_i$, and choose a neighborhood $V$ of the identity in $G_0$ such that $V = V^{-1}$ and $V \gamma_i \gamma_i^{-1}(VV) \subseteq S$, for all $i$. Now observe that if $\phi(b) = (c_1, \ldots, c_m, \alpha)$, then $\gamma_i \alpha \gamma_i^{-1} \alpha^{-1}(x) = c_i \alpha x c_i^{-1}$, for every $x \in G_0$. Since the homomorphism of $G_0$ onto $I(G_0)$ is open, we can find a neighborhood $M'$ of the identity in $A(G_0)$ such that every automorphism belonging to $I(G_0) \cap M'$ is effected by an element belonging to $V$. Choose a neighborhood $M$ of the identity in $A(G_0)$ such that, for every $\alpha \in M$, we have $\gamma_i \alpha \gamma_i^{-1} \alpha^{-1} \subseteq M'$, for each $i$, and also $\alpha(a(i, j))a(i, j)^{-1} \subseteq V$, for each pair $(i, j)$, where $a(i, j) = g_i^{-1}g_jg_i$. Now let $\alpha \in M$ and $(c_1, \ldots, c_m, \alpha) = \phi(b)$. Then the inner automorphism of $G_0$ which is effected by $c_i$ is also effected by an element $v_i \in V$, whence $c_i = v_i g_i$, with $v_i \in Z_0$. We have $\gamma_i(c_i) = g_i^{-1}b(g_i g_j) = g_i^{-1}g_i^{-1}g_k(i,j)c_k(i,j)\alpha(a(i, j))$

Note that the conclusion which now follows I am indebted to T. Nakayama. The idea of this proof is that if $\alpha$ is "small" enough the $c_i$ can be replaced by "small" elements. This possibility is due to the fact that a factor set, defined on $G/G_0$, and with sufficiently small values in $Z_0$, must be a transformation set, because unique divisibility holds near the identity in $Z_0$. Such a device has been used by Iwasawa on p. 510 of [2] in dealing with the automorphisms of a compact group.
\[ a(i,j)^{-1}c_{k(i,j)}(a(i,j)) \text{, whence } \gamma_j(t_i)t_j^{-1} = a(i,j)^{-1}v_{k(i,j)}a(a(i,j))v_j^{-1}\gamma_j(v_i^{-1}) \]

\[ = \gamma_j \gamma_j^{-1}(t_i) (v_{k(i,j)}a(a(i,j))a(i,j)^{-1})v_j^{-1}\gamma_j(v_i^{-1}) \in S^2. \]

Hence \( \prod_{i=1}^{m} \gamma_j(t_i)t_j^{-1}(t(i,i)) \in S^m \subseteq D \), i.e., \( \gamma_j(t)^{-1}t_j^m \in D \cap Z_0 \), where \( t = t_1 \cdots t_m \). By property (1) of \( D \), there are elements \( u_j \in D \cap Z_0 \) such that \( u_j^m = \gamma_j(t)^{-1}t_j^m \), and then we have

\[ (\gamma_j(u_i)u_ju_i^{-1})^m = (\gamma_j(t_i)t_j^{-1})^m. \]

By property (2) of \( D \), it follows that \( \gamma_j(u_i)u_ju_i^{-1}(t(i,j)) = \gamma_j(t_i)t_j^{-1}(t(i,j)). \) Set \( z_i = u_i \mu_i^{-1}. \) Then \( g_j^{-1}z_ig_jz_j = z_{k(i,j)}, \) and \( z_i\mu_i = u_i \mu_i \in DD \subseteq U. \) This is what we had to prove in order to establish Theorem 4.

**Corollary.** If \( G_0/Z_0 \) is compact, or if \( G_0 \) is semisimple, then the restriction homomorphism of \( A(G) \) onto \( E(G_0, G) \) is open.

**Proof.** If \( G_0/Z_0 \) is compact, so is \( I(G_0) \), and hence \( I(G_0) \) is closed in \( A(G_0) \). If \( G_0 \) is semisimple, then \( I(G_0) \) coincides with the component of the identity in \( A(G_0) \), and hence is closed in \( A(G_0) \).

**Bibliography**


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