ON RADEMACHER'S EXTENSION OF THE
GOLDBACH-VINOGRADOFF THEOREM

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1. Introduction. The Goldbach problem for odd numbers \( n \) seeks to prove that the equation

\[ n = p_1 + p_2 + p_3, \]

where the \( p_i \) are prime numbers, is always solvable. Hardy and Littlewood, using the now classical "circle" method, proved that if \( B(n) \) is the number of solutions of (1), then under certain assumptions on the zeros of Dirichlet \( L \)-functions,

\[ B(n) = \Sigma'(n) \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right), \]

where \( \Sigma'(n) \), the so-called "singular series," was proved to be greater than 0 for all odd \( n \).

Rademacher [1], using simplifications of his own as well as of Landau, proved under similar assumptions that if \( k \) be a positive integer, \( a_i \) \((i = 1, 2, 3) \) integers with \((a_i, k) = 1\), and \( A(n) \) the number of solutions of (1) with the restriction that the primes \( p_i \) belong to the residue classes \( a_i \) modulo \( k \), then

\[ A(n) = \Sigma(n) I(n) + o\left(\frac{n^2}{(\log n)^3}\right) \]

where

\[ I(n) = \int \int \frac{dudv}{\log u \log v \log (n - u - v)}, \]

and the range of integration is defined by the inequalities \( u \geq 2, v \geq 2, u + v \leq n - 2 \). He further proved that \( \Sigma(n) > 0 \) provided certain necessary arithmetic restrictions on \( n \) hold. It may be shown that

\[ I(n) = \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right). \]

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(1) Rademacher considers the more general problem of the number of representations of \( n \) as a sum of \( s \) \((s \geq 3)\) primes belonging to preassigned residue classes modulo \( k \), and obtains a precise error term.
Vinogradoff [2], using the theorems of his own on estimates of trigonometric sums together with a theorem on the uniformity of distribution of primes in an arithmetic progression due to Siegel and Walfisz, proved the Hardy-Littlewood result without assumptions.

Following the work of Vinogradoff and Rademacher, we prove the result of Rademacher without recourse to Dirichlet L-functions and the assumptions of their location of zeros.

2. Notations and preliminary results. Let \( n \) be an integer chosen sufficiently large, \( \nu = \log n \), \( \theta \) is a real number with \( |\theta| < 1 \), \( m \) is any constant \( > 3 \).

\( f(x) \ll g(x) \) means \( f(x) = O(g(x)) \), and denote \( e^{(2\pi i,q)x} \) by \( e_q(x) \).

**Theorem 2.1 (Siegel-Walfisz).** If \( \pi(n, q, t) \) denotes the number of primes \( \leq n \) in the progression \( qx+t \), \( (q, t) = 1 \), and if \( 0 < q \leq \nu^m \), then,

\[
\pi(n, q, t) = \left( \frac{1}{\phi(q)} \right) \int_2^n \frac{dx}{\log x} + O((\phi(q))^{-1}n^{\nu-15m-1}),
\]

where the constant implied by the \( O \) depends only on \( m \).

**Theorem 2.2.** Let \( (h, q) = 1 \), \( d | q \), \( (a, d) = 1 \), and denote by \( (u) \) the set of integers satisfying the conditions \( 1 \leq u \leq q, u \equiv a \pmod{d} \), \( (u, q) = 1 \). If

\[
S(q) = \sum_{(u)} e_q(hu),
\]

then,

\[
S(q) = \begin{cases} 
\mu(q/d)e_d(bah) & \text{if } ((q/d), d) = 1 \text{ and } (q/d)b \equiv 1 \pmod{k}, \\
0 & \text{if } ((q/d), d) > 1.
\end{cases}
\]

**Proof.** The proof follows Rademacher.

Let \( T(q) = \sum e_q(v) \), summed over those \( v \) such that \( 1 \leq v \leq q, v \equiv a \pmod{d} \); then

\[
T(q) = \sum_{c|q} \sum_{w} e_q \left( \frac{h}{c} \frac{q}{w} \right) = \sum_{c|q} U(c),
\]

where, for given \( c, w \) in the inner sum ranges over those integers satisfying \( (c, w) = 1, 1 \leq w \leq q, wq/c \equiv a \pmod{d} \). Moreover, if \( q_1 \mid q \) and \( T(q_1) = \sum t e_{q_1}(ht) \), where \( t \) ranges over the set \( 1 \leq t \leq q_1, t(q/q_1) \equiv a \pmod{d} \), then

\[
T(q) = \sum_{c|q_1} U(c).
\]

From (4) and (5), however, we get

\[
\sum_{q_1|q} \mu(q/q_1) T(q_1) = \sum_{q_1|q} U(q_1) \sum_{q_2|q/q_1} \mu(q/q_1q_2) = U(q) = S(q).
\]

If \( ((q/q_1), d) = 1 \), then \( T(q_1) = 0 \); moreover \( T(q_1) = e_{q_1}(hd)T(q_1) \). Conse-
quently \( T(q_1) = 0 \) for \( q_1 \mid h d \). On the other hand for \( q_1 \mid h d \), \(((q/q_1), d) = 1\). Determine \( b \) in such a way that \( bq/q_1 = i \) (mod \( d \)). Then

\[
T(q_1) = \sum_{1 \leq r \leq q_1/d} e(q_1((r/d)(vd + ba))) = (q_1/d)e(q_1(bah)).
\]

Since \( q_1 \mid d \), \( T(q_1) = 0 \) for \( q_1 < d \), and the theorem is proved.

Suppose now that

\[
(7) \quad S(x) = S_i(x) = \sum_{(p)} e(xp),
\]

where \( (p) \) denotes the set of primes satisfying the conditions \( p \leq n \), \( p \equiv a \) (mod \( k \)) with \( a = a_1, a_2, a_3 \), and \( e_1(x) = e(x) \), then

\[
(8) \quad A(n) = \int_0^1 S_1(x)S_2(x)S_3(x)e(-nx)dx = \int_0^1 f(x)dx \quad \text{(say)}.
\]

As usual, we divide the unit interval into Farey "arcs." Let \( h/q = r \) be a rational point of the unit interval with \( (h, q) = 1 \) and \( 1 \leq q \leq \nu^{\delta m} \). The major "arc" \( B_r \) belonging to \( r \) is the set of points \( x \) in \( (0, 1) \) with

\[
| x - r | \leq n^{-1}\nu^{\delta m} = r^{-1}.
\]

It is proved that no two major arcs intersect, and if \( E \) denotes the set of points not belonging to any \( B_r \), then \( x \) in \( E \) has the form

\[
x = h/q + \theta/qr \quad \text{with} \quad \nu^{\delta m} < q \leq r.
\]

Since \( f(x) \) has period 1, (8) can be written as

\[
(9) \quad A(n) = \sum_r \int_{B_r} f(x)dx + \int_E f(x)dx.
\]

3. Estimate on the major arcs.

**Theorem 3.1.** Let \( d = (k, q) \); then if \( x \) belongs to \( B_r \),

\[
(10) \quad S(x) = \frac{1}{\phi(k)} \frac{\mu(q/d)}{\phi(q/d)} e_{d(hab)} \int_2^n \frac{e(xz)}{\log x} dx + O(n\nu^{-\delta m - 1}).
\]

**Proof.** With Vinogradoff, we divide \( S(x) \) into \( O(\nu^{\delta m}) \) sums of the form

\[
(11) \quad S_u(x) = \sum_{u < \nu \leq r} e((h/q + z)p),
\]

the range of summation being further restricted by the condition \( p \equiv a \) (mod \( k \)), \( 0 < u - v \leq \nu^{\delta m} \), and since \( x \) is a point of \( B_r \), it has the form \( x = h/q + z \), with \( |z| \leq r^{-1} \). We write (11) in the form

\[
S_u(x) = \sum_j \sum_{u < \nu \leq r} e((h/q + z)p),
\]
where \( p \) in the inner sum is further restricted by \( p \equiv a \pmod{k} \) and \( p \equiv j \pmod{q} \). We deduce by the Chinese remainder theorem

\[
S_u(x) = \sum_{j \equiv a \pmod{d}} \sum_{p \equiv s \pmod{d}} e((h/q + z)p),
\]

where in the inner sum \( p \equiv s \pmod{dq/d} \). Denote the inner sum by \( S_u^j(x) \). Since \( p \equiv j \pmod{q} \), we get \( e((h/j/q + z)p) = e(hj/q + uz) + O(|z|\nu^{-9m}) \). On the other hand,

\[
S_u^j(x) = \{ e(hj/q)e(uz) + O(|z|\nu^{-9m}) \} \sum_p 1,
\]

with \( p \) satisfying the conditions of the above inner sum. Using Theorem 2.1, we deduce

\[
S_u^j(x) = (\phi(kq/d))^{-1}e(hj/q)e(uz)I_1 + O((\phi(kq/d))^{-1}(\nu^{-1-5m} + I_1|z|\nu^{-9m})),
\]

where

\[
I_1 = \int_1^v \frac{dx}{\log x}.
\]

Since \( |z||x-u| \leq |z|\nu^{-9m} \), we get

\[
I_1e(uz) = \int_1^v \frac{e(xz)}{\log x} dx + O(I_1|z|\nu^{-9m}).
\]

Consequently,

\[
S_u^j(x) = (\phi(kq/d))^{-1} \int_1^v \frac{e(xz)}{\log x} dx + O((\phi(kq/d))^{-1}(\nu^{-1-5m} + I_1|z|\nu^{-9m})).
\]

Summing over \( j \), we get,

\[
S_u(x) = (\phi(kq/d))^{-1} \int_1^v \frac{e(xz)}{\log x} dx \sum_j e(hj/q)
\]

\[+ O((\phi(kq/d))^{-1}(\nu^{-1-5m} + I_1|z|\nu^{-9m})) \sum_j 1.
\]

(12)

Here \( j \) ranges over the set \( 1 \leq j \leq q \), \( (j, q) = 1 \), \( j \equiv a \pmod{d} \). If \( ((q/d), d) = 1 \), and \( b \) is determined such that \( (q/d)b \equiv 1 \pmod{d} \), we get, by Theorem 2.3,

\[
S_u(x) = (\phi(q/d)(\phi(kq/d))^{-1})^{-1}\nu e(q/d)e(hb a/d) \int_1^v \frac{e(xz)}{\log x} dx
\]

\[+ O((\phi(q/d)(\phi(kq/d))^{-1})(\nu^{-1-5m} + I_1|z|\nu^{-9m})),
\]

while if \( ((q/d), d) > 1 \),
Continuing with (12), we observe that $l = ((q/d), d) = ((q/d), (q, k)) = ((q/d), k)$. Hence

$$S_u(x) = (\phi(k))^{-1}(\phi(q/d))^{-1}(q/d)e(hab/d) \int_0^n \frac{e(xz)}{\log x} \, dx$$

$$+ O(nq^{-1-15m} + I_1 \mid z \mid nq^{-9m}).$$

Summing over all intervals, we get

$$S(x) = (\phi(k))^{-1}(\phi(q/d))^{-1}(q/d)e(hab/d) \int_2^n \frac{e(xz)}{\log x} \, dx$$

$$+ O(nq^{-15m-1}z^m + I_1 \mid z \mid nq^{-9m} \int_2^n \frac{dx}{\log x}).$$

Since

$$\mid z \mid nq^{-9m} \int_2^n \frac{dx}{\log x} = O(nq^{-6m-1}),$$

the result follows.

**Theorem 3.2.**

$$f(x)\int_{Br} f(x) \, dx = \frac{1}{\phi(k)^3} \frac{\mu(q/d)}{\phi(q/d)^3} e((hb(a_1 + a_2 + a_3)/d) - nh/q)$$

$$\cdot \int_{-l-1}^{l-1} (I_2(z))^2 e(-nz) \, dz + O\left(\frac{1}{\phi(q/d)^2} n^2q^{-6m-3}\right),$$

where

$$I_2(z) = \left\{ \begin{array}{ll}
\int_2^n \frac{e(xz)}{\log x} \, dx & \text{if } \mid z \mid \leq n^{-1}, \\
\mid z \mid^{-1} n^{-1} & \text{if } n^{-1} < z \leq nq^{-3m}.
\end{array} \right.$$  

**Proof.** An easy calculation shows that $I_2(z) = O(\xi)$ where

$$\xi = \left\{ \begin{array}{ll}
nq^{-1} & \text{if } \mid z \mid \leq n^{-1}, \\
\mid z \mid^{-1} n^{-1} & \text{if } n^{-1} < z \leq nq^{-3m}.
\end{array} \right.$$  

We have

$$(\phi(k))^{-1}(\phi(q/d))^{-1}(q/d)e(hab/d)I_2(z) = O((\phi(q/d))^{-1}I_2(z)),$$

and since $\phi(q/d)nq^{-6m-1}(I_2(z))^{-1} = O(1)$, we deduce

$$S_1(x)S_2(x)S_3(x) = (\phi(k))^{-3}(\phi(q/d))^{-3}(q/d)e(hab(a_1 + a_2 + a_3)/d)$$

$$\cdot (I_2(z))^2 + O(\phi(q/d)^2 nq^{-6m-1}(I_2(z))^2).$$
Therefore

\[
\int_{B_r} f(x)dx = \int_{-r^{-1}}^{r^{-1}} S_1(x)S_2(x)S_3(x)e(-\frac{h}{q} + z)n)dz
\]

\[= (\phi(k)^{-3}(\phi(q/d)^{-3}\mu(q/d))e(hb(a_1 + a_2 + a_3)/d)I + I_3(z),
\]

where

\[
I = \int_{-r^{-1}}^{r^{-1}} (I_2(z))^3 e(-nz)dz,
\]

\[
I_3(z) = O\left(\left(\phi(q/d)^{-2n}\nu^{-6m-1}\right)\int_{0}^{n^{-1}z} z^2dz\right)
\]

\[= O\left(\left(\phi(q/d)^{-2n}\nu^{-6m-1}\int_{0}^{n^{-1}z} n^2\nu^{-2}dz + \int_{n^{-1}}^{n^{-1}z} n^{-2z^{-2}}dz\right)\right)
\]

\[= O((\phi(q/d)^{-2n}\nu^{-6m-3})�).
\]

**Corollary.**

\[
\sum r \int_{B_r} f(x)dx = \frac{1}{\phi(k)^3} \sum \frac{\mu(q/d)}{\phi(q/d)^3} \sum e(bh(a_1 + a_2 + a_3)/d - hn/q)I
\]

\[+ O(n^2\nu^{-3-3}).
\]

Here the inner sum ranges over the set \(i \leq h \leq q, (h, q) = 1, \) and the outer sum over the set \((d, (q/d)) = 1, q \leq \nu^{3m}.\)

4. Estimate on the minor arc.

**Theorem 4.1.** Let \((u)\) and \((v)\) be two increasing sequences of positive integers and \(w\) a positive integer. Let \(1 < N' < N_1, n_1 = \log N_1, 1 < U_0 < U_1 \leq N_1, 1 < r < N_1, x = h/q + \theta/qr, \delta = (w, q), q = \delta q_1, w = \delta w_1, \) and

\[T = \sum_u \sum_v e(xwuv),
\]

where \(u\) runs through the elements of the sequence \((u)\) satisfying the inequalities \(U_0 < u \leq U_1\) and, for given \(u, v\) ranges over those elements of the sequence \((v)\) satisfying the inequalities \(N'/u < v \leq N_1/u;\) then

\[T = O(N_1(n_1/U_0 + U_1/N_1 + q_1^3/N_1 + n_1^2/q_1 + w_1n_1^2/r)^{1/2}).
\]

**Proof.** The proof may be found in Vinogradoff [2].

Denote by \(H\) the product of all primes \(\leq n^{1/2},\) and by \((d)\) the sequence of integers satisfying the condition \(d \mid H, d \leq n.\) Using a reasoning similar to that used in the proof of Theorem 2.2, we derive the following expression for \(S(x)\)
(19) \[ S(x) = \sum_{(d)} \mu(d)S_d + O(n^{1/2}), \]

where

\[ S_d = \sum_u e(xdu). \]

Here \( u \) ranges over the sequence satisfying the conditions \( du \leq n, \ du \equiv a \pmod{k} \). We have

\[
\sum_{(d)} \mu(d)S_d = \sum_{(d_0)} S_d - \sum_{(d_1)} S_d = S_0 - S_1 \quad \text{(say)},
\]

where \((d_0)\) is the sequence of elements of \((d)\) having an even number of divisors and \((d_1)\) those elements of \((d)\) having an odd number of divisors. We estimate \(S_0; S_1\) can be estimated in exactly the same way. Write \( \lambda = \nu^{(m+1)} \), and divide \( S_0 \) into three sums,

\[ S_0 = \sum_{d \leq \lambda} S_d + \sum_{\lambda < d \leq n \lambda^{-1}} S_d + \sum_{n \lambda^{-1} \leq d \leq n} S_d = T_1 + T_2 + T_3. \]

It is understood of course that the index \( d \) ranges over the set \((d_0)\) satisfying the given inequalities.

To estimate \( T_1 \), we observe that if \( d' = d/k, d' \equiv a \pmod{k} \) and \( n_2 = n(k, d)/kd \), then

\[ S_d = \sum_{u \leq n_2} e(xd'(ku + a')). \]

Consequently, \( |S_d| \leq q \); it follows that

\[ T_1 \ll n\nu^{-m+2}. \]

To estimate \( T_2 \), we apply Theorem 4.1. We have

\[ T = \sum_d \sum_u e(xdu) \]

with the prescribed ranges of summation. We have here \( N_1 = n, U_0 = \lambda, U_1 = n\lambda^{-1}, w = 1 \). Theorem 4.1 yields

\[ T_2 \ll n(\nu^{-m-1})^{1/2} \ll n\nu^{-m+2}. \]

We turn now to the estimate of \( T_3 \). We have

\[ T_3 = \sum_d \sum_u e(xdu), \]

summed over the prescribed ranges for \( d \) and \( u \). Interchange the order summation, then

\[ T_3 = \sum_{u \leq \lambda} \sum_{n \lambda^{-1} < d \leq n/u} e(xdu) = \sum_u T(u) \]
with the inner sum further restricted by the condition \(du \equiv a \pmod{k}\). We divide the sequence \((d)\) into two sequences \((d')\) and \((d'')\) where \((d')\) is the set of \((d)\) having all prime divisors \(\leq \nu^3m\) and \((d'')\) those elements of \((d)\) having at least one prime divisor \(> \nu^3m\). \((d_0')\) is then divided into two corresponding sets \((d_0')\) and \((d_0'')\). We get \(T(u) = T'(u) + T''(u)\) where the right-hand summands correspond to the sets \((d_0')\) and \((d_0'')\) respectively. We estimate now the number of terms \(D\) of the set \((d')\) which satisfy the conditions \(d \leq n/u\) and \(1 \leq u \leq \lambda\). To this end suppose that an element \(d\) of \((d')\) have \(j\) prime divisors. Then \((\nu^3m)^j \geq n\lambda^{-1}\), and hence if \(n\) be chosen sufficiently large \(j > \nu/6m \log \nu\). If then \(\tau(d)\) be the number of divisors of \(d\), we get

\[
\tau(d) = 2^i > 2^{\nu/6m \log \nu} > n^{1/9m} \log \nu,
\]

and since

\[
\sum_{1 \leq u \leq n_1} \tau(u) \ll n_1(\nu + 1),
\]

where \(n_1 = n/u\), we conclude that

\[
Dn^{1/9m} \log \nu \ll n_1(\nu + 1) \ll n_1n^{1/9m} \log \nu^{-1/9m} \log \nu \nu^{-m}(\nu + 1).
\]

Therefore \(D \ll n\nu^{-m}\). Consequently we deduce that

\[
T(u) = T''(u) + O(\nu^{-m}nu^{-1}).
\]

For the sum \(T''(u)\) we have evidently \(j < \nu\), hence

\[
T''(u) = \sum_j T_j(u)
\]

where \(T_j(u)\) is summed over those \(d\) belonging to \((d_0'')\) satisfying the inequalities \(n\lambda^{-1} < d \leq n_1\), and having exactly \(j\) prime divisors \(> \nu^3m\). In order to estimate the sum \(T_j(u)\) we consider with Vinogradoff the more general sum

\[
T_j'(u) = \sum_v \sum_w e(xuvw)
\]

where \(v\) ranges over all primes \(> \nu^3m\) belonging to \((d)\) and, for given \(v\), \(w\) ranges over those numbers satisfying the inequalities \(n\lambda^{-1}/v < w \leq n_1/v\), the congruence \(uvw \equiv a \pmod{k}\) and containing exactly \(j-1\) prime divisors \(> \nu^3m\) and belonging to \((d_1)\). Each term \(e(xdu)\) of the sum \(T_j(u)\) is found in the sum \(T_j'(u)\) and indeed is found exactly \(j\) times. In addition, however, \(T_j'(u)\) contains terms of the form \(e(xp^2w)\) with \(n\lambda^{-1}/p^2 < w_1 \leq n_1/p^2\), where \(p > \nu^3m\), and \(w_1\) runs over elements of \((d_0)\) containing \(j-2\) prime divisors \(> \nu^3m\). These terms evidently occur without duplication. For given \(p\), the number of \(w_1p^2\) satisfying \(n\lambda^{-1}/p^2 < w_1 \leq n_1/p^2\) is \(\leq n_1/p^2\), consequently

\[
T_j'(u) = jT_j(u) + O\left(\sum_{\nu^3m < p \leq n_1/p^2} n_1/p^2\right) = jT_j(u) + O(\nu^{-3}mn\nu^{-1}).
\]
We now apply Theorem 4.1 to the sum $T'(u)$. We take $U_0 = \nu^{3m}$, $U_1 = n^{1/2}$, $N' = n\lambda^{-1}$, and conclude
\[ T(u) \ll n/u(\nu^{-3m+2})^{1/2} \ll n\nu^{-3m/2+1}u^{-1/2}. \]
Therefore, $T'(u) \ll j^{-1}nu^{-1/2}\nu^{-3m/2+1}$, from which we deduce that
\[ T''(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu, \]
and hence that
\[ T(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu + nu^{-1}\nu^{-m}. \]
Summing over $u$, we deduce that
\[ T_3 \ll n\nu^{-m+1} \log \nu + n\nu^{-m} \log \nu \ll n\nu^{-m+2}. \]
Using (19), (20), (21), (22), and (23), we conclude the following:

**Theorem 4.2.** Let $m$ be any constant $> 3$, $x = h/q + \theta/qr$, $(h, q) = 1$, $\nu^{3m} < q \leq \tau$, $\tau = n\nu^{-3m}$; then
\[ \sum_{(p)} e(xp) = O(n\nu^{-m+2}). \]

5. The asymptotic formula and proof of the theorem.

**Theorem 5.1.** If $m > 9/4$, then
\[ I = \int_{-r^{-1}}^{1} (I_2(z))^3 e(-nz)dz = \frac{n^2}{2\nu^3} + O\left(\frac{n^2}{\nu^{7/2}} \log \nu\right). \]

**Proof.** Vinogradoff [2].

Using this result, we deduce readily that
\[ \sum_{q} (\phi(q/d))^{-3}\mu(q/d) \sum_{h} e(bh(a_1 + a_2 + a_3)/d - hn/q) = O(n^2\nu^{-4}), \]
where the inner sum ranges over the set of $h$ such that $1 \leq h \leq q$, $(h, q) = 1$, and the outer sum over those $q$ satisfying $((q/d), d) = 1$, $q > \nu^{3m}$. This result, together with (17), permits us to conclude that
\[ \sum_{r} \int_{B_r} f(x)dx = \mathcal{S}(n) \frac{n}{2\nu^3} + O(n^2\nu^{-7/2} \log \nu), \]
where $\mathcal{S}(n)$, the singular series, is given by
\[ \mathcal{S}(n) = \frac{1}{\phi(k)^3} \sum_{q=1}^{\infty} \frac{\mu(a/d)}{\phi(q/d)^3} \sum_{1 \leq h \leq q} e(hb(a_1 + a_2 + a_3)/d - hn/q), \]
where as above $q$ is restricted by $((q/d), d) = 1$ and $h$ by $(h, q) = 1$. On the
other hand, using Theorem 4.2, we get
\[
\int_E f(x)dx \ll \int_E |S_1(x)S_2(x)S_3(x)| dx
\]
\[
\ll n^{\nu-m+1} \int_0^1 |S_2(x)S_3(x)| dx
\]
\[
\ll n^{\nu-m+2}\left(\int_0^1 |S_2(x)|^2 dx\right)^{1/2}\left(\int_0^1 |S_3(x)|^2 dx\right)^{1/2}
\]
\[
\ll n^{\nu-m+2}\nu^{-1} < n^{2\nu-m+1}.
\]

From (9), (26), and (28), we conclude
\[
A(n) = \mathcal{S}(n) \frac{n^2}{2(\log n)^3} + O\left(\frac{n^2}{(\log n)^{7/2}} \log \log n\right).
\]

On the other hand Rademacher has shown that if \( n \) is odd and \( n = a_1 + a_2 + a_3 \) (mod \( k \)), then
\[
\mathcal{S}(n) = \frac{C}{k^2} \prod_{p|k} \frac{p^3}{(p-1)^3 + 1} \prod_{p|n, p \nmid k} \frac{(p-1)((p-1)^2 - 1)}{(p-1)^3 + 1} \prod_{p > 2} \left(1 + \frac{1}{(p-1)^3}\right),
\]
where throughout \( p > 2 \), \( C = 2 \) for odd \( k \), and \( C = 8 \) for even \( k \). If \( n \) fails to satisfy the above conditions, then \( \mathcal{S}(n) = 0 \).

We formulate the:

**Main Theorem.** Let \( k \) be a positive integer, \( a_1, a_2, a_3 \) be residue classes modulo \( k \) with \((a_i, k) = 1\). If \( n \) is a sufficiently large odd integer satisfying the congruence \( n = a_1 + a_2 + a_3 \) (mod \( k \)), then \( n \) can be represented as a sum of three primes belonging respectively to the residue classes \( a_1, a_2, a_3 \), modulo \( k \). The asymptotic formula for the number of representations is given by (29).

**6. Concluding remarks.** The method of Linnik-Tchudakoff will provide another proof of this result. The corresponding question for the simultaneous Goldbach-Waring problem may be posed and solved.

**Bibliography**


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