A STUDY OF $\alpha$-VARIATION. I.

BY
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This paper is based on the notion of the higher variation of a function introduced by N. Wiener [9] while studying the Fourier coefficients of a function with bounded variation. L. C. Young applied this idea to derive a new existence theorem for Stieltjes integration and later collaborated with E. R. Love in publishing a number of papers on subjects related to this concept.

Preliminaries

1.1. Suppose that $f(x)$ is a real- or complex-valued function defined over $a \leq x \leq b$. For $0 < \alpha \leq 1$, we define the $\alpha$-variation of $f(x)$ over this interval as the least upper bound of the sums

$$\left\{ \sum_{n=1}^{N} \left| f(x_n) - f(x_{n-1}) \right|^{1/\alpha} \right\}^\alpha$$

taken over all subdivisions $a = x_0 < \cdots < x_N = b$, and we denote this upper bound by

$$V_\alpha\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_\alpha\{f(x); x \in I\},$$

where $I$ is the interval $a \leq x \leq b$. Similarly we define the $0$-variation (or oscillation) of $f(x)$ over this interval as the least upper bound of the difference $\left| f(x'') - f(x') \right|$ for $a \leq x' < x'' \leq b$, and we denote this upper bound by

$$V_0\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_0\{f(x); x \in I\}.$$

It is often convenient to consider the $\alpha$-variation of a function over an interval, $I$, which is open or half-open, and we can appropriately define the symbol

$$V_\alpha\{f(x); x \in I\}$$

for $0 \leq \alpha \leq 1$.

Suppose that $\{x_n\}$ is any set of real or complex numbers. For $0 < \alpha \leq 1$, we denote by

$$\left\{ \sum_n \left| \sum x_n \right|^{1/\alpha} \right\}^\alpha$$

the least upper bound of all sums.
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$$\left\{ \sum_{k} \left| x_{n_{k-1}} + \cdots + x_{n_{k-1}} \right|^{|1/\alpha|} \right\}^{\alpha},$$

where \( \{n_{k}\} \) is any appropriate finite sequence. When \( \alpha = 0 \), we let

$$\left\{ \sum_{n} \left| x_{n} \right|^{|1/\alpha|} \right\}^{\alpha} \text{ and } \left\{ \sum_{n} \sum_{n} \left| x_{n} \right|^{|1/\alpha|} \right\}^{\alpha}$$

denote the least upper bounds for \( |x_{m}| \) and \( |x_{m} + \cdots + x_{n}| \) respectively.

1.2. For \( 0 \leq \alpha \leq 1 \), we say that the function \( f(x) \) is in \( W_{\alpha} \) over the interval \( 0 \leq x \leq 1 \) if \( f(x) \) has bounded \( \alpha \)-variation over this interval; \( W_{0} \) is simply the class of bounded functions. The sum of two functions in \( W_{\alpha} \) is also in \( W_{\alpha} \) since, by Minkowski's inequality,

$$V_{\alpha}\{f(x) + g(x); x \in I\} \leq V_{\alpha}\{f(x); x \in I\} + V_{\alpha}\{g(x); x \in I\}.$$ 

Similarly, for \( 0 \leq \alpha < \beta \leq 1 \), it follows from Jensen's inequality that

$$V_{\alpha}\{f(x); x \in I\} \leq V_{\beta}\{f(x); x \in I\},$$

and hence a function in \( W_{\beta} \) is also in \( W_{\alpha} \).

If, for \( 0 \leq \alpha \leq 1 \), \( f(x) \) has period 1 and is in the class Lip (\( \alpha \)), i.e. if for some constant \( C \)

$$\left| f(x'') - f(x') \right| < C(x'' - x')^{\alpha}$$

for each \( x' < x'' \), \( f(x) \) is obviously in \( W_{\alpha} \) and the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 is less than \( C \). The converse is not true since functions in \( W_{\alpha} \) need not be continuous. However we can prove the following result.

**Theorem 1.2.1.** (Cf. [11, p. 455].) Suppose that \( 0 \leq \alpha \leq 1 \), that \( f(x) \) is real and continuous with period 1, and that the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 is less than 1. There exists a continuous increasing function \( \phi(t) \) such that \( \phi(t+1) = \phi(t) + 1 \) and such that

$$\left| f\{\phi(t'')\} - f\{\phi(t')\} \right| < (t'' - t')^{\alpha}$$

for each \( t' < t'' \).

We can suppose that \( \alpha > 0 \) and that \( f(x) \) assumes its positive maximum at \( x = 0 \). Define the function \( \gamma(x) \) so that \( \{\gamma(x)\}^{\alpha} \) is equal to the \( \alpha \)-variation of \( f(x) \) over the interval \( 0 \leq y \leq x \). \( \gamma(x) \) is continuous, \( \gamma(x+1) \geq \gamma(x) + \gamma(1) \), and \( \gamma(1) < 1 \). For each \( x > 0 \) we can select a subdivision \( 0 = y_0 < \cdots < y_N = x + 1 \) such that

$$\gamma(x + 1) < \sum_{n=1}^{N} \left| f(y_n) - f(y_{n-1}) \right|^{|1/\alpha|} + \epsilon.$$ 

If \( y_n < 1 \leq y_{n+1} \), we see by periodicity that
and hence that \( \gamma(x+1) + \gamma(1) = \gamma(x+1) + \gamma(1) + \epsilon \). For \( x > 0 \) we conclude that \( \gamma(x+1) = \gamma(x) + \gamma(1) + \epsilon \) and we extend \( \gamma(x) \) so this holds for all \( x \).

Let \( \theta(x) = \gamma(x) + x\{1 - \gamma(1)\} \). \( \theta(x) \) is continuous and increasing for all \( x \); let \( \phi(t) \) be the inverse function. Then \( \phi(t+1) = \phi(t) + 1 \) and

\[
| f\{\phi(t')\} - f\{\phi(t'')\} |^{1/\alpha} \leq \gamma\{\phi(t')\} - \gamma\{\phi(t'')\} < \theta\{\phi(t')\} - \theta\{\phi(t'')\}
\]

for each \( t' < t'' \). This completes the proof.

We also have the following

**Theorem 1.2.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( W_\alpha \). Then

\[
V_\alpha\{f(y); x \leq y \leq x + h\} = O(h^\alpha)
\]

almost everywhere, and similarly on the left.

Assume that \( \alpha > 0 \) and let \( E_k \) be the set of points in \( 0 \leq x < 1 \) for which

\[
\limsup_{h \to 0} h^{-\alpha} V_\alpha\{f(y); x \leq y \leq x + h\} > k.
\]

For each \( x \) in \( E_k \) and each \( \epsilon > 0 \) there exists an interval, \( I(x) = x \leq y \leq x + h \), such that

\[
V_\alpha\{f(y); x \leq y \leq x + h\} \geq k h^\alpha
\]

and such that \( h < \epsilon \). By Vitali's covering theorem there exists a sequence, \( \{I(x_n)\} \), of nonoverlapping intervals which cover almost all of \( E_k \) and

\[
\text{outer meas } E_k \leq \left\{ \sum_{n=1}^{\infty} |I(x_n)| \right\}^\alpha \\
\leq \frac{1}{k} \cdot \left\{ \sum_{n=1}^{\infty} V_\alpha\{f(y); y \in I(x_n)\}^{1/\alpha} \right\}^\alpha \\
\leq \frac{1}{k} \cdot V_\alpha\{f(x); 0 \leq x \leq 1\}.
\]

The set of points in \( 0 \leq x < 1 \) for which

\[
V_\alpha\{f(y); x \leq y \leq x + h\} \neq O(h^\alpha)
\]

is contained in \( E_k \) for each \( k \) and must have zero measure.

**Corollary 1.2.3.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( W_\alpha \). Then

\[
| f(x + h) - f(x) | = O(h^\alpha)
\]

almost everywhere, and similarly on the left.

Hence a function in \( W_\alpha \) satisfies a Lip (\( \alpha \)) condition almost everywhere.
but of course not uniformly.

1.3. We need the following lemma in order to study the relation between \( W_\alpha \) and the Hardy-Littlewood integrated Lipschitz classes [4, p. 612].

**Lemma 1.3.1.** Suppose that \( 0 \leq \alpha \leq 1 \), that \( f(x) \) is a real-valued function with period 1, and that the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 does not exceed 1. Then the \( \alpha \)-variation of \( f(x) \) over any interval of length \( k \) does not exceed \( k^\alpha \) for each positive integer \( k \).

We need only show for \( 0 < \alpha \leq 1 \) that

\[
V_\alpha \{ f(x) ; \ 0 \leq x \leq k \} \leq k^\alpha.
\]

Pick a subdivision \( \sigma = x_0 < \cdots < x_N = k \), so that the left-hand side of 1.3.2 is majorized by

\[
\left\{ \sum_\sigma | \Delta f |^{1/\alpha} \right\}^\alpha + \varepsilon = S^\alpha + \varepsilon.
\]

We can assume that \( \sigma \) contains two points, \( c \) and \( d \), where \( d = c + 1 \) and where \( f(d) = f(c) = \text{Max}_n f(x_n) \) for adding such points to \( \sigma \) does not decrease \( S \). (Cf. proof for 1.2.1.) We have

\[
S = \sum_{\sigma \setminus \{0, e, \}} | \Delta f |^{1/\alpha} + \sum_{\sigma \setminus \{c, d, \}} | \Delta f |^{1/\alpha} + \sum_{\sigma \setminus \{c, \}} | \Delta f |^{1/\alpha}
\]

\[
= S_1 + S_2 + S_3.
\]

\( S_2 \) is majorized by 1, \( \{ S_1 + S_3 \}^\alpha \) is majorized by the \( \alpha \)-variation of \( f(x) \) over \( 0 \leq x \leq k - 1 \), and 1.3.2 follows by induction.

**Theorem 1.3.3.** (Cf. [10, p. 259].) Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is a measurable real-valued function with period 1. If the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 never exceeds 1, then

\[
\left\{ \int_0^1 | f(x + h) - f(x) |^{1/\alpha} dx \right\}^\alpha \leq h^\alpha
\]

for every \( h > 0 \).

Assume that \( \alpha > 0 \) and let \( h = m/n \), where \( m \) and \( n \) are relatively prime positive integers. Then

\[
I(h) = \int_0^1 | f(x + h) - f(x) |^{1/\alpha} dx = \sum_{r=1}^n \int_{(r-1)/n}^{r/n} \left| f\left(x + \frac{m}{n}\right) - f(x) \right|^{1/\alpha} dx
\]

and, because \( f(x) \) has period 1, this last sum is equal to

\[
\int_0^{1/n} \left\{ \sum_{r=1}^n \left| f\left(x + \frac{r m}{n}\right) - f\left(x + \frac{(r-1)m}{n}\right) \right|^{1/\alpha} \right\} dx \leq \int_0^{1/n} m dx = h.
\]
Hence the theorem is true for rational $h$. Since $f(x)$ is bounded, $I(h)$ is continuous and the theorem holds for all $h$.

**Corollary 1.3.4.** Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is a measurable real-valued function with period 1. Then

$$\left\{ \int_0^1 |f(x + h) - f(x)|^{1/\alpha} dx \right\}^\alpha \leq 2^n V_\alpha \{f(x); 0 \leq x \leq 1\} h^\alpha$$

for every $h > 0$.

This follows from 1.3.3 since the $\alpha$-variation of $f(x)$ over any interval of length 1 is majorized by

$$V_\alpha \{f(x); 0 \leq x \leq 1\}^{1/\alpha} + V_0 \{f(x); 0 \leq x \leq 1\}^{1/\alpha}.$$

If $f(x) = e^{2\pi i x}$ and $\alpha = 1/2$, the $\alpha$-variation of $f(x)$ over any interval of length 2 exceeds $2^n$ times the $\alpha$-variation of $f(x)$ over any interval of length 1. Since 1.2.1 implies 1.3.1 when $f(x)$ is continuous, the restriction that $f(x)$ be real is essential in both of these results. The same is true for 1.3.3.

From 1.3.4 it is obvious that any measurable function with bounded $\alpha$-variation over some interval, $a \leq x \leq b$, is in the Hardy-Littlewood class $\text{Lip} (\alpha, 1/\alpha)$ over that interval. The converse is not true. Hardy and Littlewood [4, p. 621] point out that if

$$f(x) = \log \frac{1}{|x|}, \quad x \neq 0,$$

then, for $h > 0$,

$$\left\{ \int_0^1 |f(x + h) - f(x)|^{1/\alpha} dx \right\}^\alpha = O(h^\alpha), \quad 0 < \alpha < 1,$$

while $f(x)$ is not even bounded in the neighborhood of $x = 0$.

1.4. The following lemma generalizes a familiar theorem on uniform continuity.

**Lemma 1.4.1.** Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ has bounded $\alpha$-variation over $0 \leq x \leq 1$, and that $f(x)$ is continuous in this interval. For $\epsilon > 0$ there exists a $\delta > 0$ such that, for $0 \leq x_0 < x_0 + \delta \leq 1$, we have

$$V_\alpha \{f(x); x_0 \leq x \leq x_0 + \delta\} < \epsilon.$$

When $0 < \alpha \leq 1$, 1.4.1 is an immediate consequence of the following elementary result.

**Lemma 1.4.2.** Suppose that $0 < \alpha \leq 1$ and that $f(x)$ has bounded $\alpha$-variation in some right-handed neighborhood of the point $x = x_0$. Then

$$V_\alpha \{f(x); x_0 < x < x_0 + h\} = o(1)$$
as \( h \) approaches 0. A similar result holds on the left.

It follows from 1.4.2 that any function, with bounded \( \alpha \)-variation over an open interval, has right- and left-handed limits at each point of the interval.

For \( 0 \leq \alpha \leq 1 \), we say that \( f(x) \) is in \( V_\alpha \) over the interval \( 0 \leq x \leq 1 \) if, given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that, for any set of disjoint intervals \( 0 \leq x_1 < y_1 \leq \cdots \leq x_N < y_N \leq 1 \) for which

\[
\left\{ \sum_{n=1}^{N} |y_n - x_n|^{1/\alpha} \right\}^\alpha < \delta,
\]

we have

\[
\left\{ \sum_{n=1}^{N} |f(y_n) - f(x_n)|^{1/\alpha} \right\}^\alpha < \epsilon.
\]

\( V_0 \) is the class of functions continuous over \( 0 \leq x \leq 1 \) and \( V_1 \) is the class of functions absolutely continuous over this interval.

The method of proof used in 1.2.2 gives us the following result.

**Theorem 1.4.3.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( V_\alpha \). Then

\[
V_\alpha \{ f(y) : x \leq y \leq x + h \} = o(h^\alpha)
\]

almost everywhere, and similarly on the left.

**Theorem 1.4.4** [6]. Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is measurable and has period 1. \( f(x) \) is in \( V_\alpha \) if and only if

\[
V_\alpha \{ f(x + h) - f(x) ; 0 \leq x \leq 1 \} = o(1)
\]
as \( h \) approaches 0.

We see that the class \( W_\alpha \) includes \( V_\alpha \) and, for \( 0 < \alpha < 1 \), it is not difficult to show that the continuous function

\[
f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n \pi x
\]
is in \( W_\alpha - V_\alpha \). Hence a continuous function with bounded \( \alpha \)-variation over \( 0 \leq x \leq 1 \) is not necessarily in \( V_\alpha \). However, we can prove, for \( 0 \leq \alpha \leq 1 \), that any continuous function in \( W_\alpha \) which possesses a finite derivative everywhere in \( 0 \leq x \leq 1 \), except perhaps on an enumerable set, is in \( V_\alpha \) [2, Theorem 2.6].

1.5. Suppose that \( f(x) \) and \( g(x) \) are defined over the interval \( 0 \leq x \leq 1 \) and that \( g(x) \) has at most discontinuities of the 1st kind. The Stieltjes integral

\[
\int_{0}^{1} f(x)dg(x)
\]
exists in the Young sense and is equal to \( I \) if, for \( \epsilon > 0 \), there exists a finite set
E, contained in \(0 \leq x \leq 1\), such that, for any subdivision \(0 = x_0 < \cdots < x_N = 1\) which contains \(E\), we have

\[
\left| \sum_{n=1}^{N} f(\xi_n) \{ g(x_n - 0) - g(x_{n-1} + 0) \} + \sum_{n=1}^{N-1} f(x_n) \{ g(x_n + 0) - g(x_n - 0) \} 
+ f(0) \{ g(0+) - g(0) \} + f(1) \{ g(1) - g(1 - 0) \} - I \right| < \varepsilon
\]

for each set \(x_0 < \xi_1 < x_1 < \cdots < x_{N-1} < \xi_N < x_N\). L. C. Young [12] has proved the following

**Theorem 1.5.2.** Suppose that \(\alpha + \beta > 1\) and that \(f(x)\) and \(g(x)\) belong to \(W_\alpha\) and \(W_\beta\) respectively. If \(f(x)\) and \(g(x)\) have no common discontinuities, \(1.5.1\) exists in the Riemann-Stieltjes sense. In any case, \(1.5.1\) exists in the Young sense.

This theorem is not true in the limiting case when \(\alpha + \beta = 1\). The following result is also an immediate consequence of Young's work.

**Theorem 1.5.3.** Suppose that \(\alpha + \beta > 1\) and that \(f(x)\) is continuous and in \(W_\alpha\). Suppose also that \(\{ g_n(x) \} \) is a sequence of uniformly bounded functions with uniformly bounded \(\beta\)-variation over \(0 \leq x \leq 1\) which converges to \(g(x)\), a function in \(W_\beta\), on a set which includes the points \(x = 0\) and \(x = 1\) and which is dense in \(0 \leq x \leq 1\). Then

\[
\lim_{n \to \infty} \int_0^1 f(x) dg_n(x) = \int_0^1 f(x) dg(x),
\]

\[
\lim_{n \to \infty} \int_0^1 g_n(x) df(x) = \int_0^1 g(x) df(x).
\]

**Moment problems**

2.1. In this chapter we study \(W_\alpha\), the class of functions with bounded \(\alpha\)-variation over the interval \(0 \leq x \leq 1\), by considering a moment problem. Suppose that \(g(x)\) is any function in \(W_\alpha\) for \(0 < \alpha \leq 1\). We call the numbers

\[
\mu_n = \int_0^1 x^n dg(x), \quad n = 0, 1, \cdots ,
\]

the Stieltjes moments of \(g(x)\) and we say that \(g(x)\) is normalized if

\[
g(x) = \frac{1}{2} \{ g(x + 0) + g(x - 0) \} \quad \text{for} \quad 0 < x < 1.
\]

A uniqueness theorem follows from a known result [8, p. 60] after integration by parts.
Theorem 2.1.1. Suppose that $0 < \alpha \leq 1$, that $g(x)$ is normalized and in $W_\alpha$, and that

$$\int_0^1 x^n dg(x) = 0$$

for $n = 0, 1, \ldots$. Then $g(x)$ is identically constant in $0 \leq x \leq 1$.

For an arbitrary sequence of numbers, $\{\mu_n\}$, we define a linear functional over the space of polynomials by letting

$$\mu\{p\} = \mu\left\{ \sum_{n=0}^N a_n x^n \right\} = \sum_{n=0}^N a_n \mu_n.$$ 

Following Hausdorff we define, for $k = 0, 1, \ldots$ and $n = 0, 1, \ldots, k$,

$$\lambda_{k,n}(x) = C_n^{k-n} x^n (1 - x)^{k-n}$$

and

$$\lambda_{k,n} = \mu\{\lambda_{k,n}(x)\},$$

where $C_n^k$ is the binomial coefficient

$$\binom{r + s}{r} = \frac{\Gamma(r + s + 1)}{\Gamma(r + 1)\Gamma(s + 1)}.$$

Theorem 2.1.2. Suppose that $0 < \alpha \leq 1$. A necessary and sufficient condition that the set of numbers $\{\mu_n\}$ be the Stieltjes moments of a normalized function $g(x)$ in $W_\alpha$, where $V_\alpha\{g(x); 0 \leq x \leq 1\} \leq 1$, is that

$$\left\{ \sum_{n=0}^k \left( \sum_{n=0}^k \lambda_{k,n} \right)^{1/\alpha} \right\} \leq 1$$

for all $k$.

Hausdorff [5] has proved this theorem for the case where $\alpha = 1$.

2.2. In order to prove the sufficiency we derive two simple results.

Lemma 2.2.1. If $\{\mu_n\}$ is an arbitrary sequence of numbers, then

$$\sum_{n=0}^k \left( \frac{n}{k} \right)^m \lambda_{k,n} = \mu_m + O\left( \frac{1}{k} \right)$$

for $m = 0, 1, \ldots$.

Suppose that $f(x)$ is any function defined over the interval $0 \leq x \leq 1$ and consider

$$B_k\{f; x\} = \sum_{n=0}^k f\left( \frac{n}{k} \right) \lambda_{k,n}(x),$$

the Bernstein polynomial of order $k$ for $f(x)$. If $P_m(x)$ is a polynomial of degree $m$,
\[ B_k \{ P_m ; x \} = P_m(x) + \sum_{r=1}^{m-1} \frac{Q_{m,r}(x)}{k^r}, \]

where the polynomials \( Q_{m,r}(x) \) do not depend on \( k \) and are identically zero when \( P_m(x) \) is a constant [7, p. 8]. Setting \( P_m(x) = x^m \), we have

\[ \sum_{n=0}^{k-1} \left( \frac{n}{k} \right)^m \lambda_{k,n} = \mu \{ B_k \{ x^m ; x \} \} = \mu_m + \sum_{r=1}^{m-1} \frac{\mu \{ Q_{m,r} \}}{k^r}, \]

and 2.2.1 follows.

**Lemma 2.2.2.** Suppose that \( 0 < \alpha \leq 1 \), that \( \{ g_k(x) \} \) is a sequence of uniformly bounded functions, and that

\[ \lim \inf_{k \to \infty} V_a \{ g_k(x) ; 0 \leq x \leq 1 \} \leq 1. \]

There exists a function \( g(x) \) such that

\[ V_a \{ g(x) ; 0 \leq x \leq 1 \} \leq 1, \]

and a subsequence \( \{ k_j \} \) such that \( g_{k_j}(x) \to g(x) \) for every rational \( x \) in \( 0 \leq x \leq 1 \), including the end points \( x = 0 \) and \( x = 1 \).

We can use the selection principle (or diagonal process) to define \( g(x) \) on the rationals. If, for irrational \( x \), we let

\[ g(x) = \lim \sup_{r \to x} g(r), \quad r \text{ rational}, \]

then \( g(x) \) satisfies the conditions of the lemma.

To prove the sufficiency part of 2.1.2, consider the step function \( g_k(x) \) where \( g_k(0) = 0 \) and

\[ g_k(x) = \sum_{r=0}^{n} \lambda_{k,r} \quad \text{for} \quad \frac{n}{k+1} < x \leq \frac{n+1}{k+1}. \]

Since

\[ |g_k(x)| \leq V_a \{ g_k(x) ; 0 \leq x \leq 1 \} \leq 1, \]

there exists a function \( g^*(x) \) such that

\[ V_a \{ g^*(x) ; 0 \leq x \leq 1 \} \leq 1, \]

and a subsequence \( \{ k_j \} \) such that \( g_{k_j}(x) \to g^*(x) \) for each rational \( x \) in \( 0 \leq x \leq 1 \). Let

\[ g(0) = g^*(0), \quad g(1) = g^*(1), \]

2.2.3

\[ g(x) = \frac{1}{2} \{ g^*(x + 0) + g^*(x - 0) \} \quad \text{for} \quad 0 < x < 1. \]
The function \( g(x) \) is normalized and, from 1.5.3 and 2.2.1, we conclude that

\[
\mu_m = \lim_{k \to \infty} \sum_{n=0}^{k} \left( \frac{n}{k+1} \right)^m \lambda_{k,n}
\]

\[
= \lim_{k \to \infty} \int_0^1 x^m g_k(x) \ dx
\]

\[
= \int_0^1 x^m g_k(x) = \int_0^1 x^m g(x).
\]

2.3. In order to prove the necessity we require a number of lemmas.

We say that a real function \( f(x) \) is unimax in the interval \( a \leq x \leq b \) if, for \( a < c < d < b \), \( f(x) \geq \min \{ f(c), f(d) \} \).

**Lemma 2.3.1.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( X_n = \sum_{m=1}^{M} a_{n,m} x_m \) for \( n = 1, \cdots, N \), where \( a_{n,m} \) is a non-negative unimax function of \( m \) for each \( n \), and where \( \sum_{n=1}^{N} a_{n,m} \leq 1 \) for each \( m \). Then

\[
\left\{ \sum_{n=1}^{N} x_n^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{m=1}^{M} x_m^{1/\alpha} \right\}^{\alpha}.
\]

We consider 2.3.1 in a slightly different form.

**Lemma 2.3.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( X_n = \sum_{m=1}^{M} a_{n,m} x_m \) for \( n = 1, \cdots, N \), where \( a_{n,m} \) is a non-negative integer and is unimax as a function of \( m \) for each \( n \), and where \( \sum_{n=1}^{N} a_{n,m} = R \) for each \( m \). Then

\[
\left\{ \sum_{n=1}^{N} x_n^{1/\alpha} \right\}^{\alpha} \leq R \left\{ \sum_{m=1}^{M} x_m^{1/\alpha} \right\}^{\alpha}.
\]

**Lemma 2.3.3.** With the hypotheses of 2.3.2 we can write

\[
X_n = \sum_{r=1}^{R} y_{n,r} = \sum_{r=1}^{R} \left\{ \sum_{m=1}^{M} a_{n,m,r} x_m \right\},
\]

where \( a_{n,m,r} \) is either 0 or 1 and is unimax in \( m \) for each \( n \) and \( r \), and where \( \sum_{n=1}^{N} a_{n,m,r} = 1 \) for each \( m \) and \( r \).

Now 2.3.3 is true when \( R = 1 \). Suppose it true for \( R = k - 1 \) and consider the case where \( R = k \).

A. There exists \( n_1 \) such that \( a_{n_1,1} \geq 1 \). Let \( m_1 \) be the largest \( m \) for which \( a_{n_1,m} \geq 1 \) and define

\[
a_{n_1,m,1} = \begin{cases} 1 & \text{for } 1 \leq m \leq m_1, \\ 0 & \text{everywhere else}. \end{cases}
\]

It is not difficult to see that \( a_{n_1,m} - a_{n_1,m,1} \) is a non-negative integer and is unimax as a function of \( m \).
B. If \( m_1 < M \), there exists \( n_2 \neq n_1 \) such that \( a_{n_2,m_1} < a_{n_2,m_1+1} \). Let \( m_2 \) be the largest \( m \) for which \( a_{n_2,m} \geq 1 \) and define

\[
a_{n_2,m,1} = \begin{cases} 
1 & \text{for } m_1 < m \leq m_2, \\
0 & \text{everywhere else.}
\end{cases}
\]

Again \( a_{n_2,m} - a_{n_2,m,1} \) is a non-negative integer and is unimax in \( m \).

C. If \( m_2 < M \), we can find \( n_3 \neq n_2, n_1 \) such that \( a_{n_2,m_2} < a_{n_2,m_2+1} \), and we can define \( m_3 \) and the set \( \{a_{n_3,m,1}\} \) for \( m = 1, \cdots, M \). After a finite number of steps we arrive at the place where \( m_j = M \). We shall have defined a sequence of distinct integers, \( n_1, n_2, \cdots, n_j \), and a set of coefficients, \( \{a_{n,m,1}\} \), for \( n=n_1, \cdots, n_j \) and \( m=1, \cdots, M \). Let \( a_{n,m,1} = 0 \) for \( n \neq n_1, \cdots, n_j \) and \( m=1, \cdots, M \).

The set \( \{a_{n,m,1}\} \) satisfies the conditions in 2.3.3, the set \( \{a_{n,m} - a_{n,m,1}\} \) satisfies the hypotheses of 2.3.2 with \( R = k - 1 \), and 2.3.3 follows by applying the induction hypothesis to

\[
X_n' = \sum_{m=1}^{M} \{a_{n,m} - a_{n,m,1}\} x_m.
\]

For a fixed \( r \), consider the set \( \{y_{n,r}\} \). Since each element here is simply the sum of consecutive \( x_m \), we see from 2.3.3 and Minkowski's inequality that

\[
\left\{ \sum_{n=1}^{N} \left| X_n \right|^{1/a} \right\}^a \leq \sum_{r=1}^{R} \left\{ \sum_{n=1}^{N} \left| y_{n,r} \right|^{1/a} \right\}^a
\]

and

\[
\leq R \left\{ \sum_{n=1}^{N} \sum_{m=1}^{M} x_m \right\}^{1/a}.
\]

Now 2.3.2 implies 2.3.1 when all the \( a_{n,m} \) are rational and \( \sum_{n=1}^{N} a_{n,m} = \sum_{n=1}^{N} x_m = 1 \). A simple limiting process removes the restriction that the \( a_{n,m} \) be rational. When \( 0 \leq c_m \leq 1 \), we can apply our results to the set of linear forms

\[
\sum_{m=1}^{M} a_{n,m} x_m, \quad n = 1, \cdots, N,
\]

\[
(1 - c_m) x_m, \quad m = 1, \cdots, M,
\]

and show that

\[
\left\{ \sum_{n=1}^{N} \left| X_n \right|^{1/a} + \sum_{m=1}^{M} (1 - c_m) x_m \right\}^{1/a} \leq \left\{ \sum_{m=1}^{M} \left| x_m \right|^{1/a} \right\}^{1/a}.
\]

This completes the proof for 2.3.1. We can assume that no \( x_m \) is zero. Hence when \( \alpha > 0 \), we get strict inequality if, for any \( m \), \( c_m < 1 \).
Lemma 2.3.4. Suppose that \( k = 0, 1, \ldots \), and that \( 0 \leq m \leq n \leq k \). The function \( f(x) = \sum_{n-m}^{n} \lambda_{k,r}(x) \) is unimax in \( 0 \leq x \leq 1 \).

We can assume that \( k \geq 1 \) and 2.3.4 follows immediately from the identity

\[
\frac{d}{dx} \lambda_{k,n}(x) = \begin{cases} 
-k\lambda_{k-1,n}(x), & n = 0, \\
\{ k\{\lambda_{k-1,n-1}(x) - \lambda_{k-1,n}(x)\} \}, & 0 < n < k, \\
k\lambda_{k-1,n-1}(x), & n = k.
\end{cases}
\]

To complete the proof for 2.1.2, fix \( k \) and consider any finite sequence \( 0 = \nu_{0} < \cdots < \nu_{N} = k + 1 \). If

\[
\theta_{n} = \sum_{r_{n-1} \leq r < r_{n}} \lambda_{k,r}, \quad \text{and} \quad \theta_{n}(x) = \sum_{r_{n-1} \leq r < r_{n}} \lambda_{k,r}(x),
\]

then

\[
\theta_{n} = \int_{0}^{1} \theta_{n}(x)dg(x)
\]

and this integral exists in the Riemann-Stieltjes sense for each \( n \). For each subdivision \( 0 = y_{0} < \cdots < y_{M} = 1 \), let

\[
X_{n} = \sum_{m=1}^{M} \theta_{n}(y_{m}) \{ g(y_{m}) - g(y_{m-1}) \}.
\]

\( \theta_{n}(y_{m}) \) is a non-negative unimax function of \( m \) and

\[
\sum_{n=1}^{N} \theta_{n}(y_{m}) = \sum_{r=0}^{k} \lambda_{k,r}(y_{m}) = 1.
\]

Applying 2.3.1 we get

\[
\left\{ \sum_{n=1}^{N} X_{n} \right\}^{\alpha} \leq \left\{ \sum_{m=1}^{M} \sum_{n-m}^{n} \lambda_{k,r}(y_{m}) \right\}^{1/\alpha},
\]

which completes the argument.

2.4. We have discussed the Stieltjes moments of a function \( g(x) \), i.e. the sequence \( \{ \mu_{n} \} \) where

\[
\mu_{n} = \int_{0}^{1} x^{n}dg(x), \quad n = 0, 1, \cdots.
\]

We can also consider the moment sequence \( \mu_{n} = \int_{0}^{1} x^{n}g(x)dx, \quad n = 0, 1, \cdots \),

where the integral is interpreted in the Lebesgue sense. We call such a se-
sequence of numbers the Lebesgue moments of \( g(x) \) and we have the following

**Theorem 2.4.1.** Suppose that \( 0 < \alpha \leq 1 \). A necessary and sufficient condition that the set of numbers \( \{\mu_n\} \) be the Lebesgue moments of a normalized function \( g(x) \) in \( W_a \), where

\[
V_a \{g(x); 0 \leq x \leq 1\} \leq 1,
\]

is that

\[
(k + 1) \left\{ \sum_{n=1}^{b} \left| \sum \lambda_{k,n} - \lambda_{k,n-1} \right|^{1/\alpha} \right\} \leq 1
\]

for all \( k \).

The necessity follows immediately from 2.1.2 since, with the help of 2.3.5, we can write

\[
(k + 1) \left\{ \lambda_{k,n} - \lambda_{k,n-1} \right\} = \int_0^1 (k + 1) \left\{ \lambda_{k,n}(x) - \lambda_{k,n-1}(x) \right\} g(x) \, dx
\]

\[
= \int_0^1 \lambda_{k+1,n}(x) \, dg(x)
\]

for \( 0 < n \leq k \).

For the sufficiency, observe that

2.4.2 \quad \left| \mu_0 - (k + 1)\lambda_{k,m} \right| \leq \sum_{n=0}^{b} \left| \lambda_{k,n} - \lambda_{k,m} \right| \leq 1

for \( 0 \leq m \leq k \). Consider the step function \( g_k(x) \) where \( g_k(0) = (k + 1)\lambda_{k,0} \) and

\[
g_k(x) = (k + 1)\lambda_{k,n} \quad \text{for } \frac{n}{k + 1} < x \leq \frac{n + 1}{k + 1}.
\]

From 2.4.2 and 2.2.2 it follows that there exists a function \( g^*(x) \) such that

\[
V_a \{g^*(x); 0 \leq x \leq 1\} \leq 1,
\]

and a subsequence \( \{k_f\} \) such that \( g_{k_f}(x) \to g^*(x) \) for each rational \( x \) in \( 0 \leq x \leq 1 \). Define \( g(x) \) as in 2.2.3. From 1.5.3, 2.2.1, and 2.4.2 we can conclude that

\[
\mu_m = \lim_{k \to \infty} \sum_{n=0}^{b} \left( \frac{n}{k + 1} \right)^m \lambda_{k,n}
\]

\[
= \lim_{k \to \infty} \int_0^1 x^m g_k(x) \, dx
\]

\[
= \int_0^1 x^m g^*(x) \, dx = \int_0^1 x^m g(x) \, dx.
\]
2.5. Turning back to 2.3.1, we deduce two alternative forms for this inequality which are useful in later work.

**Lemma 2.5.1.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( Y_n = \sum_{m=0}^{M} b_{n,m} (y_m - y_0) \) for \( n = 0, \ldots, N \), where \( b_{n,m} \) is non-negative for all \( m \) and \( n \), where \( \sum_{m=1}^{M} b_{n,m} \) is nondecreasing in \( n \) and bounded by 1, and where, for each \( 0 \leq n' < n'' \leq N \), \( b_{n'',m} - b_{n',m} \) is at first nonpositive and then non-negative as \( m \) increases from 1 to \( M \). Then

\[
\left\{ \sum_{n=1}^{N} \left| Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^{M} \left| y_m - y_{m-1} \right|^{1/\alpha} \right\}^\alpha.
\]

**Lemma 2.5.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( Y_n = \sum_{m=0}^{M} b_{n,m} y_m \) for \( n = 0, \ldots, N \), where \( b_{n,m} \) is non-negative for all \( m \) and \( n \), where \( \sum_{m=0}^{M} b_{n,m} = 1 \), and where, for each \( 0 \leq n' < n'' \leq N \), \( b_{n'',m} - b_{n',m} \) is at first nonpositive and then non-negative as \( m \) increases from 0 to \( M \). Then

\[
\left\{ \sum_{n=1}^{N} \left| Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^{M} \left| y_m - y_{m-1} \right|^{1/\alpha} \right\}^\alpha.
\]

For 2.5.1, let \( x_m = y_m - y_{m-1} \) and pick any sequence of integers, \( 0 = k_0 < \cdots < k_{N'} = N \). If

\[
X_n = Y_{k_n} - Y_{k_{n-1}} = \sum_{m=1}^{M} a_{n,m} x_m,
\]

then

\[
a_{n,m} = \sum_{\mu=m}^{M} \{ b_{k_n,\mu} - b_{k_{n-1},\mu} \}.
\]

The difference \( a_{n,m+1} - a_{n,m} = - \{ b_{k_n,m} - b_{k_{n-1},m} \} \) is at first non-negative and then nonpositive as \( m \) increases; hence \( a_{n,m} \) is unimax in \( m \). We see that

\[
a_{n,1} = \sum_{m=1}^{M} b_{k_n,m} - \sum_{m=1}^{M} b_{k_{n-1},m} \geq 0,
\]

\[
a_{n,M} = b_{k_n,M} - b_{k_{n-1},M} \geq 0,
\]

and the unimax property ensures that \( a_{n,m} \) is non-negative for all \( m \) and \( n \). Finally

\[
\sum_{n=1}^{N'} a_{n,m} = \sum_{\mu=m}^{M} \{ b_{N',\mu} - b_{0,\mu} \} \leq 1,
\]

and we can apply 2.3.1.

For 2.5.2 observe that the set \( \{ b_{n,m} \} \), for \( n = 0, \cdots, N \) and \( m = 0, \cdots, M \), satisfies the hypotheses of 2.5.1. Since the sums

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\[
\sum_{m=0}^{M} b_{n,m}(y_m - y_0) \quad \text{and} \quad \sum_{m=0}^{M} b_{n,m}y_m
\]

differ by a constant which is independent of \( n \), the conclusion follows immediately.

From 2.5.2 we can deduce the following result concerning Bernstein polynomials.

**Theorem 2.5.3.** If \( 0 \leq \alpha \leq 1 \),
\[
V_\alpha \{ B_k \{ f; x \}; 0 \leq x \leq 1 \} \leq \left\{ \sum_{m=1}^{k} \left| \sum f \left( \frac{m}{k} \right) - f \left( \frac{m - 1}{k} \right) \right|^{1/\alpha} \right\}^\alpha .
\]

Let \( 0 < x_0 < \cdots < x_N < 1 \) be any subdivision of the interval \( 0 < x < 1 \). We see that
\[
B_k \{ f; x_n \} = \sum_{m=0}^{k} f \left( \frac{m}{k} \right) \lambda_{k,m}(x_n)
\]
where \( \lambda_{k,m}(x_n) \) is non-negative for all \( m \) and \( n \), and where \( \sum_{m=0}^{k} \lambda_{k,m}(x_n) = 1 \). For \( 0 \leq n' < n'' \leq N \),
\[
\lambda_{k,m}(x_{n''}) - \lambda_{k,m}(x_{n'}) = \lambda_{k,m}(x_{n''}) \left\{ 1 - \left( \frac{x_{n'}}{x_{n''}} \right)^m \left( 1 - \frac{x_{n'}}{x_{n''}} \right)^{m-k} \right\} .
\]
Since \( 0 < x_{n'} < x_{n''} < 1 \), the bracketed quantity is negative for \( m = 0 \), positive for \( m = k \), and strictly increasing in \( m \). Applying 2.5.2 completes the proof.

In conclusion we add that 2.3.1, 2.5.1, and 2.5.2 are valid when \( M \) and/or \( N = \infty \).

**A FALTUNG THEOREM**

3.1. In one of his papers [11], L. C. Young considered a Stieltjes Faltung of the form
\[
s(x) = \int_{0}^{1} f(x, y) dg(y).
\]

We present here a theorem suggested by Young's results.

Suppose that \( f(x, y) \) and \( g(y) \) have period 1 in \( y \) and define \( F(x, y) \) as the integral \( \int_{0}^{1} f(x, y+t)g(t)dt \) which we assume exists in the Lebesgue sense for all \( 0 \leq x, y \leq 1 \). Let
\[
s_n(x) = 2^n \{ F(x, 0) - F(x, 2^{-n}) \} \]
\[
= 2^n \int_{0}^{1} f(x, t) \{ g(t) - g(t - 2^{-n}) \} dt
\]
for \( n = 0, 1, \cdots \). Then \( s_0(x) = 0 \) and the following is easily verified.
Lemma 3.1.3. Suppose that \( g(y) \) is continuous and that the integral 3.1.1 exists in the Riemann-Stieltjes sense for \( x = x_0 \). Then \( s(x_0) = \lim_{n \to \infty} s_n(x_0) \) and, for \( n \geq 1 \), we have
\[
 s_n(x) - s_{n-1}(x) = 2^{-n-1} \int_0^1 \{ f(x, t) - f(x, t + 2^{-n}) \} \{ g(t) - g(t - 2^{-n}) \} dt.
\]

Our principal result is as follows.

Theorem 3.2. (Cf. [11, Theorem 6.1].) Suppose that \( 0 < \alpha, \beta, \gamma \leq 1, 0 < \lambda = \beta + \gamma - 1, \) and \( \mu = \alpha \lambda / \beta \). Suppose also that \( f(x, y) \) and \( g(y) \) have period 1 in \( y \), that \( g(y) \) is continuous, and that
\[
\begin{align*}
3.2.1 & \quad V_\alpha \{ f(x, y); 0 \leq x \leq 1 \} \leq A \\
3.2.2 & \quad V_\beta \{ f(x, y); 0 \leq y \leq 1 \} \leq B \\
3.2.3 & \quad V_\gamma \{ g(y); 0 \leq y \leq 1 \} \leq C.
\end{align*}
\]

If \( s(x) \) is the Stieltjes Faltung 3.1.1, then
\[
3.2.4 \quad V_\mu \{ s(x); 0 \leq x \leq 1 \} \leq k(\lambda) A^{1/\beta} B^{1-\lambda/\beta} C,
\]
where \( k(\lambda) \) is a finite constant.

If \( B = 0 \) and/or \( C = 0 \), \( s(x) \equiv 0 \) and 3.2.4 follows immediately. Hence we can assume that \( B = C = 1 \).

Obviously we can suppose that \( f(x, y) \) and \( g(y) \) are real. By an argument similar to that used in 1.2.1, we can find a strictly increasing continuous function \( \phi(t) \) such that
\[
\phi(0) = 0 \quad \text{and} \quad \phi(t+1) = \phi(t) + 1
\]
for all \( t \), and such that
\[
| g\{ \phi(t'') \} - g\{ \phi(t') \} | < 2(t'' - t')^\alpha
\]
for each \( t' < t'' \). Furthermore, for each \( 0 \leq x \leq 1 \),
\[
V_\beta \{ f(x, \phi(t)); 0 \leq t \leq 1 \} = V_\beta \{ f(x, y); 0 \leq y \leq 1 \},
\]
\[
\int_0^1 f\{ x, \phi(t) \} dg\{ \phi(t) \} = \int_0^1 f(x, y) dg(y),
\]
and, by performing a change of variable, we replace condition 3.2.3 by the condition
\[
3.2.5 \quad | g(y'') - g(y') | < 2(y'' - y')^\alpha
\]
for each \( y' < y'' \).

Since \( g(y) \) is continuous and \( \beta + \gamma > 1 \), the Faltung \( s(x) \) exists in the Riemann-Stieltjes sense for each \( x \) and is equal to \( \lim_{n \to \infty} s_n(x) \). Using 3.1.3, 3.2.5, Jensen's inequality, 3.2.2, and 1.3.4 we have
\[ |s_n(x) - s_{n-1}(x)| \leq 2^{n-1} \int_0^1 |f(x, t + 2^{-n}) - f(x, t)| \left| g(t) - g(t - 2^{-n}) \right| \, dt \]

\[ \leq 2^n (1 - \gamma) \left\{ \int_0^1 |f(x, t + 2^{-n}) - f(x, t)|^{1/\beta} \, dt \right\}^\beta \]

\[ \leq 2^\beta \cdot 2^{-n\lambda}, \]

and summing on \( n \) we get

\[ 3.2.6 \quad |s(x) - s_n(x)| \leq 2^\beta \sum_{n=n+1}^{\infty} 2^{-n\lambda} = o(\lambda)2^{-n\lambda} \]

for \( n = 0, 1, \cdots \). Fix \( n \) and consider \( 0 \leq x' < x'' \leq 1 \). From 3.1.2, 3.2.5, and Jensen's inequality we have

\[ |s_n(x'') - s_n(x')| \leq 2^n \int_0^1 |f(x'', t) - f(x', t)| \left| g(t) - g(t - 2^{-n}) \right| \, dt \]

\[ \leq 2^n (1 - \gamma) \cdot 2^\Delta \]

where

\[ \Delta = \left\{ \int_0^1 |f(x'', t) - f(x', t)|^{1/\alpha} \, dt \right\}^\alpha. \]

By considering three different cases we prove that

\[ 3.2.8 \quad |s(x'') - s(x')| \leq k(\lambda)A^{1/\alpha}, \]

where \( k(\lambda) \) is a finite constant.

A. Suppose that \( 1 < \Delta < \infty \). Setting \( n = 0 \) in 3.2.6 gives us 3.2.8 if we choose \( k(\lambda) \geq 2c(\lambda) \).

B. Suppose that \( 0 < \Delta \leq 1 \). Choose \( n \geq 1 \) so that \( 2^{-n\beta} < \Delta \leq 2^{-(n-1)\beta} \), and with 3.2.6 we have

\[ |s(x) - s_n(x)| < c(\lambda)\Delta^{1/\alpha} \]

for \( 0 \leq x \leq 1 \). From 3.2.7 we get

\[ |s_n(x'') - s_n(x')| \leq 4\Delta^{1/\alpha}, \]

and 3.2.8 follows if we choose \( k(\lambda) \geq 2(\lambda) + 4 \).

C. Suppose that \( \Delta = 0 \). We see from 3.1.3 and 3.2.7 that

\[ |s(x'') - s(x')| = \lim_{n \to \infty} |s_n(x'') - s_n(x')| = 0, \]

and 3.2.8 follows if we choose \( k(\lambda) \geq 0 \).

To complete the proof for 3.2, take any subdivision \( 0 = x_0 < \cdots < x_N = 1 \). From 3.2.8 we have
The proof introduces unnecessary restrictions. For example, Young's argument \[11, p. 459\] allows us to consider the case where \( g(y) \) is not continuous. We conclude this chapter by simply stating the following generalization of 3.2.

**Theorem 3.3.** Suppose that \( 0 < \alpha, \beta, \gamma \leq 1, 0 < \lambda = \beta + \gamma - 1, \) and \( u = \alpha \lambda / \beta. \) Suppose also that 3.2.1, 3.2.2, and 3.2.3 hold. If \( s(x) \) is the Stieltjes Faltung 3.1.1, then

\[
\left\{ \frac{\sum_{n=1}^{N} | s(x_n) - s(x_{n-1}) |^{1/\mu}}{N} \right\}^\mu \leq k(\lambda) \left\{ \int_{0}^{1} \sum_{n=1}^{N} | f(x_n, t) - f(x_{n-1}, t) |^{1/\alpha} dt \right\}^\mu
\]

\[
\leq k(\lambda) A^{\lambda/\beta}.
\]

Applications to infinite series

4.1. In this chapter we apply the notion of \( \alpha \)-variation to the study of infinite series. We say that the series

\[
\sum_{n=0}^{\infty} a_n
\]

is \( \alpha \)-convergent if, given \( \varepsilon > 0 \), there exists \( N(\varepsilon) \) such that

\[
\left\{ \sum_{r=m}^{n} | a_r |^{1/\alpha} \right\}^{\alpha} < \varepsilon
\]

for \( N(\varepsilon) \leq m < n. \) 0-convergence is ordinary convergence and 1-convergence is absolute convergence. If \( 0 \leq \alpha < \beta \leq 1, \) we have by Jensen's inequality

\[
\left\{ \sum_{r=m}^{n} | a_r |^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{r=m}^{n} | a_r |^{1/\beta} \right\}^{\beta},
\]

and thus a \( \beta \)-convergent series is always \( \alpha \)-convergent.

We can extend the notion of \( \alpha \)-convergence to sequences. We call \( \{ S_n \} \) an \( \alpha \)-convergent sequence if \( S_n \) is the \( n \)th partial sum of an \( \alpha \)-convergent series. From Minkowski's inequality we see that any finite linear combination of \( \alpha \)-convergent series (sequences) is itself an \( \alpha \)-convergent series (sequence). We also have the following result.

**Lemma 4.1.2.** (Cf. Lemma 1.4.2.) Suppose that \( 0 < \alpha \leq 1. \) The series 4.1.1 is \( \alpha \)-convergent if and only if

\[
\left\{ \sum_{n=0}^{\infty} | a_n |^{1/\alpha} \right\}^{\alpha} < \infty.
\]
The same type of result is true for sequences.

Any series derived from a 1-convergent series by a rearrangement of terms is convergent to the sum of the original series. However, when \( \alpha < 1 \), an \( \alpha \)-convergent series is "conditionally convergent" and little can be said about rearrangement.

A second important property of 1-convergent series is found in multiplication theorems. For example, it is well known that the Cauchy product of a 1-convergent series by a 0-convergent series is 0-convergent to the product of the sums of the series. We have the following extension of this result.

**Theorem 4.1.3.** Suppose that \( 0 < \alpha, \beta \leq 1 \), and that \( 0 < \gamma = \alpha + \beta - 1 \). Then the Cauchy product of an \( \alpha \)-convergent series by a \( \beta \)-convergent series is \( \gamma \)-convergent to the product of the sums.

This theorem follows easily from the following specialization of 3.3.

**Theorem 4.1.4.** Suppose that \( 0 < \alpha, \beta \leq 1 \), and that \( 0 < \gamma = \alpha + \beta - 1 \). If \( f(x) \) has bounded \( \alpha \)-variation over \( 0 \leq x < \infty \) and if \( g(x) \) has bounded \( \beta \)-variation over \( 0 \leq x < \infty \), then the Stieltjes Faltung

\[
s(x) = \int_0^x f(x - y)dg(y)
\]

exists in the Young sense for each \( x \) and has bounded \( \gamma \)-variation over \( 0 \leq x < \infty \).

Theorem 4.1.3 holds in the limiting case where \( \alpha + \beta = 1 \) if and only if \( \alpha = 0 \) or 1; 4.1.3 is also true when one considers the more general Dirichlet product [3, p. 239] instead of the Cauchy product.

4.2. We can apply our scale to the study of Cesaro and Abel summability. Suppose that \( S_n^\alpha \) is the \( n \)th Cesaro mean of order \( \alpha \) for the series 4.1.1. We say that 4.1.1 is summable \( (C, k; \alpha) \) to \( S \) if the sequence \( \{ S_n^\alpha \} \) is \( \alpha \)-convergent to \( S \). Thus \( (C, k; 0) \) summability is ordinary Cesaro summability and \( (C, k; 1) \) summability is absolute Cesaro summability. Consider the function

\[
f(x) = \sum_{n=0}^\infty a_n x^n
\]

and assume that this series converges for \( 0 \leq x < 1 \). We say that 4.1.1 is summable \( (A; \alpha) \) to \( S \) if \( f(x) \) has bounded \( \alpha \)-variation over \( 0 \leq x < 1 \) and if \( \lim_{x \to 1^-} f(x) = S \). \( (A; 0) \) summability is ordinary Abel summability and \( (A; 1) \) summability is absolute Abel summability. (See [1, p. 11] for references on absolute summability.) A linear combination of series, summable \( (C, k; \alpha) \) for some \( k \) and \( 0 \leq \alpha \leq 1 \), is itself a series summable \( (C, k; \alpha) \) and similarly for the \((A; \alpha)\) method.

When \( 0 \leq \alpha < \beta \leq 1 \), a series summable \( (C, k; \beta) \) to \( S \) is summable \( (C, k; \alpha) \) to \( S \) and a series summable \( (A; \beta) \) to some limit is summable \( (A; \alpha) \) to the
same limit. We can also establish the following consistency result.

**Theorem 4.2.1.** Suppose that $0 \leq \alpha \leq 1$ and that $k > j > -1$. A series summable $(C, j; \alpha)$ to $S$ is summable $(C, k; \alpha)$ and $(A; \alpha)$ to $S$. If $S_n^j$ and $S_n^k$ are the $n$th Cesaro means of order $j$ and $k$ respectively for 4.1.1, we have

$$
\left\{ \sum_{n=1}^{\infty} \left| \sum_{n=1}^{k} S_n^k - S_{n-1}^k \right|^{1/\alpha} \right\} \leq \left\{ \sum_{n=1}^{\infty} \left| \sum_{n=1}^{j} S_n^j - S_{n-1}^j \right|^{1/\alpha} \right\}^{\alpha},
$$

$$
V_{n} \left\{ \sum_{n=0}^{\infty} a_n x^n; 0 \leq x \leq 1 \right\} \leq \left\{ \sum_{n=1}^{\infty} \left| \sum_{n=1}^{k} S_n^k - S_{n-1}^k \right|^{1/\alpha} \right\}^{\alpha}.
$$

For the first of these inequalities let $k - j = i > 0$ and write

$$
S_n^k = \sum_{m=0}^{n} \frac{C_{n-m}^{i-1} C_m^i}{C_n^k} S_m^i = \sum_{m=0}^{\infty} b_{n,m} S_m^i,
$$

where $C_r^s$ is the binomial coefficient

$$
\binom{r + s}{r}.
$$

$b_{n,m}$ is non-negative for all $m$ and $n$ and

$$
\sum_{m=0}^{\infty} b_{n,m} = \frac{1}{C_n^k} \sum_{m=0}^{n} C_{n-m}^{i-1} C_m^i = 1.
$$

When $0 \leq n' < n'' < \infty$,

$$
b_{n'',m} - b_{n',m} = b_{n',m} \left\{ \frac{(n'' - m + i - 1) \cdots (n' - m + i)}{(n'' - m) \cdots (n' - m + 1)} \frac{n'' \cdots (n' + 1)}{(n'' + k) \cdots (n' + k + 1)} - 1 \right\}
$$

for $0 \leq m \leq n'$. If $0 < i < 1$, the bracketed quantity is negative for $0 \leq m \leq n'$ and, since $b_{n',m} = 0$ for $m > n'$, we can apply 2.5.2.

For any subdivision $0 < x_0 \cdots < x_N < 1$ let

$$
\sum_{m=0}^{\infty} a_m(x_n)^m = \sum_{m=0}^{\infty} C_m^k (1 - x_n)^{k+1} (x_n)^m S_m^k = \sum_{m=0}^{\infty} b_{n,m} S_m^k.
$$

Again $b_{n,m}$ is non-negative for all $m$ and $n$,

$$
\sum_{m=0}^{\infty} b_{n,m} = (1 - x_n)^{k+1} \sum_{m=0}^{\infty} C_m^k (x_n)^m = 1,
$$

and, for $0 \leq n' < n'' \leq N$,
The bracketed factor here is an increasing function of \( m \) and 4.2.3 follows from 2.5.2.

The rest of the theorem follows from these two inequalities and a classical result.

4.3. The direct converse of 4.2.1 is not true. The coefficients in the power series expansion for

\[
f(x) = e^{1/(1+x)}
\]

constitute a series which is summable \((A; 1)\) but which is not summable \((C, k; 0)\) for any finite \( k \). However, we can prove some "corrected converses" and the following theorems extend two results due to A. Tauber.

**Lemma 4.3.1.**
1. \( \alpha_n = \sum_{m=1}^{n-1} (1 - (1-1/n)^m)/m \) is a positive, increasing sequence bounded by \( 1 \) for \( n \geq 2 \).
2. \( \beta_n = (1-(1-1/n)^n)/n \) is a positive, decreasing sequence bounded by \( 1 \) for \( n \geq 1 \).
3. \( \gamma_n = \sum_{m=n+1}^{n} ((1-1/n)^m)/m \) is a positive, increasing sequence bounded by \( 1 \) for \( n \geq 1 \).

Consider the first sequence. If \( r > 1 \), \( x^
u - y^
u > ry^{r-1}(x-y) \) for any two positive and unequal \( x \) and \( y \). Hence for \( n \geq 2 \) we have

\[
\alpha_{n+1} - \alpha_n = \sum_{m=1}^{n-1} \frac{(1 - 1/n)^m - (1 - 1/(n + 1))^m}{m} + \frac{1 - (1 - 1/(n + 1))^n}{n} \\
\geq \frac{1}{n(n + 1)} \sum_{m=0}^{n-2} \left( \frac{1 - 1}{n + 1} \right)^m + \frac{1 - (1 - 1/(n + 1))^n}{n} \\
\geq \frac{1}{n} \left\{ \left( \frac{1 - 1}{n + 1} \right)^{n-1} - \left( \frac{1 - 1}{n + 1} \right)^n \right\} > 0.
\]

For the boundedness we see that

\[
\alpha_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} \leq \frac{n - 1}{n} < 1.
\]

The proofs of 4.3.1.2 and 4.3.1.3 follow along similar lines.

We can now generalize Tauber's first theorem.

**Theorem 4.3.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that 4.1.1 is summable \((A; \alpha)\) to \( S \). If the sequence \( \{ na_n \} \) is \( \alpha \)-convergent to 0, 4.1.1 is \( \alpha \)-convergent to \( S \).

We can assume that \( 0 < \alpha \leq 1 \). For \( 0 \leq x < 1 \) let
and let $S_n$ be the $n$th partial sum for 4.1.1, i.e.

$$S_n = \sum_{r=0}^{n} a_r.$$

For $n \geq 2$ we can write

$$S_n - f\left(1 - \frac{1}{n}\right) = A_n + B_n - C_n,$$

where

$$A_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \alpha_{n,m}(ma_m),$$

$$B_n = \frac{1 - (1 - 1/n)^n}{n} (na_n) = \beta_n(na_n),$$

$$C_n = \sum_{m=n+1}^{\infty} \frac{(1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \gamma_{n,m}(ma_m).$$

$\alpha_{n,m}$ and $\gamma_{n,m}$ are non-negative for all $m$ and $n$ and, by 4.3.1, both $\sum_{m=1}^{\infty} \alpha_{n,m}$ and $\sum_{m=1}^{\infty} \gamma_{n,m}$ are increasing in $n$ and bounded by 1. These two sets of coefficients satisfy the hypotheses of 2.5.1. Hence

$$\left\{ \sum_{n=3}^{\infty} \left| \sum A_n - A_{n-1}\right|^{1/\alpha} \right\}^\alpha$$

and

$$\left\{ \sum_{n=3}^{\infty} \left| \sum C_n - C_{n-1}\right|^{1/\alpha} \right\}^\alpha$$

are majorized by

$$\left\{ \sum_{m=1}^{\infty} \left| \sum ma_m - (m-1)a_{m-1}\right|^{1/\alpha} \right\}^\alpha,$$

and $\{A_n\}$ and $\{C_n\}$ are $\alpha$-convergent sequences. The sequence $\{B_n\}$ is also $\alpha$-convergent since $\{\beta_n\}$ is 1-convergent to 0. Because 4.1.1 is $(A; \alpha)$ summable, the sequence $\{f(1-1/n)\}$ is $\alpha$-convergent, and we conclude that $\{S_n\}$ is $\alpha$-convergent to $S$.

**Theorem 4.3.3.** Suppose that $0 \leq \alpha \leq 1$ and that the series 4.1.1 is summable $(A; \alpha)$ to $S$. 4.1.1 is $\alpha$-convergent to $S$ if and only if the sequence

$$\left\{ \sum_{n=1}^{\infty} a_1 + \cdots + na_n \right\}$$

is $\alpha$-convergent to 0.
For the necessity let $S_n^1$ be the $n$th Cesaro mean of order 1 for 4.1.1 and we see from 4.2.1 that the sequence

$$S_n - S_{n-1}^1 = \frac{a_1 + \cdots + na_n}{n}$$

is $\alpha$-convergent to 0.

For the sufficiency set $b_0 = 0$ and let

$$\sum_{m=0}^{n} b_m = \frac{a_1 + \cdots + na_n}{n}$$

for $n \geq 1$. If $a_n = b_n + c_n$ for all $n$, then

$$c_n = \frac{a_1 + \cdots + (n - 1)a_{n-1}}{n(n - 1)}$$

for $n \geq 2$. $\sum_{n=0}^{\infty} a_n$ is $(A; \alpha)$ summable to $S$, $\sum_{n=0}^{\infty} b_n$ is $\alpha$-convergent to 0, and hence the series $\sum_{n=0}^{\infty} c_n$ is also $(A; \alpha)$ summable to $S$. By 4.3.2 we see that this series is then $\alpha$-convergent to $S$ and this completes the proof.

We saw in 4.3.2 how the Tauberian condition

$\{na_n\}$ is $\alpha$-convergent to 0

allowed us to pass from summability $(A; \alpha)$ to summability $(C, 0; \alpha)$ or $\alpha$-convergence. Actually more is true and we have the following result.

**Theorem 4.3.4.** Suppose that $0 \leq \alpha \leq 1$ and that 4.1.1 is $\alpha$-convergent. If the sequence $\{na_n\}$ is $\alpha$-convergent to 0, 4.1.1 is summable $(C, k; \alpha)$ to its sum for every $k > -1$.

Pick $\delta > 0$. Let $S_n^\delta$ and $S_{n-1}^\delta$ be the $n$th Cesaro means of order $\delta$ and $\delta - 1$ respectively for 4.1.1, and let $T_n^\delta$ be the $n$th Cesaro mean of order $\delta$ for the series whose $n$th partial sum is $na_n$. From the identity $(n + \delta)C_{n-\delta} - \delta C_{n-\delta} = \nu C_{n-\delta}$, we see that

$$S_n^{\delta - 1} - S_n^\delta = \frac{1}{\delta} T_n^\delta.$$

By 4.2.1, $\{S_n^\delta\}$ and $\{T_n^\delta\}$ are $\alpha$-convergent to $S$ and 0 respectively and thus $\{S_n^{\delta - 1}\}$ is $\alpha$-convergent to $S$.

In conclusion we construct, for $0 \leq \alpha < 1$, a series which is summable $(C, k; \alpha)$ for every $k > -1$ and which is not summable $(A; \beta)$ for any $\beta > \alpha$.

Let $\{b_k\}$ be any positive decreasing sequence of numbers which approach zero such that

$$\left\{ \sum_{k=1}^{\infty} b_k^{1/\alpha} \right\}^\alpha < \infty \quad \text{and} \quad \left\{ \sum_{k=1}^{\infty} b_k^{1/\beta} \right\}^\beta = \infty.$$
for each $\beta > \alpha$. Define a sequence of integers, $1 = n_0 < n_1 < \cdots$, and a set of positive numbers, $\{c_k\}$, such that

$$\sum_{n_{k-1} \leq n < n_k} \frac{1}{n} \geq 2^k b_k = c_k \sum_{n_{k-1} \leq n < n_k} \frac{1}{n}$$

for $k = 1, 2, \cdots$. Set $a_0 = 0$ and $a_n = ((-1)^k / n)2^{-k}c_k$ for $n_{k-1} \leq n < n_k$. Then

$$\left\{ \sum_{n=n_k}^{\infty} |a_n|^{1/p} \right\}^p = \left\{ \sum_{j=k+1}^{\infty} b_j^{1/p} \right\}^p$$

for each $0 \leq \alpha \leq 1$, and the series is $\alpha$-convergent but not $\beta$-convergent for any $\beta > \alpha$. Since

$$\sum_{n=1}^{\infty} |na_n - (n - 1)a_n| = 2 \sum_{k=1}^{\infty} 2^{-k}c_k < 2,$$

the sequence $\{na_n\}$ is 1-convergent to 0, and we see from 4.3.2 and 4.3.4 that this series has the desired properties.

REFERENCES

2. J. C. Burkill and F. W. Gehring, A scale of integrals from Lebesgue’s to Denjoy’s, Quarterly Journal vol. 4 (1953) pp. 210-220.