SEMI-PRIME RINGS

BY

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Following Nagata [2], we call an ideal of a ring *semi-prime* if and only if it is an intersection of prime ideals of the ring. A *semi-prime ring* is one in which the zero ideal is semi-prime. In view of the definition of the prime radical of a ring given by McCoy [1, p. 829], we have that a ring is semi-prime if and only if it has a zero prime radical. The semi-simple rings of Jacobson [6] are also semi-prime.

In the first section of this paper, general properties of a semi-prime ring R are developed. The discussion centers around the concept of the component I^c of a right ideal I of R. The component I^c of I is just the left annihilator of the right annihilator of I.

The second section is devoted to a study of the prime right ideals of a semi-prime ring R. Prime right ideals are defined in R much as they are in a prime ring [7]. Associated with each right ideal I of R is a least prime right ideal $p(I)$ containing I. Some generalizations of results of [7] are obtained. Thus if I and I' are any right ideals of R and if a is any element of R, it is proved that $p(I\cap I')=p(I)\cap p(I')$ and that $p((I:a))=(p(I):a)$.

In the third section, the structure of a semi-prime ring R is introduced along the lines of the structure of a prime ring given in [7] and [8]. If $\mathfrak{P}_r(R)$ is the algebra of all prime right ideals of R, then a subalgebra $\mathfrak{R}$ of $\mathfrak{P}_r(\mathfrak{R})$ is a *structure* of R if (A1) to (A3) given below are satisfied. For each right ideal I of R, I^* is the least element of $\mathfrak{R}$ containing I.

(A1) $0, R\in\mathfrak{R}$.

(A2) $(I\cap I')^*=I^*\cap I'^*$, I, I' any right ideals of R.

(A3) $(I:a)^*=(I^*:a)$, $a\in R$, I any right ideal of R.

(A4) The algebra $\mathfrak{R}$ has atoms.

In case (A4) is also satisfied by a structure $\mathfrak{R}$, this structure is called an a-structure of R. It is proved that if R has an a-structure $\mathfrak{R}$, then any ideal S of R also has an a-structure provided that some atom of $\mathfrak{R}$ intersects S. Conversely, if the ideal S of R has an a-structure $\mathfrak{S}$, then R has an a-structure $\mathfrak{R}$ induced by that of S. If, furthermore, $S^c=R$, then $\mathfrak{S}$ and $\mathfrak{R}$ are isomorphic.

The final section is devoted to the determination of all semi-prime rings having an a-structure. If I is an atom of the a-structure $\mathfrak{R}$ of R, then I^c is a prime ring. The union of all such prime rings contained in R is called the *base* of R. It is shown that the base B of R is the direct sum of the prime rings I^c, I an atom of $\mathfrak{R}$, and that $R=(B+B^r)^c$ where B^r is the right annihilator of B. Associated with each semi-prime ring R is a universal extension ring $N(R)$

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containing R as an ideal and having the property that $R^c = N(R)$ [9]. The final result is that R is a semi-prime ring with a-structure if and only if $B \oplus B' \subseteq R \subseteq N(B) \oplus N(B')$, where B is the direct sum of prime rings each having an a-structure and B' is a semi-prime ring having a non-atomic structure.

1. **The component of an ideal.** Our discussion throughout the paper is limited to semi-prime rings. For any ring R, the sets of ideals, right ideals, and left ideals are denoted respectively by $\mathfrak{J}(R)$, $\mathfrak{J}_r(R)$, and $\mathfrak{J}_l(R)$. If S is a subset of R, then S^r (S^l) denotes the right (left) annihilator of S in R. Clearly $S^r \in \mathfrak{J}_r(R)$ and $S^l \in \mathfrak{J}_l(R)$; S^r (S^l) is called an *annihilating right* (*left*) *ideal*. We denote by $\mathfrak{A}_r(R)$ ($\mathfrak{A}_l(R)$) the set of all annihilating right (left) ideals of R, and let $\mathfrak{A}(R) = \mathfrak{A}_r(R) \cap \mathfrak{A}_l(R)$.

If $\mathfrak{B}$ is a set of subsets of a given set B that is closed under the operation of infinite intersection, then $\{\mathfrak{B}; \subseteq, \cap\}$ is called an *algebra*. This is the only sense in which the word algebra is used in this paper. For example, each of the sets of ideals defined in the previous paragraph is an algebra. If B is in $\mathfrak{B}$, then $\mathfrak{B}$ defines a closure operation on any set $\mathfrak{C}$ of subsets of B containing $\mathfrak{B}$. Thus for each $S \in \mathfrak{C}$, define S^* as the intersection of all elements of $\mathfrak{B}$ containing S. The mapping $*: S \rightarrow S^*$ is a closure operation on $\mathfrak{C}$; that is, $S \subseteq S^*$; $S^{**} = S^*$; if $S \subseteq S'$ then $S^* \subseteq S'^*$.

Levitzki [3, p. 29] and Nagata [2, Proposition 8] have recently proved that a ring is semi-prime if and only if it contains no nonzero nilpotent ideals. Clearly, then, a ring is semi-prime if and only if it contains no nonzero nilpotent right (left) ideals. Every ideal of a semi-prime ring is also a semi-prime ring [1, Theorem 4].

The results of this section have obvious duals obtained by interchanging the roles of the right and left ideals.

1.1 Lemma. *If $I \in \mathfrak{J}_r(R)$ and $S \in \mathfrak{J}(R)$, then $IS = 0$ if and only if $I \cap S = 0$.*

Proof. This follows from the chain $(I \cap S)^2 \subseteq IS \subseteq I \cap S$.

A consequence of 1.1 is that $I \cap I^r = 0$ for each $I \in \mathfrak{J}_r(R)$. If $S \in \mathfrak{A}(R)$ then $S = S^{lr}$, and since $(S^r S)^2 = 0$, evidently $S^r S = 0$. We conclude that $S^r = S^l$ for each $S \in \mathfrak{A}(R)$. We also note that S^r is the unique complement of S in that S^r is the largest ideal of R such that $S \cap S^r = 0$.

For each $I \in \mathfrak{J}_r(R)$, the *component* I^c of I in R is defined by

$$I^c = I^{rl}.$$

Clearly $I^c \in \mathfrak{A}(R)$, and the mapping $^c: I \rightarrow I^c$ is a closure operation on $\mathfrak{J}_r(R)$. The component I^c of I is not in general the least ideal of R containing I, nor is it in general even the least semi-prime ideal of R containing I. However, I^c is a semi-prime ideal of R as we now show.

1.2 Theorem. *For each $I \in \mathfrak{J}_r(R)$, I^c is a semi-prime ideal of R.*

Proof. If, for $S \in \mathfrak{J}(R)$, R/S is not a semi-prime ring, then there exists $T \in \mathfrak{J}(R)$ such that

$$T \supset S, \qquad T^2 \subseteq S.$$

Now $(T \cap S^r)^2 \subseteq T^2 \cap S^r = 0$, and therefore $TS^r = 0$. This proves that $T \subseteq S^c$, and hence that $S \neq S^c$.

Since $I^{cc} = I^c$, it follows that R/I^c is a semi-prime ring, that is, that I^c is a semi-prime ideal of R. This proves 1.2.

According to this lemma, $\mathfrak{A}(R)$ is an algebra of semi-prime ideals of R. We shall give now another useful characterization of the ideals of $\mathfrak{A}(R)$.

For $S \in \mathfrak{J}(R)$ and $I \in \mathfrak{J}_r(R)$ with $S \supseteq I$, we shall say that S is *close* to I if for each $T \in \mathfrak{J}(R)$,

$$T \cap S \neq 0 \quad \text{implies} \quad T \cap I \neq 0.$$

1.3 THEOREM. *For each $I \in \mathfrak{J}_r(R)$, I^c is the unique maximal close ideal of I.*

Proof. We first show that I^c is close to I. If $T \cap I^c = S \neq 0$, $T \in \mathfrak{J}(R)$, then $IS \neq 0$. For if $IS = 0$, then $S \subseteq I^r$ and $S^2 = 0$ contrary to the assumption that $S \neq 0$. Thus $I \cap S \neq 0$ and I^c is close to I.

If $S \in \mathfrak{J}(R)$, $S \supseteq I$, and $SI^r \neq 0$, then $S \cap I^r = T$ has the property that $T \cap S \neq 0$, $T \cap I = 0$. Hence S is not close to I. We conclude that S is close to I if and only if $S \subseteq I^c$. This proves 1.3.

1.4 THEOREM. *If $I \in \mathfrak{J}_r(R)$ and $S \in \mathfrak{J}(R)$, then*

$$(IS)^c = (I \cap S)^c = I^c \cap S^c.$$

Proof. Since $IS \subseteq I \cap S \subseteq I^c \cap S^c$, and $I^c \cap S^c \in \mathfrak{A}(R)$, clearly

$$(IS)^c \subseteq (I \cap S)^c \subseteq I^c \cap S^c.$$

If $T' \in \mathfrak{J}(R)$, $T' \cap (I^c \cap S^c) = T \neq 0$, then $T \cap S^c \neq 0$ and $T \cap S \neq 0$. Since $T \cap S \subseteq I^c$, evidently $T \cap (I \cap S) \neq 0$. Now

$$(T \cap (I \cap S))^2 \subseteq T \cap IS,$$

and therefore $T \cap IS \neq 0$. This proves that $(IS)^c = I^c \cap S^c$ in view of 1.3.

1.5 COROLLARY. *If $I, I' \in \mathfrak{J}_r(R)$, then $(II')^c = I^c \cap I'^c$.*

Proof. We first note that $(RI')^c = I'^c$. Hence

$$(II')^c \supseteq I^c(RI')^c = I^c \cap I'^c.$$

Evidently $(II')^c \subseteq I^c \cap I'^c$, and the corollary follows.

For each $I \in \mathfrak{J}_r(R)$, $(I + I^r)^r \subseteq I^r \cap I^c = 0$ and therefore

$$(I + I^r)^c = R.$$

More generally, we have the following result.

1.6 THEOREM. *If* $I \in \mathfrak{J}_r(R)$ *and* $S \in \mathfrak{A}(R)$ *with* $I \subseteq S$, *then*

$$(I + I^r \cap S)^c = S.$$

Proof. By the modular law, $I+I^r\cap S=(I+I^r)\cap S$. Hence, by 1.4, $(I+I^r\cap S)^c=(I+I^r)^c\cap S=S$.

1.7 THEOREM. *For each* $S \in \mathfrak{J}(R)$, $\mathfrak{A}_r(S)=\{I\cap S;\ I\in\mathfrak{A}_r(R)\}$.

Proof. For each subset A of S, the right annihilator of A in S is simply $A^r\cap S$. If $A\in\mathfrak{A}_r(S)$, then it is evident from the equation $A^lA=0$ that $A\in\mathfrak{J}_r(R)$ and $A^l\in\mathfrak{J}_l(R)$. Also, if $A\in\mathfrak{J}_l(R)$, $c\in S$ with $(A\cap S)c=0$, then $(Ac)^2=(AcA)c=0$ and $Ac=0$. Thus $(A\cap S)^r\cap S=A^r\cap S$, and 1.7 follows.

A corollary of 1.7 is that $\mathfrak{A}(S)=\{I\cap S;\ I\in\mathfrak{A}(R)\}$.

2. **Prime right ideals.** The concept of a prime right ideal was introduced for prime rings in [7]. We show now that this concept is fruitful in a semi-prime ring also.

The right ideal I of R is called a *prime right ideal* if and only if

2.1 $\mathfrak{a}\mathfrak{b}\subseteq I$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ with $\mathfrak{b}^c=R$, implies $\mathfrak{a}\subseteq I$.

The condition $\mathfrak{b}^c=R$ is equivalent to the condition $\mathfrak{b}\neq 0$ in case R is a prime ring, for in a prime ring $\mathfrak{A}(R)=\{0, R\}$.

It is clear that $\mathfrak{b}$ in 2.1 might just as well be assumed to be an ideal of R, since $\mathfrak{a}\mathfrak{b}\subseteq I$ implies $\mathfrak{a}(R\mathfrak{b})\subseteq I$, and $(R\mathfrak{b})^c=\mathfrak{b}^c=R$. We shall henceforth assume this fact whenever it is convenient.

If $a\in R$, $\mathfrak{b}\in\mathfrak{J}(R)$ with $\mathfrak{b}^c=R$, and if $a\mathfrak{b}\subseteq I$, a prime right ideal of R, then evidently $a\in I$. In particular, $aR\subseteq I$ implies $a\in I$.

If $I\in\mathfrak{J}_r(R)$ and $\mathfrak{a}\mathfrak{b}\subseteq I$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ with $\mathfrak{b}^c=R$, then $\mathfrak{a}^c\cap\mathfrak{b}^c\subseteq I^c$ and $\mathfrak{a}^c\subseteq I^c$. Thus $\mathfrak{a}\subseteq I^c$ regardless of whether or not I is prime.

2.2 LEMMA. *If I is a prime right ideal of R and if* $\mathfrak{a}\mathfrak{b}\subseteq I$, *where* $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ *with* $\mathfrak{a}^c\subseteq\mathfrak{b}^c$, *then necessarily* $\mathfrak{a}\subseteq I$.

Proof. Evidently $\mathfrak{a}(\mathfrak{b}+\mathfrak{b}^r)=\mathfrak{a}\mathfrak{b}\subseteq I$. Since $(\mathfrak{b}+\mathfrak{b}^r)^c=R$ by 1.6, necessarily $\mathfrak{a}\subseteq I$ according to 2.1.

It is evident that prime left ideals may be defined similarly, and that our results on prime right ideals also apply to prime left ideals. We shall denote the set of all prime right (left) ideals of R by $\mathfrak{P}_r(R)$ ($\mathfrak{P}_l(R)$). It is clear that $\mathfrak{P}_r(R)$ and $\mathfrak{P}_l(R)$ are algebras. In view of our remarks before 2.2, we see that

$$\mathfrak{A}(R) \subseteq \mathfrak{P}_r(R) \cap \mathfrak{P}_l(R).$$

The *bound* of a right (left) ideal I of R is defined to be the largest ideal of R contained in I. We denote the bound of I by I^b.

2.3 THEOREM. *For each* $I\in\mathfrak{P}_r(R)$ *we have* $I^b\in\mathfrak{A}(R)$.

Proof. If $\mathfrak{a}\mathfrak{b}\subseteq I^b$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ with $\mathfrak{b}^c=R$, then $\mathfrak{a}'\mathfrak{b}\subseteq I^b$ where $\mathfrak{a}'$ is the ideal of R generated by $\mathfrak{a}$. Since $I\in\mathfrak{P}_r(R)$, we must have $\mathfrak{a}'\subseteq I$; and since $\mathfrak{a}'\in\mathfrak{J}(R)$, $\mathfrak{a}'\subseteq I^b$. Thus $I^b\in\mathfrak{P}_r(R)$. Now $I^{bc}I^b\subseteq I^b$, and therefore $I^{bc}\subseteq I^b$ by 2.2. Hence $I^b\in\mathfrak{A}(R)$ as desired.

It is obvious from 2.3 that an ideal S of R is in $\mathfrak{P}_r(R)$ if and only if $S\in\mathfrak{A}(R)$. It also follows that

$$\mathfrak{A}(R) = \mathfrak{P}_r(R) \cap \mathfrak{P}_l(R).$$

2.4 Theorem. *If $S\in\mathfrak{A}(R)$, then $\mathfrak{P}_r(S)\subseteq\mathfrak{P}_r(R)$ and $\mathfrak{P}_l(S)\subseteq\mathfrak{P}_l(R)$ so that $\mathfrak{A}(S)\subseteq\mathfrak{A}(R)$.*

Proof. If $I\in\mathfrak{P}_r(S)$, then $IR\subseteq I$ since $(IR)S\subseteq I$. Thus $\mathfrak{P}_r(S)\subseteq\mathfrak{J}_r(R)$. If $I\in\mathfrak{P}_r(S)$ and $\mathfrak{a}\mathfrak{b}\subseteq I$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ with $\mathfrak{b}^c=R$, then $\mathfrak{a}(\mathfrak{b}\cap S)\subseteq I$, and since $\mathfrak{a}\subseteq S$, $\mathfrak{a}\subseteq I$. Thus $I\in\mathfrak{P}_r(R)$ and the theorem follows.

The following theorem is analogous to [9, 2.5] which applies to prime rings.

2.5 Theorem. *Assume that $S\in\mathfrak{J}(R)$ and that $S^c=R$. For each $I\in\mathfrak{P}_r(S)$, define*

$$I' = \{a;\, a \in R,\, aS \subseteq I\}.$$

Then $I=I'\cap S$ and the mapping $'$: $I\to I'$ is an isomorphism between the algebras $\mathfrak{P}_r(S)$ and $\mathfrak{P}_r(R)$.

Proof. We see immediately that $\mathfrak{P}_r(S)\subseteq\mathfrak{J}_r(R)$ and that $I=I'\cap S$ for each $I\in\mathfrak{P}_r(S)$. If $\mathfrak{a}\mathfrak{b}\subseteq I'$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(R)$ with $\mathfrak{b}^c=R$, then $(\mathfrak{a}S)(\mathfrak{b}S)\subseteq I$, $\mathfrak{a}S\subseteq I$, and finally $\mathfrak{a}\subseteq I'$. Thus $I'\in\mathfrak{P}_r(R)$.

If $I'\in\mathfrak{P}_r(R)$ and $I=I'\cap S$, and if $\mathfrak{a}\mathfrak{b}\subseteq I$, $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(S)$ with $\mathfrak{b}^c=S$, then $(\mathfrak{a}S)(\mathfrak{b}S)\subseteq I$. Since $\mathfrak{a}S$, $\mathfrak{b}S\in\mathfrak{J}_r(R)$ and $(\mathfrak{b}S)^c=R$, this implies that $\mathfrak{a}S\subseteq I'\cap S$ due to the primeness of I'. Hence $\mathfrak{a}\subseteq I$ by remarks following 2.1 and $I\in\mathfrak{P}_r(S)$.

If $I'\cap S=I''\cap S$ for some I', $I''\in\mathfrak{P}_r(R)$, then $I'S\subseteq I''$ and $I''S\subseteq I'$. Thus $I'=I''$. This completes the proof of 2.5.

We derive from 2.4 and 2.5 that for each $S\in\mathfrak{J}(R)$, $\mathfrak{P}_r(S)$ is isomorphic to the subalgebra $\mathfrak{P}_r(S^c)$ of $\mathfrak{P}_r(R)$. The mapping of 2.5 also is an isomorphism between $\mathfrak{A}(S)$ and $\mathfrak{A}(R)$.

2.6 Theorem. *If $\{S_\alpha\}$ is a set of semi-prime rings and if $S=\bigcup_\alpha S_\alpha$ is the (finite) direct sum of the rings of this set, then*

$$\mathfrak{P}_r(S) = \{\bigcup_\alpha I_\alpha;\, I_\alpha \in \mathfrak{P}_r(S_\alpha)\}.$$

Proof. Let $I=\bigcup_\alpha I_\alpha$, $I_\alpha\in\mathfrak{P}_r(S_\alpha)$. If $\mathfrak{a}\mathfrak{b}\subseteq I$, where $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{J}_r(S)$ and $\mathfrak{b}^c=S$, then $(\mathfrak{a}S_\alpha)(\mathfrak{b}S_\alpha)\subseteq I_\alpha$, and since $(\mathfrak{b}S_\alpha)^c=S_\alpha$, $\mathfrak{a}S_\alpha\subseteq I_\alpha$. Thus for each $a=\bigcup_\alpha a_\alpha\in\mathfrak{a}$, $a_\alpha S_\alpha\subseteq I_\alpha$ and $a_\alpha\in I_\alpha$. This proves that $a\in I$, and hence that $\mathfrak{a}\subseteq I$. Therefore $I\in\mathfrak{P}_r(S)$.

If, conversely, $I\in\mathfrak{P}_r(S)$, then $IS_\alpha\subseteq I\cap S_\alpha$. Since $I\cap S_\alpha\in\mathfrak{P}_r(S_\alpha)$, an argument similar to that of the previous paragraph proves that $I \subseteq \bigcup_\alpha (I \cap S_\alpha)$. Thus $I = \bigcup_\alpha (I \cap S_\alpha)$, and the proof of 2.6 is completed.

If $\mathfrak{S}$ is a partially-ordered set with a least element 0, then a least nonzero element of $\mathfrak{S}$ is called an *atom* of $\mathfrak{S}$. The atoms of $\mathfrak{A}(R)$ are described by the following theorem.

2.7. THEOREM. *The element S of $\mathfrak{A}(R)$ is an atom if and only if S is a prime ring.*

Proof. If S is an atom of $\mathfrak{A}(R)$, and if $\mathfrak{a}\mathfrak{b}=0$, $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{J}(S)$, then $\mathfrak{a}^c \cap \mathfrak{b}^c = 0$, $\mathfrak{a}^c$, $\mathfrak{b}^c \in \mathfrak{A}(S)$. Since $\mathfrak{A}(S) \subseteq \mathfrak{A}(R)$ by 2.4, and since S is an atom of $\mathfrak{A}(R)$, clearly either $\mathfrak{a}^c = 0$ or $\mathfrak{b}^c = 0$. Hence S is a prime ring.

On the other hand, if S is a prime ring and $S \in \mathfrak{A}(R)$, and if $0 \neq T \subseteq S$ for some $T \in \mathfrak{A}(R)$, then $T(T^r \cap S) = 0$ and $T^r \cap S = 0$. Hence $S \subseteq T$, and we conclude that S is an atom of $\mathfrak{A}(R)$.

With each $I \in \mathfrak{J}_r(R)$ we can associate its *prime cover* $p(I)$, $p(I)$ being the least element of $\mathfrak{P}_r(R)$ containing I. The mapping

$$p: I \to p(I)$$

of $\mathfrak{J}_r(R)$ onto $\mathfrak{P}_r(R)$ is a closure operation on $\mathfrak{J}_r(R)$, and since $I \subseteq p(I) \subseteq I^c$, clearly $(p(I))^c = I^c$.

2.8 THEOREM. *For each I_1, $I_2 \in \mathfrak{J}_r(R)$ we have*

$$p(I_1 \cap I_2) = p(I_1) \cap p(I_2).$$

Proof. Let $\mathfrak{B}$ be the set of all right ideals I of R such that

$$I \supseteq I_1, \qquad I \cap I_2 \subseteq p(I_1 \cap I_2).$$

Since Zorn's lemma applies to $\mathfrak{B}$, we may select a maximal element I from $\mathfrak{B}$. Let us prove that $I \in \mathfrak{P}_r(R)$.

If $\mathfrak{a}\mathfrak{b} \subseteq I$, $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{J}_r(R)$ with $\mathfrak{b}^c = R$ and $\mathfrak{a} \supseteq I$, then

$$(\mathfrak{a} \cap I_2)\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b} \cap I_2 \subseteq I \cap I_2 \subseteq p(I_1 \cap I_2).$$

Hence $\mathfrak{a} \cap I_2 \subseteq p(I_1 \cap I_2)$ and $\mathfrak{a} = I$ since I is maximal in $\mathfrak{B}$. Thus $I \in \mathfrak{P}_r(R)$. Clearly, then, $p(I_1) \cap I_2 \subseteq p(I_1 \cap I_2)$.

A continuation of the same process obviously yields that $p(I_1) \cap p(I_2) \subseteq p(I_1 \cap I_2)$. Since it is evident that $p(I_1 \cap I_2) \subseteq p(I_1) \cap p(I_2)$, the proof of 2.8 is complete.

We recall that for $I \in \mathfrak{J}_r(R)$ and $a \in R$,

$$(I:a) = \{b;\, b \in R,\, ab \in I\}.$$

2.9 LEMMA. *If $I \in \mathfrak{P}_r(R)$ and $a \in R$, then $(I:a) \in \mathfrak{P}_r(R)$.*

Proof. If $\mathfrak{a}\mathfrak{b} \subseteq (I:a)$, $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{J}_r(R)$ with $\mathfrak{b}^c = R$, then $(a\mathfrak{a})\mathfrak{b} \subseteq I$ and $a\mathfrak{a} \subseteq I$. Thus $\mathfrak{a} \subseteq (I:a)$, which proves that $(I:a) \in \mathfrak{P}_r(R)$.

2.10 Lemma. *If* $a \in R$ *and* $I \in \mathfrak{J}_r(R)$ *are such that* $(I:a) \in \mathfrak{P}_r(R)$, *then* $(I:a) = (p(I):a)$.

Proof. If $(I:a) = (I_\alpha:a)$ for some linear sequence $\{I_\alpha\} \subseteq \mathfrak{J}_r(R)$, then $ax \in \bigcup_\alpha I_\alpha$ implies that $ax \in I_\alpha$ for some α, and this in turn implies that $x \in (I:a)$. Thus $(I:a) = (\bigcup_\alpha I_\alpha : a)$. Let $\mathfrak{B}$ consist of all $I' \in \mathfrak{J}_r(R)$ such that $I' \supseteq I$, $(I:a) = (I':a)$. By Zorn's lemma, $\mathfrak{B}$ contains a maximal element I_1. Let us prove that $I_1 \in \mathfrak{P}_r(R)$.

If $\mathfrak{a}\mathfrak{b} \subseteq I_1$, $\mathfrak{a}, \mathfrak{b} \in \mathfrak{J}_r(R)$ with $\mathfrak{b}^c = R$ and $\mathfrak{a} \supseteq I_1$, then

$$a(\mathfrak{a}:a)\mathfrak{b} \subseteq I_1 \quad \text{and} \quad (\mathfrak{a}:a)\mathfrak{b} \subseteq (I:a).$$

Since $(I:a) \in \mathfrak{P}_r(R)$, we have $(\mathfrak{a}:a) \subseteq (I:a)$. The maximality of I_1 in $\mathfrak{B}$ therefore implies that $\mathfrak{a} = I_1$. Thus $I_1 \in \mathfrak{P}_r(R)$.

Clearly $I_1 \supseteq p(I)$, and $(I:a) = (p(I):a) = (I_1:a)$ as desired.

2.11 Lemma. *If* $a \in R$ *and* $I \in \mathfrak{J}_r(R)$, *then* $ap(I) \subseteq p(aI)$.

Proof. Since $(p(aI):a) \supseteq I$, evidently $(p(aI):a) \supseteq p(I)$. Hence $ap(I) \subseteq p(aI)$.

2.12 Lemma. *If* $a \in R$ *and* $I \in \mathfrak{J}_r(R)$ *with* $I \supseteq (0:a)$, *then*

$$aR \cap p(aI) = ap(I).$$

Proof. Since $I \supseteq (0:a)$, we have $I = (aI:a)$. Similarly, $(ap(I):\ a) = p(I)$. Now $ap(I) \subseteq p(aI)$ by 2.11, and therefore

$$p(aI) \subseteq p(ap(I)) \subseteq p(p(aI)) = p(aI).$$

Thus $p(aI) = p(ap(I))$. We have by 2.10 that $(p(ap(I)):a) = p(I)$, and hence that $(p(aI):a) = p(I)$. Thus $aR \cap p(aI) = ap(I)$, and 2.12 is proved.

2.13 Theorem. *For each* $a \in R$ *and* $I \in \mathfrak{J}_r(R)$, *we have*

$$p((I:a)) = (p(I):a).$$

Proof. Since $(I:a) \supseteq (0:a)$, $aR \cap I = a(I:a)$ and, by 2.8,

$$p(aR) \cap p(I) = p(a(I:a)).$$

Thus $aR \cap p(I) = aR \cap p(a(I:a))$ and $aR \cap p(I) = ap((I:a))$ by 2.12. Hence $(p(I):a) = p((I:a))$ and the theorem is proved.

Since $p((I:a)) = (p(I):a) = R$ if and only if $a \in p(I)$, it is clear that $p(I)$ may be characterized as follows.

2.14 Corollary. *For each* $I \in \mathfrak{J}_r(R)$,

$$p(I) = \{a;\, a \in R,\, p((I:a)) = R\}.$$

3. **Structures of semi-prime rings.** Consider a semi-prime ring R. A subalgebra $\mathfrak{R}$ of $\mathfrak{P}_r(R)$ is called a (right) *structure* of R if the following three

axioms are satisfied. For each $I\in\mathfrak{J}_r(R)$, we denote by I^* the least element of $\mathfrak{N}$ containing I.

(A1) $0, R\in\mathfrak{N}$.

(A2) $(I\cap I')^*=I^*\cap I'^*$, $I, I'\in\mathfrak{J}_r(R)$.

(A3) $(I:a)^*=(I^*:a)$, $I\in\mathfrak{J}_r(R)$, $a\in R$.

We shall restrict ourselves to right structures, although obviously one could make similar remarks about left structures.

In view of the results §2, every semi-prime ring R has at least one structure, namely $\mathfrak{P}_r(R)$. It is not difficult to show that a ring can have more than one structure.

If $\mathfrak{N}$ is a structure of R, then we note from (A1) and (A3) that $(0:a)\in\mathfrak{N}$ for each $a\in R$. It is clear then that

$$\mathfrak{P}_r(R)\supseteq\mathfrak{N}\supseteq\mathfrak{A}_r(R)\supseteq\mathfrak{A}(R).$$

Also, since $I\subseteq p(I)\subseteq I^*$ for each $I\in\mathfrak{J}_r(R)$, we have $(p(I))^*=I^*$.

For each $I\in\mathfrak{J}_r(R)$, the following analogue of 2.13 clearly holds in $\mathfrak{N}$.

3.1 $$I^* = \{a;\, a\in R,\ (I:a)^* = R\}.$$

A structure $\mathfrak{N}$ of R is called an *a-structure* of R if the following additional axiom is satisfied by $\mathfrak{N}$.

(A4) The algebra $\mathfrak{N}$ has atoms.

The structure $\mathfrak{P}_r(R)$ is an a-structure of R provided that R has minimal nonzero prime right ideals. In particular, any semi-simple ring (in the sense of Jacobson [6]) with minimal nonzero right ideals has $\mathfrak{P}_r(R)$ as an a-structure, since it is easily shown that the minimal right ideals of R are in $\mathfrak{P}_r(R)$.

A rather trivial example of a semi-prime ring R that is neither a prime ring nor a semi-simple ring with minimal right ideals and that has an a-structure is given now. Let I be the ring of integers, and

3.2 $$R = \{(a, b);\, (a, b)\in I\oplus I,\ a+b\in 2I\}.$$

If

$$S_1 = \{(a, 0);\, a\in 2I\},\qquad S_2 = \{(0, a);\, a\in 2I\},$$

then $\mathfrak{P}_r(R)=\{0, S_1, S_2, R\}$ since R is commutative and $\mathfrak{P}_r(R)=\mathfrak{A}(R)$ by 2.3. Clearly $\mathfrak{P}_r(R)$ is an a-structure of R. We note that if $S=S_1+S_2$, then $R/S=I/(2)$.

3.3 THEOREM. *If $\mathfrak{N}$ is an a-structure of R and I is an atom of $\mathfrak{N}$, then I^c is an atom of $\mathfrak{A}(R)$.*

Proof. If $0\neq S\subseteq I^c$, $S\in\mathfrak{A}(R)$, then $S\cap I\neq 0$ and $I\subseteq S$. Hence $I^c\subseteq S$, and we conclude that I^c is an atom of $\mathfrak{A}(R)$.

This theorem together with 2.7 implies that if I is an atom of $\mathfrak{N}$, then I^c is a prime ring.

The remainder of this section will be devoted to the following problems. If S is an ideal of R, does an a-structure of R induce an a-structure of S? And, conversely, does an a-structure of S induce an a-structure of R? The answer to each of these questions, with certain obvious restrictions, is yes as we shall now show.

3.4 THEOREM. *Let $\mathfrak{R}$ be an a-structure of R and $S \in \mathfrak{A}(R)$, with at least one atom of $\mathfrak{R}$ being contained in S. Then $\mathfrak{S} = \mathfrak{J}_r(S) \cap \mathfrak{R}$ is an a-structure of S.*

Proof. It is evident from 2.4 that $\mathfrak{S}$ is a subalgebra of $\mathfrak{P}_r(S)$. If I^* denotes the least element of $\mathfrak{S}$ containing $I \in \mathfrak{J}_r(S)$, then $I^* = (IS)^*$ since $p(I) = p(IS)$. Note that $IS \in \mathfrak{J}_r(R)$ and that $(IS)^*$ is the least element of $\mathfrak{R}$ containing IS. It is immediate that (A1) and (A4) hold for $\mathfrak{S}$; and (A2) is a consequence of the following chain of inclusion relations:

$$(I \cap I')^* \supseteq (IS \cap I'S)^* = I^* \cap I'^* \supseteq (I \cap I')^*, \qquad I, I' \in \mathfrak{J}_r(S).$$

For any ideal T of R, we shall use the notation $(I:a)_T$ to mean

$$(I:a)_T = \{b; b \in T, ab \in I\}, \qquad a \in R, I \subseteq R.$$

Clearly

$$(I \cap T:a)_T = (I:a)_R \cap T.$$

In order to prove that (A3) holds for $\mathfrak{S}$, let $I \in \mathfrak{J}_r(S)$ and $a \in S$. Then

$$(I^*:a)_R = ((IS)^*:a)_R = (IS:a)_R^* \subseteq (I:a)_R^* \subseteq (I^*:a)_R.$$

Hence $(I^*:a)_R = (I:a)_R^*$ and $(I^*:a)_S = (I:a)_S^*$. This completes the proof of 3.4.

If $S \in \mathfrak{J}(R)$ such that $S^c = R$, then we proved in 2.5 that the structures $\mathfrak{P}_r(R)$ and $\mathfrak{P}_r(S)$ are isomorphic under the mapping

3.5 $$I' \to I' \cap S, \qquad I' \in \mathfrak{P}_r(R).$$

If $\mathfrak{R}$ and $\mathfrak{S}$ are subsets of $\mathfrak{P}_r(R)$ and $\mathfrak{P}_r(S)$ respectively that are in a 1-1 correspondence by the above mapping, then evidently $\mathfrak{R}$ is an algebra if and only if $\mathfrak{S}$ is an algebra.

3.6 THEOREM. *Let $S \in \mathfrak{J}(R)$ such that $S^c = R$. If $\mathfrak{R}$ and $\mathfrak{S}$ are subalgebras of $\mathfrak{P}_r(R)$ and $\mathfrak{P}_r(S)$ respectively that are isomorphic under the mapping* 3.5, *then $\mathfrak{R}$ is an a-structure of R if and only if $\mathfrak{S}$ is an a-structure of S.*

Proof. It is clear that if either $\mathfrak{R}$ or $\mathfrak{S}$ satisfies (A1) and (A4) then they both do. So let us assume that $\mathfrak{R}$ and $\mathfrak{S}$ are algebras satisfying both (A1) and (A4). For each I in $\mathfrak{J}_r(R)$ ($\mathfrak{J}_r(S)$), let I^* ($I^\blacktriangle$) be the least element of $\mathfrak{R}(\mathfrak{S})$ containing I. Evidently

$$I^\blacktriangle = (IS)^\blacktriangle = (IS)^* \cap S, \qquad I \in \mathfrak{J}_r(S).$$

If (A2) holds for $\mathfrak{R}$, then for each $I, I' \in \mathfrak{J}_r(S)$,

$$(I \cap I')^{\blacktriangle} \supseteq (IS \cap I'S)^{\blacktriangle} = (IS \cap I'S)^* \cap S = (IS)^* \cap (I'S)^* \cap S$$
$$= I^{\blacktriangle} \cap I'^{\blacktriangle} \supseteq (I \cap I')^{\blacktriangle}.$$

Hence (A2) holds for $\mathfrak{S}$ also. Conversely, if (A2) holds for $\mathfrak{S}$, then for each $I, I' \in \mathfrak{J}_r(R)$,

$$(I \cap I')^* \cap S \supseteq (IS \cap I'S)^* \cap S = (IS \cap I'S)^{\blacktriangle} = I^{\blacktriangle} \cap I'^{\blacktriangle}$$
$$= (IS)^* \cap (I'S)^* \cap S = I^* \cap I'^* \cap S \supseteq (I \cap I')^* \cap S.$$

Hence $(I \cap I')^* = I^* \cap I'^*$ and (A2) holds for $\mathfrak{R}$.

Let us now assume that (A1), (A2), and (A4) hold for both $\mathfrak{R}$ and $\mathfrak{S}$, and prove that if (A3) holds for either $\mathfrak{R}$ or $\mathfrak{S}$ then it holds for both. If (A3) is satisfied by $\mathfrak{R}$, and if $I \in \mathfrak{J}_r(S)$, $a \in S$, then

$$(I^{\blacktriangle}:a)_S = ((IS)^* \cap S:a)_S = ((IS)^*:a)_R \cap S = (IS:a)_R^* \cap S$$
$$= [(IS:a)_R \cap S]^* \cap S = (IS:a)_S^{\blacktriangle} \subseteq (I:a)_S^{\blacktriangle} \subseteq (I^{\blacktriangle}:a)_S.$$

Thus (A3) is satisfied by $\mathfrak{S}$.

Finally, let us assume that (A3) is satisfied by $\mathfrak{S}$, and prove that it also holds for $\mathfrak{R}$. First, let us prove that for each $I \in \mathfrak{S}$ and $a \in R$ we have $(I:a)_S \in \mathfrak{S}$. To that end, let $J = (I:a)_S$, $c \in J^{\blacktriangle}$, and $L = (J:c)_S$. Then $L^{\blacktriangle} = S$, and since $(I:ac)_S \supseteq L$, $(I:ac)_S = (I:ac)_S^{\blacktriangle} = S$. Hence $ac \in I$ and we conclude that $aJ^{\blacktriangle} \subseteq I$, that is, that $J = J^{\blacktriangle}$ and $(I:a)_S \in \mathfrak{S}$.

If $I \in \mathfrak{J}_r(S)$, $a \in R$, and $c \in S$, then

$$((I:a)_S:c)_S = (I:ac)_S$$

and

$$(I^{\blacktriangle}:ac)_S = ((I^{\blacktriangle}:a)_S:c)_S = (I:ac)_S^{\blacktriangle} = ((I:a)_S:c)_S^{\blacktriangle} = ((I:a)_S^{\blacktriangle}:c)_S.$$

Thus $((I^{\blacktriangle}:a)_S:c)_S = ((I:a)_S^{\blacktriangle}:c)_S$ for each $c \in S$, and since $(I^{\blacktriangle}:a)_S$, $(I:a)_S^{\blacktriangle} \in \mathfrak{S}$, this clearly implies that

$$(I^{\blacktriangle}:a)_S = (I:a)_S^{\blacktriangle}, \qquad I \in \mathfrak{J}_r(S),\ a \in R.$$

To complete the proof that (A3) holds for $\mathfrak{R}$, let $I \in \mathfrak{J}_r(R)$ and $a \in R$. Then

$$(I:a)_R^* \cap S = [(I:a)_R \cap S]^{\blacktriangle} = (I \cap S:a)_S^{\blacktriangle} = ((I \cap S)^{\blacktriangle}:a)_S$$
$$= (I^* \cap S:a)_S = (I^*:a)_R \cap S.$$

Thus $(I:a)_R^* = (I^*:a)_R$ and (A3) holds for $\mathfrak{R}$. This completes the proof of 3.6.

3.7 THEOREM. *If $\{S_\alpha\}$ is a set of rings such that each S_α of the set has a structure $\mathfrak{S}_\alpha$, and if $\mathfrak{S} = \cup_\alpha S_\alpha$ is the (finite) direct sum of the rings of this set,* *then*

$$\mathfrak{S} = \left\{ \bigcup_\alpha I_\alpha ; I_\alpha \in \mathfrak{S}_\alpha \right\}$$

is a structure of S. This structure is an a-structure if and only if some $\mathfrak{S}_\alpha$ is an a-structure.

Proof. If $I_\beta = \bigcup_\alpha I_{\beta\alpha} \in \mathfrak{S}$, then $\bigcap_\beta I_\beta = \bigcup_\alpha (\bigcap_\beta I_{\beta\alpha}) \in \mathfrak{S}$ and we conclude that $\mathfrak{S}$ is an algebra; it is a subalgebra of $\mathfrak{P}_r(S)$ by 2.6. For each $I \in \mathfrak{J}_r(S)$, it is clear that the least element I^* of $\mathfrak{S}$ containing I is given by

$$I^* = \bigcup_\alpha (I \cap S_\alpha)^*,$$

where $(I \cap S_\alpha)^*$ is the least element of $\mathfrak{S}_\alpha$ containing $I \cap S_\alpha$. Since

$$\left(\bigcup_\alpha I_\alpha \right) \cap \left(\bigcup_\alpha I'_\alpha \right) = \bigcup_\alpha (I_\alpha \cap I'_\alpha)$$

for any I_α, $I'_\alpha \in \mathfrak{J}_r(S_\alpha)$, evidently (A2) holds for $\mathfrak{S}$. In order to prove (A3) for $\mathfrak{S}$, note first that

$$\left(\bigcup_\alpha I_\alpha : a \right) = \bigcup_\alpha (I_\alpha : a)_\alpha,$$

where $a \in S$ and $I_\alpha \in \mathfrak{J}_r(S_\alpha)$, and

$$(I:a)_\alpha = \{ b; b \in S_\alpha, ab \in I \}.$$

Hence for each $I \in \mathfrak{J}_r(S)$, $a \in S$,

$$\begin{aligned}(I^*:a) &= \bigcup_\alpha ((I \cap S_\alpha)^*:a)_\alpha = \bigcup_\alpha (I \cap S_\alpha : a)^*_\alpha = \left(\bigcup_\alpha (I \cap S_\alpha : a)_\alpha \right)^* \\ &= \left(\bigcup_\alpha I \cap S_\alpha : a \right)^* \subseteq (I:a)^* \subseteq (I^*:a).\end{aligned}$$

Thus (A3) holds for $\mathfrak{S}$, and $\mathfrak{S}$ is a structure of S. Clearly, $\mathfrak{S}$ is an a-structure if and only if some $\mathfrak{S}_\alpha$ has atoms. This proves 3.7.

If now S is any ideal of the semi-prime ring R, then a structure $\mathfrak{R}$ of R induces a structure of S^c by 3.4. In turn, the structure of S^c induces an isomorphic structure of S by 3.6. The resulting structure of S is an a-structure if and only if $\mathfrak{R}$ is an a-structure having at least one of its atoms contained in S^c.

If, conversely, the ideal S of the semi-prime ring R has a structure $\mathfrak{S}$, then a structure of R is induced by $\mathfrak{S}$. To see this, we take an arbitrary structure $\mathfrak{S}'$ of S^r. For example, we might select $\mathfrak{S}' = \mathfrak{P}_r(S^r)$. Then $\mathfrak{S}$ and $\mathfrak{S}'$ induce a structure of $S + S^r$ by 3.7. Finally, this structure of $S + S^r$ induces an isomorphic structure of R by 3.6, since $R = (S + S^r)^c$. Clearly the induced structure of R is an a-structure if $\mathfrak{S}$ is.

4. **Determination of all semi-prime rings with a-structures.** If R is a semi-prime ring with a-structure $\mathfrak{R}$, and if

$$\mathfrak{B}(R) = \{I^c;\ I \in \mathfrak{R},\ I \text{ an atom}\},$$

then the *base* of R is defined to be the ring union B of all elements of $\mathfrak{B}$: $B = \bigcup_\alpha S_\alpha$, $S_\alpha \in \mathfrak{B}(R)$. In view of 2.7 and 3.3, the elements of $\mathfrak{B}(R)$ are prime rings.

4.1 THEOREM. *The base B of a ring R is the (finite) direct sum of the distinct elements of $\mathfrak{B}(R)$.*

Proof. If $\{S_\alpha;\ \alpha\in\Delta\} \subseteq \mathfrak{B}(R)$ and $T \in \mathfrak{B}(R)$ with

$$T \cap \left(\bigcup_{\alpha\in\Delta} S_\alpha\right) \neq 0,$$

then

$$T \cap \left(\bigcup_{i=1}^{n} S_i\right) \neq 0$$

for some finite subset of $\{S_\alpha\}$. Hence

$$T\left(\bigcup_{i=1}^{n} S_i\right) \neq 0,$$

and therefore $TS_i \neq 0$ for some $S_i \in \{S_\alpha\}$. Since S_i and T are atoms of $\mathfrak{A}(R)$, evidently $T = S_i$. Thus B is a direct sum of the elements of $\mathfrak{B}(R)$.

The base B of a ring R is similar to the socle of a semi-simple ring with minimal right ideals as defined by Dieudonné [4]. The socle is defined to be the union of all minimal right ideals of the ring. Dieudonné shows that the socle of a ring is a direct sum of simple rings. It is clear that if R is a semi-simple ring with minimal right ideals, then the socle of R is contained in the base of R.

If the ring R has an a-structure $\mathfrak{R}$, then the component B^c of the base B in R has an a-structure induced by $\mathfrak{R}$ according to 3.4. Now B is the base of B^c and B itself has an a-structure isomorphic to that of B^c by 3.6. In turn, each prime ring $B_\alpha \in \mathfrak{B}(R)$ has an a-structure induced by $\mathfrak{R}$. Since the a-structure of a prime ring is unique [7, p. 806], the a-structure of each B_α induced by $\mathfrak{R}$ must be the intrinsic a-structure of B_α. The base B of R has a structure induced by the prime rings $B_\alpha \in \mathfrak{B}(R)$ according to 3.7. Let us now prove that this is the unique a-structure of B.

4.2 THEOREM. *If B is the base of R so that $B = \bigcup_\alpha B_\alpha$, $B_\alpha \in \mathfrak{B}(R)$, and if $\mathfrak{S}_\alpha$ is the a-structure of B_α, then*

$$\mathfrak{S} = \left\{ \bigcup_{\alpha} I_\alpha; \quad I_\alpha \in \mathfrak{S}_\alpha \right\}$$

is the unique a-structure of B.

Proof. Let $\mathfrak{S}'$ be an a-structure of B such that the base of B relative to $\mathfrak{S}'$ is again B. Then $\mathfrak{S}' \cap \mathfrak{J}_r(B_\alpha) = \mathfrak{S}_\alpha$ for each α in view of 3.4 and the uniqueness of the a-structure of a prime ring. For each $I \in \mathfrak{J}_r(B)$, let $I^\blacktriangle$ (I^*) be the least element of $\mathfrak{S}'$ ($\mathfrak{S}$) containing I. If $I \in \mathfrak{J}_r(B_\alpha)$, then $I^\blacktriangle = I^*$. Now for any $I \in \mathfrak{J}_r(B)$, $(I \cap B_\alpha)^* \subseteq I^\blacktriangle$ and therefore $I^* = \bigcup_\alpha (I \cap B_\alpha)^* \subseteq I^\blacktriangle$. Thus

$$I^{\blacktriangle *} = I^\blacktriangle = \bigcup_\alpha (I^\blacktriangle \cap B_\alpha)^* = \bigcup_\alpha (I \cap B_\alpha)^{\blacktriangle *} = I^*,$$

and we conclude that $\mathfrak{S}' = \mathfrak{S}$.

We know now that if R is any semi-prime ring with a-structure $\mathfrak{R}$ and base B, then $R = (B + B^r)^c$ where B has an intrinsic a-structure induced by its prime components and B^r has a structure induced by $\mathfrak{R}$ that definitely is not atomic. Let us now try to find all semi-prime rings that possess a-structures.

For a given ring R, such as a semi-prime ring, having the property that $R^r = 0$, it was shown in [9, 1.2] that there exists a universal extension ring $N(R)$ containing R as an ideal and having the property that $R^r = 0$ in $N(R)$. Another way of looking at it is that $N(R)$ is a maximal ring containing R as an ideal for which $R^c = N(R)$.

The ring $N(R)$ may be defined in the following way. Consider the additive group R^+ of R as a left R-module and let $E(R)$ be the total endomorphism ring of this module. Then R^+ may be considered as an $(R, E(R))$-module. It is clear that R is a right ideal of $E(R)$; one has only to select $N(R)$ as the normalizer of R in $E(R)$.

4.3 Theorem. *If R is a semi-prime ring, then so is any ring T such that $R \subseteq T \subseteq N(R)$.*

Proof. If $S^2 = 0$, $S \in \mathfrak{J}(T)$, then $(RS)^2 = 0$, and, since $RS \in \mathfrak{J}_l(R)$, $RS = 0$. Hence $S = 0$, and we conclude that T is a semi-prime ring.

For any set $\{T_\alpha\}$ of rings, we have been using the notation $\bigcup_\alpha T_\alpha$ for the finite direct sum of these rings. In the following theorem, the notation $\sum_\alpha T_\alpha$ is used to signify the full direct sum of the rings of this set.

4.4 Theorem. *If $\{S_\alpha\}$ is a set of semi-prime rings, then $R = \bigcup_\alpha S_\alpha$ also is a semi-prime ring and $N(R) = \sum_\alpha N(S_\alpha)$.*

Proof. It is clear that R is semi-prime. Since $(S_\alpha N(R) S_\beta)^2 = 0$ if $\alpha \neq \beta$, $S_\alpha N(R) S_\beta = 0$ and we conclude that each $S_\alpha \in \mathfrak{J}(N(R))$. Hence for each $a \in N(R)$, $S_\alpha a \subseteq S_\alpha$ and a has the same effect on S_α as some $a_\alpha \in N(S_\alpha)$. Clearly $a = \sum_\alpha a_\alpha$ so that $N(R) \subseteq \sum_\alpha N(S_\alpha)$. That $\sum_\alpha N(S_\alpha) \subseteq N(R)$ is evident, and 4.4 is proved.

4.5 THEOREM. *If R is a semi-prime ring with a-structure $\mathfrak{R}$ and base B, then*

$$B \oplus B^r \subseteq R \subseteq N(B) \oplus N(B^r).$$

Furthermore, the rings $B+B^r$ and $N(B)+N(B^r)$ have a-structures that are isomorphic to $\mathfrak{R}$.

This theorem follows readily from 4.4, 4.3, and 3.6.

In Example 3.2, $B^r=0$, $B=S_1\oplus S_2$, and $N(B)=I\oplus I$. Clearly $B\subset R \subset N(B)$, and $\mathfrak{P}_r(R)$ is isomorphic to $\mathfrak{P}_r(N(B))$.

The final result of this paper is the following partial converse of 4.5. This result is in a sense a generalization of the results of Jaffard [5] and Dieudonné [4] on semi-simple rings with minimal right ideals.

4.6 THEOREM. *If $\{S_\alpha\}$ is a set of prime rings each of which has an a-structure, if $S=\bigcup_\alpha S_\alpha$, and if S' is a semi-prime ring with a non-atomic structure, then $S\oplus S'$ has an a-structure $\mathfrak{S}$ induced by its components. Any ring R such that*

$$S \oplus S' \subseteq R \subseteq N(S) \oplus N(S')$$

has an a-structure $\mathfrak{R}$ isomorphic to $\mathfrak{S}$ and the base of R relative to $\mathfrak{R}$ is $\bigcup_\alpha(R\cap N(S_\alpha))$.

The structure $\mathfrak{S}$ of $S\oplus S'$ is as given in 3.7. The base of $S\oplus S'$ relative to $\mathfrak{S}$ is just S, while the base of $N(S)\oplus N(S')$ relative to the structure induced by (and isomorphic to) $\mathfrak{S}$ is just $\bigcup_\alpha N(S_\alpha)$. Clearly then the base of R is as stated in 4.6.

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