AREAS OF $k$-DIMENSIONAL NONPARAMETRIC SURFACES IN $k$+1 SPACE

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Introduction. Suppose $X \subset E_k \ (k \geq 2)$ is a $k$ cell, $g$ is a real-valued continuous function on $E_k$, $j$ is a positive integer not greater than $k$, and $f$ is the mapping defined by the relation

$$f(x) = (x_1, \ldots, x_{j-1}, g(x), x_{j+1}, \ldots, x_k) \quad \text{for } x \in E_k.$$  

For this class of mappings on $E_k$ into $E_k$ (denoted by $\Omega_k$) we concern ourselves with the validity of the formula

$$\int_X |D_jg(x)| \, d\mathcal{L}_k x \, dx = L(f \mid X) = \int_{E_k} S(f, X, y) \, d\mathcal{L}_k y = \int_{E_k} N(f, X, y) \, d\mathcal{L}_k y$$

where $D_jg$ is the partial derivative of $g$ in the direction of the $j$th base vector, $L(f \mid X)$ is the Lebesgue area of the surface $f \mid X$, $S(f, X, y)$ and $N(f, X, y)$ are respectively the stable multiplicity and multiplicity of $f$ on $X$, and $\mathcal{L}_k$ denotes the $k$-dimensional Lebesgue measure.

The main results, which comprise a complete theory of area for the class $\Omega_k$, are embodied in Theorems 2.6, 2.13, 2.14, and 2.15.

For $k=2$ the results are in most part known (see [T1], [T2], and [R]).

The theory of area of the class $\Omega_k$ is intimately connected with the theory of area of $k$-dimensional nonparametric surfaces in $E_{k+1}$. In fact if $\bar{g}$ is the function defined by

$$\bar{g}(x) = (x_1, x_2, \ldots, x_k, g(x)) \quad \text{for } x \in E_k,$$

and $\pi$ is an orthogonal projection of $E_{k+1}$ onto $E_k$, then

$$\bar{g} \mid X$$

is a $k$-dimensional nonparametric surface in $E_{k+1}$, and

$$\pi \circ \bar{g}$$

is a mapping of $E_k$ into $E_k$. For some, but not all, orthogonal projections $\pi$, $\pi \circ \bar{g}$ is a member of $\Omega_k$.

A reduction procedure is devised whereby the theory of area of the class $\Omega_k$ extends, in great part, to the mappings $\pi \circ \bar{g}$. A theory of area for $k$-dimensional nonparametric surfaces in $E_{k+1}$ evolves; a theory in which complete information is obtained concerning the validity of the relation

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\[ \int_X Jg(x) dL_k x = L(g | X) \]

\[ = \beta(k + 1, k)^{-1} \int_{g_{k+1}} \int_{E_k} S(p_{k+1} \circ R \circ g, X, y) dL_k y d\phi_{k+1} R \]

\[ = \beta(k + 1, k)^{-1} \int_{g_{k+1}} \int_{E_k} N(p_{k+1} \circ R \circ g, X, y) dL_k y d\phi_{k+1} R \]

where \( Jg \) is the Jacobian associated with \( g \) by means of its approximate differential, and where the last two members of the string are respectively the stable integralgeometric and integralgeometric areas of the surface \( g | X \).

The main results are contained in Theorems 3.8, 3.11, 3.15, 3.16, and 3.17.

For \( k = 2 \) the theory of area for \( k \)-dimensional nonparametric surfaces in \( E_{k+1} \) is well established (see [F4, 6] and [S, V]).

1. Definitions.

1.1 Definition. If \( f \) is a function, then \( \text{inv } f \) is its inverse, and for any set \( A, f|A \) is the function with domain \( (A \cap \text{domain } f) \) for which

\[ (f|A)(x) = f(x) \]

whenever \( x \in (A \cap \text{domain } f) \).

Furthermore

\[ N(f, A, x) \]

is the number (possibly \( \infty \)) of elements of the set \( A \cap \{ z | f(z) = x \} \), and

\[ f^*(A) = \{ x | x = f(z) \text{ for some } z \in A \} \]

If \( g \) is also a function, then

\[ f \circ g, \]

the superposition of \( f \) on \( g \), is defined by the formula

\[ (f \circ g)(x) = f(g(x)) \]

for all \( x \).

1.2 Definition. Euclidean \( n \) space will be denoted by \( E_n \). The usual metric and inner product (denoted by \( \cdot \)) are assumed for this \( n \)-dimensional vector space. We write

\[ x = (x_1, x_2, \cdots, x_n) \]

for \( x \in E_n \).

For \( m \) and \( n \) positive integers we shall often identify \( E_m \times E_n \) with \( E_{m+n} \). Lebesgue \( n \) dimensional measure over \( E_n \) is denoted by \( L_n \).

\[ \alpha(n) = L_n(E_n \cap \{ x | | x | < 1 \}) \]

1.3 Definition. If \( f \) is a function on \( E_n \) to \( E_1 \) and \( x \in E_n \), then we define

\[ \limsup_{\varepsilon \to 0} \text{ ap } f(z) \]
to be the infimum of the numbers of the form
\[ \limsup_{z \to x} f(z) \]
where \( A \) is a Lebesgue measurable subset of \( E_n \) with density one at \( x \).

1.4 Definition. For \( m \) and \( n \) positive integers a function \( L \) on \( E_m \) to \( E_n \) is linear if and only if
\[
L(x + y) = L(x) + L(y) \quad \text{for} \quad x \in E_m, \ y \in E_m,
\]
\[
L(\lambda x) = \lambda L(x) \quad \text{for} \quad x \in E_m \text{ and } \lambda \text{ a real number}.
\]
For \( 1 \leq i \leq m \) let \( m^i \) denote the \( i \)th unit vector of \( E_m \). Then if \( L \) is a linear function on \( E_m \) to \( E_n \)
\[
L(m^i) = L^i = (L_1^i, L_2^i, \ldots, L_n^i) \in E_n.
\]
The matrix of \( L \), which we identify with \( L \), is the \( n \) by \( m \) matrix whose entry in the \( j \)th row and \( i \)th column (\( 1 \leq j \leq n \), \( 1 \leq i \leq m \)) is \( L_j^i \). If \( m \leq n \) the square root of the sum of the squares of the determinants of all \( m \) by \( m \) minors will be denoted by \( \Delta(L) \).

If \( 1 \leq m \leq n \), \( f \) is a function on \( E_m \) to \( E_n \), and \( L \) is a linear function on \( E_m \) to \( E_n \) for which
\[
\limsup_{z \to x} \frac{|f(z) - f(x) - L(z - x)|}{|z - x|} = 0,
\]
then \( L \) is unique and is termed the \textit{approximate differential of} \( f \) at \( x \). If \( x \) is a point at which \( f \) has the approximate differential \( L \) we denote
\[
Jf(x) = \Delta(L).
\]
If \( f \) is a function on \( E_1 \) into \( E_n \) and \( a < b \), then \( T_{a \to b} f(t) \) is the supremum of the numbers of the form
\[
\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,
\]
where \( a = t_0 \leq t_1 \leq \cdots \leq t_n = b \).

If \( f \) is a function on \( E_n \) to \( E_1 \) and \( j \) is a positive integer no greater than \( n \), then \( D_j f \) is the function on \( E_n \) such that
\[
D_j f(x) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{j-1}, x_{j} + h, x_{j+1}, \ldots, x_n) - f(x)}{h} \quad \text{for} \quad x \in E_n.
\]

1.5 Definition. If \( m \leq n \) are positive integers, then \( p_m^n \) is the function on \( E_n \) onto \( E_m \) defined by
\[
p_m^n(x) = (x_1, \ldots, x_m) \quad \text{for} \quad x \in E_n.
\]
1.6 Definition. If \( n \) is a positive integer, then \( G_n \) will denote the set of all linear transformations \( R \) on \( E_n \) to \( E_n \) for which
\[
| R(x) | = | x | 
\]
whenever \( x \in E_n \).

With respect to the topology of uniform convergence and the operation of superposition, \( G_n \) is a compact topological group, in fact, the orthogonal group of \( E_n \).

The identity element of \( G_n \) will be designated by \( *I \), and \( \phi_n \) will be the unique Haar measure over \( G_n \) for which \( \phi_n(G_n) = 1 \).

The following fact may be inferred (see \([W, 8]\)):

If \( f \) is a \( \phi_n \) measurable function on \( G_n \) then
\[
\int_{G_n} f(R)d\phi_nR = \int_{G_n} f(\text{inv } R)d\phi_nR = \int_{G_n} f(R \circ S)d\phi_nR = \int_{G_n} f(S \circ R)d\phi_nR
\]
whenever \( S \in G_n \).

1.7 Definition. If \( m \leq n \) are positive integers, then
\[
\beta(n, m) = \frac{\alpha(m) \cdot \alpha(n - m)}{\alpha(n) \cdot \binom{n}{m}}.
\]

1.8 Definition. A function \( g \) on \( E_n \) is said to be a gauge over \( E_n \) if and only if \( \text{domain } g \subseteq \{ X \mid X \subseteq E_n \} \), range \( g \subseteq \{ t \mid 0 \leq t \leq \infty \} \).

If \( g \) is a gauge over \( E_n \) and \( 0 < r \leq \infty \), the function \( \hat{g}_r \) is defined by the relation
\[
\hat{g}_r(A) = \inf_{F \in B} \sum_{S \in F} g(S) \quad \text{for } A \subseteq E_n,
\]
where \( F \subseteq B \) if and only if \( F \) is a countable subfamily of domain \( g \) for which
\[
A \subseteq \bigcup_{S \in F} S, \quad \text{diameter } S < r, \text{ whenever } S \in F.
\]

One says that \( \phi \) is generated by \( g \) if and only if \( g \) is a gauge over \( E_n \) and \( \phi \) is the function defined by
\[
\phi(A) = \lim_{r \to 0^+} \hat{g}_r(A) \quad \text{for } A \subseteq E_n.
\]

It may be shown that \( \phi \) is a (Carathéodory outer) measure over \( E_n \) and that closed subsets of \( E_n \) are \( \phi \) measurable.

1.9 Definition. If \( m \leq n \) are positive integers and \( \gamma_n^m \), \( \tau_n^m \), and \( \chi_n^m \) are the gauges over \( E_n \) defined by
\[
\gamma_n^m(S) = \sup_{R \in G} \mathcal{L}_k[(\phi_n^m \circ R)^*(S)]
\]
whenever \( S \) is a Borel subset of \( E_n \),

\[
\zeta^m_n(S) = \beta(n, m)^{-1} \int_{a_n} L^k_n[(\phi_n^m \circ R)^*(S)]d\phi_nR
\]

whenever \( S \) is an analytic subset of \( E_n \),

\[
\chi^m_n(S) = \alpha(m)2^{-m} (\text{diameter } S)^m \quad \text{whenever } S \subseteq E_n,
\]

then \( \mathfrak{C}^m_n \) generated by \( \chi^m_n \), \( \mathfrak{J}^m_n \) generated by \( \zeta^m_n \), and \( \Gamma^m_n \) generated by \( \gamma^m_n \) are respectively the Hausdorff, the integralgeometric, and the Gross \( k \)-dimensional measures over \( n \) space (see \([F4] , [H] , [C]\)).

One may easily check that \( \mathfrak{C}^m_n \), \( \mathfrak{J}^m_n \), and \( \Gamma^m_n \) are invariant under isometries of \( E_n \) and that any subset of \( E_n \) is contained in a \( G_\delta \) set of equal \( \mathfrak{C}^m_n \) measure, in an analytic set of equal \( \mathfrak{J}^m_n \) measure, and in a Borel set of equal \( \Gamma^m_n \) measure. The equality of \( \mathfrak{J}^m_n \) and \( \mathfrak{L}^m_n \) is apparent from the definition. It is also true that \( \mathfrak{C}^m_n = \mathfrak{L}^m_n \) (see \([SD]\)).

1.10 Definition. If \( m \leq n \) are positive integers and \( X \subseteq E_m \) is an \( m \) cell or its interior, then \( C_n(X) \) will denote the set of continuous functions on \( X \) to \( E_n \).

If \( g \in C_n(X) \), then \( g \) is a polyhedron if and only if \( X \) can be so triangulated that \( g \) maps each simplex baracentrically onto a rectilinear simplex of \( E_n \).

It is to be noted that relative to the topology of uniform convergence the class of polyhedra is dense in \( C_n(X) \), and also that all areas used in this paper are equivalent on the class of polyhedra.

1.11 Definition. Suppose \( m \leq n \) are positive integers and \( f \) is a continuous function on \( E_m \) to \( E_n \).

If \( X \subseteq E_m \) is an \( m \) cell, then

\[
L(f \mid X), \text{ the } m\text{-dimensional Lebesgue area of } f \mid X,
\]

is the lower limit of the areas of polyhedra approximating \( f \mid X \).

If \( X \subseteq E_m \) is the interior of an \( m \) cell, then

\[
L(f \mid X), \text{ the } m\text{-dimensional Lebesgue area of } f \mid X,
\]

is the supremum of \( L(f \mid Y) \) for all subsets \( Y \) of \( X \) which are \( m \) cells.

If \( X \subseteq E_m \) is either an \( m \) cell or its interior, then

the \textit{m-dimensional Hausdorff area of } \( f \mid X = \int_{E_n} N(f, X, \gamma)d\mathfrak{C}^m_n \gamma; \)

the \textit{m-dimensional integralgeometric area of } \( f \mid X = \int_{E_n} N(f, X, \gamma)d\mathfrak{J}^m_n \gamma = \beta(n, m)^{-1}\int_{a_n} \int_{E_m} N(\phi_n^m \circ R \circ f, X, x)d\mathfrak{J}_mxd\phi_nR; \)
the m-dimensional integralgeometric stable area of $f \mid X$

$$\beta(n, m)^{-1} \int_{E_n} \int_{E_m} S(p^n \circ R \circ f, X, x) d\mathcal{L}_n x d\mathcal{L}_m X;$$

where $S(f, X, x)$ is the stable multiplicity as defined in [F7].

1.12 Definition. If $n$ is a positive integer exceeding one, $j$ is a positive integer no greater than $n$, $A \subset E_n$, $z \in E_n$, and $u \in E_{n-1}$, then

$$A_z = E_n \cap \{x + z \mid x \in A\},$$
$$A_{(j)} = E_{n-1} \cap \{w \mid (w_1, \ldots, w_{j-1}, v, w_{j+1}, \ldots, w_n) \in A \text{ for some } v\},$$
$$A^{u}_{(j)} = E_1 \cap \{v \mid (u_1, \ldots, u_{j-1}, v, u_{j+1}, \ldots, u_n) \in A\}.$$

If $f$ is a continuous real-valued function on $E_n$ and $i$ is a positive integer, then

$$K_i = E_n \cap \{x \mid x = i^{-1}\},$$

and $f_i$, the $i$th integral mean, of $f$, is the real-valued function on $E_n$ defined by the formula

$$f_i(x) = \alpha(n)^{-i} \int_{K_i} f(x + z) d\mathcal{L}_n z \quad \text{for } x \in E_n.$$

2. On a certain class of mappings of $E_k$ into $E_k$ ($k \geq 2$). Let $\Omega_k$ denote the class of mappings on $E_k$ into itself defined by: $f \in \Omega_k$ if and only if for some positive integer $j$ not exceeding $k$ and for some continuous real-valued function $g$ on $E_k$, $f$ is defined by the formula

$$f(x) = (x_1, \ldots, x_{j-1}, g(x), x_{j+1}, \ldots, x_k) \quad \text{for } x \in E_k.$$

For such a function $f \in \Omega_k$ and $i$ a positive integer, $f_i$ will be defined as that element of $\Omega_k$ for which

$$f_i(x) = (x_1, \ldots, x_{j-1}, g_i(x), x_{j+1}, \ldots, x_k) \quad \text{for } x \in E_k.$$

2.1 Sectional assumption. In the development of a theory of area for the class $\Omega_k$ it will be convenient to fix a function $f \in \Omega_k$ which will be defined by $f(x) = (x_1, \ldots, x_{k-1}, g(x))$ for $x \in E_k$, where $g$ is a (fixed) real-valued continuous function on $E_k$.

No restriction in generality will be effected by this assumption (see Remark 2.20).

2.2 Lemma. If $Y \subset E_k$ is a bounded open convex set and $i$ is a positive integer, then

$$\int_{E_k} N(f_i, Y, x) d\mathcal{L}_k x \leq \alpha(k)^{-1} \int_{K_i} \int_{E_k} N(f, Y, z) d\mathcal{L}_k x d\mathcal{L}_k z.$$
Proof. We let
\[ a(u) = \inf_{Y(k)} Y_u, \quad b(u) = \sup_{Y(k)} Y_u \quad \text{for } u \in Y(k), \]
and for each \( u \in Y(k) \) we define the function
\[ h_u: \text{closure } Y_u \to E_1, \]
\[ h_u(v) = g(u_1, u_2, \ldots, u_{k-1}, v) \quad \text{for } v \in \text{closure } Y_u. \]
The remainder of the proof is divided into four parts.

Part 1. \( N(f, Y, (u, v)) = N(h_u, Y_u, v) \) for \( (u, v) \in E_{k-1} \times E_1 \).

Proof. If \( A \) is a subset of \( E_k \) we shall denote by \( q(A) \) the number (possibly \( \infty \)) of elements of \( A \). Whence
\[
N(f, Y, (u, v)) = q \left( (E_{k-1} \times E_1) \cap \{(w, t) \mid f(w, t) = (u, v)\} \right) \\
= q \left( (E_{k-1} \times E_1) \cap \{(w, t) \mid w = u, t \in Y_u, h_u(t) = v\} \right) \\
= q \left( E_1 \cap \{t \mid t \in Y_u, h_u(t) = v\} \right) \\
= N(h_u, Y_u, v).
\]

Part 2. \( \int_{E_k} N(f, Y, (u, v)) d\mathcal{L}_k(u, v) = \int_{Y(k)} T_{v=a(u)} g(u, v) d\mathcal{L}_{k-1} u. \)

Proof. Using Part 1 and [F1, 4.3] we compute:
\[
\int_{E_k} N(f, Y, (u, v)) d\mathcal{L}_k(u, v) = \int_{E_{k-1}} \int_{E_1} N(f, Y, (u, v)) d\mathcal{L}_1 v d\mathcal{L}_{k-1} u \\
= \int_{Y(k)} \int_{E_1} N(h_u, Y_u, v) d\mathcal{L}_1 v d\mathcal{L}_{k-1} u \\
= \int_{Y(k)} \int_{E_1} N(h_u, \text{closure } Y_u, v) d\mathcal{L}_1 v d\mathcal{L}_{k-1} u \\
= \int_{Y(k)} T_{v=a(u)} h_u(v) d\mathcal{L}_{k-1} u \\
= \int_{Y(k)} T_{v=a(u)} g(u, v) d\mathcal{L}_{k-1} u.
\]

Part 3. If \( c \) and \( d \) are real numbers with
\( c < d \) and \( z = (z', z'') \in E_{k-1} \times E_1, \)
then
\[
T_{v=c} g(u, v) \leq \alpha(k)^{-1} \beta^k \int_{E_k} T_{v=c} g(u + z', v + z'') d\mathcal{L}_k z.
\]

Proof. Let \( c = v_0 < v_1 < \cdots < v_m = d \). Then
\[
\sum_{p=1}^{m} \left| g_i(u, v_p) - g_i(u, v_{p-1}) \right|
\]
\[
= \sum_{p=1}^{m} \alpha(k)^{-1} i^k \int_{K_i} \left\{ g(u + z', v_p + z'') - g(u + z', v_{p-1} + z'') \right\} d\mathcal{L}_{kz}
\]
\[
\leq \alpha(k)^{-1} i^k \int_{K_i} \sum_{p=1}^{m} \left| g(u + z', v_p + z'') - g(u + z', v_{p-1} + z'') \right| d\mathcal{L}_{kz}
\]
\[
\leq \alpha(k)^{-1} i^k \int_{K_i} T_{v=0} g(u + z', v + z'') d\mathcal{L}_{kz}.
\]

**Part 4.** \( \int_{\mathcal{B}_k} N(f, Y, x) d\mathcal{L}_{kx} \leq \alpha(k)^{-1} i^k \int_{K_i} \int_{\mathcal{B}_k} N(f, Y, x) d\mathcal{L}_{kx} d\mathcal{L}_{kz}. \)

**Proof.** Suppose \( z = (z', z'') \in E_{k-1} \times E_1 \). We know that
\[ Y = (E_{k-1} \times E_1) \cap \{(u, v) \mid v \in Y^{(k)}(u)\}. \]

It is easy to check that
\[ Y_z = (E_{k-1} \times E_1) \cap \{(u, v) \mid v - z'' \in Y^{(k)}(u)\}, \quad [Y^{(k)}]_{z'} = [Y_z]_{(k)}. \]

Letting \( x = (u, v) \in E_{k-1} \times E_1 \) and applying Part 2 to \( \tilde{f} \), we infer with the help of Parts 3 and 2 that
\[
\int_{\mathcal{B}_k} N(f, Y, x) d\mathcal{L}_{kx} = \int_{Y^{(k)}} d\mathcal{L}_{kx} \sum_{j=1}^{n} \alpha(k)^{-1} i^k \int_{K_i} T_{v=0} g_i(u, v) d\mathcal{L}_{k-1} d\mathcal{L}_{kz}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_i} \int_{Y^{(k)}} T_{v=0} g_i(u, v + z') d\mathcal{L}_{k-1} u d\mathcal{L}_{kz}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_i} \int_{Y^{(k)}} T_{v=0} g_i(u, v + z') d\mathcal{L}_{k-1} u d\mathcal{L}_{kz}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_i} \int_{Y^{(k)}} T_{v=0} g_i(u, v + z') d\mathcal{L}_{k-1} u d\mathcal{L}_{kz}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(f, Y, x) d\mathcal{L}_{kx} d\mathcal{L}_{kz}.
\]

**2.3 Lemma.** If \( X \subseteq E_k \) is a \( k \) cell with boundary \( \tilde{X}, F \in C_k(X), G \in C_k(X), \) and if \( F \mid \tilde{X} \) is homotopic to \( G \mid \tilde{X} \) where
\[ F \mid \tilde{X} \text{ is a map of } \tilde{X} \text{ into } E_k - \{ y \}, \]
\[ G \mid \tilde{X} \text{ is an essential map of } \tilde{X} \text{ into } E_k - \{ y \}, \]
then \( y \) is a stable value of \( F \mid X - \tilde{X} \).
Proof. Let \( r = \inf_{x \in X} |F(x) - y| \), and let \( \pi \) denote the function on \( E_k - \{ y \} \) into the \( k-1 \) sphere defined by the formula
\[
\pi(x) = \frac{x - y}{|x - y|} \quad \text{for } x \in E_k - \{ y \}.
\]
The proof will be divided into three parts.

Part 1. If degree \((\pi \circ F \mid X) \neq 0\), then there exists a \( p \in X - \tilde{X} \) for which \( F(p) = y \).

Proof. Since \( X \) is a \( k \) cell, for \( q \in X \) we may define a continuous contraction \( \phi \) of \( X \) into \( \{ q \} \):
\[
\phi: \{ t \mid 0 \leq t \leq 1 \} \to \{ H \mid H \text{ is a continuous function on } X \text{ into } X \},
\]
\[
(\phi(0))(x) = x \quad \text{for } x \in X,
\]
\[
(\phi(1))(x) = q \quad \text{for } x \in X.
\]
Suppose that the statement is false. Then for \( 0 \leq t \leq 1 \)
\[
F \circ \phi(t) \mid X : X \to E_k - \{ y \},
\]
\[
F \circ \phi(t) \mid X \text{ is homotopic to } F \circ \phi(0) \mid \tilde{X},
\]
\[
\pi \circ F \circ \phi(t) \mid X \text{ is homotopic to } \pi \circ F \circ \phi(0) \mid \tilde{X}.
\]
Accordingly we arrive at the contradiction
\[
0 = \text{degree } (\pi \circ F \circ \phi(1) \mid \tilde{X}) = \text{degree } (\pi \circ F \circ \phi(0) \mid \tilde{X}) = \text{degree } (\pi \circ F \mid \tilde{X}) \neq 0.
\]

Part 2. If \( H \in C_k(X) \) and \( |H - F| \tilde{X} < r \), then
\[
\text{degree } (\pi \circ H \mid \tilde{X}) = \text{degree } (\pi \circ F \mid \tilde{X}).
\]

Proof. Since for \( x \in \tilde{X} \)
\[
|F(x) - H(x)| < |F(x) - y|,
\]
it follows that
\[
y \in E_k \cap \{ z \mid z = F(x) + t(H(x) - F(x)) \text{ for } 0 \leq t \leq 1 \}.
\]
Hence it suffices to define the continuous function
\[
\phi: \tilde{X} \times \{ t \mid 0 \leq t \leq 1 \} \to E_k - \{ y \},
\]
\[
\phi(x, t) = F(x) + t(H(x) - F(x)) \quad \text{for } (x, t) \in \tilde{X} \times \{ t \mid 0 \leq t \leq 1 \},
\]
to establish that \( F \mid \tilde{X} \) and \( H \mid \tilde{X} \) are homotopic. Consequently \( \pi \circ F \mid \tilde{X} \) and \( \pi \circ H \mid \tilde{X} \) are homotopic and
\[
\text{degree } (\pi \circ F \mid \tilde{X}) = \text{degree } (\pi \circ H \mid \tilde{X}).
\]

Part 3. \( y \) is a stable value of \( F \).

Proof. We infer from the hypothesis that
degree \((\pi \circ F \mid X)\) = degree \((\pi \circ G \mid X)\) \(\neq 0\).

If \(L \subseteq C_k(X)\) and \(|L - F| < r\), we may use Part 2 to establish that

degree \((\pi \circ L \mid X)\) = degree \((\pi \circ F \mid X)\),

and Part 1 to show the existence of a \(p \in X - \hat{X}\) for which \(L(p) = y\).

2.4 Theorem. If \(Y\) is a bounded open convex subset of \(E_k\), then

\[ N(f, Y, x) = S(f, Y, x) \quad \text{for } \mathcal{L}_k \text{ almost all } x \in E_k. \]

Proof. We assume \(h_u\) to have the same meaning as in Lemma 2.2. Let \(C\) be the set of points \((u, v)\) in \(Y\) for which \(h_u\) has either a relative maximum or a relative minimum at \(v\). We may check that \(C\) is a \(F_e\) set and

\[ h_u^*(C_{(k)}) \text{ is countable for } u \in Y_{(k)}. \]

If follows that \(f^*(C)\) is \(\mathcal{L}_k\) measurable and

\[ \mathcal{L}_k[f^*(C)] = 0. \]

We know that

\[ N(f, Y - C, x) = N(f, Y, x) \quad \text{for } x \in f^*(Y - C) - f^*(C), \]
\[ N(f, Y, x) = S(f, Y, x) \quad \text{for } x \notin f^*(Y), \]

hence we may complete the proof by showing that \(f\) is stable \([F4, 6.6]\) at every point of \(Y - C\). For if this were so, then

\[ S(f, Y, x) \leq N(f, Y, x) = N(f, Y - C, x) \leq S(f, Y, x) \]

whenever \(x \in f^*(Y - C) - f^*(C)\).

Let \(\epsilon > 0\).

If \((u^0, v^0) = (u^0_1, u^0_2, \ldots, u^0_{k-1}, v^0) \in Y - C\), then using the continuity of \(g\) we can select

\[ u^1 \in Y_{(k)}, \quad u^2 \in Y_{(k)}, \quad v^1 \in E_1, \quad v^2 \in E_1, \]

satisfying

\[ u^0_i - \epsilon < u^1_i < u^0_i < u^2_i < u^0_i + \epsilon \quad \text{for } i = 1, 2, \ldots, k - 1, \]
\[ v^0 - \epsilon < v^1 < v^0 < v^2 < v^0 + \epsilon, \]

and such that if we denote

\[ P = E_{k-1} \cap \{ u \mid u^1_i \leq u_i \leq u^2_i \text{ for } i = 1, 2, \ldots, k - 1 \}, \]
\[ X = (E_{k-1} \times E_1) \cap \{ (u, v) \mid u \in P \text{ and } v^1 \leq v \leq v^2 \}, \]

then \(X \subseteq Y\) and either

\[ (1) \quad h_u(v^1) < h_u(v^0) < h_u(v^2) \quad \text{whenever } u \in P, \]
or else

\[(2) \quad h_u(v^1) > h_u(v^0) > h_u(v^2) \quad \text{whenever } u \in P.\]

Observe that \(X\) is a \(k\) cell with diameter less than \(2k^{1/2}\varepsilon\) and, denoting its boundary by \(\hat{X}\) and \(f(u^0, v^0)\) by \(y\), that \(f| \hat{X}\) is a map of \(\hat{X}\) into \(E_k - \{y\}\).

If (1) occurs we can define the function \(G\) on \(\hat{X}\) into \(E_k - \{y\}\) by

\[
G(u, v) = (u, y_k + v - v^0) \quad \text{for } (u, v) \in (E_{k-1} \times E_1) \cap \hat{X},
\]

and the function \(\phi\) on \(\hat{X} \times \{t| 0 \leq t \leq 1\}\) into \(E_k - \{y\}\) by

\[
\phi(u, v, t) = (u, (1 - t)(y_k + v - v^0) + th_u(v))
\]

for \((u, v, t) \in ((E_k \times E_1) \cap \hat{X}) \times \{t| 0 \leq t \leq 1\}\), and infer that

\(G\) is an essential map of \(\hat{X}\) into \(E_k - \{y\}\),

\(G\) is homotopic to \(f| \hat{X}\).

Lemma 2.3 implies that \(y\) is a stable value of \(f| X - \hat{X}\), and from the arbitrary nature of \(\varepsilon\) we conclude that \(f\) is stable at \((u^0, v^0)\).

Whenever (2) occurs a similar treatment is employed. The proof is complete.

**2.5 Lemma.** If \(X \subseteq E_k\) is a \(k\) cell with boundary \(\hat{X}\), then \(\mathcal{L}_k[f^*(X)] = 0\).

**Proof.** Let

\[h(x) = (x_1, x_2, \ldots, x_k, g(x)) \quad \text{for } x \in X.\]

Then observing that \(\hat{X}\) lies on \(2k\) \(k - 1\) cells \(A_i\) \((i = 1, 2, \ldots, 2k)\) it is apparent that the \(h\) image of each such \(k - 1\) cell lies on a \(k\) plane in \(E_{k+1}\), and that

\[\mathfrak{C}_{k+1}^k [h^*(\hat{X})] \leq \sum_{i=1}^{2k} \mathfrak{C}_{k+1}^k [h^*(A_i)].\]

If \(j\) is a positive integer no greater than \(2k\), let \(\pi\) be an isometric projection onto \(E_k\) of the \(k\) plane containing \(A_j\) and \(h^*(A_j)\), which satisfies

\[\pi^*(A_j) \subseteq E_k \cap \{x| x_k = 0\}.\]

It follows that

the number of elements in \(\{(\pi \circ h)^*(A_j)\}_{(h)}\) = 1

for \(u \in [(\pi \circ h)^*(A_j)]_{(h)}\). We apply Fubini’s theorem to obtain

\[
\mathfrak{C}_{k+1}^k [h^*(A_j)] = \mathcal{L}_k [(\pi \circ h)^*(A_j)]
\]

\[= \int_{(\pi \circ h)^*(A_j)} \mathcal{L}_j \{(\pi \circ h)^*(A_j)\}_{(h)} d\mathcal{L}_{k-1} u = 0,
\]

\[\mathfrak{C}_{k+1}^k [h^*(\hat{X})] = 0.\]
Finally letting $R$ be the element of $G_k$ for which
$$R(w) = (w_1, \ldots, w_{k-1}, w_{k+1}, w_k) \quad \text{for } w \in E_{k+1},$$
we conclude from the Lipschitzian character of $p_{k+1}^* \circ R$ that
$$\mathcal{L}_k[f^*(\check{X})] = \mathcal{L}_k[(p_{k+1}^* \circ R \circ h)^*(\check{X})] = 0.

2.6 Theorem. If $X \subseteq E_k$ is a $k$ cell, then
$$\int_{E_k} N(f, X, x) d\mathcal{L}_k x = \int_{E_k} S(f, X, x) d\mathcal{L}_k x = L(f \mid X) = \lim_{f \to \infty} L(f \mid X).$$

Proof. Let $Y$ denote the interior of $X$ and let $A$ be an open interval of $E_k$ for which
$$\text{closure } A \subseteq Y.$$ Select a positive integer $i$ so large that
$$A_z \subseteq Y \quad \text{for } z \in K_i.$$ Then using \cite{F4, 6.13}, \cite{F2, 4.5} and Lemma 2.2 we compute
$$L(f_i \mid \text{closure } A) = \int_A J f_i(x) d\mathcal{L}_k x$$
$$= \int_{E_k} N(f_i, A, x) d\mathcal{L}_k x$$
$$\leq \alpha(k)^{-i} \int_{K_i} \int_{E_k} N(f, A_z, x) d\mathcal{L}_k x d\mathcal{L}_k z$$
$$\leq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x.$$ It follows using the lower semi-continuity of $L$ on $C_k(X)$ that
$$L(f \mid \text{closure } A) \leq \liminf_{i \to \infty} L(f_i \mid \text{closure } A) \leq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x.$$ From the arbitrary nature of $A$ and Theorem 2.4 we may conclude
$$L(f \mid Y) \leq \int_{E_k} N(f, Y, x) d\mathcal{L}_k x = \int_{E_k} S(f, Y, x) d\mathcal{L}_k x \leq L(f \mid Y).$$ In view of Lemma 2.5 it is seen that this relation holds with $Y$ replaced by $X$.

For the other part of the statement we recall in general that
$$L(f \mid X) \leq \liminf_{j \to \infty} L(f_i \mid X).$$
Therefore under the assumption that $L(f\mid X)$ is finite we need to prove that

$$\limsup_{j \to \infty} L(\bar{f}_j \mid X) \leq L(f \mid X).$$

Pick $\epsilon > 0$. Utilizing the foregoing results we can select an open interval $U$ of $E_k$ such that

$$X \subseteq U, \quad L(f \mid U) \leq L(f \mid X) + \epsilon.$$ 

Let $j$ be so large a positive integer that

$$(\text{interior } X)_x \subseteq U \quad \text{whenever } z \subseteq K_j.$$ 

Then we can show just as before that

$$L(\bar{f}_j \mid X) \leq \int_{E_k} S(f, U, x) dL_k x \leq L(f \mid U) \leq L(f \mid X) + \epsilon.$$ 

Accordingly

$$\limsup_{j \to \infty} L(\bar{f}_j \mid X) \leq L(f \mid X) + \epsilon.$$ 

Since $\epsilon$ was arbitrary this completes the proof.

2.7 Remark. Using the results of the preceding theorem and the notation of [F 6] we find for $X \subseteq E_k$ a $k$ cell that

$$L(f \mid X) = M**(f \mid X) = M*(f \mid X) = S**(f \mid X) = S*(f \mid X)$$

$$= V**(f \mid X) = V*(f \mid X) = U**(f \mid X) = U*(f \mid X)$$

$$= N**(f \mid X) = N*(f \mid X).$$

2.8 Definition. If $X \subseteq E_k$ is a $k$ cell and $h \in C_i(X)$, then

(i) for $i$ a positive integer no greater than $k$, $h$ is said to be of bounded variation (i) in the sense of Tonelli (BVT (i)) on $X$ if and only if

$$\int_{X_{(i)}} T^\sup_{\nu=\int X_{(i)}} h(u_1, \ldots, u_i, v, u_{i+1}, \ldots, u_k) dL_{k-1} u < \infty;$$

(ii) $h$ is said to be of bounded variation in the sense of Tonelli (BVT) on $X$ if and only if $h$ is BVT (i) on $X$ whenever $i = 1, 2, \ldots, k$.

2.9 Definition. If $X \subseteq E_k$ is a $k$ cell and $h \in C_i(X)$, then

(i) for $i$ a positive integer no greater than $k$, $h$ is said to be absolutely continuous (i) in the sense of Tonelli (ACT (i)) on $X$ if and only if $h$ is BVT (i) on $X$ and the function

$$v \mapsto h(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_k)$$

is absolutely continuous in the classical sense on $X_{(i)}$, for $L_{k-1}$ almost all $u \in X_{(i)}$.
(ii) $h$ is said to be absolutely continuous in the sense of Tonelli (ACT) on $X$ if and only if $h$ is ACT (i) on $X$ whenever $i = 1, 2, \ldots, k$.

2.10 Definition. If $X \subset E_k$ is a $k$ cell and $h \in C_k(X)$, then $h$ is said to be absolutely continuous on $X$ if and only if

$$\int_{E_k} N(h, X, y) d\mathcal{L}_ky < \infty,$$

and $h$ transforms subsets of $X$ of $\mathcal{L}_k$ measure zero into sets of $\mathcal{L}_k$ measure zero.

2.11 Sectional Assumption. For the rest of this section $X \subset E_k$ will denote a $k$ cell.

2.12 Lemma. If $g$ is BVT$(k)$ on $X$, then

$$\int_Y |D_kg(x)| d\mathcal{L}_kx = \int_{Y(k)} \sup_{U \in \mathcal{F}}_{Y(k)} g(u, v) d\mathcal{L}_{k-1}U \leq \int_Y T_{\sup_{U \in \mathcal{F}}_{Y(k)} g(u, v)} d\mathcal{L}_{k-1}U$$

$$= \int_{E_k} N(f, Y, (u, v)) d\mathcal{L}_k(u, v)$$

$$= \int_{E_k} N(f, Y, y) d\mathcal{L}_ky$$

whenever $Y \subset X$ and $Y$ is a $k$ cell.

Proof. This statement follows directly from Definition 2.8 and Part 2 of Lemma 2.2.

2.13 Theorem. The following statements are equivalent:

(i) $g$ is BVT$(k)$ on $X$,

(ii) $L(f \mid X) < \infty$.

Proof. By virtue of Part 2 of Lemma 2.2 and Theorem 2.6

$$\int_{X(k)} T_{\sup_{U \in \mathcal{F}}_{X(k)} g(u, v)} d\mathcal{L}_{k-1}U = L(f \mid X).$$

The theorem is an immediate consequence of this equality.

2.14 Theorem. If $g$ is BVT$(k)$ on $X$, $U$ is the set of those points $x$ in $X$ for which $D_kg(x)$ exists, $Y$ is an $\mathcal{L}_k$ measurable subset of $X$ and $Z$ is an analytic subset of $E_k$ contained in $X$, then

(i) $U$ is a Borel subset of $E_k$ and $\mathcal{L}_k(X - U) = 0$,

(ii) $\int_Y |D_kg(x)| d\mathcal{L}_kx = \int_{E_k} N(f, Y \cap U, y) d\mathcal{L}_ky < \infty$,

(iii) $\int_Z |D_kg(x)| d\mathcal{L}_kx \leq \int_{E_k} N(f, Z, y) d\mathcal{L}_ky < \infty$.
Proof. We know from \([S, V4.1]\) that \(D_kg\) is a Borel measurable function. We may infer that

\[ U \text{ is a Borel subset of } E_k, \]

and since

\[ \mathcal{L}_1[(X - U)_{(k)}] = 0 \quad \text{for } \mathcal{L}_{k-1} \text{ almost all } u \in X_{(k)}, \]

may use Fubini's theorem to conclude that

\[ \mathcal{L}_k(X - U) = 0. \]

Using the notation of Lemma 2.2 the statement

\[ |D_kg(u, v)| = |D_1h_u(v)| = Jh_u(v) \quad \text{for } v \in U_{(k)}^w \]

is true for \(u \in X_{(k)}\). Also we know that

\[ (Y \cap U)_{(k)}^w \]

is an \(\mathcal{L}_3\) measurable subset of \(E_k\) for \(u \in X_{(k)}\).

Upon combining these facts and using \([F2, 5.2]\) (applied to \(h_u\), Part 1 of Lemma 2.2, and Theorem 2.13, we find that

\[
\int_Y |D_kg(x)| \, d\mathcal{L}_{k-1}x = \int_{Y \cap U} |D_kg(x)| \, d\mathcal{L}_{k-1}x
\]

\[ = \int_{(Y \cap U)_{(k)}} \int_{(Y \cap U)_{(k)}} |D_kg(u, v)| \, d\mathcal{L}_1vd\mathcal{L}_{k-1}u
\]

\[ = \int_{(Y \cap U)_{(k)}} \int_{(Y \cap U)_{(k)}} Jh_u(v) \, d\mathcal{L}_1vd\mathcal{L}_{k-1}u
\]

\[ = \int_{(Y \cap U)_{(k)}} \int_{E_k} N(h_u, (Y \cap U)_{(k)}^w, v) \, d\mathcal{L}_1vd\mathcal{L}_{k-1}u
\]

\[ = \int_{E_k} \int_{E_k} N(f, Y \cap U, (u, v)) \, d\mathcal{L}_1vd\mathcal{L}_{k-1}u
\]

\[ = \int_{E_k} N(f, Y \cap U, y) \, d\mathcal{L}_{k}y < \infty. \]

For (iii) it is only necessary in addition to observe that

\[ N(f, Z, y) \text{ is } \mathcal{L}_k \text{ measurable in } y, \]

\[ \int_{E_k} N(f, Z, y) \, d\mathcal{L}_{k}y < \infty. \]

2.15 Theorem. If \(g\) is \(BVT(k)\) on \(X\), then the following statements are equivalent:
(i) \( f \) is absolutely continuous on \( X \),
(ii) \( \int_Y |D_k g(x)| \, d\mathcal{L}_k x = \int_{E_k} N(f, Y, y) \, d\mathcal{L}_k y < \infty \) whenever \( Y \) is an \( \mathcal{L}_k \) measurable subset of \( X \),
(iii) \( \int_X |D_k g(x)| \, d\mathcal{L}_k x = \int_{E_k} N(f, X, y) \, d\mathcal{L}_k y < \infty \),
(iv) \( g \) is \( ACT(k) \) on \( X \).

**Proof.** The proof is divided into five parts.

**Part 1.** (i) implies (ii).

**Proof.** Let \( U \) be the set of points \( x \) in \( X \) for which \( D_k g(x) \) exists, and \( Y \) be an \( \mathcal{L}_k \) measurable subset of \( X \). Then

\[
N(f, Y, y) \text{ is } \mathcal{L}_k \text{ measurable in } y,
\]

\[
\mathcal{L}_k(Y - U) = 0,
\]

and we may infer, using the preceding theorem, that

\[
\int_Y |D_k g(x)| \, d\mathcal{L}_k x = \int_{E_k} N(f, Y \cap U, y) \, d\mathcal{L}_k y
\]

\[
= \int_{E_k} N(f, Y, y) \, d\mathcal{L}_k y < \infty.
\]

**Part 2.** (ii) implies (iii).

**Part 3.** (iii) implies (iv).

**Proof.** Lemma 2.12 implies that

\[
\sup_{x \in X} D_k^u g(u, v) = \int_{X(v)} |D_k g(u, v)| \, d\mathcal{L}_k v
\]

for \( \mathcal{L}_{k-1} \) almost all \( u \in X(v) \). Whence

\[
g \text{ is } ACT(k) \text{ on } X.
\]

**Part 4.** (iv) implies (ii).

**Proof.** If \( Y \subseteq X \) and \( Y \) is a \( k \) cell, then (iv) being true means

\[
\sup_{x \in Y(v)} D_k^u g(u, v) = \int_{Y(v)} |D_k g(u, v)| \, d\mathcal{L}_k v
\]

for \( \mathcal{L}_{k-1} \) almost all \( u \in Y(v) \),

\[
\int_{E_k} N(f, Y, y) \, d\mathcal{L}_k y < \infty.
\]

We infer from Lemma 2.12 that

\[
\int_Y |D_k g(x)| \, d\mathcal{L}_k x = \int_{E_k} N(f, Y, y) \, d\mathcal{L}_k y < \infty.
\]
This equality may be extended to hold whenever $Y$ is a Borel subset of $E_k$ contained in $X$.

Now suppose $W$ is an $L_k$ measurable subset of $X$. Borel subsets $S$ and $T$ of $E_k$ may be chosen in such a way that

$$S \subset W \subset T \subset X, \quad L_k(T - S) = 0.$$  

Since

$$N(f, S, y) \leq N(f, W, y) \leq N(f, T, y) \quad \text{for } y \in E_k,$$

$$\int_{B_k} N(f, S, y) dL_k y = \int_{S} |D_k g(x)| dL_k x = \int_{W} |D_k g(x)| dL_k x$$

$$= \int_{T} |D_k g(x)| dL_k x = \int_{B_k} N(f, T, y) dL_k y < \infty,$$

it follows that

$$N(f, S, y) = N(f, T, y) \quad \text{for } L_k \text{ almost all } y \in E_k,$$

$$N(f, W, y) \text{ is } L_k \text{ measurable in } y,$$

$$\int_{W} |D_k g(x)| dL_k x = \int_{B_k} N(f, W, y) dL_k y < \infty.$$

**PART 5. (ii) implies (i).**

**2.16 Remark.** A combination of Theorems 2.6, 2.14, and 2.15 reveals that $g$ is $BVT(k)$ on $X$ implies

$$\int_{X} |D_k g(x)| dL_k x \leq L(f \mid X) = \int_{B_k} S(f, X, y) dL_k y = \int_{B_k} N(f, X, y) dL_k y,$$

equality holding if and only if $g$ is $ACT(k)$.

**2.17 Theorem.** If $h$ is an $L_k$ measurable function satisfying

$$-\infty \leq h(x) \leq \infty$$

for $L_k$ almost all $x \in E_k$, then

$$g$$

is $ACT(k)$ on $X$ implies

$$\int_{Y} (h \circ f)(x) \cdot |D_k g(x)| dL_k x = \int_{B_k} h(y) \cdot N(f, Y, y) dL_k y$$

whenever $Y$ is an $L_k$ measurable subset of $X$.

**Proof.** This statement is an immediate consequence of Theorem 2.15 and [F3, 2.1].

**2.18 Theorem.** If for $x \in E_k$ and $r > 0$
\[ C(x, r) = E_k \cap \{ z \mid |z - x| \leq r \}, \]

then

\[ g \text{ is BVT}(k) \text{ on } X \implies \lim_{r \to 0^+} \frac{L[f | C(x, r)]}{\alpha(k)r^k} = |D_k g(x)| \quad \text{for } L_k \text{ almost all } x \in \text{interior } X. \]

**Proof.** If \( U \) is the set of those points \( x \) in \( X \) for which \( D_k g(x) \) exists, we know that \( U \) is a Borel subset of \( E_k \) and may define functions \( \beta, \gamma, \) and \( \delta \) as follows:

\[
\begin{align*}
\beta(Y) &= \int_{E_k} N(f, Y, \gamma) dL_k \gamma, \\
\gamma(Y) &= \int_Y |D_k g(x)| \, dL_k x, \\
\delta(Y) &= \int_{E_k} N(f, Y - U, \gamma) dL_k \gamma,
\end{align*}
\]

for \( Y \) a Borel subset of \( E_k \) contained in \( X \).

Employing the terminology of Saks [S, p. 30], \( \beta, \gamma, \) and \( \delta \) are additive set functions on the class of Borel subsets of \( E_k \) contained in \( X \),

\[ \gamma \text{ is } L_k \text{ absolutely continuous,} \]

and since \( L_k(X - U) = 0 \)

\[ \delta \text{ is } L_k \text{ singular.} \]

Because of Theorem 2.14

\[ \beta = \gamma + \delta. \]

Hence using Theorem 2.6 and [S, IV 5.4, 6.3, 7.1] we conclude that

\[
\lim_{r \to 0^+} \frac{L[f | C(x, r)]}{\alpha(k)r^k} = \lim_{r \to 0^+} \frac{\beta[C(x, r)]}{\alpha(k)r^k} + \lim_{r \to 0^+} \frac{\delta[C(x, r)]}{\alpha(k)r^k} = |D_k g(x)|
\]

for \( L_k \text{ almost all } x \in \text{interior } X. \)

2.19 **Remark.** If \( g \) is BVT, then a computation shows that

\[ Jf(x) = |D_k g(x)| \quad \text{for } L_k \text{ almost all } x \text{ in } X. \]

In this case the validity of Theorems 2.14, 2.15, 2.17, and 2.18 and Remark
2.16 holds whenever $|D_{kg}|$ is replaced by $Jf$.

2.20 Remark. If $f \in \Omega_k$ were of the form

$$f(x) = (x_1, \ldots, x_{j-1}, g(x), x_{j+1}, \ldots, x_k) \quad \text{for } x \in E_k,$$

for $j$ some positive integer less than $k$ and $g$ a continuous real-valued function on $E_k$, then the foregoing theory would hold, provided only that the conditions $\text{BVT}(k)$ and $\text{ACT}(k)$ be replaced by $\text{BVT}(j)$ and $\text{ACT}(j)$, and the function $D_{kg}$ be replaced by $D_{jg}$.

3. $k$ dimensional nonparametric surfaces in $E_{k+1}$ ($k > 1$). Each orthogonal projection of $E_{k+1}$ onto $E_k$ is of the form

$$\hat{p}_{k+1}^k \circ R$$

for some $R \in G_{k+1}$. In fact $\hat{p}_{k+1}^k \circ R$ maps the $k$-dimensional subspace of $E_{k+1}$ spanned by the $k$ tuple of vectors

$$\langle (\text{inv } R)^1, \ldots, (\text{inv } R)^k \rangle$$

onto $E_k$, transforming $(\text{inv } R)^i$ into $^kI^i$ for $i = 1, 2, \ldots, k$.

Since for $x \in E_{k+1}$ and $R \in G_{k+1}$,

$$(\hat{p}_{k+1}^k \circ R)(x) = (x \circ (\text{inv } R)^1, \ldots, x \circ (\text{inv } R)^k),$$

two orthogonal projections $\hat{p}_{k+1}^k \circ S$ and $\hat{p}_{k+1}^k \circ T$ are equal if and only if

$$(\text{inv } S)^i = (\text{inv } T)^i \quad \text{for } i = 1, 2, \ldots, k.$$

For $m$ a positive integer greater than two, $\Lambda_m$ will denote the set of all those elements $R$ of $G_m$ for which there exists some $S \in G_2$ satisfying

$$R^i = mI^i \quad \text{for } i = 1, 2, \ldots, m - 2;$$

$$R_i^i = 0 \quad \text{for } i = m - 1, m; j = 1, 2, \ldots, m - 2;$$

$$R_i^j = S_{j-m+2}^i \quad \text{for } i = m - 1, m; j = m - 1, m.$$

If $f$ is a continuous real-valued function on $E_k$ and $i$ is a positive integer, then $\vec{f}$ and $\vec{f}_i$ will be the functions defined by the relations

$$\vec{f}(x) = (x_1, x_2, \ldots, x_k, f(x)) \quad \text{for } x \in E_k,$$

$$\vec{f}_i(x) = (x_1, x_2, \ldots, x_k, f_i(x)) \quad \text{for } x \in E_k.$$

3.1 Sectional assumptions. We let $f$ be a fixed continuous real-valued function on $E_k$, and for $j$ a positive integer no greater than $k+1$, $iR$ will denote that element of $G_{k+1}$ for which

$$iR(x) = (x_1, \ldots, x_{j-1}, x_{j+1}, x_{j+1}, \ldots, x_k, x_i) \quad \text{for } x \in E_k.$$

Observe that

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3.2 Lemma. If $Y$ is a bounded open convex subset of $E_k$ and $i$ is a positive integer, then

$$S \in \Lambda_{k+1} \text{ implies}$$

(i) $\int_{E_k} N(p_{k+1}^k \circ S \circ f, Y, x) d\mathcal{L}_k x$

$$\leq c(k)^{-1} i \int_{E_k} N(p_{k+1}^k \circ S \circ f, Y, x) d\mathcal{L}_k x$

(ii) $N(p_{k+1}^k \circ S \circ f, Y, x) = S(p_{k+1}^k \circ S \circ f, Y, x)$ for $\mathcal{L}_k$ almost all $x$ in $E_k$.

Proof. We know that

$$p_{k+1}^k \circ S(w) = (w \cdot (\text{inv} S), \ldots, w \cdot (\text{inv} S)^k)$$

for $w \in E_{k+1}$. Thus

$$p_{k+1}^k \circ S \circ f(x) = (p_{k+1}^k \circ S)(x_1, x_2, \ldots, x_k, f(x))$$

$$= (x_1, x_2, \ldots, x_{k-1}, S_k \cdot x_k + S_{k+1}^{k+1} \cdot f(x))$$

whenever $x \in E_k$.

We may infer then that $p_{k+1}^k \circ S \circ f \in \Omega_k$. Moreover upon letting $F(x) = S_k \cdot x_k + S_{k+1}^{k+1} \cdot f(x)$ for $x \in E_k$, a check reveals that

$$F_i(x) = S_k^i \cdot x_k + S_{k+1}^{k+1} \cdot f_i(x)$$

for every positive integer $i$. Accordingly statements (i) and (ii) follow from Lemma 2.2 and Theorem 2.4.

3.3 Lemma. If $R \in G_{k+1}$, then there exist functions $S$, $T$ and $U$ for which

$$S \in \Lambda_{k+1}, \quad T \in G_{k+1}, \quad U \in G_k,$$

$$T(k+1 I^{k+1}) = k+1 I^{k+1},$$

$$p_{k+1}^k \circ R = U \circ p_{k+1}^k \circ S \circ T.$$

Proof. We know for any two vector subspaces of a vector space that the sum of their dimensions is equal to the sum of the dimensions of the vector spaces generated respectively by their union and intersection.

Accordingly if $Z$ is the subspace of $E_{k+1}$ spanned by the $k$-tuple of vectors

$$\langle k+1 I^1, k+1 I^2, \ldots, k+1 I^k \rangle$$

and $K$ the subspace of $E_{k+1}$ spanned by the $k$-tuple of vectors

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then the dimension of the subspace $K \cap Z$ equals either $k$ or $k - 1$.

If dimension $(K \cap Z) = k$, then $K$ coincides with $Z$, and we may select $T \in G_{k+1}$ so that $T$ is the rotation of $Z$ which transforms $(\text{inv } R)_i$ into $k^{i+1} i^{i}$ for $i = 1, 2, \ldots, k$. Letting $U$ denote the identity map of $E_k$ and $S$ the identity map of $E_{k+1}$ we may easily check that

$$p^k_{k+1} \circ R = U \circ (p^k_{k+1} \circ R \circ \text{inv } T) \circ T,$$

$$p^k_{k+1} \circ R \circ \text{inv } T = p^k_{k+1} \circ S.$$

Now suppose that the dimension of $K \cap Z$ is $k - 1$.

Using the transitivity of the orthogonal group, we can choose functions $A$, $U$, and $T$ for which

$$A \in G_{k+1}, \quad A^*(K) = K,$$

$$((A \circ \text{inv } R)_1, (A \circ \text{inv } R)_2, \ldots, (A \circ \text{inv } R)_{k-1}) \text{ spans } K \cap Z,$$

$$U \in G_k,$$

$$U[(p^k_{k+1} \circ R)((\text{inv } A \circ \text{inv } T)^i)] = k^i \quad \text{for } i = 1, 2, \ldots, k,$$

$$T \in G_{k+1}, \quad T^*(Z) = Z,$$

$$T[(A \circ \text{inv } R)^i] = k^{i+1} i^{i} \quad \text{for } i = 1, 2, \ldots, k - 1.$$

Since $(T \circ A \circ \text{inv } R)^k$ lies in the orthogonal complement of the space generated by $\langle k^{i+1} i^{i}, k^{i+1} i^{i}, \ldots, k^{i+1} i^{i-1} \rangle$, an element $B$ of $G_2$ may be picked such that

$$(T \circ A \circ \text{inv } R)^k_{k-1+i} = B^1_i \quad \text{for } i = 1, 2.$$

Then defining $S$ to be the element of $G_{k+1}$ satisfying

$$(\text{inv } S)^i = k^{i+1} i^{i} \quad \text{for } i = 1, 2, \ldots, k - 1;$$

$$(\text{inv } S)^i = 0 \quad \text{for } i = k, k + 1; j = 1, 2, \ldots, k - 1;$$

$$(\text{inv } S)^i = B^i_{j-k+1} \quad \text{for } i = k, k + 1; j = k, k + 1;$$

we may infer that $S$ (along with $\text{inv } S$) is an element of $A_{k+1}$.

Inasmuch as

$$(\text{inv } (S \circ T))^i = (\text{inv } T \circ \text{inv } S)^i$$

$$= \text{inv } T[(\text{inv } S)^i]$$

$$= \text{inv } T[(T \circ A \circ \text{inv } R)^i]$$

$$= (A \circ \text{inv } R)^i$$

$$= (\text{inv } (R \circ \text{inv } A))^i \quad \text{for } i = 1, 2, \ldots, k,$$
it follows that
\[ p^k_{k+1} \circ S \circ T = p^k_{k+1} \circ R \circ \text{inv } A. \]

Let \( i \) be a positive integer no greater than \( k \). To complete the proof it is sufficient to show the equality of the functions \( p^k_{k+1} \circ R \) and \( U \circ p^k_{k+1} \circ S \circ T \) on the vector
\[ (\text{inv } R)^i. \]

We compute:
\[
(U \circ p^k_{k+1} \circ S \circ T) [ (\text{inv } R)^i ] = (U \circ p^k_{k+1} \circ R \circ \text{inv } A) [ (\text{inv } R)^i ]
\]
\[ = U [ (p^k_{k+1} \circ R) ( (\text{inv } A \circ \text{inv } R)^i ) ] = \|^i. \]

3.4 Lemma. If \( Y \) is a bounded open convex subset of \( E_k \) and \( i \) is a positive integer, then
\[
R \subseteq G_{k+1} \implies 
\int_{E_k} N(p^k_{k+1} \circ R, f_i, Y, x) d\mathcal{L}^k x
\]
\[
\leq \alpha(k)^{-1} i^k \int_{E_i} \int_{E_k} N(p^k_{k+1} \circ S \circ f, Y, x) d\mathcal{L}^k x d\mathcal{L}^k z.
\]

**Proof.** Suppose \( R \subseteq G_{k+1} \). We use the preceding lemma to select functions \( U, S, T, t \) such that
\[
U \subseteq G_k, \quad S \subseteq A_k, \quad T \subseteq G_{k+1}, \quad t \subseteq G_k,
\]
\[
T^{(k+1)} I^{k+1} = k+1 I^{k+1},
\]
if \( w \in E_k, z \in E_k, \) and \( v \in E_k, \) then \( t(z) = w \) if and only if \( T(z_1, z_2, \ldots, z_k, v) = (w_1, w_2, \ldots, w_k, v), \)
\[
p^k_{k+1} \circ R = U \circ p^k_{k+1} \circ S \circ \text{inv } T.
\]

We shall denote
\[ \text{inv } t(z) = z' \] whenever \( z \in E_k, \)
and for \( i \) a positive integer, we define the functions \( F, F_i, \nabla, \nabla_i \) by the formulae
\[
F = f \circ t, \quad F_i = f_i \circ t,
\]
\[
\nabla(x) = (x_1, x_2, \ldots, x_k, F(x)) \quad \text{for } x \in E_k,
\]
\[
\nabla_i(x) = (x_1, x_2, \ldots, x_k, F_i(x)) \quad \text{for } x \in E_k.
\]

The rest of the proof is divided into three parts.
Part 1. If $V \in G_k$, $B \subseteq E_k$, and $z \in E_k$, then
$$[(\text{inv } V)^*(B)]_z = (\text{inv } V)^*[B_{V(z)}].$$

Proof.

$$(\text{inv } V)^*[B_{V(z)}] = (\text{inv } V)^*\{y \mid y - V(z) \in B\}$$
$$= \{x \mid V(x) - V(z) \in B\}$$
$$= \{x \mid x - z \in (\text{inv } V)^*(B)\}$$
$$= [(\text{inv } V)^*(B)]_z.$$ 

Part 2. (i) $p_{k+1}^k \circ R \circ f = U \circ p_{k+1}^k \circ S \circ \bar{F} \circ \text{inv } t,$

(ii) $p_{k+1}^k \circ R \circ f = U \circ p_{k+1}^k \circ S \circ \bar{F} \circ \text{inv } t.$

Proof. It is sufficient to verify (ii). Let $z \in E_k,$ then

$$(p_{k+1}^k \circ R \circ f)(z) = (p_{k+1}^k \circ R)(z_1, z_2, \ldots, z_k, f_i(z))$$
$$= (U \circ p_{k+1}^k \circ S \circ \text{inv } T)(z_1, z_2, \ldots, z_k, f_i(z))$$
$$= (U \circ p_{k+1}^k \circ S)(z_1', z_2', \ldots, z_k', f_i(z'))$$
$$= (U \circ p_{k+1}^k \circ S)(z_1', z_2', \ldots, z_k', F_t(z'))$$
$$= (U \circ p_{k+1}^k \circ S \circ \bar{F}_i)(z')$$
$$= (U \circ p_{k+1}^k \circ S \circ \bar{F}_i \circ \text{inv } t)(z).$$

Part 3.

$$\int_{E_k} N(p_{k+1}^k \circ R \circ f_i, Y, x)d\mathcal{L}_k x$$

$$\leq \alpha(k)^{-1} i^k \int_{K_i} \int_{E_k} N(p_{k+1}^k \circ S \circ \bar{F}_i, Y, x)d\mathcal{L}_k xd\mathcal{L}_k z.$$

Proof. With the aid successively of Part 2, Lemma 3.2 applied to $F_i$, Part 1, the fact that

$$J_t(z) = 1 \quad \text{whenever } z \in E_k,$$

and Part 2 again, we check that

$$\int_{E_k} N(p_{k+1}^k \circ R \circ f_i, Y, x)d\mathcal{L}_k x$$

$$= \int_{E_k} N(U \circ p_{k+1}^k \circ S \circ \bar{F}_i \circ \text{inv } t, Y, x)d\mathcal{L}_k x$$

$$= \int_{E_k} N(p_{k+1}^k \circ S \circ \bar{F}_i \circ \text{inv } t, Y, x)d\mathcal{L}_k x$$
AREAS OF \( k \)-DIMENSIONAL NONPARAMETRIC SURFACES

\[
\int_{E_k} N(\phi_{k+1}^k \circ S \circ \overline{F}, (\text{inv } t)^*(Y), x) \, d\mathcal{L}_k x
\]
\[
\leq \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(\phi_{k+1}^k \circ S \circ \overline{F}, [(\text{inv } t)^*(Y)]_t, x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(\phi_{k+1}^k \circ S \circ \overline{F}, (\text{inv } t)^*[Y_t(z)], x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(\phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y_t(z), x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(\phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y_t(z), x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(U \circ \phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y_t(z), x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]
\[
= \alpha(k)^{-1} i^k \int_{K_k} \int_{E_k} N(U \circ \phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y_t(z), x) \, d\mathcal{L}_k x \, d\mathcal{L}_{k+1}
\]

3.5 Theorem. If \( Y \) is a bounded open convex subset of \( E_k \), then

\( R \in G_{k+1} \) implies \( N(\phi_{k+1}^k \circ R \circ \overline{f}, Y, x) = S(\phi_{k+1}^k \circ R \circ \overline{f}, Y, x) \) for \( \mathcal{L}_k \) almost all \( x \in E_k \).

Proof. Let \( R \in G_{k+1} \). Just as in the preceding lemma we may select functions \( U, S, t \) and define functions \( F \) and \( \overline{F} \) so that

\[
S \in \Lambda_{k+1}, \quad U \in G_k, \quad t \in G_k,
\]
\[
F = f \circ t,
\]
\[
\phi_{k+1}^k \circ R \circ \overline{f} = U \circ \phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t.
\]

Then with the help of Lemma 3.2 we find that

\[
N(\phi_{k+1}^k \circ R \circ \overline{f}, Y, x)
\]
\[
= N(U \circ \phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y, x)
\]
\[
= N(\phi_{k+1}^k \circ S \circ \overline{F}, (\text{inv } t)^*(Y), x)
\]
\[
= S(\phi_{k+1}^k \circ S \circ \overline{F}, (\text{inv } t)^*(Y), x)
\]
\[
= S(U \circ \phi_{k+1}^k \circ S \circ \overline{F} \circ \text{inv } t, Y, x)
\]
\[
= S(\phi_{k+1}^k \circ R \circ \overline{f}, Y, x)
\]

3.6 Remark. If \( Y \subset E_k \) is the interior of a \( k \) cell and \( R \in G_{k+1} \), then
\[
\int_{E_k} N(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x = \int_{E_k} S(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x \\
= L(p^k_{k+1} \circ R \circ f | Y).
\]

For if \(S, U, t, \text{ and } F\) are chosen as in Theorem 3.5, it then follows by Theorems 2.6 and 3.5 and certain invariance properties of the Lebesgue area that

\[
L(p^k_{k+1} \circ R \circ f | Y) = L(U \circ p^k_{k+1} \circ S \circ \overline{F} \circ \text{inv} t | Y) = L(p^k_{k+1} \circ S \circ \overline{F} \circ \text{inv} t | Y)
\]

\[
= L(p^k_{k+1} \circ S \circ \overline{F} | (\text{inv} t)^*(Y))
\]

\[
= \int_{E_k} N(p^k_{k+1} \circ S \circ \overline{F}, (\text{inv} t)^*(Y), x) d\mathcal{L}_k x
\]

\[
= \int_{E_k} N(U \circ p^k_{k+1} \circ S \circ \overline{F} \circ \text{inv} t, Y, x) d\mathcal{L}_k x
\]

\[
= \int_{E_k} N(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x
\]

\[
= \int_{E_k} S(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x.
\]

### 3.7 Lemma
If \(X \subset E_k\) is a \(k\) cell with boundary \(X\), then

\[
\mathfrak{c}_{k+1} \left[ f^*(X) \right] = 0.
\]

**Proof.** The method of proof selected for Lemma 2.5 included an explicit proof of this statement.

### 3.8 Theorem
If \(X \subset E_k\) is a \(k\) cell, then

\[
\beta(k+1, k)^{-1} \int_{\partial_{k+1}} \int_{E_k} N(p^k_{k+1} \circ R \circ f, X, x) d\mathcal{L}_k x d\phi_{k+1} R
\]

\[
= \beta(k+1, k)^{-1} \int_{\partial_{k+1}} \int_{E_k} S(p^k_{k+1} \circ R \circ f, X, x) d\mathcal{L}_k x d\phi_{k+1} R
\]

\[
= L(f | X)
\]

\[
= \lim_{f \to \infty} L(f | X).
\]

**Proof.** Let \(Y=\text{interior } X\). Then from Lemma 3.5 we know that

\[
\int_{E_k} N(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x = \int_{E_k} S(p^k_{k+1} \circ R \circ f, Y, x) d\mathcal{L}_k x
\]
for $R \subseteq G_{k+1}$. Therefore

$$
\beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R
$$

$$
= \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} S(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.
$$

We suppose that $A$ is an open interval of $E_k$ for which

$$
\text{closure } A \subseteq Y,
$$

and select a positive integer $\iota$ so large that

$$
A \cap \Lambda Y \text{ whenever } \eta \in K_\iota.
$$

Then with the help of [F4, 6.13], [F4, 4.5], and Lemma 3.4 we obtain

$$
L(\vec{f} \mid \text{closure } A) = \int_A \vec{f}(x) d\mathcal{L}_k x
$$

$$
= \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, A, x) d\mathcal{L}_k x d\phi_{k+1} R
$$

$$
\leq \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} \alpha(k) \iota^k \int_{E_k} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, A, x) d\mathcal{L}_k x d\mathcal{L}_k x d\phi_{k+1} R
$$

$$
\leq \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.
$$

From the lower semi-continuity of $L$ on $C_k(X)$ it follows that

$$
L(\vec{f} \mid \text{closure } A) \leq \liminf_{\iota \to \infty} L(\vec{f} \mid \text{closure } A)
$$

$$
\leq \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R.
$$

As a consequence of the arbitrary nature of $A$

$$
L(\vec{f} \mid Y) \leq \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R
$$

$$
= \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} S(p_{k+1}^k \circ R \circ \vec{f}, Y, x) d\mathcal{L}_k x d\phi_{k+1} R
$$

$$
\leq L(\vec{f} \mid Y).
$$

Because of Lemma 3.7 this last relationship holds when $Y$ is replaced by $X$.

Using the foregoing results the other part of the statement may be proved exactly as the corresponding statement was proved in Theorem 2.6.
3.9 Remark. For $X \subset E_k$ a $k$ cell, the results of the last theorem allow us, using the notation of [F6], to state

$$L(\bar{f} | X) = M^{**}(\bar{f} | X) = S^{**}(\bar{f} | X) = U^{**}(\bar{f} | X) = V^{**}(\bar{f} | X) = N^{**}(\bar{f} | X).$$

3.10 Remark. If $Y$ is an analytic subset of $E_k$, then

$$\gamma_{k+1}^k [\bar{f}^*(Y)] = \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \bar{f}, Y, x) d\mathcal{L}_{k} x d\phi_{k+1} R.$$

Since $\bar{f}^*(Y)$ is an analytic subset of $E_{k+1}$, the univalence of $\bar{f}$ implies

$$\gamma_{k+1}^k [\bar{f}^*(Y)] = \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R, \bar{f}^*(Y), x) d\mathcal{L}_{k} x d\phi_{k+1} R$$

$$= \beta(k + 1, k)^{-1} \int_{G_{k+1}} \int_{E_k} N(p_{k+1}^k \circ R \circ \bar{f}, Y, x) d\mathcal{L}_{k} x d\phi_{k+1} R.$$

3.11 Theorem. If $X \subset E_k$ is a $k$ cell and $A$ is a Borel subset of $E_{k+1}$, then

$$\gamma_{k+1}^k [A \cap \bar{f}^*(X)] = \Gamma_{k+1}^k [A \cap \bar{f}^*(X)].$$

Proof. Since in general $\gamma_{k+1}^k$ is dominated by $\Gamma_{k+1}^k$, it is sufficient to assume

$$\gamma_{k+1}^k [A \cap \bar{f}^*(X)] < \infty,$$

and to show that

$$\gamma_{k+1}^k [A \cap \bar{f}^*(X)] \geq \Gamma_{k+1}^k [A \cap \bar{f}^*(X)].$$

We divide the proof of this into three parts.

Part 1. If $R \in G_{k+1}$, then

$$L(\bar{f} | X) \geq \int_{E_k} S(p_{k+1}^k \circ R \circ \bar{f}, X, x) d\mathcal{L}_{k} x.$$

Proof. Let $R \in G_{k+1}$. Select a sequence

$$P_1, P_2, \ldots,$$

of polyhedra of $C_k(X)$ for which

$$\lim_{t \to \infty} P_i = \bar{f} | X, \quad \liminf_{t \to \infty} L(P_i) = L(\bar{f} | X).$$

For each positive integer $i$, $p_{k+1}^k \circ R \circ P_i$ is a polyhedron and

$$\lim_{t \to \infty} (p_{k+1}^k \circ R \circ P_i) = p_{k+1}^k \circ R \circ \bar{f} | X.$$

Suppose $\bigcup_{i=1}^n T_i$ is the simplicial triangulation of $X$ associated with $P_i$. 

From the Lipschitzian character of $\rho_{k+1} \circ R$ we may infer that

$$L(\bar{f} \mid X) = \liminf_{t \to \infty} \sum_{i=1}^{m_i} \mathcal{A}_{k+1}^i [P_i (T_i)]$$

$$\geq \liminf_{t \to \infty} \sum_{i=1}^{m_i} \mathcal{A}_{k+1}^i [(\rho_{k+1} \circ R \circ P_i)^* (T_i)]$$

$$\geq L(\rho_{k+1} \circ R \circ \bar{f} \mid X).$$

We know that the $k$-dimensional Lebesgue area and the $k$-dimensional stable area are lower semi-continuous functions on $C_k(X)$, and that both are extensions of the classical area integral over the class of polyhedra. The Lebesgue area being numerically the largest of all such lower semi-continuous extensions, we may conclude that

$$L(\rho_{k+1} \circ R \circ \bar{f} \mid X) \geq \int_{E_k} S(\rho_{k+1} \circ R \circ \bar{f}, X, x) d\mathcal{L}_k x.$$

**Part 2.** If $B$ is a Borel subset of $E_k$ and $R \subseteq G_{k+1}$, then

$$\mathcal{J}_{k+1}^k [f^*(B)] \geq \int_{E_k} N(\rho_{k+1} \circ R \circ \bar{f}, B, x) d\mathcal{L}_k x.$$

**Proof.** Using Remark 3.10, Theorem 3.8, and Part 1,

$$\mathcal{J}_{k+1}^k [f^*(Y)] = L(\bar{f} \mid Y) \geq \int_{E_k} S(\rho_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x$$

$$= \int_{E_k} N(\rho_{k+1} \circ R \circ \bar{f}, Y, x) d\mathcal{L}_k x$$

whenever $Y \subseteq E_k$ is a $k$ cell.

The inequality between the first and last members of this string may be extended (by the customary methods) to hold whenever $Y$ is a Borel subset of $E_k$.

**Part 3.** $\mathcal{J}_{k+1} [A \cap f^*(X)] \geq \Gamma_{k+1}^k [A \cap f^*(X)].$

**Proof.** Suppose $B$ is a Borel subset of $E_{k+1}$ and $B' = (\text{inv} \bar{f})^*[B \cap f^*(X)]$.

Then using Part 2

$$\mathcal{J}_{k+1}^k [B \cap f^*(X)] = \mathcal{J}_{k+1}^k [f^*(B')] \geq \int_{E_k} N(\rho_{k+1} \circ R \circ \bar{f}, B', x) d\mathcal{L}_k x$$

$$\geq \mathcal{L}_k [(\rho_{k+1} \circ R \circ \bar{f})^*(B')]$$

$$= \mathcal{L}_k [(\rho_{k+1} \circ R)^*(B \cap f^*(X))]$$

for $R \subseteq G_{k+1},$

$$\mathcal{J}_{k+1}^k [B \cap f^*(X)] \geq \mathcal{J}_{k+1} [B \cap f^*(X)].$$

It follows that
3.12 Lemma. If $B$ is a Borel subset of $E_k$ and $R \subseteq G_{k+1}$, then

$$\int_{E_k} N(p_{k+1}^k \circ R \circ \tilde{f}, B, y) d\mathcal{L}_k y \leq \int_{E_k} \tilde{f}^*(B)$$

$$\leq \sum_{i=1}^{k+1} \int_{E_k} N(p_{k+1}^k \circ R \circ \tilde{f}, B, y) d\mathcal{L}_k y.$$

Proof. The first inequality is a restatement of Part 2 of the preceding theorem.

Let $Y \subseteq E_k$ be a $k$ cell with boundary $\tilde{Y}$. With the help of [F4, 6.13] we infer that

$$L(\tilde{f} | Y) = \int_{Y_{-\tilde{Y}}} J\tilde{f}_i(x) d\mathcal{L}_k x$$

$$= \int_{Y_{-\tilde{Y}}} \left[ \sum_{i=1}^{k+1} \left\{ J(p_{k+1}^k \circ R \circ \tilde{f}_i)(x) \right\} ^2 \right]^{1/2} d\mathcal{L}_k x$$

$$\leq \int_{Y_{-\tilde{Y}}} \sum_{i=1}^{k+1} J(p_{k+1}^k \circ R \circ \tilde{f}_i)(x) d\mathcal{L}_k x$$

$$= \sum_{i=1}^{k+1} L(p_{k+1}^k \circ R \circ \tilde{f}_i | Y)$$

whenever $j$ is a positive integer. Letting $j \to \infty$, it follows from Theorems 3.10 and 2.8 that $L(\tilde{f} | Y) \leq \sum_{i=1}^{k+1} L(p_{k+1}^k \circ R \circ \tilde{f} | Y)$. We see by reference to Theorems 2.6, 3.8, and Remark 3.10 that

$$\mathcal{F}_{k+1}^k[\tilde{f}^*(Y)] \leq \sum_{i=1}^{k+1} \int_Y N(p_{k+1}^k \circ R \circ \tilde{f}, Y, y) d\mathcal{L}_k y.$$

The proof is completed by extending this inequality to hold whenever $Y$ is a Borel subset of $E_k$.

3.13 Sectional assumption. For the rest of this section $X \subseteq E_k$ will denote a $k$ cell.

3.14 Lemma. If $j$ is a positive integer not exceeding $k$, then

$$\int X(\phi)^* T_{p_{k+1}^k \circ R \circ \tilde{f}}(u_1, \ldots, u_{i-1}, v, u_{j+1}, \ldots, u_k) d\mathcal{L}_{k-1} u$$

$$\leq L(\tilde{f} | X) \leq \sum_{i=1}^{k} \int X(\phi)^* T_{p_{k+1}^k \circ R \circ \tilde{f}}(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_k) d\mathcal{L}_{k-1} u$$

$$+ \mathcal{L}_k(X).$$
Proof. We recall for $i = 1, 2, \ldots, k$, that

\[ (p_{k+1}^i \circ R \circ f)(x) = (x_1, \ldots, x_{i-1}, f(x), x_{i+1}, \ldots, x_k) \quad \text{for } x \in E_k, \]

\[ p_{k+1}^i \circ R \circ f \in \Omega_k. \]

Consequently we may use Part 2 of Lemma 2.2, Theorem 3.8, and the preceding lemma to compute:

\[ \int_{X(\Theta)} \sup_{x(\Theta)} \inf_{x(\Theta)} f(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_k) d\mathcal{L}_{k-1} \]

\[ = \int_{E_k} N(p_{k+1}^i \circ R \circ f, x, x) d\mathcal{L}_k x \leq \mathcal{F}_{k+1}^k [f^*(X)] \]

\[ = L(f \mid X) \leq \sum_{i=1}^{k+1} \int_{E_k} N(p_{k+1}^i \circ R \circ f, X, x) d\mathcal{L}_k x \]

\[ = \sum_{i=1}^{k} \int_{X(\Theta)} \sup_{x(\Theta)} \inf_{x(\Theta)} f(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_k) d\mathcal{L}_{k-1} + \mathcal{L}_k(X). \]

3.15 Theorem. The following statements are equivalent:

(i) $f$ is BVT on $X$.

(ii) $L(f \mid X) < \infty$.

Proof. This statement is an immediate consequence of the preceding lemma.

3.16 Theorem. If $f$ is BVT on $X$ and $U$ is the subset of $X$ on which $\tilde{f}$ is approximately differentiable, then

(i) $\mathcal{L}_k(X - U) = 0$,

(ii) $\int_Y f(x) d\mathcal{L}_k x = \mathcal{F}_{k+1}^k [f^*(Y \cap U)] \leq \mathcal{F}_{k+1}^k [\tilde{f}^*(Y)] < \infty$

whenever $Y$ is an $\mathcal{L}_k$ measurable subset of $X$.

Proof. Let $j$ be a positive integer no greater than $k$. Just as in Theorem 2.14 we can show that $D_j f(x)$ exists for $\mathcal{L}_k$ almost all $x$ in $X$.

Whence

\[ D_j f(x) \text{ exists for } \mathcal{L}_k \text{ almost all } x \text{ in } X, \]

and Stepanoff’s theorem (see [S, IX 12.2]) implies that $\mathcal{L}_k(X - U) = 0$.

We may apply [F2, 5.2], with the measure $\Phi$ replaced by $\mathcal{F}_{k+1}^k$ (see [F5, 5.10]), and Theorem 3.15 to obtain

\[ \int_Y \tilde{f}(x) d\mathcal{L}_k x = \int_{Y \cap U} \tilde{f}(x) d\mathcal{L}_k x = \int_{E_k} N(\tilde{f}, Y \cap U, y) d\mathcal{F}_{k+1}^k y \]

\[ = \mathcal{F}_{k+1}^k [\tilde{f}^*(Y \cap U)] \leq \mathcal{F}_{k+1}^k [\tilde{f}^*(Y)] < \infty. \]
3.17 Theorem. If $f$ is BVT on $X$, then the following statements are equivalent:

(i) $\mathcal{J}_{k+1}^k[f^*(V)] = 0$ whenever $V \subset X$ and $\mathcal{L}_k(V) = 0$,

(ii) $\int_Y f^*(x) d\mathcal{L}_k x = \mathcal{J}_{k+1}^k[f^*(Y)] < \infty$ whenever $Y$ is an $\mathcal{L}_k$ measurable subset of $X$,

(iii) $\int_Y f(x) d\mathcal{L}_k x = \mathcal{J}_{k+1}^k[f^*(X)] < \infty$,

(iv) $f$ is ACT on $X$.

Proof. Let $V \subset X$ and $\mathcal{L}_k(V) = 0$. Select a Borel subset $A$ of $E_k$ so that $V \subset A \subset X$, $\mathcal{L}_k(A) = 0$.

The proof will be divided into five parts.

Part 1. (i) implies (ii).

Proof. Let $F$ be an $\mathcal{L}_k$ measurable subset of $X$ and $U$ be the subset of $X$ on which $f$ is approximately differentiable. Then

$$\mathcal{J}_{k+1}^k[f^*(Y - U)] = 0,$$

and Theorem 3.16 implies

$$\int_Y f^*(x) d\mathcal{L}_k x = \mathcal{J}_{k+1}^k[f^*(Y)] < \infty.$$

Part 2. (ii) implies (iii).

Part 3. (iii) implies (i).

Proof. Let $\epsilon > 0$. Choose on $X$ a non-negative continuous function $c$ and a number $M$ satisfying

$$\int_X |f(x) - c(x)| d\mathcal{L}_k x \leq \epsilon, \quad M = \sup_{x \in X} c(x).$$

We can select an open set $G$ of $E_k$ for which

$$V \subset G, \quad \mathcal{L}_k(G) < \epsilon \cdot M^{-1},$$

and a grating of $k - 1$ planes which defines such a family $\{X_i \mid i = 1, 2, \ldots \}$ of subsets of $E_k$ that

$$V \subset \bigcup_{i=1}^{\infty} X_i \subset G \cap X,$$

interior $X_i \cap \text{interior } X_j = 0$ for $i \neq j$.

Then
\[ J_{k+1}^k[f^\ast(V)] \leq J_{k+1}^k\left[ f^\ast \left( \bigcup_{i=1}^\infty X_i \right) \right] \leq \sum_{i=1}^\infty J_{k+1}^k[f^\ast(X_i)] \]
\[ = \sum_{i=1}^\infty \int_{X_i} Jf(x) d\mathcal{L}_k x = \sum_{i=1}^\infty \int_{\text{interior } X_i} Jf(x) d\mathcal{L}_k x \]
\[ \leq \int_{\partial \cap X} Jf(x) d\mathcal{L}_k x \leq \int_{\partial \cap X} c(x) d\mathcal{L}_k x + \varepsilon < 2\varepsilon. \]

Because of the arbitrary nature of \( \varepsilon \) the proof is complete.

**Part 4. (ii) implies (iv).**

**Proof.** Suppose \( j \) is a positive integer no greater than \( k \).

The truth of (ii) means

\[ J_{k+1}^k[f^\ast(A)] = 0, \]

and it follows from Lemma 3.12 and Theorem 2.15 that

\[ \mathcal{L}_k[(p_{k+1} \circ f^\ast A)^\ast(A)] = 0, \quad \mathcal{L}_k[(p_{k+1} \circ f^\ast A)^\ast(V)] = 0, \]

\( p_{k+1} \circ f^\ast A \) is absolutely continuous on \( X \),

\( f \) is ACT \( (j) \) on \( X \).

Consequently

\( f \) is ACT on \( X \).

**Part 5. (iv) implies (i).**

**Proof.** Using Theorem 2.15 we know that

\( f \) is ACT \( (i) \) on \( X \),

\( p_{k+1} \circ f^\ast A \) is absolutely continuous on \( X \),

\[ \int_{B_k} N(p_{k+1} \circ f^\ast A, x) d\mathcal{L}_k x = 0, \]

for \( i = 1, 2, \cdots, k \). Recalling that

\[ \int_{B_k} N(p_{k+1} \circ f^\ast A, x) d\mathcal{L}_k x = \mathcal{L}_k(A), \]

we may conclude from Lemma 3.12 that

\[ J_{k+1}^k[f^\ast(A)] = 0, \quad J_{k+1}^k[f^\ast(V)] = 0. \]

**3.18 Remark.** If

(i) \( h \) is a continuous function on \( X \) into \( X \),

(ii) \( \int_{B_k} N(h, x) d\mathcal{L}_k x < \infty \),

(iii) \( f \) is ACT on \( X \),

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then \( h \) is absolutely continuous on \( X \) if and only if

\[
\mathcal{L}^{k+1}[\tilde{f} \circ h]^*(V) = 0 \quad \text{whenever} \quad V \subset X \quad \text{and} \quad \mathcal{L}^k(V) = 0.
\]

This is a consequence of Theorem 3.17.

3.19 Remark. Using the notation of \([F6]\) we may restate the results of Theorems 3.8, 3.16, and 3.17:

\[
L(\tilde{f} \mid X) < \infty \implies \int_X J\tilde{f}(x) d\mathcal{L}^k x \leq L(\tilde{f} \mid X) = M^*(\tilde{f} \mid X) = S^*(\tilde{f} \mid X) = U^*(\tilde{f} \mid X) = V^*(\tilde{f} \mid X) = N^*(\tilde{f} \mid X),
\]

equality holding if and only if \( f \) is ACT on \( X \).

3.20 Theorem. If for \( x \in E_k \) and \( r > 0 \)

\[
C(x, r) = E_k \cap \{ z \mid |z - x| \leq r \},
\]

then

\[
f \text{ is BVT on } X \implies \lim_{r \to 0^+} \frac{L[\tilde{f} \mid C(x, r)]}{\alpha(k) r^k} = J\tilde{f}(x) \quad \text{for } \mathcal{L}^k \text{ almost all } x \in \text{interior } X.
\]

Proof. Letting Theorems 3.8 and 3.16 play the respective roles of Theorems 2.6 and 2.14, the proof is similar to that of Theorem 2.18. In fact only the verification of the additivity of the singular part of the decomposition is different. For this the following general property of Carathéodory outer measures proves useful:

if \( \phi \) is a Carathéodory outer measure, \( A, B, \) and \( T \) are elements of domain \( \phi \), and \( A \) is \( \phi \) measurable, then

\[
\phi(T \cap (A \cup B)) + \phi(T \cap A \cap B) = \phi(T \cap A) + \phi(T \cap B).
\]

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