COMPOUND GROUP EXTENSIONS. II

BY

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1. Introduction. Consider a system of groups and homomorphisms

\[
\begin{array}{c}
\ldots \\
R_0 \xrightarrow{\rho_0} R_1 \xrightarrow{\beta_1} R_2 \xrightarrow{\rho_3} \ldots \\
S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_3} \ldots \\
\end{array}
\]

(1.1)

satisfying the following conditions:

(1.2) each horizontal sequence is exact\(^{(2)}\),
(1.3) commutativity holds in each rectangle,
(1.4) \(\sigma_1 \beta_1(R_1) = \sigma_1(S_1)\).

Diagram (1.1) may be thought of as extending to infinity in both directions, with only one gap, as shown. It will soon be apparent, however, that the "infinite part" has no bearing on our problem.

By a completion of (1.1) we mean a group \(R_3\), together with three homomorphisms

\[
\beta_0: R_1 \rightarrow P_1, \quad \beta_1: R_2 \rightarrow P_1, \quad \beta_2: R_2 \rightarrow S_2,
\]

such that when these are inserted in (1.1), conditions (1.2) and (1.3) still hold.

It is easily seen that if (1.1) admits a completion, then

(1.5) \(\rho_0(R_0)\) is a normal subgroup of \(R_1\),
(1.6) \(\beta_3[\rho_3^{-1}(0)] \subseteq \sigma_2(S_2)\).

Henceforth we shall assume that (1.1) satisfies conditions (1.2)–(1.6).

Two completions \((R_2, \rho_1, \rho_2, \beta_2)\) and \((R'_2, \rho'_1, \rho'_2, \beta'_2)\) will be called isomorphic if there exists an isomorphism \(W: R_2 \rightarrow R'_2\) such that

\[
W \rho'_1 = \rho_1, \quad W \rho'_2 = \rho'_2, \quad W \beta_2 = \beta'_2.
\]

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\(^{(1)}\) This work was partially supported by the Office of Scientific Research (ARDC). The terminology and notational conventions of [6] are freely used throughout. Numbers in brackets refer to the list of references at the end of the paper.

\(^{(2)}\) The statement that a sequence is exact is to be understood as applying only in those groups which are both ranges and domains for indicated homomorphisms. Thus, condition (1.2) is not meant to imply that \(\rho_0(R_0) = R_1\) or that \(\rho_3^{-1}(0) = 0\). When we wish to indicate that the initial homomorphism of an exact sequence is an isomorphism-into, or that the terminal homomorphism is onto, we attach appropriate zeros.

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If \( R_0 = S_0 = R_4 = S_4 = 0 \) and \( \beta_3 \) is an isomorphism of \( R_3 \) onto \( S_3 \), then conditions (1.5) and (1.6) are redundant, (1.4) becomes \( \beta_1(R_1) = S_1 \), and by comparing the present situation with diagram (1.2) of [6], we see that a "completion" is now virtually the same thing as a continuation of the normal homomorphism \( \sigma_i \beta_1 : R_1 \rightarrow S_2 \). Thus the problem of classifying the completions of (1.1) is a generalization of the problem of classifying the continuations of a normal homomorphism. The case "\( R_0 = S_0 = R_4 = S_4 = 0, \beta_3 \) iso-onto" will be called the "normal homomorphism case."

In this paper we shall show that the general problem represented by (1.1) can be reduced to the normal homomorphism case. That is, with each diagram (1.1) we shall associate a certain normal homomorphism whose continuations are in a natural 1-1 correspondence with the completions of (1.1). This, in conjunction with the results of [6], constitutes the solution of the more general problem. An application will be found in [7].

2. The categorical case. In this section we consider a diagram (1.1) satisfying conditions (1.2) and (1.3), and also satisfying

\[
\begin{align*}
(2.1) & \quad R_0 = S_0 = R_4 = S_4 = 0, \\
(2.2) & \quad \beta_1 \text{ is an isomorphism of } R_1 \text{ onto } S_1.
\end{align*}
\]

We note that (2.1) implies (1.5) and (1.6), while (2.2) implies (1.4).

Theorem 2.3. Let (1.1) satisfy conditions (1.2), (2.1), and (2.2). Then there exists one and only one isomorphism-class of completions. Moreover, any two completions are isomorphic in only one way.

Proof. This result has been observed (in essence) by Baer(4) and Mac-Lane(4) in the following context: \( S_2 \) is the group \( A(K) \) of all automorphisms of some group \( K \); \( R_1 = S_1 = \) the group \( I(K) \) of all inner automorphisms of \( K \); \( S_3 = A(K)/I(K) \); \( \beta_1 \) is the identity mapping of \( I(K) \) onto itself; \( \sigma_1 \) is the identity injection of \( I(K) \) into \( A(K) \); \( \sigma_2 \) is the natural homomorphism of \( A(K) \) onto \( A(K)/I(K) \). (In this context, \( R_3 \) is commonly denoted \( Q \), and \( \beta_3 \) is denoted \( \theta \).) For our purposes, however, the special nature of these groups is irrelevant. The following reproduces Baer's construction in the notation of (1.1).

We may assume without loss of generality that \( R_1 = S_1 \subset S_2 \), \( \beta_1 \) = identity mapping of \( S_1 \) onto itself, \( \sigma_1 \) = identity injection of \( S_1 \) into \( S_2 \). Let \( (\Gamma, \epsilon, \pi_1, \pi_2) \) denote the graph of \( \beta_3 \) rel. \( \sigma_2 \), in the sense of [6, §6]. That is, \( \Gamma \) denotes the subgroup of the direct sum(4) \( R_3 + S_4 \) consisting of those ordered pairs \( (r_3, s_2) \) such that \( \beta_3(r_3) = \sigma_2(s_2) \); the homomorphisms \( \epsilon, \pi_1, \pi_2 \) displayed in the diagram

(4) Since an empty set is never regarded as an isomorphism-class (or as an equivalence class of any kind), there exists an isomorphism-class of completions if and only if there exists a completion.

(4) [1]; see also [4, §12], and [5, §3].

(4) We speak of the direct sum rather than direct product in keeping with our convention of writing most groups additively.
are defined by

\[(2.5) \quad \epsilon(s_1) = (0, s_1); \quad \pi_1(r_3, s_2) = r_3; \quad \pi_2(r_3, s_2) = s_2.\]

It follows immediately from the definitions that both rectangles of (2.4) are commutative, and that \(\epsilon^{-1}(0) = 0\). Given any \(r_3 \in R_3\), since \(\sigma_2(S_2) = S_3\), we may choose \(s_2 \in S_2\) such that \(\sigma_2(s_2) = \beta_3(r_3)\). Then \((r_3, s_2) \in \Gamma,\) and \(\pi_1(r_3, s_2) = r_3\).

Therefore \(\pi_1(\Gamma) = R_3\). \(\pi_1^{-1}(0)\) consists of all pairs \((r_3, s_2)\) such that \(\beta_3(r_3) = \sigma_2(s_2)\) and \(r_3 = 0\); that is, all pairs \((0, s_2)\) such that \(\sigma_2(s_2) = 0\); that is, all pairs \((0, s_1)\), where \(s_1 \in S_1\). Clearly this equals \(\epsilon(S_1)\). Therefore \((\Gamma, \epsilon, \pi_1, \pi_2)\) is a completion.

Now let \((R_2, \rho_1, \rho_2, \beta_2)\) be any completion; then conditions (1.2) and (1.3) hold in the diagram

\[(2.6) \quad 0 \rightarrow S_1 \xrightarrow{\rho_1} R_2 \xrightarrow{\rho_2} R_3 \rightarrow 0 \]

From commutativity in the right-hand rectangle of (2.6), it follows that for every \(r_2 \in R_2\), \([\rho_2(r_2), \beta_2(r_2)] \in \Gamma\). Hence we can define \(W: R_2 \rightarrow \Gamma\) by

\[(2.7) \quad W(r_2) = [\rho_2(r_2), \beta_2(r_2)].\]

Obviously \(W\) is a homomorphism. It follows immediately from the definition that \(\pi_1W = \rho_2\), and \(\pi_2W = \beta_2\); it follows from commutativity in the left-hand rectangle of (2.6), and from \(\rho_2\rho_1 = 0\), that \(W \rho_1 = \epsilon\). Hence the appropriate versions of (1.7) are satisfied.

By the well known “five-lemma,” which holds also for nonabelian groups, \(W\) is an isomorphism of \(R_2\) onto \(\Gamma\).

Formula (2.7) is “forced” by conditions (1.7b) and (1.7c), which, as we indicated above, are written \(\pi_1W = \rho_2\) and \(\pi_2W = \beta_2\), respectively, in the present notation. This proves that there is only one isomorphism\(^{(6)}\) of the arbitrary completion \((R_2, \rho_1, \rho_2, \beta_2)\) onto the particular completion \((\Gamma, \epsilon, \pi_1, \pi_2)\). It follows by an obvious argument that any two completions are isomorphic in only one way. Q.E.D.

In view particularly of the final assertion of Theorem 2.3, we may say that for any use to which the graph \((\Gamma, \epsilon, \pi_1, \pi_2)\) may be put, an entirely

\(^{(6)}\) In fact, there is only one “homomorphism,” in the obvious sense; moreover, condition (1.7a) is redundant in the concept of a “homomorphism” of one completion into another, under the conditions of Theorem 2.3.
arbitrary completion will serve just as well. However, we shall adhere to the particular completion \((\Gamma, e, \pi_1, \pi_2)\) as a matter of convenience.

3. The case \(R_0 = S_0 = R_1 = S_4 = 0\). In this section we assume that (1.1) satisfies (1.2)–(1.4) and (2.1). In the presence of (2.1), (1.4) may be rewritten \(\beta_1(R_1) = S_1\). As before, we let \((\Gamma, e, \pi_1, \pi_2)\) denote the graph of \(\beta_2: R_3 \to S_3\) rel. \(\sigma_2: S_2 \to S_3\); we assume, for simplicity of notation, that \(S_1 \subseteq S_2\) and that \(\sigma_1\) is the identity injection of \(S_1\) into \(S_2\). Then conditions (1.2) and (1.3) hold in the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & R_1 \\
\downarrow & & \downarrow \beta_1 \\
S_1 & \rightarrow & \Gamma \\
\downarrow \sigma_1 & & \downarrow \pi_1 \\
S_1 & \rightarrow & S_2 \\
\downarrow & & \downarrow \pi_2 \\
S_2 & \rightarrow & S_3 \\
\downarrow & & \downarrow \beta_3 \\
0 & \rightarrow & S_3 \\
\end{array}
\]

Let \((3.1')\) denote diagram (3.1) with the bottom row deleted. By a completion of (3.1) we mean a completion of \((3.1')\). But as we indicated in §1, a completion of \((3.1')\) may be regarded as the same thing as a continuation of the normal homomorphism \(\epsilon \beta_1: R_1 \to \Gamma\). It turns out that the completions of (1.1) [assuming (2.1)] are in a natural 1-1 correspondence with the completions of (3.1). Hence we may say that the case \(R_0 = S_0 = R_1 = S_4\) "factors," in the manner indicated by (3.1), into the normal homomorphism case and the categorical case. Formally:

**Theorem 3.2.** Let (1.1) satisfy conditions (1.2)–(1.4) and (2.1). Let \((\Gamma, e, \pi_1, \pi_2)\) denote the graph\(^\star\) of \(\beta_2: R_3 \to S_3\) rel. \(\sigma_2: S_2 \to S_3\). Then the completions of (1.1) are in a natural 1-1 correspondence with the completions of (3.1).

**Corollary 3.3.** Under the hypotheses of Theorem 3.2, the isomorphism-classes of completions of (1.1) are in a natural 1-1 correspondence with the isomorphism-classes of continuations of the normal homomorphism \(\epsilon \beta_1: R_1 \to \Gamma\).

By way of proof of Theorem 3.2, we shall describe the required constructions, omitting detailed verification.

Let \((R_2, \rho_1, \rho_2, \delta)\) be a completion of (3.1). Then conditions (1.2) and (1.3) hold in

\(^\star\) Instead of requiring that \((\Gamma, e, \pi_1, \pi_2)\) be the actual graph, it is sufficient, by Theorem 2.3, to demand that \(\Gamma\) be a group, and \(e, \pi_1, \pi_2\) be homomorphisms, such that conditions (1.2) and (1.3) hold throughout (3.1).
It follows immediately that \((R_3, \rho_1, \rho_2, \pi_3)\) is a completion of (1.1).
Conversely, let \((R_3, \rho_1, \rho_2, \beta_3)\) be a completion of (1.1). Define \(\delta: R_2 \to \Gamma\) by
\[
\delta(r_2) = \left[\rho_2(r_2), \beta_3(r_2)\right].
\]

Then conditions (1.2) and (1.3) may be verified throughout the top half of (3.4). Therefore \((R_3, \rho_1, \rho_2, \delta)\) is a completion of (3.1).

It is easily seen that these constructions are inverses of each other. Q.E.D.

4. The general case. When (1.1) satisfies only conditions (1.2)—(1.6), it is easily seen that its completions are in a natural 1-1 correspondence with the completions of the following diagram derived from (1.1):

\[
\begin{array}{c}
0 \longrightarrow R_1/\rho_0(R_0) \qquad \rho_3^{-1}(0) \longrightarrow 0 \\
\sigma_1^{-1}(0) \quad \sigma_2^{-1}(0) \rightarrow S_2 \quad \sigma_3^{-1}(0) \rightarrow 0,
\end{array}
\]

where \((\sigma_1\beta_1)_*: R_1/\rho_0(R_0) \to \sigma_2^{-1}(0)\) is the homomorphism induced by \(\sigma_1\beta_1: R_1 \to \sigma_2^{-1}(0)\).

Let \((\Gamma, e, \pi_1, \pi_2)\) denote the graph of \(\beta_3|\rho_3^{-1}(0)\) rel. \(\sigma_2: S_2 \to \sigma_3^{-1}(0)\), in the sense of [6, §6]. Then, combining the obvious reduction of (1.1) to (4.1) with Corollary 3.3, we obtain

**Theorem 4.2.** The isomorphism-classes of completions of (1.1) are in a natural 1-1 correspondence with the isomorphism-classes of continuations of the normal homomorphism \(\epsilon(\sigma_1\beta_1)_*: R_1/\rho_0(R_0) \to \Gamma\).

5. Modular structures. We supplement the above results with a theorem concerning modular structures in a system (1.1). For simplicity of notation, we shall confine attention to the case \(R_0 = S_0 = R_4 = S_4 = 0\); the theorem will be obviously applicable to the general case via diagram (4.1). We shall also assume, for simplicity of notation, that \(S_1 \subset S_2\) and that \(\sigma_1\) is the identity mapping of \(S_1\) into \(S_2\); then \(S_1 = \beta_1(R_1) = \sigma_2^{-1}(0)\).

Let \((\Gamma, e, \pi_1, \pi_2)\) denote the graph of \(\beta_2: R_3 \to S_3\) rel. \(\sigma_2: S_2 \to S_3\). Let \(X\) denote the subgroup \((e\beta_1)^{-1}(0) = \beta_1^{-1}(0)\) of \(R_1\); let \(N = e\beta_1(R_1) = \epsilon(S_1) = \pi_1^{-1}(0) \subset \Gamma\). Let

\[\text{Note that } \sigma_2\sigma_1\beta_1 = 0 \text{ and } \sigma_1\beta_1\rho_0 = \sigma_1\sigma_0\beta_0 = 0.\]
\[ \lambda: A X(R_i) \to A X(R_i)/c(X), \]
\[ (\beta_1)_!: A X(R_i) \to A(S_1), \]
\[ (\beta_1)_*: A X(R_i)/c(X) \to A(S_1), \]
\[ (\epsilon_1)_!: A X(R_i) \to A(N), \]
\[ (\epsilon_1)_*: A X(R_i)/c(X) \to A(N), \]
\[ C': S_2 \to A(S_1), \]
\[ C'': \Gamma \to A(N), \]
\[ \epsilon_1: A(S_1) \approx A(N) \]
denote the natural homomorphisms, in the manner of [6, §2]. Then the following relations are easily verified:

\[(5.1) \quad \epsilon_1(\beta_1)_! = (\epsilon_1)_{\ast}, \quad \epsilon_1(\beta_1)_* = (\epsilon_1)_{\ast}, \quad \epsilon_1 C' \pi_2 = C''. \]

Given a completion \((R_2, \rho_1, \rho_2, \beta_2)\), we let \(C: R_2 \to A X(R_1)\) denote the homomorphism provided by (a) conjugation of elements of \(\rho_1(R_1)\) by elements of \(R_2\), (b) the isomorphism of \(A[\rho_1(R_1)]\) onto \(A(R_1)\) induced by \(\rho_1: R_1 \approx \rho_1(R_1)\). We say that a modular structure \(\theta: \Gamma \to A X(R_1)/c(X)\) on the normal homomorphism \(\epsilon_1: R_1 \to \Gamma\) is associated with the completion \((R_2, \rho_1, \rho_2, \beta_2)\) if it is associated with the continuation of \(\epsilon_1: R_1 \to \Gamma\) which corresponds to \((R_2, \rho_1, \rho_2, \beta_2)\) via Theorem 3.2: that is, if

\[(5.2) \quad \theta[\rho_2(r_2), \beta_2(r_2)] = \lambda C(r_2) \quad \text{for every } r_2 \in R_2. \]

According to [6], the continuations of \(\epsilon_1: R_1 \to \Gamma\) divide into classes according to the modular structures with which they are associated. Hence the completions of (1.1) divide into classes according to the modular structures on the normal homomorphism \(\epsilon_1: R_1 \to \Gamma\) with which they are associated, in the sense of (5.2). In particular, we might be given a definite modular structure \(\theta\) on \(\epsilon_1: R_1 \to \Gamma\), and we might wish to classify those completions which are associated with \(\theta\). By Corollary 3.3, this is equivalent to classifying the extensions of the pseudomodule \((\epsilon_1, \theta\) ); the answer to the latter problem is given by the results of [6, §§6–10].

However, it frequently happens (as for example in [7]) that instead of a modular structure on \(\epsilon_1: R_1 \to \Gamma\), we are given a modular structure \(\bar{\theta}: S_2 \to A X(R_1)/c(X)\) on the normal homomorphism \(\beta_1: R_1 \to S_2\), and we are interested in those completions which are "associated" with \(\bar{\theta}\), by which we mean those completions \((R_2, \rho_1, \rho_2, \beta_2)\) which satisfy

\[(5.3) \quad \bar{\theta}\beta_2 = \lambda C. \]

**Theorem 5.4.** Let (1.1) satisfy conditions (1.2)–(1.4) and (2.1), and let \(\bar{\theta}: S_2 \to A X(R_1)/c(X)\) be a modular structure on the normal homomorphism \(\beta_1: R_1 \to S_2\). Then

(a) the homomorphism \(\bar{\theta}\pi_2: \Gamma \to A X(R_1)/c(X)\) is a modular structure on the normal homomorphism \(\beta_1: R_1 \to \Gamma\); 

(b) a completion of (1.1) is associated with \(\bar{\theta}\), in the sense of (5.3), if and only if it is associated with \(\bar{\theta}\pi_2\), in the sense of (5.2);
(c) the isomorphism-classes of those completions of (1.1) which are associated with \( \bar{\theta} \) are in a natural 1-1 correspondence with the isomorphism-classes of extensions of the pseudo-module \((\epsilon \beta_1, \bar{\theta} \pi_2)\).

**Proof.** Part (b) turns out to be a tautology; part (c) follows from (a), (b), and Corollary 3.3. It only remains to prove part (a).

By hypothesis, commutativity holds throughout

\[
\begin{array}{ccc}
R_1 & \xrightarrow{\beta_1} & S_2 \\
\downarrow c_{R_1} & & \downarrow \bar{\theta} \\
A_x(R_1) & \xrightarrow{\lambda} & A_x(R_1)/c(x) \\
& \downarrow & \downarrow (\beta_1)_* \\
& A(S_1) & A(N).
\end{array}
\]

Assertion (a) is that commutativity holds throughout

\[
\begin{array}{ccc}
R_1 & \xrightarrow{\epsilon \beta_1} & \Gamma \\
\downarrow c_{R_1} & & \downarrow \bar{\theta} \pi_2 \\
A_x(R_1) & \xrightarrow{\lambda} & A_x(R_1)/c(x) \\
& \downarrow & \downarrow (\epsilon \beta_1)_* \\
& A(N) & A(N).
\end{array}
\]

Commutativity in the rectangle of (5.6) follows immediately from commutativity in the rectangle of (5.5), since \( \pi_2 \epsilon \) is the identity mapping of \( S_1 \) onto itself.

Using formulae (5.1b) and (5.1c), and commutativity in the triangle of (5.5), we have \((\epsilon \beta_1)_* \bar{\theta} \pi_2 = \epsilon \beta(\beta_1)_* \bar{\theta} \pi_2 = \epsilon \beta C' \pi_2 = C''\), establishing commutativity in the triangle of (5.6). Q.E.D.

**References**


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