

ON SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY THE HAHN-BANACH THEOREM

BY

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1. Introduction. In this paper we show how to construct the Green's and Neumann's functions of a linear elliptic partial differential equation by a method based on linear functionals. The construction depends essentially on the Hahn-Banach extension theorem [1], which has been used recently in a similar connection by Lax [5]. Lax's approach differs from ours in that, while his proof centers about a bounded linear functional based on inhomogeneous boundary conditions, our proof centers rather about a functional based on the solution of an inhomogeneous differential equation. This latter point of view has the advantage that in constructing the Green's and Neumann's functions we need only a fundamental solution of the differential equation in the small, and, indeed, we can succeed with merely a parametrix, which is readily obtained for a linear elliptic partial differential equation, whereas all earlier existence proofs of the type studied here [3; 4; 5] are based on the previous knowledge of a fundamental solution in the large. Our considerations are also advantageous for domains with general boundaries and for Riemannian manifolds.

For earlier work on application of the Hahn-Banach theorem to boundary value problems, we refer to the papers of Caccioppoli [2] and of Miranda [6]. It should be mentioned at the outset that the proof of the Hahn-Banach theorem in the form in which we shall need it in this paper does not require transfinite induction, since the Banach space of continuous functions which we use is separable.

2. The Neumann's function. Let $P = P(z) \geq \epsilon > 0$ be a positive, continuously differentiable function of the point z in a plane domain D bounded by a finite number of simple closed analytic curves C . Denote by B the Banach space of all functions f which are continuous in $D + C$, with the norm

$$\|f\| = \max |f|,$$

and denote by B' the subspace of B of functions

$$(1) \quad f = \Delta u - Pu,$$

where u is any function, continuous with its first and second derivatives in $D + C$, whose inner normal derivative vanishes on C ,

$$(2) \quad \frac{\partial u}{\partial \nu} = 0.$$

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For any point w in D , we define a linear functional $L_w\{f\}$ on B' by the formula

$$(3) \quad u(w) = L_w\{\Delta u - Pu\}.$$

That this linear functional is uniquely defined and bounded can be shown by the following argument.

First, if

$$(4) \quad \Delta U - PU \leq 0$$

in D and

$$(5) \quad \frac{\partial U}{\partial \nu} = 0$$

on C , then $U \geq 0$ in D . For if $U < 0$ in some subregion of D , we could apply Green's theorem to that subregion to obtain

$$(6) \quad \iint [(\nabla U)^2 + PU^2]d\tau = - \iint U[\Delta U - PU]d\tau - \int U \frac{\partial U}{\partial \nu} ds,$$

where $d\tau$ is the area element and where s denotes the arc length along the boundary curves of the region $U < 0$. By (4), (5), and the condition $U < 0$, it is clear that the right-hand side of (6) is nonpositive, whereas it is equally clear that the left-hand side of (6) is positive. This contradiction yields the original statement $U \geq 0$. Uniqueness of the functional L_w for given $\Delta u - Pu$, as defined in (3), is an immediate consequence of this.

Second, let $v_1 = 1/\epsilon$, so that

$$(7) \quad \Delta v_1 - Pv_1 \leq -1,$$

and let $M = \|f\|$. Then in D

$$\Delta(Mv_1 - u) - P(Mv_1 - u) \leq -M - f = -\|f\| - f \leq 0,$$

and on C

$$\frac{\partial}{\partial \nu}(Mv_1 - u) = 0,$$

whence $Mv_1 - u \geq 0$. It follows from the definition (3) that for f in B' ,

$$(8) \quad L_w\{f\} \leq \frac{1}{\epsilon} \|f\|,$$

which proves that L_w is a bounded functional.

We conclude from the Hahn-Banach theorem that the bounded linear functional L_w can be extended from the vector space B' to operate on all elements of the Banach space B and to satisfy (8) there. By applying the ex-

tended functional in a suitable way, we shall set up an expression which we shall prove to be the Neumann's function of the linear elliptic partial differential equation

$$(9) \quad \Delta u - Pu = 0$$

in D .

For the construction, we need a so-called parametrix $S(w, z)$ for (9). Such a parametrix is a function of w which has a logarithmic singularity at $w = z$ and is such that $\Delta S - PS$ is continuous, even at the point z . More precisely, we shall require certain regularity properties of $\Delta S - PS$ as a function of z , and for our present purposes we select $S(w, z)$ to be a symmetric function of w and z . Let

$$(10) \quad g(w, z) = \log \frac{1}{|w - z|} + \dots$$

be a fundamental solution of Laplace's equation in a region E . When E is a semicircle, it is an elementary problem to construct g explicitly in such a way that g vanishes on the circular arc bounding E and has zero normal derivatives on the diameter bounding E . We set

$$I_1(w, z) = \frac{1}{2\pi} \iint_E P(t)g(w, t)g(z, t)d\tau,$$

$$I_2(w, z) = \frac{1}{2\pi} \iint_E P(t)g(w, t)I_1(z, t)d\tau,$$

and we set

$$(11) \quad S_0(w, z) = g(w, z) - I_1(w, z) + I_2(w, z),$$

whence by Poisson's equation

$$\Delta S_0 - PS_0 = -PI_2,$$

which shows, for our purposes, that S_0 is a parametrix for (9). If E is the above semicircle and g is explicitly constructed in the manner described for a semicircle, then S_0 will have a normal derivative which vanishes on the diameter bounding E . For z in the neighborhood of a point z_0 in which we are interested, we obtain for all w a suitable parametrix S whose normal derivative vanishes on C by multiplying S_0 by a function δ , with continuous derivatives of the first few orders, which is identically 1 near z_0 , but which vanishes identically outside a small circle about z_0 . For example, with sufficiently small $\epsilon > 0$, we can take $\delta = f_\epsilon(|z - z_0|)f_\epsilon(|w - z_0|)$, where $f_\epsilon(r)$ is a continuous function of r with continuous derivatives of the first few orders which is equal to 1 for $0 \leq r \leq \epsilon$ and vanishes for $r \geq 2\epsilon$.

Having obtained a parametrix $S(w, z)$ in this manner, we define the Neu-

mann's function $N(w, z)$ for (9) in D by the formula

$$(12) \quad N(w, z) = S(w, z) - L_w \{ [\Delta - P(t)] S(t, z) \},$$

and we proceed to show that N has, as a function of the point z , the characteristic properties of the Neumann's function. It is easily verified from the definition (3) that N is independent of the choice of the parametrix S , since the normal derivative of S vanishes on C and the difference of two such parametrices is regular.

First, since L_w is a bounded functional, the quantity $L_w [\{ \Delta - P(t) \} S(t, z)]$ depends continuously on z , and, indeed, we can differentiate with respect to z under the functional L_w . We obtain

$$(13) \quad \begin{aligned} [\Delta - P(z)] N(w, z) &= [\Delta - P(z)] S(w, z) \\ &\quad - L_w \{ [\Delta - P(t)] [\Delta - P(z)] S(t, z) \} = 0. \end{aligned}$$

The expression on the right vanishes, according to (3), because, as a function of w , the expression $[\Delta - P(z)] S(w, z)$ has continuous second derivatives in $D + C$ and has a normal derivative which vanishes on C . Thus N satisfies the partial differential equation (9) as a function of z .

Second, $N(w, z)$ has a logarithmic singularity at the point $z = w$, since the second term on the right in (12) is bounded.

Third, we can calculate the normal derivative of N at a point z_0 of C conveniently by making a special choice of E and g in (11). The curves C have an analytic representation $z = z(s)$ in terms of a parameter s , and we choose E to lie in D and to correspond in the s -plane to a small semicircle whose bounding diameter lies on the real axis and whose center maps into z_0 . We choose g to be the fundamental solution of the Laplace equation in the semicircle which vanishes on the circular boundary and has normal derivative zero on the rectilinear boundary. Since $z = z(s)$ is a conformal mapping, g satisfies Laplace's equation in E also, and substitution into (11) yields a parametrix S_0 with vanishing normal derivatives with respect to both z and w on an arc of C including the point z_0 . Multiplication by a function δ which is identically 1 in a small neighborhood of z_0 , identically zero outside a slightly larger neighborhood of z_0 , and has a vanishing normal derivative on C , yields our choice of S . (A selection of δ as a function of $|s|$ will satisfy these conditions.)

With this choice of S in (12), it is permissible to compute the normal derivative of N with respect to z at z_0 by differentiating under the bounded functional L_w . We find

$$(14) \quad \frac{\partial N}{\partial \nu} = \frac{\partial S}{\partial \nu} - L_w \left\{ (\Delta - P) \frac{\partial S}{\partial \nu} \right\} = 0,$$

since $\partial S / \partial \nu = \delta \cdot \partial S_0 / \partial \nu \equiv 0$ throughout D .

We have thus shown that N has the three characteristic properties of the

Neumann's function of (9) in D , as a function of z , and we conclude therefore that the Neumann's function exists and is equal to N .

We remark that it is possible to generalize the above existence proof to partial differential equations other than (9). Also, the restriction to two independent variables is unnecessary, and, indeed, for more independent variables the only additional complication comes in deriving the boundary condition (14), which has been based here on conformal mapping. However, for general elliptic equations in several variables we can treat the easy case of a plane boundary without any loss of generality, since we can always find a transformation of coordinates which takes a small portion of an analytic surface into a portion of a plane and this transformation will merely carry one elliptic equation into another which can be discussed equally well. In another direction, we can generalize the existence proof by replacing the boundary condition (2) by

$$(15) \quad \frac{\partial u}{\partial \nu} - \lambda u = 0,$$

where λ is a given positive function on C . In this case, we obtain the existence of a Green's function of the third kind. Also, we note that the domain D may be a multisheeted Riemann surface over the plane, instead of a schlicht region.

3. The Green's function. In this section we study the Dirichlet problem for a linear elliptic partial differential equation in several independent variables by a technique analogous to that of §2. Our objective is to give a more penetrating analysis of the boundary condition, with hypotheses on the boundary far less restrictive than analyticity. This can be achieved by appropriate use of the principle of majorization and by a stronger use of the Hahn-Banach theorem. We present the material for three independent variables in space for the sake of simplicity, and we merely remark that for any finite number of dimensions the steps to be carried out are similar.

Let D be a bounded domain in space with boundary C having the following property: Each point of C can be touched by a solid sphere R which is otherwise disjoint from $D+C$. Let $P(z) \geq 0$ be a non-negative continuously differentiable⁽¹⁾ function of the point z in a neighborhood of $D+C$, and let B' denote the linear space of all functions

$$f = \Delta u - Pu$$

in $D+C$ with u twice continuously differentiable in $D+C$ and zero on C . We let B denote the Banach space of all continuous functions in $D+C$ with the norm

$$\|f\| = \max |f|.$$

⁽¹⁾ The considerations apply if, in place of continuous derivatives, $P(z)$ satisfies a Lipschitz condition $|P(z) - P(z')| \leq K|z - z'|$.

For any point w in D we define a linear functional L_w over B' by the formula

$$(16) \quad L_w[\Delta u - Pu] = u(w).$$

That this functional is uniquely defined follows from the theorem that a solution of $\Delta u - Pu = 0$ in D which vanishes on C must be identically 0, since it cannot have a positive maximum or negative minimum in the interior of D .

We prove that, for f in B' ,

$$(17) \quad L_w\{f\} \leq b(f),$$

where

$$(18) \quad b(f) = \inf v(w)$$

among all twice continuously differentiable functions v with

$$(19) \quad \Delta v - Pv \leq f$$

in D , and $v \geq 0$ on C . For the proof, we note that since

$$\Delta(v - u) - P(v - u) \leq f - f = 0,$$

$v - u$ cannot have a negative minimum inside D . Therefore, since $v - u \geq 0$ on C , we must have $v - u \geq 0$ in D also, which yields the desired estimate (17). In fact, since u is itself an admissible function in the determination of $b[f]$, we have equality in (17) for f in B' . In particular, $b[0] = 0$.

The functional $b[f]$ defined in (18), (19) has a meaning for any f in the entire space B . For, let v_2 be the function

$$v_2 = K - \frac{1}{6}(x^2 + y^2 + z^2),$$

where K is a large constant such that $v_2 \geq 0$ in $D + C$. We have

$$\Delta v_2 - Pv_2 \leq -1,$$

and $\|f\|v_2$ is an admissible function for (18). Thus

$$(20) \quad b[f] \leq v_2(w) \cdot \|f\| \leq K \cdot \|f\|.$$

Also, we notice that

$$(21) \quad b[f] \leq 0$$

if $f \geq 0$ throughout D , since the function $v \equiv 0$ is then admissible in (18).

The functional $b[f]$ is positively homogeneous and subadditive; i.e.,

$$b[tf] = tb[f] \quad \text{if } t > 0, \quad b[f_1 + f_2] \leq b[f_1] + b[f_2].$$

The latter inequality follows from the fact that if v' and v'' are admissible functions for the determination of $b[f_1]$ and $b[f_2]$, respectively, then $v' + v''$ is admissible for $b[f_1 + f_2]$. We remark that since we already know that $b[0] = 0$, the subadditivity shows that $b[f] \geq -b[-f] \geq -K\|f\|$, whence $b[f]$ is

finite and likewise bounded from below.

We now use the Hahn-Banach theorem to extend $L_w[f]$ as a linear functional over the entire space B still satisfying the inequality (17). Applying the extended functional to a parametrix for the equation

$$\Delta u - Pu = 0$$

in a suitable way, we shall obtain the Green's function in D for this equation.

Let E be a (small) region containing z , and let $g(w, z) = 1/|w - z| + \dots$ be a fundamental solution for Laplace's equation which is non-negative in E and symmetric in w and z ($|w - z|$ means the distance between w, z). For example, $g(w, z)$ could be the Green's function for Laplace's equation for a sphere enclosing E . Set

$$(22) \quad I_j(w, z) = \frac{1}{4\pi} \iiint_E P(t)g(w, t)I_{j-1}(z, t)d\tau,$$

$j = 1, 2, \dots$, where $I_0(w, z) = g(w, z)$ and $d\tau$ is the volume element. We have $I_j(w, z) \geq 0$ in E . Then

$$(23) \quad S_0(w, z) = \sum_{j=0}^3 (-1)^j I_j(w, z)$$

is a parametrix for the equation $\Delta u - Pu = 0$, and

$$(24) \quad \Delta S_0 - PS_0 = PI_3 \geq 0.$$

For z in the neighborhood of a point z_0 in which we are interested, we obtain for all w in D a suitable parametrix $S_1(w, z)$ by multiplying S_0 by a function δ , with continuous derivatives of the first few orders, which is identically 1 near z_0 , but which vanishes identically outside a small sphere about z_0 .

We now define the Green's function $G(w, z)$ in D by the formula

$$(25) \quad G(w, z) = S(w, z) - L_w[\{\Delta - P(t)\}S(t, z)],$$

where $S(w, z)$ is a parametrix in D which vanishes when w is on C . We show directly that G has as a function of z in D the characteristic properties of the Green's function. Note that the formula (25) is independent of the particular $S(w, z)$ used, by virtue of (16).

First, the second term on the right in (25) is bounded for z near w by virtue of (17) and (20), so that G must have a pole at $z = w$. Second, since the functional L_w is a bounded linear functional, the second term on the right in (25) is continuous in z for z in D . Indeed, we can differentiate with respect to the coordinates of z under the operator L_w , and we find

$$\{\Delta - P(z)\}G = \{\Delta - P(z)\}S - L_w[\{\Delta - P(t)\}\{\Delta - P(z)\}S] = 0.$$

The last equality follows from (16), since $\{\Delta - P(z)\}S(w, z)$ is a twice con-

tinuously differentiable function of w which vanishes for w on C .

Third, we prove that $G \geq 0$. Let σ be a large positive number and let $F(S)$ be a function of S which is identically S for $S \leq \sigma$, which is continuous with its first two derivatives for all S , which has a finite limit as $S \rightarrow +\infty$, and which has a nonpositive second derivative $F''(S)$ for $S > \sigma$. With the parametrix S_0 constructed in (23), we have⁽²⁾

$$(26) \quad \begin{aligned} \{\Delta - P\}F(S_0) - \{\Delta - P\}S_0 &= F''(\nabla S_0)^2 + (F' - 1)(\Delta S_0 - PS_0) \\ &+ (F'S_0 - F)P \leq 0, \end{aligned}$$

by (24) and the inequalities $F'S - F \leq 0$ and $F' - 1 \leq 0$, which follow from $F'' \leq 0$.

Using the parametrix S_1 constructed in the paragraph immediately following (24), we therefore have for sufficiently large σ

$$(\Delta - P)F(S_1) - (\Delta - P)S_1 \leq 0.$$

It now follows that

$$F(S_1) = L_w[(\Delta - P)F(S_1)] \geq L_w[(\Delta - P)S_1],$$

since the linear functional $L_w[f] \leq 0$ when f is everywhere non-negative, by virtue of (21) and (17). Therefore

$$G = S_1 - L_w[(\Delta - P)S_1] \geq S_1 - F(S_1) \geq 0,$$

which is the desired result.

Finally, we prove that G has boundary values 0. Let z_0 be a point of C , let R be a small sphere in the exterior of D and touching C at z_0 , and let R' be a small sphere containing z_0 and orthogonal to R . Denote the region outside R and inside R' by E , and let $g(w, z)$ be the Green's function for Laplace's equation in E . For this choice of E and g , define $I_j(w, z)$ as in (22) and set

$$S_0^*(w, z) = \sum_{j=0}^4 (-1)^j I_j(w, z).$$

Then,

$$(27) \quad \Delta S_0^* - PS_0^* = -PI_4(w, z) \leq 0.$$

Also, for w on the boundary of E , we have $g(w, z) = 0$ and $I_j(w, z) = 0$ from (22). Therefore, $S_0^*(w, z) = 0$ on the boundary of E . Since S_0^* cannot have a negative minimum in the interior of E by (27), it follows that

$$S_0^* \geq 0$$

throughout E .

⁽²⁾ For the general differential equation, F can always be constructed so that the first term corresponding to $F''(\nabla S_0)^2$ is the dominant term, and we still have the inequality.

Multiply S_0^* by a non-negative function δ , continuous with its first few derivatives, which is identically 1 in a small neighborhood of z_0 and which is identically 0 in a slightly larger neighborhood of z_0 . This yields a parametrix S_1^* throughout D which is non-negative on C and approaches 0 as $z \rightarrow z_0$.

For any parametrix S which vanishes on C , the difference $S_1^* - S$ is twice continuously differentiable in D and is non-negative on C . Hence, by (17), (18), and (19), with $f = (\Delta - P)(S_1^* - S)$ and $v = S_1^* - S$, we obtain

$$L_w[(\Delta - P)(S_1^* - S)] \leq S_1^* - S,$$

or

$$(28) \quad G \leq S_1^* - L_w[(\Delta - P)S_1^*].$$

Letting $z \rightarrow z_0$, we have $S_1^*(w, z) \rightarrow 0$ and $(\Delta - P(t))S_1^*(t, z) \rightarrow 0$ uniformly in t ; this last limiting relation is clear outside any neighborhood of z_0 , while inside a small neighborhood of z_0 it follows from (27). Consequently

$$(29) \quad \limsup_{z \rightarrow z_0} G(w, z) \leq 0,$$

and since $G \geq 0$, we can make the final desired conclusion

$$(30) \quad \lim_{z \rightarrow C} G(w, z) = 0.$$

These calculations show that G is the Green's function of D for the linear elliptic partial differential equation $\Delta u = Pu$. This construction of G gives a remarkably simple proof of the existence of the Green's function, without previous knowledge of a fundamental solution in the large. In the special case $P \equiv 0$ we obtain the Green's function of Laplace's equation.

Incidentally, the symmetry of the Green's function is also contained in the above. Since $G(w, z)$ is the Green's function for $\Delta u = Pu$ as a function of the second variable, with singularity at the first variable, the function $G(z, w)$ is an admissible singularity function $S(w, z)$ in (25). We immediately obtain, therefore,

$$(31) \quad G(w, z) = G(z, w).$$

Of course, solvability of the Dirichlet problem is obtained once G is known.

4. General domains. The argument of the preceding section can easily be generalized to the case of a domain D , bounded or unbounded, with an arbitrary boundary C . Let D lie in n -dimensional Euclidean space, where $n \geq 3$, and let $P(z) \geq 0$ be an arbitrary non-negative function over D , continuous with continuous first derivatives in D . In 2 dimensions, we suppose either that D is bounded, or if D is unbounded, that $P(z) > 0$. Consider the function

$$v_3 = g(|z|),$$

where $|z| = r$ is the distance from z to the origin, and

$$g(r) = \int_r^\infty \frac{r dr}{1 + r^n},$$

so that

$$\Delta g(r) = -\frac{n}{(1 + r^n)^2} < 0.$$

The function v_3 is positive, is continuous and has continuous first and second derivatives, and

$$-w_3 = (\Delta - P)v_3 < 0,$$

so that $w_3 > 0$. In 2 dimensions, with D unbounded and $P > 0$, select $v_3 \equiv 1$, and $w_3 = P > 0$. Define the norm of a function f which is continuous in D by

$$(32) \quad \|f\| = \sup \left| \frac{f}{w_3} \right|,$$

and let B denote the Banach space of the functions f with finite norm. Let B' denote the subspace of functions f in B given by

$$f = \Delta u - Pu,$$

where u is continuous together with first and second derivatives in D , and $u \rightarrow 0$ as the boundary or the point at infinity (if D is unbounded) is approached. Finally, define the linear functional L_w over B as in (16).

Proceeding as in §3, define the bound $b[f]$ for any f in B as in (18), (19), with the condition $v \geq 0$ on C replaced by $\liminf v \geq 0$ as the boundary C or the point at infinity is approached. Inequality (17) applies. An admissible function in (19) is $\|f\|v_3$. For,

$$(\Delta - P)\|f\|v_3 = -\|f\|w_3 \leq -\frac{|f|}{w_3} w_3 \leq f.$$

Thus

$$b[f] \leq v_3(w)\|f\|$$

as in (20), and we also have (21).

Now extend the functional $L_w[f]$ throughout B so that the inequality (17) is retained, and define the function $G(w, z)$ as in §3 by (25), where $S(w, z)$ is a parametrix which vanishes on C and at infinity. §3 shows that $G(w, z)$, as a function of z , satisfies the equation

$$\Delta u - Pu = 0,$$

and has a fundamental singularity at $z = w$. As a function of the second vari-

able z , G is therefore a fundamental solution over D of $\Delta u = Pu$.

We shall prove that, among all fundamental solutions, G is characterized by the property of being the smallest positive fundamental solution. This replaces the condition $G=0$ on the boundary C because of the arbitrariness of C .

First, $G(w, z) \geq 0$ throughout D , just as in §3. Second, we shall show that the important symmetry property

$$(33) \quad G(w, z) = G(z, w)$$

is easily contained in our formulation. Let $S'(w, z)$ be a parametrix with non-negative boundary values and non-negative values at infinity (i.e., let $\liminf S'(w, z) \geq 0$ as $w \rightarrow C$ or the point at infinity, if D is unbounded). Then, by (17) and (18),

$$S'(w, z) - S(w, z) \geq L_w[(\Delta - P)(S'(t, z) - S(t, z))],$$

or

$$(34) \quad S'(w, z) - L_w[(\Delta - P)S'(t, z)] \geq G(w, z).$$

Now, we have already proved that G , as a function of its second point, is a fundamental solution with singularity at the first point. Thus, $G(z, w)$ (note the location of z and w) is a function of w which is continuous and has continuous first and second derivatives throughout D , except at the point z . Since $G(z, w) \geq 0$, the function $G(z, w)$ is therefore admissible as a choice for the parametrix $S'(w, z)$ in (34), and we obtain

$$G(z, w) \geq G(w, z).$$

This inequality holds for any location of the points z, w , so that by interchanging z, w the reverse inequality follows and the symmetry property (33) is established.

The symmetry property (33) shows that $G(w, z)$ is a well-behaved function of its first point w as well, and is a fundamental solution as a function of this first point with singularity at the second point.

Third, $G(w, z)$ is the smallest positive fundamental solution in D with singularity at z . For, if $F(w, z)$ is such a positive fundamental solution, it can be used as the parametrix $S'(w, z)$ in (34), with the desired result

$$F(w, z) \geq G(w, z).$$

We mention the following additional property of $G(w, z)$: If we exclude a neighborhood Ω of the singular point z , the function $G(w, z)$ is bounded for w outside this neighborhood. If this were not so, use the function $F(S_0)$ constructed in connection with (26), for sufficiently large σ . Define the function $\phi(w, z)$ as equal to $G(w, z)$ for w in Ω , and equal to $F(G(w, z))$ for w outside Ω . The function $\phi(w, z)$ is an admissible parametrix in (34) with the result

$$\phi(w, z) - L_w[(\Delta - P)\phi(t, z)] \geq G(w, z).$$

But $(\Delta - P)\phi(t, z) \leq 0$ by (26), and (21) now gives

$$\phi(w, z) \geq G(w, z).$$

Selecting a point w outside Ω where $\phi(w, z) \neq G(w, z)$, we contradict the inequality $F(S_0) < S_0$, or $\phi(w, z) < G(w, z)$, at such a point. Therefore, $G(w, z)$ is bounded outside Ω . (Indeed, we have proved that $G(w, z)$, for w outside Ω , attains its maximum on the boundary of Ω .)

We shall conclude with several remarks concerning the generality of the method. First, the considerations apply equally well to an arbitrary elliptic partial differential equation satisfying a maximum principle. All that is required is the construction of a parametrix, and this is done in a standard manner using a number of terms of a Neumann expansion analogous to (23). Also, the construction of a particular function analogous to v_3 can be performed under reasonable assumptions concerning the behavior of the coefficients of the differential equation on the boundary C . The fact that the boundary values of G are zero, as in §3, can be obtained by using the infinite Neumann series for S_0^* . Of course, if the equation is not self-adjoint, the adjoint equation will enter into the considerations.

Second, the considerations apply to Riemannian manifolds in place of domains in Euclidian space. Integrals of the first, second, and third kinds, with or without periods, can be constructed.

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