ON MONOTONE INTERIOR MAPPINGS IN THE PLANE

BY

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The purpose of this paper is to show two associated results: (I) that there exists a nonhomeomorphic monotone interior mapping of the plane onto itself(1) and (II) that there exists a monotone interior dimension-raising mapping of a one-dimensional continuous curve in the plane onto the plane. These results will henceforth be referred to as I and II respectively.

In 1929, Roberts [1] showed that there exists an upper semi-continuous collection of mutually exclusive nondegenerate compact continua (no one of which separates the plane) filling up the plane and, in 1936, he [2] showed that there exists an upper semi-continuous collection of mutually exclusive nondegenerate compact continuous curves (no one of which separates the plane) filling up the plane but there does not exist an upper semi-continuous collection of mutually exclusive arcs filling up the plane. The collections of continua described are not lower semi-continuous. It follows from a result of Moise [3] that if G is an upper semi-continuous and lower semi-continuous collection of mutually exclusive compact continua filling up the plane, there is no arc in G regarded as space with the property that every element of G on the arc is itself an arc in the original space.

In 1937, Kolmogoroff [4] showed that there exists an interior mapping of a one-dimensional continuum onto a two-dimensional continuum and in 1947 Kazdan [5] showed that there exists an interior mapping of a one-dimensional continuous curve onto a 2-cell. Neither of these mappings were monotone.

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(1) In 1949 Rozanskaya [8] published a statement without a detailed proof to the effect that there does not exist an open at least one-dimensional mapping of R^n (an n-cell) onto itself. (A mapping f is said to be at least one-dimensional if for each point p of the image space, $f^{-1}(p)$ is at least one-dimensional.) This "result" of Rozanskaya is contradicted by an example of such a mapping defined over R^3 based on Theorem I or immediately by such a mapping defined over R^2 by Theorem III of this paper. (The proof of Theorem III is not given in full detail but if the proof of Theorem I is understood the argument for Theorem III is easy to see.) From Theorem I, and without further recourse to the proof of Theorem I, it is fairly easy (using spiralling techniques, for example) to prove a theorem similar to Theorem I for the 2-sphere and from the theorem for the 2-sphere by similar techniques to show a monotone interior one-dimensional mapping of R^3 onto R^3 and hence of R^p onto R^p, $p>2$. Apparently as an immediate corollary of her result above (which it is), Rozanskaya in [8] also asserts that there does not exist an open mapping of R^p onto R^q for $p<q$. The question of the existence of such a mapping remains unsettled. In a separate paper, I shall show the existence of an open mapping from R^p $(p>2)$ onto a space of infinite dimension. The methods of proof, however, do not seem to lead to a technique for showing an open mapping of R^p onto R^q for $p<q$.

In order to demonstrate I (or II) it is sufficient to show the existence of a continuous, i.e. both upper and lower semi-continuous, collection G of mutually exclusive compact continua filling up the plane (or in the case of II a one-dimensional continuous curve in the plane) such that G with respect to its elements as points is topologically equivalent to the plane.

It will be understood that the terms "link" and "chain" will be used in this paper in their usual sense with the added convention that every link of a chain will be the sum of the elements of a finite collection of mutually exclusive 2-cells and that if a 2-cell of such a collection of one link intersects a 2-cell of such a collection of another link (of the same or of some other chain) their intersection is the sum of a finite collection of mutually exclusive 2-cells. The "link diameter" of a chain will be the largest number which is the diameter of some link of the chain. If C is a simple chain, a subchain of C is a simple chain whose links are links of C. A chain will be said to be "linkwise connected" if each link of the chain is connected. A chain which is not linkwise connected will be said to be "non-linkwise connected." If Y is a collection of point sets, let Y^* denote the sum of the elements of Y.

Definition. An A-chain will be a linkwise connected simple chain the sum of whose links does not separate the plane.

Definition. A B-chain will be a non-linkwise connected simple chain B such that the collection of all components of links of B is an A-chain A with the further property that A contains three sub-A-chains no two of which have a link in common and, for any end link b of B, each of which contains a link which is a subset of b.

Definition. The terms α-set and β-set will denote point sets which can be considered to be the sums of the links of an A-chain and a B-chain respectively and it will be understood that the statement that α (or β) is an α-set (or β-set) implies the existence of a particular A-chain A (or B-chain B) of which α (or β) is the sum of the links.

Consider a B-chain B and an A-chain A_B whose links are the components of the links of B. Let A_{B1}, A_{B2}, and A_{B3} denote three sub-A-chains of A_B, each of A_{B1} and A_{B3} having an end link in common with A_B, all three of which contain links in both end links of B, and no two of which have any link in common. Let the chains A_{B1} and A_{B3} be known as "fundamental end A-chains" of B and the corresponding sets $\alpha_{\beta 1}$ and $\alpha_{\beta 3}$ be "fundamental end α-sets" of β. Let chain A_{B2} be known as a "fundamental central A-chain" of B and the corresponding set $\alpha_{\beta 2}$ be a "fundamental central α-set" of β (with respect to A_{B1} and A_{B3}). It will be understood that if two chains are fundamental end A-chains of a B-chain, then a fundamental central A-chain of this B-chain exists and conversely.

Definition. A γ-set will be a point set γ such that (1) γ is the sum of three particular β-sets β_1, β_2, and β_3 plus those points separated from infinity by $\beta_1+\beta_2+\beta_3$, (2) $\beta_1\cdot\beta_2\cdot\beta_3$ does not exist, and (3) $\beta_i\cdot\beta_j$ ($i, j=1, 2, 3, i\neq j$) is a fundamental end α-set of each of the sets β_i and β_j.

Let β_1, β_2, and β_3 be called "fundamental β-sets" of γ, the three fundamental end α-sets of the various β_k which are $\beta_1 \cdot \beta_2$, $\beta_1 \cdot \beta_3$, and $\beta_2 \cdot \beta_3$ be called "fundamental end α-sets" of γ, and let three fundamental central α-sets of the various β_k with respect to the end α-sets just mentioned be called "fundamental central α-sets" of γ. Let the corresponding B-chains and A-chains be called "fundamental B-chains" and "fundamental end or central A-chains" of γ.

If C_z is a set of α-sets (or β-sets), let C_z' be a set of A-chains (or B-chains) in one-to-one correspondence with C_z in such a way that every element of C_z is the sum of the links of the corresponding element of C_z'.

Definition. If x is a chain, a chain y (or a point set z) lying in x is said to be straight with respect to x if for any link w of x the common part of y^* (or z) and w is connected.

Definition. A W-septet is a set W of seven collections E, E', F, F', G, G', and H such that

(1) G' is a collection of B-chains and G is a collection of β-sets with a one-to-one correspondence between the elements of G' and of G, each element of G being the sum of the links of the corresponding element of G';

(2) F' is a collection of A-chains and F is a collection of α-sets with a one-to-one correspondence between the elements of F' and of F, each element of F being the sum of the links of the corresponding element of F';

(3) E' is a collection of A-chains and E is a collection of α-sets with a one-to-one correspondence between the elements of E' and of E, each element of E being the sum of the links of the corresponding element of E';

(4) H is a locally finite minimal collection of γ-sets covering the plane;

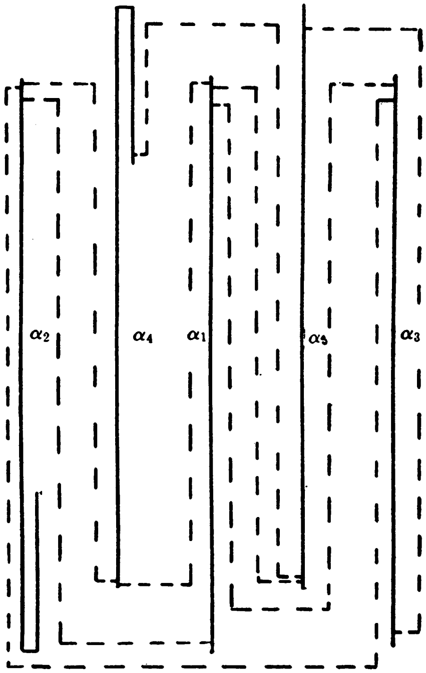
(5) Each element of H contains exactly three elements of G which are fundamental β-sets of it and each element of G is contained in exactly two elements of H;

(6) F is a minimal collection of mutually exclusive α-sets with every element of G containing exactly two elements of F which are fundamental end α-sets of such element and every element of H containing exactly three elements of F which are fundamental end α-sets of such element;

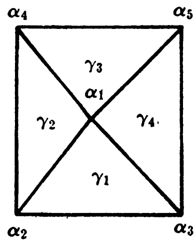
(7) E is a minimal collection of α-sets such that every element of E is contained in exactly one element of G and is a fundamental central α-set of such element (with respect to the contained elements of F');

(8) If two elements of H intersect, the intersection is an element of G or an element of F and if two elements of G intersect the intersection is an element of F.

Lemma A (*known as* FLA). *There exists a sequence W_1, W_2, W_3, $\cdots$ such that for each i, (1) W_i is a W-septet, E_i, E_i', F_i, F_i', G_i, G_i', and H_i; (2) Each element of G_i is of diameter greater than $1+1/i$; (3) Each element h_{i+1} of H_{i+1} is a subset of an element h_i of H_i, contains no point of the boundary of at least* *one element of H_i containing it, and each element of F'_{i+1} or E'_{i+1} in h_{i+1} has a link in each link of some one of the elements of F'_i lying in h_i; and* (4) *Every element of E'_i or F'_i is of link diameter less than* $1/(20+2i)$ *and no element of H_i contains a point at a distance of more than* $1/(10+i)$ *from any element of E_i or F_i contained in it.*



Schematically



Each arc drawn is to be thought of as being thick, i.e. containing a domain.
Each solid arc represents an α-set.
Each dotted arc together with the solid arcs it intersects represents a β-set.
Four γ-sets are represented, γ_1 containing α_1, α_2, and α_3; γ_2 containing α_1, α_2, and α_4; γ_3 containing α_1, α_4, α_5; and γ_4 containing α_1, α_3, and α_5.

LEMMA B (*known as* FLB). FLB *is identical with* FLA *except that in lieu of condition* (4) *read condition* (4'): *There exist a set K_i of non end links $k_{i,1}, k_{i,2}, \cdots, k_{i,10^i}$ of 10^i distinct elements of F'_i, a set of positive numbers $\delta_1, \delta_2, \cdots, \delta_{10^i}$ with $1/2>\delta_1>\delta_2>\delta_3> \cdots >\delta_{10^i}<1/i$, and a set of convex 2-cells $Y_{i,1}, Y_{i,2}, \cdots, Y_{i,10^i}$ with for each j, $Y_{i,j}$ containing exactly one point at a distance of δ_j from $S-Y_{i,j}$, such that $Y_{i,j}$ is $Y_{i+1,j}$ for all $j\leqq 10^i$, the interior of $k_{i,j}$ contains $Y_{i,j}$, $k_{i+1,j}$ is a subset of $k_{i,j}$ (and contains $Y_{i,j}$), no point in the interior of the circle with center at the origin and radius i is at a distance of more than $1/i$ from $Y_{i,1}+Y_{i,2}+ \cdots +Y_{i,10^i}$, all links of elements belonging to E'_i or F'_i not in K_i are of diameter less than* $1/(20+2i)$, *and no element of H_i contains a point not in the set $Y_{i,1}+Y_{i,2}+ \cdots +Y_{i,10^i}$ at a distance of more* *than* $1/(10+i)$ *from any element belonging to* E_i *or* F_i *contained in it.*

The FLA implies I. Consider the collection G of compact continua such that in order that x should belong to G it is necessary and sufficient that x be the common part of a sequence $h_{x,1}, h_{x,2}, h_{x,3}, \cdots$ where for each i, $h_{x,i}$ is an element of H_i and $h_{x,i+1}$ is contained in $h_{x,i}$ and must therefore contain points of every link of some element f' of F_i' such that f' is contained in $h_{x,i}$. Then G covers the plane (see, for example, Theorem 78 of Chap. 1 of [6]) and no element of G separates the plane. No two distinct elements of G intersect, for if y and z, elements of G, had a point in common, then corresponding elements of $h_{y,1}, h_{y,2}, \cdots$ and $h_{z,1}, h_{z,2}, \cdots$ would have the same point in common. For any i, if two elements $h_{y,i}$ and $h_{z,i}$ of H_i intersect and $h_{y,i+1}$ and $h_{z,i+1}$ elements of H_{i+1} are continua in $h_{y,i}$ and $h_{z,i}$ respectively, no point of either one is more than $3/(20+2i)$ from the other, which statement implies that no two distinct elements of G intersect. Similarly, for any element y of G, no point of any element of H_i containing y is at a distance of more than $3/(20+2i)$ from y. With condition 4 of FLA, this statement implies that G is both upper and lower semi-continuous as the sum of all elements of H_i containing y can be thought of as containing a neighborhood of y. But if G is an upper semi-continuous collection of mutually exclusive compact continua filling up the plane and no element of G separates the plane, then by the well known theorem of R. L. Moore [7], G with respect to its elements as points is topologically equivalent to the plane.

The FLB implies II by an analogous argument except that a collection G' is to be considered for the purposes of the argument where G' is the collection of boundaries of elements of G. G in this case will be an upper semi-continuous collection but will not be lower semi-continuous, G' will be a continuous collection, i.e. both upper and lower semi-continuous, G will fill up the plane, and G and G' with respect to their elements as points will be topologically equivalent to each other.

Proof of FLA and FLB. An argument will be given for FLA, with modifications of the argument indicated at the conclusion to establish FLB. Suppose the existence of W_i. It is desired to show the existence of W_{i+1}. In order to show the existence of W_1 a somewhat similar but much simpler argument can be given. The essential structure of such an argument will be given at the conclusion of the argument for W_{i+1}. We can consider the collection H_i to cover the plane by means of overlapping triangular type 2-cells whose "sides" are the contained β-sets of G_i, whose "vertices" are the contained α-sets of F_i, and whose interiors, i.e. non overlapped portions, are the γ-sets of H_i minus in each case the three contained β-sets of G_i. But from a consideration of a locally finite covering C of the plane by triangles plus their interiors such that two elements of C intersect in a vertex of both, an edge of both, or not at all, it is clear that there exists a simple sequence ρ of all the γ-sets of H_i such that for no n does the sum of the first n elements of ρ separate the plane and for each n is such sum connected. Let the sum of the first n elements of ρ be denoted by ρ_n.

Let $W_{i,n}$ denote a collection of subcollections of W_i ($E_{i,n}$ etc.) such that in order that an element of a collection of W_i should belong to the corresponding collection of $W_{i,n}$ it is necessary and sufficient that it should be in ρ_n. We shall understand that by W_z, for any set z of subscripts, is meant a set of seven collections (E_z etc.) satisfying properties similar to those of a W-septet, except possibly for condition 4, and such additional properties as may be specified at the introduction of the set W_z. If H_z of W_z is any finite collection such that H_z^* is connected and does not separate the plane, we let $\omega_{z,\alpha}$ denote the collection of elements of F_z containing points of the boundary of H_z^* and $\omega_{z,\beta}$ denote the collection of elements of G_z containing points of the boundary of H_z^* not in $\omega_{z,\alpha}^*$.

Suppose the existence of a finite collection $W_{i+1,n}$ such that (1) with respect to the elements of $W_{i,n}$ and the set $H_{i,n}^*$ the collection $W_{i+1,n}$ satisfies all applicable properties of FLA (some of which are detailed below); (2) $H_{i+1,n}^*$ is connected, lies in ρ_n, and does not separate S; (3) if p is a point of the closure of $\rho_n-\omega_{i,n,\beta}^*$, p is contained in an element of $H_{i+1,n}$; (4) no element of $\omega_{i+1,n,\alpha}$ or $\omega_{i+1,n,\beta}$ contains points of two of the elements of $\omega_{i,n,\beta}$, such points not being in $\omega_{i,n,\alpha}^*$; (5) every element x belonging to $E'_{i+1,n}$ or $F'_{i+1,n}$ is of link diameter less than $1/(20+2(i+1))$ and no point belonging to any element of $H_{i+1,n}$ containing x is at a distance of as much as $1/(10+(i+1))$ from x^*; (6) if A_1, A_2 and A_3 denote the elements of $F'_{i,n}$ in an element of $H_{i,n}$ containing x^*, then for some one A_j of the A_i each link of A_j contains a link of x, and if C is a component of the intersection of x^* and a link of A_j, x^*-C is not connected or C contains an end link of x in the interior of such link of A_j; (7) no component of the common part of the boundary of $H_{i+1,n}^*$ (a simple closed curve) and any link of any element of $F'_{i,n}$ is a subset of the boundary of such link, and (8) if two elements b_1 and b_2 of $G_{i+1,n}$ intersect in a set a_1, then the closure of b_1-a_1 and the closure of b_2-a_1 do not intersect. We desire to show that there exists a collection $W_{i+1,n+1}$ which has comparable properties with respect to $W_{i,n+1}$. The existence of $W_{i+1,1}$ follows by a simplified version of the same argument.

The collection $\omega_{i,n,\beta}$ forms a cyclic (not necessarily simple cyclic) chain with respect to its elements as links which preserves order on the simple closed curve which is the boundary of ρ_n. The intersection of any two successive links is an element of $\omega_{i,n,\alpha}$. But the $(n+1)$st element ρ'_{n+1} of ρ intersects ρ_n either in two successive elements of $\omega_{i,n,\beta}$, in exactly one element of $\omega_{i,n,\alpha}$, or in exactly one element of $\omega_{i,n,\beta}$. The existence of $W_{i+1,n+1}$ for the first (and most involved) of these three cases will be shown. In a simple manner, i.e., by the addition of two chains or one chain of B-chains in a natural way, either of the others cases can be reduced, in effect, to the first. Considered on their own merits these latter cases admit simpler arguments.

Let β_1 and β_2 denote the two elements of $\omega_{i,n,\beta}$ contained in ρ'_{n+1}. Let $\beta_1 \cdot \beta_2$ be α_2. Let α_1 denote the element of F_i in β_1 distinct from α_2, let α_3 denote the element of F_i in β_2 distinct from α_2, and let β_3 denote the third element of G_i in ρ'_{n+1}. Let A_j be the element of F'_i corresponding to α_j as an element of F_i. There exist two finite sets (a) $f_1, f_2, \cdots, f_m$ of linkwise connected simple chains of link diameter less than $1/(100+10i)$ and (b) $\phi_1, \phi_2, \cdots, \phi_m$ of point sets with, for each k, ϕ_k the sum of the links of f_k such that (1) ϕ_1 lies in $\omega^*_{i,n,\beta}$ and covers that portion of the boundary of $H^*_{i+1,n}$ which lies in the closure of $\beta_1+\beta_2-(\alpha_1+\alpha_3)$, (2) ϕ_1 contains no point of $H^*_{i+1,n}$ not in the closure of $S-H^*_{i+1,n}$ (note: ϕ_1 and the links of f_1 do not intersect the links of chains of $E'_{i+1,n}$, $F'_{i+1,n}$, or $G'_{i+1,n}$ in 2-cells which forms an exception to the stated understanding), (3) $\phi_1, \phi_2, \cdots, \phi_m$ form a simple chain in that order, (4) ϕ_1 is the only element of $\phi_1, \cdots, \phi_m$ which has a point in common with $H^*_{i+1,n}$, (5) $\phi_1+\cdots+\phi_m$ lies in ρ'_{n+1} and covers the closure of $\rho'_{n+1}-\beta_3-(\beta_1+\beta_2)\cdot H^*_{i+1,n}$, (6) the end links of $f_1, \cdots, f_m$ form two simple chains, (7) ϕ_m lies in the interior of β_3, (8) for each k, ϕ_k is a 2-cell intersecting each link of f_{k-1}, each link of f_{k+1}, and ϕ_{k+1} in a 2-cell, (9) any two successive links of f_k intersect in a 2-cell, (10) for each k, the number of links in f_k is equal to the number of links in f_{k+1} and in $\phi_k \cdot \phi_{k+1}$ the respective links of f_k and f_{k+1} coincide, (11) if z is a component of the intersection of ϕ_1 and a link of A_1, A_2, or A_3, z is the sum of the links of a subchain of f_1 having an end link in common with f_1 or ϕ_1-z is the sum of two mutually exclusive connected sets, and (12) there exist positive integers v and v' such that the first v links of f_k lie in α_1, the last v' links of f_k lie in α_3, for any link of A_1, some one of the first v links of f_k is a subset of such link, and for any link of A_3 some one of the last v' links of f_k is a subset of such link. The existence of the sets (a) and (b) can be seen by consideration of successive small deformations of the boundary of $\omega_{i+1,n,\beta}$ in ρ'_{n+1} from $\beta_1+\beta_2$ across ρ'_{n+1} to β_3. Condition 10 can be shown by supposing the condition to be impossible and then by use of an inductive type argument.

It is sufficient to show that there exists an extension $W_{i+1,n,1}$ of $W_{i+1,n}$ preserving the basic properties of $W_{i+1,n}$ in which $H_{i+1,n,1}$ covers the closure of $\phi_1-\phi_1\cdot\phi_2-\phi_1\cdot(\alpha_1+\alpha_3)$, and to show how an extension $W_{i+1,n,2}$ covering the closure of $\phi_1+\phi_2-\phi_2\cdot\phi_3-\phi_1\cdot\phi_2(\alpha_1+\alpha_3)$ and satisfying the basic properties of $W_{i+1,n}$ can be defined from $W_{i+1,n,1}$ in a way which is clearly sufficient to imply the existence of such successive extensions from ϕ_j to ϕ_{j+1} and therefore to imply the existence of $W_{i+1,n+1}$. In each case the notion of an extension is to be considered to imply the carrying over of all elements of the collections of the collection to be extended and to preserve all applicable properties indicated in the assumptions on $W_{i+1,n}$ and earlier extensions.

Consider a chain $\omega'_{i+1,n,\beta}$ of elements of $\omega_{i+1,n,\beta}$ each containing a point of ϕ_1 such that $\omega^*_{i+1,n,\beta}-\omega'^*_{i+1,n,\beta}$ is connected, no element of $\omega_{i+1,n,\beta}$ not in $\omega'_{i+1,n,\beta}$ intersects $\beta_1+\beta_3-(\alpha_1+\alpha_3)$, and the end elements of $\omega'_{i+1,n,\beta}$ lie in α_1 and

α_3 respectively. Consider the maximal arc t' of the boundary of $\omega'^{*}_{i+1,n,\beta}$ which lies in ϕ_1 and intersects every element of $\omega'_{i+1,n,\beta}$. Along t' in the order from α_1 to α_3 there exists a simple chain T' of subarcs $t'_1, t'_2, \cdots, t'_g$ of t', each link of which is the component of the intersection of t' and an element of $\omega'_{i+1,n,\beta}$ which contains a point in an element of $G_{i+1,n}$ not in any element of $F_{i+1,n}$. Each two successive elements of T' have an arc in common. But there exists a set T of mutually exclusive A-chains $t_1, t_2, \cdots, t_g$ in the interior of ϕ_1, straight in f_1, whose links are subsets of the links of f_1 such that for each element of T' there is exactly one element of T the sum of whose links intersects exactly those links of f_1 as such element of T' and all links of A_1, A_2, or A_3, and which has the property that on no link of f_1 is an element τ of T separated from its corresponding element of T' by any element of T corresponding to an element of T' following τ in the order from t_1 to t_g. For all $k<g$ there must also exist sets U_k of mutually exclusive A-chains $t_{k,1}, t_{k,2}, \cdots, t_{k,j_k}$, each straight in f_1, whose links are subsets of the links of f_1 such that (1) for each z, the sum of the links of $t_{k,z}$ intersects every link of A_1, A_2, or A_3, is in the interior of ϕ_1, and $t_{k,1}$ is t_k, t_{k,j_k} is t_{k+1}, (2) on any link of f_1, $t_{k,q}$ separates $t_{k,p}$ from $t_{k,r}$ only if q is between p and r and (3) for some g' with $1<g'<g$, every element $t_{k,y}$ of U_k with $y<g'$ is such that $t^*_{k,y}$ intersects the same links of f_1 as $t^*_{k,y+1}$ except for the last link of f_1 in the order from α_1 to α_3 intersected by $t^*_{k,y+1}$ and with $g>y>g'$, $t_{k,y}$ is such that $t^*_{k,y}$ intersects the same links of f_1 as $t^*_{k,y-1}$ except for the first link in the order from α_1 to α_3 intersected by $t^*_{k,y-1}$. But then there must exist two collections of β-sets, the one such that each element contains as fundamental end α-sets $t^*_{z,y}$ and $t^*_{z,y+1}$ for all possible y and z and lies on and between such sets on f_1 joining the α_1 end of $t^*_{z,y}$ to the α_3 end of $t^*_{z,y+1}$, and the other such that each element contains as fundamental end α-sets one element $t^*_{z,y}$ and one element u of $\omega_{i+1,n,\alpha}$ for which t_z is an element of T corresponding to an element of T' lying in an element of $\omega_{i+1,n,\beta}$ which contains u as the first of its two elements of $\omega_{i+1,n,\alpha}$ in the order from α_1 to α_3 for all possible $t^*_{z,y}$ and u. Any β-set of these collections is to intersect no link of f_1 not intersected by one of the fundamental end α-sets of such β-set. In this manner there exists an extension W_{i+1,n,λ_1} of $W_{i+1,n}$ to cover part of ϕ_1. Of the collection of α-sets used, any element is seen to be close to all points of the γ-sets containing it because it lies close to some element of $E_{i+1,n}$ in $\omega^*_{i+1,n,\beta}$ and along f_1.

It will be understood that extensions of W_{i+1,n,λ_1} to be defined in ϕ_1 will be set up by the introduction of ordered sets of mutually exclusive A-chains each A-chain of each set being a straight subchain of f_1 such that each of these sets corresponds to the collection $\omega'_{i+1,n,\lambda_1,\alpha}$ of all elements of $\omega_{i+1,n,\lambda_1,\alpha}$ in ϕ_1 in number and order of elements and that each new set will have the previously indicated separation properties on the links of f_1. Each element of each set is also to have the property that the sum of its links intersects every link of A_1, A_2, or A_3.

We desire to show the existence of an extension $W_{i+1,n,1}$ in which every element of $H_{i+1,n,1}$ not in H_{i+1,n,λ_1} is in ϕ_1 and in which every element of $\omega'_{i+1,n,1,\alpha}$ in ϕ_1 is in $\phi_1 \cdot \phi_2$ and contains points of the end link of f_1 in α_1 or the end link of f_1 in α_3 and points of every link of A_1 or every link of A_3. To accomplish this we require that some element of $\omega'_{i+1,n,1,\alpha}$ contain points of every link of f_1 so that the transition on the set $\omega'_{i+1,n,1,\alpha}$ from those elements containing points of the end link of f_1 in α_1 to those elements containing points of the end link of f_1 in α_3 can be effected. We select one element s of $\omega'_{i+1,n,\lambda_1,\alpha}$ in α_2 as a transition element for the construction and consider a set Ω' of mutually exclusive A-chains corresponding to the set $\omega'_{i+1,n,\lambda_1,\alpha}$ such that for each element r of $\omega'_{i+1,n,\lambda_1,\alpha}$ distinct from s there is exactly one element r' of Ω' the sum of whose links intersects all links of f_1 intersected by r^* and exactly one additional link which is on the A_1 side of r on the chain f_1 if and only if r precedes s on $\omega'_{i+1,n,\lambda_1,\alpha}$ in the natural order from A_1 to A_3, and is on the A_3 side of r on f_1 if and only if r follows s on $\omega'_{i+1,n,\lambda_1,\alpha}$ in the natural order from A_1 to A_3. Corresponding to s there is an element s' of Ω' the sum of whose links intersects all links of f_1 intersected by s and in addition one link on the A_1 side of s and one link on the A_3 side of s.

There exist two collections of β-sets one joining, in effect, successive elements of the collection Ω of α-sets corresponding to the collection Ω' of A-chains and the other joining an element of Ω and the corresponding element of $\omega'_{i+1,n,\lambda_1,\alpha}$ or an element of Ω and the element of $\omega'_{i+1,n,\lambda_1,\alpha}$ following its corresponding element (if any exists) by means of which an extension W_{i+1,n,λ_2} may be obtained. It is clear that by reapplication of the technique just used the extension $W_{i+1,n,1}$ can be obtained. But by use of a similar ordered set of mutually exclusive A-chains in $\phi_2 \cdot \phi_3$ in one-to-one correspondence with the set $\omega'_{i+1,n,1,\alpha}$ and thus also the set $\omega'_{i+1,n,\lambda_1,\alpha}$ where each chain of the new set has the property that the sum of its links intersects exactly the same links of f_2 as the corresponding element in $\omega'_{i+1,n,1,\alpha}$, it is clearly possible to define an extension $W_{i+1,n,2}$ and by reapplication of such ideas to define an extension from $W_{i+1,n,e}$ to $W_{i+1,n,e+1}$ as was to be shown. We have then demonstrated the existence of $W_{i+1,n+1}$ given $W_{i+1,n}$.

Finally to suggest a construction for W_1 we consider a collection C: c_0, c_1, c_{-1}, $\cdots$ of collections of point sets with, for each i, c_i the collection of all squares plus their interiors with centers at $(p/40,\ i/40)$ for p any positive, negative, or zero integer, and with sides parallel to the coordinate axes and of length 1/25. Let C'_i be the sum of the elements of c_i. Every element of H_1 is to have the property that it lies entirely in a bounded portion of exactly one set C'_i. If an element h of H_1, with h in C'_i, intersects an element c of c_i, then every element of F_1 or E_1 contained in h is to intersect an element of c_i intersecting $c \cdot h$. It is straightforward to define inductively a collection $W_{1,q}$, $q \geqq 0$, with $H_{1,q}$ lying in $\sum_{k=-q}^{q} C'_k$ and covering $S - \sum_{k<-q} C'_k - \sum_{k>q} C'_k$, with $H_{1,q+1}$ containing the elements of $H_{1,q}$, and with $H_1 = \sum_{q=0}^{\infty} H_{1,q}$.

The proof of FLA is now complete.

It should be noted that it could be easily required that every element of the collection G used in demonstrating I be indecomposable, and in fact by imposing sufficiently strong conditions of a sort employed by Moise in defining a certain type of hereditarily indecomposable continuum it could be required that all elements of the collection G be hereditarily indecomposable and chained and therefore by a current result of Bing homeomorphic to a pseudo arc and one to another. The author will show in a separate paper that there does exist such a collection G, i.e., that there exists a continuous decomposition of the plane in pseudo arcs.

Proof of FLB. The modifications in the argument for FLA sufficient to imply FLB will now be noted. In establishing the induction, we suppose $W_{i+1,n}$ as before, but with reference to the conditions of FLB, and also a set K_i of exactly 10^i links $k_{i,1}, k_{i,2}, \cdots, k_{i,10^i}$ of elements of F_i' of diameter not subject to the restriction that they be less in diameter than $1/(20+2i)$. No element of F_i contains two elements of K_i. But each element of K_i is subject to restrictions now to be specified. There exists a set of positive proper fractions $\delta_1, \delta_2, \cdots, \delta_{10^i}$ with $\delta_1 > \delta_2 > \cdots > \delta_{10^i}$ and $\delta_{10^i} < 1/i$ and a set of convex 2-cells $Y_1, Y_2, \cdots, Y_{10^i}$ such that for each m, $k_{i,m}$ contains Y_m which contains exactly one point at a distance equal to δ_m from $S - Y_m$, $k_{i,m} - Y_m$ contains no point at a distance of more than $1/(10+i)$ from any element of E_i or F_i which is contained in some element of H_i containing $k_{i,m}$. The set K_i also satisfies the property that no point of the plane point set $x^2+y^2<i^2$ is at a distance of more than $1/i$ from K_i^*. There is no difficulty in extending this last property to K_{i+1} by arbitrarily selecting $10^{i+1}-10^i$ elements of F_{i+1} to contain elements of K_{i+1}. But to achieve the induction, particularly with respect to lower semi-continuity of G', i.e. to show that condition (4') of the lemma can be met, careful attention must be given to those elements of F_i, G_i, and H_i containing elements of K_i. We observe that in defining $H_{i+1,n+1}$ to cover ρ_{n+1}, if an element of K_i, say $k_{i,m}$, occurs in any element of F_i in ρ_{n+1}' and no point of Y_m is covered by $H_{i+1,n}$, which is assumed, it is always possible by means of the technique explained for FLA to get a covering $H_{i+1,n+1}$ of ρ_{n+1} which does not cover any point of Y_m except in the one case where $k_{i,m}$ lies in the set α_2 of the argument for FLA and in this case it is never possible to do so. That such a covering can be obtained in the case where $k_{i,m}$ is not in α_2 and therefore that the case for $k_{i,m}$ in α_2 is the only one requiring further consideration can be seen by the joining of Y_m to an end link of the element of F_i' corresponding to the element of F_i in which it lies by means of an arc lying in such element of F_i and by means of joining this arc to the point at infinity by means of an open curve where no point of the arc or the open curve is covered by either $H_{i+1,n}$ or the set to be defined $H_{i+1,n+1}$.

In the event Y (we omit the subscript) is in α_2 we proceed with the induction. No element of $H_{i+1,n}$ intersects Y. Let b_1 and b_2 be the end links of A_2.

Consider the set $\omega'_{i+1,n,\beta}$ of the elements of $\omega_{i+1,n,\beta}$ in the interior of α_2 and the set $\omega'_{i+1,n,\alpha}$ of elements of $\omega_{i+1,n,\alpha}$ in $\omega'^*_{i+1,n,\beta}$. Every element of $\omega'_{i+1,n,\alpha}$ contains points of b_1 and of b_2.

Let t be an element of $\omega'_{i+1,n,\beta}$ such that, in $\alpha_2 - H^*_{i+1,n}\cdot\alpha_2$, Y is accessible from a point of an element of $E_{i+1,n}$ in t. There exists an extension $W_{i+1,n,\lambda}$ of $W_{i+1,n}$ satisfying the properties of $W_{i+1,n}$ in which exactly two new γ-sets are added, exactly two new fundamental end α-sets are added, these lying in the sum of the links of a linkwise connected simple chain M of link diameter less than $1/(100+10i)$ lying in $\alpha_2 - (H^*_{i+1,n}\cdot\alpha_2) - Y$ and intersecting the end links of M such that one of the new Y-sets contains exactly one of the new fundamental end α-sets, the other contains both of these sets, the two have a fundamental β-set in common, and one of them contains t. Y is accessible from M^*.

There exist three 2-cells, θ_1, θ_2, and θ_3, and three linkwise connected simple chains, c_1, c_2, and c_3, c_1 and c_3 of link diameter less than $1/(1000+100i)$ with c_2 containing one large link containing Y, being except for this link of link diameter less than $1/(1000+100i)$, having end links d_1 and d_2, and having the property that the link containing Y contains no point at a distance of more than $1/(1000+100i)$ from Y, such that θ_q is c_q^*, $\theta_1+\theta_2+\theta_3$ is a subset of $\alpha_2 - H^*_{i+1,n,\lambda}\cdot\alpha_2$, $\theta_1\cdot\theta_2$ is a 2-cell, $\theta_2\cdot\theta_3$ is a 2-cell, θ_1 and θ_3 do not intersect, $\theta_1+\theta_2+\theta_3$ does not separate the plane, $\theta_1+d_1+d_2+\theta_3$ separates Y from the point at infinity, every point of θ_{z_1} ($z_1=1$, 3) is at a distance of less than $1/(200+20i)$ from θ_{z_2} ($z_1+z_2=4$), and the end links of c_1, c_2, and c_3 form two simple chains. These conditions require each of the sets θ_1 and θ_3 to "wrap around" Y but in different directions, so to speak.

There exist a set of linkwise connected simple chains of link diameter less than $1/(100+10i)$, $M_1, M_2, M_3, \cdots, M_p$, and a set of 2-cells $N_1, N_2, N_3, \cdots, N_p$ such that for each j, N_j is the sum of the links of M_j, $N_j\cdot N_{j+1}$ is a 2-cell, N_j does not intersect N_k for $|j-k|>1$, M_1 is M, N_j contains points of both end links of A_2, $N_2+N_3+\cdots+N_p$ lies in $\alpha_2 - H^*_{i+1,n,\lambda}\cdot\alpha_2 - (\theta_1+\theta_2+\theta_3)$ except that N_p contains θ_1 and d_1 and d_2, c_1 being straight in M_p with links subsets of the links of M_p, and the end links of the chains $M_1, M_2, \cdots, M_p$ form two simple chains. That such a set of chains as $M_1, M_2, \cdots, M_p$ exists can be seen by a consideration of a gradual deformation of N_1 held almost fixed at the end links of M_1 and following the path of an arc from N_1 to θ_1 in $\alpha_2 - (H_{i+1,n,\lambda}\cdot\alpha_2) - (\theta_1+\theta_2+\theta_3)$.

An extension $W_{i+1,n,\lambda'}$ of $W_{i+1,n,\lambda}$ can be defined satisfying the properties of $W_{i+1,n}$ such that $F_{i+1,n,\lambda'}$ contains exactly two elements in every N_j, no element of $F_{i+1,n,\lambda'}$ separates $H_{i+1,n,\lambda'}$ or intersects N_j and $N_k (j\neq k)$, each element of $F_{i+1,n,\lambda'}$ contains points of both end links of the M_j in which it lies, no element of $H_{i+1,n,\lambda'}$ contains points of three different sets N_{j_1}, N_{j_2}, and N_{j_3}, and the two elements of $F_{i+1,n,\lambda'}$ in N_p intersect every link of c_1 but do not intersect d_1 or d_2.

But then there can be defined an extension $W_{i+1,n,\mu}$ of $W_{i+1,n,\lambda'}$ satisfying the properties of $W_{i+1,n}$ such that $F_{i+1,n,\mu}$ contains exactly three new elements not in $F_{i+1,n,\lambda'}$, all contained in $N_p+\theta_2+\theta_3$, all containing points of both end links of M_p, no one containing any point of a link of M_p which contains a point of a link of c_1 but no point of an end link of c_1, one not intersecting θ_3 and containing Y, the other two not intersecting any link of c_2 except d_1 and d_2, $\omega_{i+1,n,\mu}$ does not contain the element of $F_{i+1,n,\mu}$ which contains Y, and $H_{i+1,n,\mu}$ contains exactly four new elements each containing Y and each lying in $M_p+\theta_2+\theta_3$.

From $W_{i+1,n,\mu}$, the argument for FLA implies the existence of $W_{i+1,n+1}$ which in turn implies FLB.

The following two theorems have now been demonstrated:

THEOREM I. *There exists a monotone interior mapping f of the plane S onto itself with, for each s in S, $f^{-1}(s)$ nondegenerate.*

THEOREM II. *There exists a monotone interior mapping of a one-dimensional continuous curve in the plane onto the plane.*

By a rather simple and straightforward modification of the argument for FLA it is possible to demonstrate

THEOREM III. *If M is a 2-cell, there exists a monotone interior mapping f of M onto itself such that for any point x of M, $f^{-1}(x)$ is nondegenerate.*

The bounding elements of H_i will contain two fundamental end α-sets intersecting the boundary of M and will contain only two fundamental β-sets instead of three.

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