THE INTEGRAL GEOMETRY DEFINITION OF ARC LENGTH
FOR TWO-DIMENSIONAL FINSLER SPACES

BY

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Introduction. A curve $C^*$ in a 2-dimensional Finsler space, metrized by
$ds = F(x, y, dx, dy)$, is admissible if it is a simple, closed, regular, extremal
convex curve, where extremal convexity means locally $C^*$ lies in the closure
of a component (side) of any tangent geodesic.

A conjecture of Hans Lewy is that an "integral geometry" relation, proved
by Legendre for euclidean space, can be generalized to the following:

Theorem. Specify the geodesics by suitable parameters $\theta, p$ and let $D$ denote
the cartesian region consisting of all points $(\theta, p)$ for which the corresponding
geodesic intersects $C^*$. Then there exists a contact transformation $(x, y, dx, dy)
\rightarrow(\theta, p, d\theta, dp)$ and a density function $\sigma(\theta, p)$ such that the Finsler length of any
admissible $C^*$, extremal convex in the large, is

\[ \int_{C^*} F(x, y, dx, dy) = \int \int_D \sigma(\theta, p) d\theta dp. \]

The integral equality (1) will be established for admissible $C^*$ contained
in a region $U$, where: (a) Certain regularity and boundedness conditions hold
for $F$ and specified partial derivatives of $F$ on $U$; for example, $F$ is to define
a regular variational problem. (b) The geodesics of $F$ satisfy a "field hy-
pothesis," see Part II, (3.4). (c) An "embedment hypothesis" holds for $C^*$,
see Part II, (9.1).

Assuming the geodesics are straight lines, Eberhard Hopf devised an
elegant proof of (1) and with no restriction as to the sign of $F$. I thank
Professor Hopf for allowing me to present his proof as Part I of this paper.
In Part II the author resolves the technical difficulties met in generalizing
the Hopf procedure and establishes the theorem and a generalization ap-
licable to $C^*$ not necessarily extremal convex in the large.

Part I

We shall assume that the Finsler metric, $ds = F(x, y, dx, dy)$, has straight
lines for geodesics, is of class $C^\infty$ in its arguments, and that $U$ is any convex
region on which $F$ is defined.

1. Variation of Finsler length. Impose the restriction, later to be removed,
that $C^*$ encloses the origin of coordinates, and let $\phi$ denote the positive
angle formed by the directed tangent with the polar axis. Then,
\[
\frac{dx}{dt} = \cos \phi, \quad \frac{dy}{dt} = \sin \phi, \quad \frac{dy}{dx} = \tan \phi,
\]
where \(dt = ((dx)^2 + (dy)^2)^{1/2}\). Hence, as \(F(x, y, dx, dy)\) is homogeneous of degree one in its differentials,

\[
ds = F \left( x, y, \frac{dx}{dt}, \frac{dy}{dt} \right) dt = L(x, y, \phi) dt,
\]
defining \(L(x, y, \phi)\).

If \(\delta\) denotes the first variation, then

\[
\delta dt = (\delta dx) \cos \phi + (\delta dy) \sin \phi.
\]
Furthermore, since

\[
\sec^2 \phi \delta \phi = \frac{dx(\delta dy) - dy(\delta dx)}{(dx)^2},
\]
it follows that

\[
\delta t \delta \phi = (\delta dy) \cos \phi - (\delta dx) \sin \phi.
\]
Assuming fixed end points \(P_0\) and \(P_1\), we now compute the variation of

\[
I = \int_{P_0}^{P_1} L(x, y, \phi) dt.
\]

By means of (1.3), (1.4), and the vanishing of the variations at \(P_0\) and \(P_1\), we find

\[
\delta I = \int_{P_0}^{P_1} \left[ L_x dt - d(L \cos \phi) + d(L \sin \phi) \right] \delta x
\]
\[
+ \int_{P_0}^{P_1} \left[ L_y dt - d(L \sin \phi) - d(L \cos \phi) \right] \delta y.
\]
Therefore, if

\[
X = L_x dt - d[-L_\phi \sin \phi + L \cos \phi]
\]
and

\[
Y = L_y dt - d[L_\phi \cos \phi + L \sin \phi],
\]
then

\[
\delta I = \int_{P_0}^{P_1} [X \delta x + Y \delta y].
\]

This result also holds for closed curves, \(P_0 \equiv P_1\), as well as curvilinear seg-
ments bounded by distinct end points.

2. Simplification of $\delta I$. The differential forms $X$ and $Y$ satisfy the following relations:

\begin{align*}
(2.1) & \quad X \cos \phi + Y \sin \phi = 0, \\
(2.2) & \quad X \sin \phi - Y \cos \phi = S(x, y, \phi)dt + T(x, y, \phi)d\phi,
\end{align*}

where

\begin{align*}
(2.3) & \quad T(x, y, \phi) = L + L_{\phi\phi} \\
(2.4) & \quad S(x, y, \phi) = L_{x\phi} \cos \phi + L_{y\phi} \sin \phi + L_x \sin \phi - L_y \cos \phi.
\end{align*}

A consequence of (2.3) and (2.4) is

\begin{equation}
(2.5) \quad \frac{dS}{d\phi} = \frac{dT}{dx} \cos \phi + \frac{dT}{dy} \sin \phi.
\end{equation}

Furthermore, as the extremals of the variational problem, $I = \text{minimum}$, are straight lines, $X = Y = 0$, whenever $d\phi = 0$, which implies

\begin{equation}
(2.6) \quad S(x, y, \phi) = 0.
\end{equation}

Thus, because of (2.1), (2.2), and (2.6),

\begin{equation}
(2.7) \quad X = T \sin \phi d\phi, \quad Y = -T \cos \phi d\phi.
\end{equation}

Substituting (2.7) into (1.9) gives

\begin{equation}
(2.8) \quad \delta I = \oint_{C^*} T[\delta x \sin \phi - \delta y \cos \phi]d\phi.
\end{equation}

3. Identity of $I$ with double integral. Since the closed convex curve $C^*$ encircles the origin, there is a unique $p = p(\phi)$ such that $C^*$ is the envelope of its tangent lines:

\begin{equation}
(3.1) \quad x \sin \phi - y \cos \phi = p(\phi).
\end{equation}

The parametric equations of $C^*$ are

\begin{align*}
(3.2) \quad x &= p \sin \phi + p' \cos \phi, \\
&= -p \cos \phi + p' \sin \phi,
\end{align*}

\begin{align*}
\left( p' = \frac{dp}{d\phi} \right).
\end{align*}

Thus, because of (2.5), (2.6), and (3.2),

\begin{equation}
(3.3) \quad \frac{\partial T}{\partial p'} = \frac{\partial T}{\partial x} \cos \phi + \frac{\partial T}{\partial y} \sin \phi = \frac{\partial S}{\partial \phi} \equiv 0,
\end{equation}

so $T = T(\phi, p)$ is not a function of $p'$. Now, let the variation of $C^*$ be that obtained by varying the support function $p$. Then,
\[
\delta \rho = \delta x \sin \phi - \delta y \cos \phi, 
\]
and (2.8) becomes
\[
\delta I = \int_0^{2\pi} T \delta \rho \, d\phi = \delta \int \int_D T(\phi, \rho) \, d\rho \, d\phi,
\]
where \(D\) consists of all points \((\phi, \rho)\) for which \([0 \leq \phi \leq 2\pi; 0 \leq \rho \leq \rho(\phi)\], or all points \((\phi, \rho)\) such that the corresponding line \(3.1\) intersects \(C^*\). Consequently, as the two integrals in \((3.5)\) have equal variations and as both vanish when \(C^*\) is contracted to the origin, they are equal. That is,
\[
I = \oint_{C^*} L(x, y, \phi) \, dt = \int \int_D \left[ L + L_{\phi \phi} \right] d\rho \, d\phi,
\]
where the integrand of the double integral depends only on the variables \((\phi, \rho)\).

Consider now the case that \(C^*\) does not enclose the origin. Let \(\alpha\) and \(\beta\) be the smaller and larger, respectively, of the positive angles formed by the polar axis and the two directed tangents to \(C^*\) from the origin. The map of \(C^*\) under \((3.1)\) will consist of two separate curvilinear segments:
\[
P = p_1(\phi), \quad p = p_2(\phi),
\]
where \(p_1(\phi) \leq p_2(\phi)\) and \(\alpha \leq \phi \leq \pi + \beta\). If \(D\) is taken as the set of points \((\phi, \rho)\) satisfying the inequalities
\[
\alpha \leq \phi \leq \pi + \beta, \quad p_1(\phi) \leq \rho \leq p_2(\phi),
\]
then \((3.6)\) still holds and has the same "integral geometry" interpretation.

Part II

We shall assume that the Finsler metric, \(ds = F(x, y, dx, dy)\), is of class \(C^3\) in its arguments and defines a regular variational problem on a region \(U\) which is extremal convex in the large. Furthermore, on \(U\), a "field hypothesis," \((3.4)\), is satisfied by the geodesics and an "embedment hypothesis," \((9.1)\), by the admissible curves \(C^*\).

1. The Euler equations. Any geodesic \(C: [x(t), y(t)]\) of the space is a solution of the Euler equations:
\[
F_x - \frac{dF_{\dot{x}}}{dt} = 0, \quad F_y - \frac{dF_{\dot{y}}}{dt} = 0,
\]
where \((x, y, \dot{x}, \dot{y}) = (x(t), y(t), dx/dt, dy/dt)\). Furthermore, the homogeneity condition
\[
F(x, y, k\dot{x}, k\dot{y}) = kF(x, y, \dot{x}, \dot{y}) \quad (k \geq 0)
\]
implies the following:
(1.3) \[ F_1(x, y, \dot{x}, \dot{y}) = \frac{F_{\dot{x}\dot{x}}}{\dot{x}^2} = - \frac{F_{\dot{x}\dot{y}}}{\dot{x}\dot{y}} = \frac{F_{\dot{y}\dot{y}}}{\dot{y}^2}, \]

(1.4) \[ F_1(x, y, k\dot{x}, k\dot{y}) = k^{-3}F_1(x, y, \dot{x}, \dot{y}). \]

Hence, for regular curves \( C \), that is,

(1.5) \[ \rho \equiv (\dot{x}^2 + \dot{y}^2)^{1/2} \neq 0, \]

the Euler equations are equivalent to the single equation:

(1.6) \[ (\dot{x}\dot{y} - \dot{y}\dot{x})F_1 + F_{x\dot{y}} - F_{\dot{y}x} = 0 \quad (F_1 > 0). \]

Thus, if

(1.7) \[ G(x, y, \dot{x}, \dot{y}) \equiv (F_{y\dot{z}} - F_{x\dot{y}})/F_1, \]

which implies the homogeneity relation

(1.8) \[ G(x, y, k\dot{x}, k\dot{y}) = k^3G(x, y, \dot{x}, \dot{y}), \]

the differential equation (1.6) is equivalent to

(1.9) \[ \dot{x} = - \frac{\dot{y}G(x, y, \dot{x}, \dot{y})}{G_{\dot{x}}} = + \frac{\dot{x}G(x, y, \dot{y})}{G_{\dot{y}}}, \]

the parametrization \( \rho = 1 \) being assumed.

From (1.8) follows

(1.10) \[ GF_{\dot{y}\dot{y}} + F_{x\dot{y}} - F_{y\dot{x}} + G_{\dot{y}}F_1 = 0, \quad GF_{x\dot{x}} + F_{x\dot{y}} - F_{x\dot{y}} + G_{x}F_1 = 0, \]

which, because of (1.4), can be written as

(1.11) \[ GF_{\dot{y}\dot{y}} + \dot{x}^2F_{x\dot{x}} + \dot{x}F_{x\dot{y}} + \dot{y}G_{\dot{y}}F_1 = 0, \quad GF_{x\dot{x}} - \dot{x}\dot{y}F_{x\dot{x}} - \dot{y}^2F_{x\dot{y}} + G_{x}F_1 = 0. \]

Thus, since \( \rho = 1 \) by hypothesis,

(1.12) \[ G(\dot{x}F_{\dot{y}\dot{y}} - \dot{y}F_{x\dot{x}}) + \dot{x}F_{x\dot{x}} + \dot{y}F_{x\dot{y}} + (\dot{x}G_{\dot{y}} - \dot{y}G_{\dot{x}})F_1 = 0. \]

2. Variation of Finsler length. For a simple, regular arc \( C: [(x(t), y(t)), \quad t_0 \leq t \leq t_1] \), bounded by prescribed end points \( P_0 = (x(t_0), y(t_0)) \) and \( P_1 = (x(t_1), y(t_1)) \), the variation of the Finsler length

(2.1) \[ I = \int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y})dt \]

is

(2.2) \[ \delta I = \int_{t_0}^{t_1} \left\{ \left[ F_x - \frac{dF_x}{dt} \right] \delta x + \left[ F_y - \frac{dF_y}{dt} \right] \delta y \right\} dt, \]

where we assume that

(2.3) \[ F_x^2 + F_y^2 < A, \]
a uniform bound for all \( \rho \neq 0 \). Moreover, as (2.2) also holds for closed arcs, that is, \( P_0 = P_1 \),

\[
\delta I = \oint_C \left[ (\dot{x}\dot{y} - \dot{y}\dot{x}) F_1 + F_{x\dot{y}} - F_{y\dot{x}} \right] [dy\delta x - dx\delta y]
\]

for closed curves \( C \). Now, because of (1.9), (2.4) becomes

\[
\delta I = \oint_C \left[ (\dot{x}\dot{y} - \dot{y}\dot{x}) - G(x, y, \dot{x}, \dot{y}) \right] F_1(x, y, \dot{x}, \dot{y}) [dy\delta x - dx\delta y].
\]

3. **Transformation of \( F_i(x, y, \dot{x}, \dot{y}) \).** Here a transformation of the variables \((x, y, \dot{x}, \dot{y})\), with \( x^2 + y^2 = 1 \), is defined and the altered form of \( F_i(x, y, \dot{x}, \dot{y}) \) is determined. The existence of the transformation will be assured by invoking a “field hypothesis,” (3.4).

Denote by \( E \) an arbitrarily directed lineal element at the origin, which makes a positive angle \( \theta \) with the positive \( X \)-axis, and assign to the geodesic tangent to \( E \) the orientation of \( E \). Let this extremal be defined by the parametric equations

\[
x = X(0, \theta, p), \quad y = Y(0, \theta, p) \quad (0 \leq \theta < 2\pi),
\]

where the parameter \( p \) is signed euclidean arc length, measured from the origin along the geodesic, the sign being positive or negative according as the sense of measurement coincides with the orientation of the extremal or not.

Consider the extremal which is transverse to the geodesic (3.1) at the point (3.1). Orient it by defining its positive sense to be from the point (3.1) to the positive side, the left, of the geodesic (3.1). This extremal will be defined by parametric equations

\[
x = X(s, \theta, p), \quad y = Y(s, \theta, p),
\]

the parameter \( s \) being signed euclidean arc length, where

\[
\left( \frac{dX}{ds} \right)^2 + \left( \frac{dY}{ds} \right)^2 = 1,
\]

measured from the point (3.1) along the curve (3.2) and, as above, the sign being positive or negative according as the sense of measurement coincides with the orientation of the extremal or not.

The need for certain field properties, holding in the large for the geodesics (3.2), forces us to postulate the following:

**(3.4) Field hypothesis.** There is a neighborhood, \( U \), of the origin which possesses the following properties: (a) \( U \) is extremal convex in the large. (b) Any direct lineal element of \( U \) is tangent to a unique equally directed extremal (3.2). (c) The jacobians

\[
J_1 = J_1 \left( \frac{X, Y, X_s}{s, \theta, \rho} \right), \quad J_2 = J_2 \left( \frac{X, Y, Y_s}{s, \theta, \rho} \right)
\]
never vanish simultaneously on $U$.

Hence, a unique triplet $(s, \theta, \rho)$ is associated with any given directed lineal element $(x, y, \dot{x}, \dot{y})$ of $U$, and

$$
\begin{align*}
    x &= X(s, \theta, \rho) & \dot{x}/\rho &= X_s(s, \theta, \rho), \\
    y &= Y(s, \theta, \rho) & \dot{y}/\rho &= Y_s(s, \theta, \rho),
\end{align*}
$$

$$\rho = (x^2 + y^2)^{1/2}.$$

Consequently,

$$
(3.5) \quad F_i = F_i(X, Y, X_s, Y_s)
$$

is a function of $(s, \theta, \rho)$ which when partially differentiated with respect to $s$ yields—recalling (1.9)—

$$
(3.6) \quad \frac{dF_i}{ds} = X_s F_{1s} + Y_s F_{1y} + G[X_s F_{1y} - Y_s F_{1x}].
$$

Therefore, equation (1.12) becomes

$$
(3.7) \quad \frac{dF_i}{ds} + (X_s G_y - Y_s G_x) F_i = 0,
$$

yielding

$$
F_i(X, Y, X_s, Y_s) = M(\theta, \rho) \exp \left[ -\int_0^s (X_s \dot{G}_y - Y_s \dot{G}_x) ds \right],
$$

for suitably chosen $M = M(\theta, \rho)$. Finally, because of (1.4),

$$
(3.8) \quad J_i = X_s G[X_s Y_s - X_s Y_p] + Y_s G[X_p Y_s - X_s Y_p] + X_s [X_s Y_\theta - X_\theta Y_s] + Y_s [X_p Y_\theta - X_\theta Y_p] = X_s G[X_s Y_s - X_s Y_p] + Y_s G[X_p Y_s - X_s Y_p] + X_s [X_s Y_\theta - X_\theta Y_s] + Y_s [X_p Y_\theta - X_\theta Y_p].
$$

**4. Evaluations for $J_1$ and $J_2$.** By reason of (1.9) the geodesics (3.2) are integral curves for the following differential equations:

$$
(4.1) \quad X_{ss} = -Y_s G(X, Y, X_s, Y_s), \quad Y_{ss} = X_s G(X, Y, X_s, Y_s).
$$

It follows that

$$
(4.2) \quad J_1 \left( \frac{X, Y, X_s}{s, \theta, \rho} \right) = Y_s G[X_p Y_\theta - X_\theta Y_s] + X_s [X_p Y_s - X_s Y_p] + X_s [X_s Y_\theta - X_\theta Y_s],
$$

$$
(4.3) \quad J_2 \left( \frac{X, Y, Y_s}{s, \theta, \rho} \right) = X_s G[Y_p X_\theta - X_\theta Y_s] + Y_s [X_p Y_s - X_s Y_p] + Y_s [X_s Y_\theta - X_\theta Y_s].
$$
We now deduce from (3.3) that

\[ (4.4) \quad X_sX_{sp} + Y_sY_{sp} = 0 \]

and

\[ (4.5) \quad X_sX_{s\theta} + Y_sY_{s\theta} = 0, \]

which imply

\[ (4.6) \quad X_{sp}Y_{s\theta} - X_{s\theta}Y_{sp} = 0. \]

These above equations immediately yield

\[ (4.7) \quad X_sJ_1 + Y_sJ_2 = 0. \]

We shall now show that \((Y_sJ_1 - X_sJ_2)\) is the solution of a certain differential equation. A rather tedious direct calculation shows that

\[
\frac{\partial J_1}{\partial s} = (X_sG_\theta - Y_sG_\theta)J_1 + X_sG(X_pX_{s\theta} + Y_pY_{s\theta} - Y_sY_{sp} - X_sX_{sp}) \\
+ X_sG^2(X_pY_\theta - X_\theta Y_p),
\]

where the simplification only involves using (4.4), (4.5), and (4.6). Another like calculation shows that

\[
\frac{\partial J_2}{\partial s} = (X_sG_\theta - Y_sG_\theta)J_2 + Y_sG(Y_pY_{s\theta} + X_pX_{s\theta} - Y_sY_{sp} - X_sX_{sp}) \\
- Y_sG^2(Y_pX_\theta - X_\theta Y_p).
\]

We now infer from (4.8) and (4.9) the relation

\[ (4.10) \quad Y_s \frac{\partial J_1}{\partial s} - X_s \frac{\partial J_2}{\partial s} = (X_sG_\theta - Y_sG_\theta)(Y_sJ_1 - X_sJ_2), \]

and from (4.1) and (4.7)

\[ (4.11) \quad \frac{\partial}{\partial s} (Y_sJ_1 - X_sJ_2) = Y_s \frac{\partial J_1}{\partial s} - X_s \frac{\partial J_2}{\partial s}. \]

Thus, because of (4.10) and (4.11),

\[ (4.12) \quad \frac{\partial}{\partial s} (Y_sJ_1 - X_sJ_2) = (X_sG_\theta - Y_sG_\theta)(Y_sJ_1 - X_sJ_2), \]

which implies

\[ (4.13) \quad Y_sJ_1 - X_sJ_2 = N(\theta, p) \exp \left[ \int_0^\theta (X_sG_\theta - Y_sG_\theta) \, ds \right]. \]
for suitably chosen $N = N(\theta, \rho)$.

An important relation immediately derivable from (3.9) and (4.13) is

$$F_1(x, y, x, y) = \frac{M(\theta, \rho)N(\theta, \rho)}{\rho^3[Y_1J_1 - J_2J_3]}.$$  \hspace{1cm} (4.14)

Since, by hypothesis, $F_1(x, y, x, y) > 0$ and $(\rho J_2 - J_1J_3) \neq 0$, (4.14) implies that the density $MN \neq 0$.

5. **A contact transformation.** Henceforth, the simple, closed, regular curve $C: [(x(t), y(t)), t_0 \leq t \leq t_1]$ will be traced in the positive sense, that is, $C$ is so traversed, for increasing values of $t$, that its finite component is to the left. Because of the “field hypothesis,” associated with $(x, y, x, y)$ is a unique triplet $(s, \theta, \rho)$, and

$$x = X(s, \theta, \rho), \quad \dot{x} = \rho X(s, \theta, \rho); \quad y = Y(s, \theta, \rho), \quad \dot{y} = \rho Y(s, \theta, \rho); \quad \rho = (x^2 + y^2)^{1/2}.$$  \hspace{1cm} (5.1)

An easily established consequence of (5.1) is

$$\theta = \tau[X_pY_r - X_rY_p], \quad \dot{\theta} = \tau[X_pY_r - X_rY_p],$$  \hspace{1cm} (5.2)

where

$$\tau = \frac{(\rho - s)}{[X_pY_r - Y_rX_p]}.$$  \hspace{1cm} (5.3)

Equations (5.1) and (5.2) define a contact transformation of directed lineal elements $(x, y, x, y) \subset U$ into lineal elements $(\theta, \rho, \theta, \dot{\rho})$. Furthermore, as seen in §8, any regular, extremal convex curve $C$ is mapped by these equations into a regular curve $F_*: [(\theta(t), \rho(t)), t_0 \leq t \leq t_1]$ in the cartesian $(\theta, \rho)$-plane.

6. **Transformation of $\delta I$.** The altered form of $\delta I$, (2.5), under the contact transformation will now be computed.

Relations (1.9), (5.1), and (5.2) imply

$$(\dot{x}\dot{y} - \dot{y}\dot{x}) = \rho^2G(X, Y, X_\theta, Y_\theta)$$  \hspace{1cm} (6.1) \hspace{1cm} + \rho \tau[X_pY_rX_\theta - Y_rX_p\theta] + X_\theta(X_pY_r^2 - X_pY_rY_p) + \rho \tau[Y_pY_rX_\theta - Y_rX_p\theta] + Y_\theta(X_rY_p^2 - X_rY_pX_r)].$$

Now, computing the value of $X_\theta J_2 - Y_\theta J_1$ and using it in the simplification of (6.1) yields

$$\begin{align*}
(\dot{x}\dot{y} - \dot{y}\dot{x}) &= \rho^2[\dot{s} + \tau(X_pY_r - Y_pX_\theta)]G + \tau \rho^2[Y_sJ_1 - J_2J_3].
\end{align*}$$  \hspace{1cm} (6.2)

Finally, by reason of (1.8) and (5.3), (6.2) reduces to

$$\begin{align*}
(\dot{x}\dot{y} - \dot{y}\dot{x}) &= G(x, y, x, y) + \tau \rho^2[Y_sJ_1 - J_2J_3].
\end{align*}$$  \hspace{1cm} (6.3)

Consequently, by reason of (2.5), (4.14), (5.1), and (6.3), the variation...
δI can be expressed as

\[
(6.4) \quad \delta I = \int_{F^*} M(\theta, \varphi) N(\theta, \varphi) \left[ \frac{Y_s \delta x - X_s \delta y}{(\delta \theta^2 + \delta \varphi^2)^{1/2}} \right] ((d\theta)^2 + (d\varphi)^2)^{1/2}
\]

7. Map of admissible curves $C^*$. If the admissible curve $C^* \subset U$ encloses the origin, a restriction which is removed in §9, then, as we now show, the map of $C^*$, under the contact transformation, is a regular arc $F^*$: $[(\theta(t), \varphi(t)), t_0 \leq t \leq t_1]$, in the cartesian $(\theta, \varphi)$-plane, bounded by end points $Q_0 = (\theta(t_0), \varphi(t_0))$, and $Q_1 = (\theta(t_1), \varphi(t_1)) = (\theta(t_0) + 2\pi, \varphi(t_0))$.

The end point condition for $F^*$ is automatically fulfilled, because of the nature of the contact transformation. Furthermore, the regularity of the arc $F^*$ follows from (5.2) if it can be asserted that

\[
(7.1) \quad [X_\theta Y_\varphi - X_\varphi Y_\theta]^2 + [X_\varphi Y_\theta - X_\theta Y_\varphi]^2 \neq 0
\]

and

\[
(7.2) \quad \tau \neq 0.
\]

The negation of (7.1) gives

\[
(7.3) \quad X_\theta Y_\varphi - X_\varphi Y_\theta = 0, \quad X_\varphi Y_\theta - X_\theta Y_\varphi = 0,
\]

which, in turn, implies

\[
(7.4) \quad X_\theta Y_\varphi - Y_\varphi X_\theta = 0,
\]

since $X_\theta^2 + Y_\theta^2 = 1$. Therefore, $J_1$ and $J_2$, see (4.2) and (4.3), both vanish, which contradicts the "field hypothesis."

We now establish (7.2). Extremal convexity of $C^*$ implies that on $C^*$

\[
\frac{\dot{x} \ddot{y} - \dot{y} \ddot{x}}{\rho^3} - G\left( x, y, \frac{\dot{x}}{\rho}, \frac{\dot{y}}{\rho} \right) \neq 0.
\]

Consequently, because of (6.3), $\tau \neq 0$ on $C^*$.

Furthermore, if $C^*$ is extremal convex in the large, then the image $F^*$ is a simple arc and has the unique representation $\varphi = \varphi(\theta)$.

8. Double integral with variation $\delta I$. Let

\[
(8.1) \quad K(D^*) = - \int_{\Omega} n(\theta, \varphi, D^*) M(\theta, \varphi) N(\theta, \varphi) d\varphi d\theta,
\]

where $\Omega$ is the entire cartesian $(\theta, \varphi)$-space and $n(\theta, \varphi, D^*)$ is the topological order of the point $(\theta, \varphi)$ with respect to the closed oriented curve $D^*$. The existence of a $D^* \supset F^*$ will be established for which the variation of $I(C^*)$ is identical with the induced variation of $K(D^*)$.

$D^*$ is defined as follows. The contact transformation maps $C^*$ into $F^*$ with an orientation induced by that of $C^*$. The projections of the end points
of $F^*$, $Q_0$ and $Q_1$, on the $\theta$-axis will be denoted by $R_0 = (\theta(t_0), 0)$ and $R_1 = (\theta(t_1), 0)$. Then, $D^*$ is the oriented curvilinear polygon consisting of the directed line-segments $Q_1 R_1, R_1 R_0, R_0 Q_0$ and the directed arc $F^*$.

The verification of $\delta I(C^*) = \delta K(D^*)$ is now considered. Embed $F^* \equiv F_0^*$ in a 1-parameter family of curves

$$
\begin{align*}
F_k^*; \quad \theta &= \theta(t, k); \quad t_0 \leq t \leq t_1; \quad \theta(t_0, k) = \theta(t_0), \\
p &= p(t, k); \quad 0 \leq k \leq k_0; \quad \theta(t_1, k) = \theta(t_1),
\end{align*}
$$

for sufficiently small $k_0$. Introduce the notation $K(k) \equiv K(D_k^*)$, where $D_k^*$ and $F_k^*$ are related in the same manner as $D^*$ and $F^*$, and observe that since $D_0^*$ has at most a finite number of singularities,

$$
\delta K(D_0^*) \equiv \lim_{\epsilon \to 0} \frac{K(\epsilon) - K(0)}{\epsilon}
$$

(8.4)

$$
= - \int_{F_0^*} M(\theta, p) N(\theta, p) \left[ \frac{\delta \theta \theta \theta - \delta \theta \theta \theta}{(\theta^2 + \theta^2)^{1/2}} \right] ((d\theta)^2 + (d\theta)^2)^{1/2}.
$$

Therefore, by reason of (6.4) and (8.4), $\delta I(C^*) = \delta K(D^*)$ if

$$
\frac{\tau_0 [Y_0 \delta x - X_0 \delta y]}{(\theta^2 + \theta^2)^{1/2}} = 0
$$

(8.5)

where $(\delta \theta, \delta p) \equiv (\theta_k(l, 0), p_k(l, 0))$. Consequently,

$$
\delta K(D_0^*) \equiv \lim_{\epsilon \to 0} \frac{K(\epsilon) - K(0)}{\epsilon}
$$

(8.4)

$$
= - \int_{F_0^*} M(\theta, p) N(\theta, p) \left[ \frac{\delta \theta \theta \theta - \delta \theta \theta \theta}{(\theta^2 + \theta^2)^{1/2}} \right] ((d\theta)^2 + (d\theta)^2)^{1/2}.
$$

Therefore, by reason of (6.4) and (8.4), $\delta I(C^*) = \delta K(D^*)$ if

$$
\frac{\tau_0 [Y_0 \delta x - X_0 \delta y]}{(\theta^2 + \theta^2)^{1/2}} = 0
$$

(8.5)

where $(\delta \theta, \delta p)$ is the variation induced in $(\theta(t), p(t))$ by the variation $(\delta x, \delta y)$ of $(x(t), y(t))$. We now consider the proof of (8.5). Since

$$
\delta x = \delta sX_s + \delta \theta X_\theta + \delta pX_p
$$

(8.6)

$$
\delta y = \delta sY_s + \delta \theta Y_\theta + \delta pY_p,
$$

it follows that

$$
Y_0 \delta x - X_0 \delta y = (Y_0 X_p - X_0 Y_p) \delta p + (Y_0 X_\theta - X_0 Y_\theta) \delta \theta.
$$

Thus, because of (5.2),

$$
Y_0 \delta x - X_0 \delta y = \frac{[\delta \theta \theta \theta - \delta \theta \theta \theta]}{\tau},
$$

(8.8)

which completes the verification of (8.5).

9. The identity $I(C^*) = K(D^*)$. We now impose the following condition:

(9.1) Embedment hypothesis. The admissible curves $C^*$ can be embedded in a 1-parameter family of closed curves $C_k^*$: $(x(t, k), y(t, k))$; $t_0 \leq t \leq t_1$,
which possess the following properties: (a) \( C_k^* = C^* \) (b) The \( C_k^* \) are admissible curves which are sufficiently differentiable in the variables \((t, k)\). (c) \( C_k^* \) tends uniformly to the origin as \( k \) tends to zero. (d) The map \( F_k^* \) of \( C_k^* \), under the contact transformation, is such that \( \theta(t_0, k) \) and \( \theta(t_1, k) \) are independent of \( k \).

The identity will now be proved. Since the curves (3.2) were so defined that \( X(0, \theta, 0) = Y(0, \theta, 0) = 0 \), the "field and embedment hypothesis" imply that \( D_k^* \) tends uniformly to a finite segment on the \( \theta \)-axis, as \( k \) tends to zero. Therefore, since \( \delta I = \delta K \),

\[
I(C^*) = \lim_{k \to 0} \int_0^1 \delta I dk = \lim_{k \to 0} \int_0^1 \delta K dk = K(D^*),
\]

which is the same as

\[
\int_{c^*} F(x, y, dx, dy) = -\int_\Omega n(\theta, p, D^*)M(\theta, p)N(\theta, p)d\theta;
\]

and, if \( C^* \) is extremal convex in the large, (9.3) reduces to

\[
\int_{c^*} F(x, y, dx, dy) = -\int_{D} M(\theta, p)N(\theta, p)d\theta,
\]

since \( F^* \) now admits the unique, continuously differentiable, nonparametric representation \( p = p(\theta) \).

If \( C^* \) does not enclose the origin, \( C^* \) will lie in a curvilinear sector defined by two tangent geodesics (3.1) having minimum and maximum values of \( \theta \). If an "embedment hypothesis" can be invoked which guarantees the contraction of \( C^* \) in the sector to the origin by means of admissible curves whose end points lie on the bounding tangents, then the validity of (9.3) and (9.4) is assured.

10. Results for special metrics. If the geodesics of the metric are straight lines, then

\[
x = X(s, \theta, p) = p \cos \theta - s \sin \theta,
\]

\[
y = Y(s, \theta, p) = p \sin \theta + s \cos \theta,
\]

and

\[
\int_{c^*} F(x, y, dx, dy) = \int_{D} \rho F_1(x, y, \dot{x}, \dot{y})d\rho d\theta.
\]

Let

\[
2u = (1/p + p), \quad 2v = (1/p - p),
\]

and \( C^* \subset U: x^2 + y^2 < 1 \). Then, for the hyperbolic metric,
\[ x = X(s, \theta, \phi) = u \cos \theta - v \cos (s/v - \theta), \]
\[ y = Y(s, \theta, \phi) = u \sin \theta + v \sin (s/v - \theta), \]
and
\[
\oint_{C^*} \frac{(dx)^2 + (dy)^2}{1 - x^2 - y^2} = \int \int_D \frac{1 + p^2}{(1 - p^2)^2} \, dp \, d\theta;
\]
while in the case of the spherical metric,
\[ x = X(s, \theta, \phi) = -v \cos \theta + u \cos (s/u + \theta), \]
\[ y = Y(s, \theta, \phi) = -v \sin \theta + u \sin (s/u + \theta), \]
and
\[
\oint_{C^*} \frac{(dx)^2 + (dy)^2}{1 + x^2 + y^2} = \int \int_D \frac{1 - p^2}{(1 + p^2)^2} \, dp \, d\theta.
\]