SYMPELCTIC MODULAR COMPLEMENTS
BY
IRVING REINER

Introduction. Let $\Omega_n$ denote the group of $n \times n$ integral matrices of determinant $\pm 1$ (the unimodular group), and let $I^{(n)}$ be the identity matrix in $\Omega_n$. We use $X'$ to represent the transpose of $X$, and $X \oplus Y$ for the direct sum of $X$ and $Y$.

The symplectic modular group $\Gamma_{2n}$ is the group of $2n \times 2n$ integral matrices $\mathcal{M}$ such that

\[
\mathcal{M} \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix} \mathcal{M}' = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}.
\]

A primitive integral $(j+k) \times 2n$ matrix

\[
\begin{pmatrix} A_1^{(i,n)} & B_1^{(i,n)} \\ C_1^{(k,n)} & D_1^{(k,n)} \end{pmatrix}
\]

in which

\begin{enumerate}
  \item $A_1 B_1'$ and $C_1 D_1'$ are symmetric
  \item $A_1 D_1' - B_1 C_1' = (I^{(i)} 0)$ or $(I^{(k)} 0)$
\end{enumerate}

(depending on whether $j \leq k$ or $j \geq k$) will be called a normal $(j, k)$ array. A normal $(j, 0)$ array will be called a normal pair. Then $\Gamma_{2n}$ is known to consist of all normal $(n, n)$ arrays.

In this paper we shall consider the problem of completing a normal $(j, k)$ array to an element of $\Gamma_{2n}$ by placing $(n-j)$ rows after the first $j$ rows, and $(n-k)$ rows after the last $k$ rows. Since a sub-array of a normal array is normal, it is clear that an array cannot be completed unless it is normal. It will be shown that every normal array may be so completed, and a parametrization of the general completion will be obtained. These results will generalize those due to C. L. Siegel(2) for the special case $j=n, k=0$, but the proofs given here will not depend on his results.

Presented to the Society, September 3, 1953; received by the editors August 4, 1953.


1. Let \( \mathbf{X}_1, \mathbf{X}_2 \) be arrays of the type given in (1); we write \( \mathbf{X}_1 \sim \mathbf{X}_2 \) if there exists \( \mathbf{Y} \in \Gamma_{2n} \) such that \( \mathbf{X}_1 = \mathbf{X}_2 \mathbf{Y} \). This relationship is an equivalence relationship, and we have:

**Lemma 1.** Let \( \mathbf{X}_1 \) be a normal array and \( \mathbf{X}_2 \sim \mathbf{X}_1 \). Then \( \mathbf{X}_2 \) is also a normal array, and \( \mathbf{X}_2 \) can be completed if and only if \( \mathbf{X}_1 \) can be completed.

**Proof.** Clear.

Before proceeding to the next lemma, it will be convenient to single out certain elements of \( \Gamma_{2n} \) which play the same role in \( \Gamma_{2n} \) as do the elementary transformations in \( \Omega_n \). Specifically, we define three types of elements of \( \Gamma_{2n} \):

(I) **Translations**

\[
\mathbf{X}_S = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},
\]

where \( S \) is symmetric.

(II) **Rotations**

\[
\mathbf{R}_U = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix},
\]

where \( U \in \Omega_n \).

(III) **Semi-involutions**

\[
\mathbf{E}_J = \begin{pmatrix} J & I - J \\ J - I & J \end{pmatrix},
\]

where \( J \) is diagonal with elements 0's and 1's.

**Lemma 2.** If \( \text{G.C.D.} \ (a_1, \ldots, a_n, b_1, \ldots, b_n) = 1 \), then

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n) \sim (1, 0, \ldots, 0, 0, \ldots, 0).
\]

**Proof.** We first observe that

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n) \mathbf{R}_U = ((a_1, \ldots, a_n) U, (b_1, \ldots, b_n) U^{-1})
\]

for \( U \in \Omega_n \). If we set \( a_0 = \text{G.C.D.} \ (a_1, \ldots, a_n) \), by proper choice of \( U \) we obtain

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n) \sim (a_0, 0, \ldots, 0, b_1, \ldots, b_n)
\]

for some integers \( b_1, \ldots, b_n \). Let \( b_0 = \text{G.C.D.} \ (b_2, \ldots, b_n) \); then the above reasoning with \( U = 1 + U_1, U_1 \in \Omega_{n-1} \), shows that

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n) \sim (a_0, 0, \ldots, 0, b_1, b_0, 0, \ldots, 0),
\]

and furthermore \( \text{G.C.D.} \ (a_0, b_1, b_0) = 1 \).

We now note the formulas

\[
(x_1, x_2, 0, \ldots, 0, y_1, y_2, 0, \ldots, 0) \mathbf{X}_S = (x_1, x_2, 0, \ldots, 0, y_1 + \lambda x_1, y_2, 0, \ldots, 0),
\]

where \( S = \lambda + 0^{(n-1)} \), and
\[(x_1, x_2, 0, \cdots, 0, y_1, y_2, 0, \cdots, 0) \otimes_J = (-y_1, -y_2, 0, \cdots, 0, x_1, x_2, 0, \cdots, 0),\]

where \(J = 0^{(n)}\). The alternate use of these formulas has the effect of setting up a Euclidean algorithm on the elements in the first and \((n+1)\)st positions. Therefore after a finite number of steps we have either

\[(a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \sim (a, 0, \cdots, 0, c, 0, \cdots, 0),\]

(where \(c \) occurs in the \((n+2)\)nd position) or

\[(a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \sim (a, b, 0, \cdots, 0, 0, 0, \cdots, 0),\]

for some integers \(a, b, \) and \(c\). In the former case observe that

\[(a, 0, \cdots, 0, c, 0, \cdots, 0) \otimes_J = (a, -c, 0, \cdots, 0, 0, 0, \cdots, 0),\]

where \(J = 1 + 0^{(n-1)}\). In either case, therefore,

\[(a_1, \cdots, a_n, b_1, \cdots, b_n) \sim (a, d, 0, \cdots, 0, 0, \cdots, 0),\]

where \(a\) and \(d\) are relatively prime. Now choose \(V \in \Omega_2\) so that \((a, d) V = (1, 0),\) and set \(U = V + I^{(n-2)}\). Then

\[(a, d, 0, \cdots, 0, 0, \cdots, 0) \otimes_U = (1, 0, \cdots, 0, 0, \cdots, 0).\]

This proves the result.

**Theorem 1.** Let \(A_1\) and \(B_1\) be \(j \times n\) integral matrices, \(j \leq n\). Then \((A_1, B_1)\) can be completed to an element of \(\Gamma_{2n}\) by placing \(2n-j\) rows below \((A_1, B_1)\) if and only if \((A_1, B_1)\) is a normal pair.

**Proof.** If \((A_1, B_1)\) is completable, trivially \((A_1, B_1)\) form a normal pair. We now prove the converse by induction on \(n\). The result for \(n = 1\) is an immediate consequence of Lemma 2; let \(n > 1\), and assume that a normal pair of \(t \times (n-1)\) integral matrices can be completed to an element of \(\Gamma_{2(n-1)}\) for \(t \leq n-1\).

Since \((A_1, B_1)\) is primitive, the G.C.D. of the elements of its first row is 1. By Lemma 2 we have therefore

\[(A_1, B_1) \sim \left(\begin{array}{c|c|c}
1 & n' & 0 \\
\hline
\tau & A_2 & \eta \\
\eta' & B_2
\end{array}\right),\]

where \(\tau\) and \(\eta\) are \((j-1) \times 1\) vectors, and \(n\) represents a null column vector whose size depends on the context. Since the right-hand side is a normal pair, the matrix

\[
\left(\begin{array}{c|c|c}
1 & n' & 0 \\
\hline
\tau & A_2 & \eta' \\
\eta & B_2
\end{array}\right) = \left(\begin{array}{c|c|c}
0 & \eta' & 0 \\
\hline
\eta & B_2 & \tau \eta' + A_2 B_2
\end{array}\right)
\]

must be symmetric; therefore \(\eta = n\), and consequently \((A_2, B_2)\) form a normal
pair of \((j-1) \times (n-1)\) matrices. By the induction hypothesis there exists a matrix

\[
\begin{pmatrix}
R & S \\
T & U
\end{pmatrix} \in \Gamma_{2(n-1)}
\]

with its first \((j-1)\) rows given by \((A_2, B_2)\). Define

\[
\bar{x} = \begin{pmatrix}
1 + R & 0 & S \\
0 + T & 1 + U
\end{pmatrix}.
\]

Then \(\bar{x} \in \Gamma_{2n}\), and

\[
\begin{pmatrix}
1 & n' & 0 & n' \\
1 & A_2 & n & B_2
\end{pmatrix} \bar{x}^{-1} = \begin{pmatrix}
1 & n' & 0 & n' \\
1 & I^{(j-1)} & 0 & 0
\end{pmatrix}.
\]

But the right-hand matrix consists of the first \(j\) rows of \(R\), where

\[
V = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & I^{(j-1)} & 0 \\
n & 0 & I^{(n-j)}
\end{pmatrix}.
\]

The theorem now follows by the use of Lemma 1.

**Theorem 2.** The array given by (1) is computable if and only if it is normal.

**Proof.** As observed before, a completable array is obviously normal. Assume hereafter that the array given in (1) is normal, and (without loss of generality) that \(j \geq k\).

For, if the given array is a normal \((j, k)\) array with \(j < k\), then

\[
\begin{pmatrix}
C_1 & D_1 \\
-A_1 & -B_1
\end{pmatrix}
\]

is a normal \((k, j)\) array; if this latter array is completed to an element \(Y \in \Gamma_{2n}\), then the original array is completable to

\[
\begin{pmatrix}
0 & -I^{(n)} \\
I^{(n)} & 0
\end{pmatrix} \cdot Y \in \Gamma_{2n}.
\]

By Theorem 1 we have

\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \sim \begin{pmatrix} I^{(j)} & 0 & 0 \\ 0 & C_2 & D_2 \end{pmatrix}
\]

for some \(C_2, D_2\). By Lemma 1 we see that \((C_2, D_2)\) form a normal pair, and furthermore

\[
(I^{(i)} \ 0)D'_2 = (I^{(k)} \ 0)',
\]
so that we have

\[ D_2 = (I^{(k)} \ 0^{(k,i-k)} \ X^{(k,n-i)}) \]

for some \( X \). Now set

\[ U = \begin{pmatrix} I^{(k)} & 0 & 0 \\ 0 & I^{(i-k)} & 0 \\ X' & 0 & I^{(n-i)} \end{pmatrix}, \]

and observe that

\[
\begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ C_2 & I^{(k)} & 0 & X \end{pmatrix} U = \begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ C_3 & C_4 & I^{(k)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

where \( C_3 \) is a \( k \times j \) matrix, and \( C_4 \) a \( k \times (n-j) \) matrix. Again using Lemma 1, the matrix

\[(C_3 \ C_4)(I^{(k)} \ 0 \ 0)'\]

must be symmetric, so that

\[ C_3 = (C_{31}^{(k,k)} \ C_{32}) \]

with symmetric \( C_{31} \). But now

\[
\begin{pmatrix} I^{(i)} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_4 & I^{(k)} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

consists of the first \( j \) rows and the \((n+1)\)st, \( \cdots \), \((n+k)\)th rows of

\[
\begin{pmatrix} I^{(i)} & 0 & 0 & 0 & 0 \\ 0 & I^{(n-i)} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_4 & I^{(k)} & 0 \\ C_{32}' & 0 & 0 & 0 & I^{(i-k)} \\ C_4' & 0 & 0 & 0 & 0 \end{pmatrix} \in \Gamma_{2n}.
\]

This proves the result.

2. We now turn to the problem of finding an expression for the general completion \( \mathcal{C} \) of a given normal \((j, k)\) array, and we again assume without loss of generality that \( j \geq k \). If \( \mathcal{C}_0 \) is a specific completion of the given array, then \( \mathcal{C} \) is a completion if and only if \( \mathcal{C}_0^{-1} \) is an element of \( \Gamma_{2n} \) whose first \( j \) rows are given by

\[(I^{(i)} \ 0^{(i,2n-i)})\]

and its \((n+1)\)st, \( \cdots \), \((n+k)\)th rows by

\[(0^{(k,n)} \ I^{(k)} \ 0^{(k,n-k)}).\]
Thus if $\hat{x}$ represents the general such element of $\Gamma_{2n}$, then $\hat{x}C_0$ is the general completion of the given array.

Let us write

$$
\hat{x} = \\
\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
E_1 & E_2 & E_3 & F_1 & F_2 & F_3 \\
0 & 0 & 0 & I & 0 & 0 \\
G_4 & G_5 & G_6 & H_4 & H_5 & H_6 \\
G_1 & G_2 & G_3 & H_1 & H_2 & H_3 \\
\end{bmatrix}
\begin{bmatrix}
k \\
j - k \\
n - j \\
k \\
j - k \\
\end{bmatrix}
$$

where we have indicated the numbers of rows and columns in the various submatrices. Then $\hat{x} \in \Gamma_{2n}$ if and only if $\hat{x}$ is an integral matrix for which

$$
\begin{bmatrix}
I & 0 & 0 & F_1' \\
0 & I & 0 & F_2' \\
E_1 & E_2 & E_3 & F_3' \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
G_4 & G_5 & G_6 & 0 \\
G_1 & G_2 & G_3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
I & H_4' & H_1' \\
G_4' & G_1' & 0 \\
G_6' & G_2' & 0 \\
G_6 & G_3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
F_1 & F_2 & F_3 \\
G_6 & G_2 & G_3 \\
\end{bmatrix} = I.
$$

These conditions give:

$$
F_1 = 0, F_2 = 0, E_3F_3' \text{ symmetric}, G_4 = 0, G_1 = 0, G_5H_6' + G_6H_5' \text{ symmetric},
$$

$$
G_2H_1' + G_3H_3' \text{ symmetric}, G_6H_2' + G_5H_3' = H_6G_2' + H_5G_3',
$$

$$
E_1 = 0, H_4 = 0, H_1 = 0, H_5 = I, H_2 = 0,
$$

$$
E_3H_6' + E_3H_6' - F_5G_5' = 0, E_2H_2' + E_3H_3' - F_4G_3' = I.
$$

Hence we have

$$
\hat{x} = \\
\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & E_2 & E_3 & 0 & 0 & F_3 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & G_5 & G_6 & 0 & I & H_6 \\
0 & G_2 & G_3 & 0 & 0 & H_3 \\
\end{bmatrix}.
$$

1 License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
with symmetric \( E_2 F_3', G_6 + G_6 H_6', G_4 H_4' \), where
\[
E_2 = F_3 G_6' - E_3 H_6', \quad G_2 = H_2 G_6' - G_3 H_6', \quad \text{and} \quad E_2 H_6' - F_3 G_4' = I.
\]
Therefore
\[
\begin{pmatrix}
E_3 & F_3' \\
G_3 & H_3'
\end{pmatrix} \subseteq \Gamma_2(n-1),
\]
and so
\[
\begin{pmatrix}
E_3 \\
G_3
\end{pmatrix} \begin{pmatrix}
F_3' \\
H_3'
\end{pmatrix} = \begin{pmatrix}
E_3 & F_3' \\
G_3 & H_3'
\end{pmatrix} \begin{pmatrix}
-H_6' \\
G_6'
\end{pmatrix}
\]
is true if and only if
\[
\begin{pmatrix}
-H_6' \\
G_6'
\end{pmatrix} = \begin{pmatrix}
H_3' & -F_3' \\
-G_3' & E_3'
\end{pmatrix} \begin{pmatrix}
E_2
\end{pmatrix}.
\]
We find easily that
\[
G_6 + G_6 H_6' = G_6 + (-E_3 G_3 + G_3 H_3')(-H_6' E_2 + F_3' G_2)
\]
\[
= G_6 + E_3 G_3 H_3'E_2 + G_3 E_2 F_3' G_2 - G_3 H_3' E_2 - E_2 G_3 F_3' G_2.
\]
But \( E_2 H_6' = I + F_3 G_4' \), so
\[
G_6 + G_6 H_6' = G_6 - G_2 E_2 + \text{symmetric matrix},
\]
and therefore \( G_6 + G_6 H_6' \) is symmetric if and only if \( G_6 - G_2 E_2 = S \) is symmetric.

We now observe that
\[
\begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
l + E_4 & 0 + F_3 \\
0 + E_3 & I \\
0 & 0 + E_2 & I + H_3
\end{pmatrix}
\]

\[
\chi = \begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
l + E_4 & 0 + F_3 \\
0 + E_3 & I \\
0 & 0 + E_2 & I + H_3
\end{pmatrix}
\]

Hence, if \( \xi_0 \) is a specific completion of a given normal \((j, k)\) array, the general completion equals \( \xi_0 \) multiplied on the left by the above expression for \( \chi \), where
\[
\begin{pmatrix}
E_3 & F_3' \\
G_3 & H_3'
\end{pmatrix}
\]
is an arbitrary element of \( \Gamma_2(n-1) \), where \( S \) is an arbitrary symmetric \((j-k) \times (j-k)\) matrix, and where \( G_2 \) and \( E_2 \) are arbitrary \((n-j) \times (j-k)\) matrices.
For the special case $j=n$, $k=0$ we obtain from the above Siegel's result that

$$C = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} C_0,$$

with symmetric $S$.

We finally note that the above reasoning holds true for any Euclidean ring.

University of Illinois,
Urbana, Ill.