BANACH SPACES WITH THE EXTENSION PROPERTY

BY

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It is the object of this note to complete a characterization of those Banach spaces $B$ with the Hahn-Banach extension property: each bounded linear function $F$ on a subspace of any Banach space $C$ with values in $B$ has a linear extension $F'$ carrying all of $C$ into $B$ such that $\|F'\| = \|F\|$. It is shown here that:

**Theorem.** Each such space $B$ is equivalent to the space $C_X$ of continuous real-valued functions on an extremally disconnected compact Hausdorff space $X$, $C_X$ having the usual supremum norm.

Recently, in these Transactions, Nachbin [N] and, independently, Goodner [G] have shown that if $B$ has the extension property and if its unit sphere has an extreme point, then $B$ is equivalent to a function space of this sort; both authors have also proved that such a function space has the extension property. The above theorem simply omits the extreme point hypothesis, and so establishes the equivalence.

My original proof, of which the proof given here is a distillate, depends on an idea of Jerison [J]. Briefly, letting $X$ be the weak* closure of the set of extreme points of the unit sphere of the adjoint $B^*$, $B$ can be shown equivalent to the space of all weak* continuous real functions $f$ on $X$ such that $f(x) = -f(-x)$, and then properties of $X$ are deduced which imply the theorem. The same idea occurs implicitly in the proof below.

Note. Goodner asks [G, p. 107] if every Banach space having the extension property is equivalent to the conjugate of an abstract (L)-space. It is known (this is not my contribution) that the Birkhoff-Ulam example ([B, p. 186] or [HT, p. 490]) answers this question in the negative, the pertinent Banach space being the bounded Borel functions on $[0, 1]$ modulo those functions vanishing except on a set of the first category, with $\|f\| = \inf \{K : |f(x)| \leq K \text{ save on a set of first category}\}$.

1. Preliminary definitions and remarks. A point $x$ is an extreme point of a convex subset $K$ of a real linear space if $x$ is not an interior point of any line segment contained in $K$ (i.e., if $x=ty+(1-t)z$, $0 < t < 1$, $y \in K$, and $z \in K$, then $x=y=z$). A set $L$ is a support of $K$ if $L$ is a convex, nonvoid subset of $K$ such that each line segment contained in $K$ which has an interior point in $L$ is contained in $L$. If $x$ is an extreme point of $L$ and $L$ is a sup-
port of $K$, then $x$ is an extreme point of $K$. If $F$ is a linear function carrying a convex set $K$ into a convex set $M$ and $L$ is a support of $M$, then $F^{-1}(L) \cap K$ is either void or a support of $K$.

For each Banach space $B$ the adjoint space is denoted by $B^*$ and the weak* topology for $B^*$ is the topology of pointwise convergence of functionals. Each convex, norm-bounded, weak* closed set $K$ (=$\text{convex, weak* compact subset}$) is, according to the classic theorem of Krein and Milman, the smallest convex weak* closed set which contains all extreme points of $K$. (See, for example, [K].) If $F$ is a bounded linear function on $B$ to a Banach space $C$, then $F^*$, the adjoint function, carries $C^*$ into $B^*$ in a weak* continuous fashion, and in particular, the image of the unit sphere of $C^*$ is weak* compact.

A compact Hausdorff space is extremally disconnected if the closure of each open set is open. If $X$ is a compact Hausdorff space, then $C_X$ is the Banach space of all real-valued continuous functions on $X$, with the usual supremum norm. For each $x \in X$ there is assigned a functional $e_x$, by setting $e_x(f) = f(x)$ for $f \in C_X$. This functional $e_x$ is the evaluation at $x$. It is known (see [AK]) that the set of extreme points of the unit sphere of $C_X^*$ is precisely $E \cup (-E)$, where $E$ is the set of all evaluations. Moreover, if $E$ has the relativized weak* topology, then the function $e$ carrying $x$ into $e_x$ maps $X$ homeomorphically onto $E$.

2. **Proof of the theorem.** Let $B$ be a Banach space with the property: if $H$ is a linear isometry of $B$ into a Banach space $C$, then there is a linear map $G$ of norm one carrying $C$ onto $B$ such that $GH$ is the identity map of $B$ onto itself. Let $X$ be the weak* closure of the set of all extreme points of the unit sphere of $B^*$. Then $X$ is weak* compact. In what follows, a subset of $X$ is “open” if it is “open in the relativized weak* topology for $X$,” and the closure $V_e$ of a subset $U$ of $X$ is the weak* closure of $U$.

Suppose, now, that $U$ and $V$ are open subsets of $X$ such that both $U \cap V$ and $[-(U \cup V)] \cap (U \cup V)$ are void, and $[-(U \cup V)] \cup (U \cup V)$ is dense in $X$. We construct a space $Y$, by setting $Y = (\{0\} \times U)^c \cup (\{1\} \times V)^c$, so that $Y$ consists of disjoint copies of $U^c$ and $V^c$. The set $Y$ is topologized by agreeing that if $U_1$ is open in $U^c$ and $V_1$ is open in $V^c$, then $\{0\} \times U_1$ and $\{1\} \times V_1$ are each open in $Y$. Let $H$ be the map of $B$ into $C_Y$ defined, for $b \in B$, $u \in U^c$, $v \in V^c$ by: $H(b)((0, u)) = u(b)$, $H(b)((1, v)) = v(b).$ The basic result about this construction is:

**Lemma.** The map $H$ is a linear isometry of $B$ onto $C_Y$. Moreover, $U^c \cap V^c$ and $[-(U^c \cup V^c)] \cap (U^c \cup V^c)$ are void, and $H^*$ maps the set of evaluations in $C_Y^*$ weak* homeomorphically onto $U^c \cap V^c$.

**Proof.** We first verify that $H$ is a linear isometry. The unit sphere $S$ of $B^*$ is weak* compact and, for each $b \in B$, the linear functional $b'$, whose value at $z \in B^*$ is $z(b)$, is weak* continuous, and maps $S$ onto the closed interval
The set of points at which the functional $b'$ assumes the value $\|b\|$ is a support of $S$ and hence contains an extreme point $x$, which is a member of $X$. Either $x$ or $-x$ belongs to $U^c \cup V^c$, and consequently $\|H(b)\| \geq |x(b)| = \|b\|$. On the other hand, since $U^c \cup V^c$ is a subset of the unit sphere of $B^*$, $\|H(b)\| \leq \|b\|$, so that $H$ is an isometry.

Next, a small calculation. Suppose $e_{(0,u)} \in C^*_F$ is the evaluation at $(0, u)$, and that $\bar{b} \in B$. Then $H^*(e_{(0,u)})(\bar{b})$ is, by definition of $H^*$, $e_{(0,u)}(H(b))$, which from the definition of $e_{(0,u)}$ is $H(b)((0, u))$, and using the definition of $H$ this is $u(b)$. Consequently, the valuation at $(0, u)$ maps under $H^*$ onto $u$, and similarly the evaluation at $(1, v)$ maps onto $v$.

If $u \in U$ and $u$ is an extreme point of the unit sphere $S$ of $B^*$, then $H^{*-1}(u)$ intersects the unit sphere $T$ of $C^*_F$ in a set which is a support of $S$. This support, being weak* compact, consists of a single point or else contains at least two extreme points (the Krein-Milman theorem). Each extreme point of the support is also an extreme point of $T$. But the extreme points of $T$ are ± evaluations, and since $u \in V^c$, the only extreme point which can map onto $u$ under $H^*$ is $e_{(0,u)}$, in view of the preceding paragraph. Consequently, $H^{*-1}(u) \cap T$ consists of the single point $e_{(0,u)}$ and similarly, if $v \in V$ and $v$ is an extreme point of $S$, then $H^{*-1}(v) \cap T = \{e_{(1,v)}\}$.

Now let $G$ be a linear function of norm one carrying $C_Y$ onto $B$ so that $GH$ is the identity on $B$. Then $G^*$ carries the unit sphere $S$ of $B^*$ into the unit sphere $T$ of $C^*_F$ and $(GH)^* = H^*G^*$ is the identity on $B^*$. If $u \in U$ and $u$ is an extreme point of $S$, then necessarily $G^*(u) = e_{(0,u)}$, in view of the preceding paragraph, and if $v \in V$ and $v$ is an extreme point of $S$, then $G^*(v) = e_{(1,v)}$. Because such points are dense in $U$ and in $V$ the function $G^*$ carries a dense subset of $X$ onto a weak* dense subset of $E \cup (-E)$, where $E$ is the set of evaluations. Because $X$ and $E \cup (-E)$ are weak* compact $G^*$ carries $X$ onto $E \cup (-E)$. Now $H^*G^*$ is the identity on $B^*$, and if $u \in U$ and $u$ is an extreme point of $S$, then $G^*H^*(e_{(0,u)}) = G^*(u) = e_{(0,u)}$, and similarly for $v \in V$ and $v$ extreme, so that $G^*H^*$ is the identity on a dense subset of $E \cup (-E)$. Consequently $G^*$ is, on $X$, a homeomorphism, and $H^*$ is, on $E \cup (-E)$, the inverse of this homeomorphism. From the structure of $E \cup (-E)$ it follows (see preliminary remarks) that $U^c \cap V^c$ and $[-(U^c \cup V^c)] \cap (U^c \cup V^c)$ are void, and it is also clear that $H^*$ maps $E$ homeomorphically onto $U^c \cup V^c$.

It remains to show that $H$ maps $B$ onto $C_Y$. The image $G^*(S)$ of the unit sphere $S$ of $B^*$ is convex and weak* compact, and each extreme point of the unit sphere $T$ of $C^*_F$, as was shown in the preceding paragraph, belongs to $G^*(S)$. From the Krein-Milman theorem it follows that $T = G^*(S)$, and since $G^*$ has norm one, $T = G^*(S)$. Since $H^*G^*$ is the identity on $B^*$ and since $G^*$ maps $B^*$ onto $C^*_F$, it follows that $H^*$ is 1-1. Because $H^*$ is 1-1 it is true that $H$ maps $B$ onto $C_Y$, for otherwise there is a nonzero linear functional on $C_Y$ which vanishes on the range of $H$ (a closed subspace) and $H^*$ applied to this functional gives the zero of $B^*$. The proof of the lemma is then complete.
The theorem is now established as follows. Choose, using Zorn's Lemma, an open subset $W$ of $X$ maximal with respect to the property that $(-W) \cap W$ be void. Then $(-W) \cup W$ is dense in $X$. Applying the lemma to $U = W$, $V = \text{void set}$, it follows that $(-W^c) \cap (W^c)$ is void, and that $W^c$ is open as well as closed in $X$. Moreover $H$ is an isometry of $B$ onto $C_Y$, where $Y$ is homeomorphic to $W^c$. Proceeding, let $U$ be any open subset of $W^c$ and let $V = W^c \setminus U^c$. Applying the lemma again, we see that $U^c \cap V^c$ is void so that $U^c$ is open and it is proven that $W^c$ is extremally disconnected, which establishes the theorem.

References


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