NOTE ON THE BESSEL POLYNOMIALS

BY

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1. This note can be considered as an addendum to the comprehensive study of the class of Bessel polynomials carried on by H. L. Krall and O. Frink [1]. In fact I study here the expansion of particular functions in terms of Bessel polynomials as well as the location of the zeros of these polynomials. Write \( p_n(z) = \sum_k \rho_{nk}z^k \), so that \([1, \text{p. 101}]
\[
(1) \quad \rho_{nk} = 2^{-k}(n + k)!/(k!(n - k)!), \quad 0 \leq k \leq n; \ n \geq 0.
\]
The first result is the following

**Theorem A.** Let \( f(z) \) be a function regular in \( |z - a| \leq R \), where \( R > 0 \) and \( a \) is any point of the plane. Then \( f(z) \) can be expanded in a series of Bessel polynomials of the form
\[
f(z) \sim \sum c_n p_n(z - a),
\]
where
\[
c_n = 2^n(2n + 1) \sum_{r=0}^{\infty} (-2)^r f^{(n+r)}(a)/(r!(2n + r + 1)!),
\]
and the series is convergent uniformly in \( |z - a| \leq R \).

We first suppose that \( f(z) \) is regular in \( |z| \leq R \) and prove that \( f(z) \) can be expanded in a series \( \sum \gamma_n p_n(z) \) where
\[
(2) \quad \gamma_n = 2^n(2n + 1) \sum_{r=0}^{\infty} (-2)^r f^{(n+r)}(0)/(r!(2n + r + 1)!),
\]
and that the series is uniformly convergent in \( |z| \leq R \). Theorem A follows readily when \( z - a \) is written for \( z \).

Since the set \( \{p_n(z)\} \) of Bessel polynomials is basic in the sense of J. M. Whittaker [3, Chap. II], we shall appeal in the proof to the theory of basic series of polynomials as given by J. M. Whittaker and B. Cannon. In fact suppose that \( z^n \) admits the representation
\[
z^n = \sum \pi_n i p_i(z),
\]
so that the matrix \( (\pi_{ni}) \) is the unique reciprocal of the matrix \( (p_{ni}) \) (see Whittaker [3, T21, p. 40]). Hence
\[
(4) \quad p_i, i\pi_{ii} = 1,
\]
and

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(5) \[ \sum_{k=0}^{n} p_{nk} p_{ki} = 0, \quad n > i, \ i \geq 0. \]

Setting \( n = m + i \) in (5) and inserting the values of \( (p_{nk}) \) from (1) we obtain

(6) \[ \pi_{m+i,i} = - \sum_{\nu=0}^{m-1} \frac{2^{m-\nu} (2i + m + \nu)! (m + i)!}{(i + \nu)! (m - \nu)! (2i + 2m)!} \pi_{i+i,i}. \]

Applying (6) and the identity

\[ \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} \binom{k + m + \nu}{m - 1} = 0 \]

(this is in fact the coefficient of \( x^{l+m+1} \) in the expansion of \( (1+x)^{m}(1+x)^{-m} = 1 \)), we can easily deduce by induction that

\[ \pi_{m+i,i} = \frac{(-2)^{m}(m + i)!(2i + 1)!}{m!i!(2i + m + 1)!} \pi_{i,i}; \quad m, \ i \geq 0. \]

Substituting for \( \pi_{i,i} \) from (4) it follows that

(7) \[ \gamma_{n} = \sum_{\nu=0}^{\infty} \pi_{n+n+n} f^{(n)}(0)/(n + \nu)!. \]

Hence inserting the value of \( \pi_{n+n+n} \) from (7), (2) follows at once.

In order to prove that the series \( \sum \gamma_{n} p_{n}(z) \) is convergent in \( |z| \leq R \) we form the sum (see [3, Chap. II, III])

\[ \omega_{n}(R) = \sum_{i} |\pi_{ni}| M_{i}(R), \]

where \( M_{i}(R) = \max_{|z| = R} |\pi_{i}(z)| = \sum_{k=0}^{i} \pi_{ik} R^{k} \). Applying (1) and (7) we obtain after simple reduction

(8) \[ \omega_{n}(R) = 2^{n} \sum_{k=0}^{n} \binom{n}{k} (R/2)^{k} \frac{(n - k)! (2k + 2j + 1)(2k + j)!}{(n + k + j + 1)!} \]

\[ < (2n + 1)R^{n} \sum_{k=0}^{n} \binom{n}{k} (4/R)^{k}/k! = (2n + 1)B_{n}R^{n}, \]

say. Effecting the transformation \( y = x (1+x)^{-1} \) on the function \( x \exp(4x/R) = \sum_{n=0}^{\infty} (4/R)^{n} x^{n+1}/n! \) it follows that

\[ F(y) = (y/(1 - y)) \exp \{4y/(R(1 - y))\} = \sum_{n=0}^{\infty} B_{n}y^{n+1}. \]
This function is regular in $|y| < 1$; hence by Cauchy’s inequality we have

$$B_n < K/\alpha^{n+1} \quad (0 < \alpha < 1),$$

where $K = \max_{|y|=\alpha} |F(y)| < \infty$. Inserting this in (8) and making $n$ tend to infinity we obtain

$$\lambda(R) = \limsup_{n \to \infty} \left\{ \omega_n(R) \right\}^{1/n} \leq R/\alpha,$$

and since $\alpha$ can be taken as near to 1 as we please we conclude that $\lambda(R) = R.$ According to Cannon [3, T6, p. 11] we infer that the series $\sum \gamma_n p_n(z)$ is uniformly convergent in $|z| \leq R$, as required.

2. As for the location of the zeros of Bessel polynomials the following result is established.

THEOREM B. All the zeros of the Bessel polynomial $p_n(z)$, for $n > 1$, lie within or on the circle $|z| = ((n - 1)/(2n - 1))^{1/2}$.

We shall suppose that $n \geq 3$, as the zeros of $p_2(z)$ are of modulus $(1/3)^{1/2}$. Write $r = ((2n - 1)/(n - 1))^{1/2}$. We shall first show that the real zeros of $p_n(z)$, if any, are of modulus less than $1/r$. Let $x$ be any real number not less than $1/r$, then the formula (1) for the coefficients $(p_{nk})$ yields

$$p_{nk}x^k < p_{n,k+1}x^{k+1}, \quad 0 \leq k \leq n - 4,$$

and furthermore

$$R_n(x) = p_{nn}x^n - p_{n,n-1}x^{n-1} + p_{n,n-2}x^{n-2} - p_{n,n-3}x^{n-3}$$

$$= (n - 1)^{-1}p_{n,n-2}x^{n-3}\{(2n - 1)(x^3 - x^2) + (n - 1)x - (n - 2)/3\}.$$

It can be easily shown that the cubic polynomial inside the brackets has, for $n \geq 3$, one real positive zero less than $1/r$. Hence for $x \geq 1/r$ we have

$$R_n(x) > 0, \quad (n \geq 3).$$

Since the coefficients of $p_n(z)$ are all positive we need only consider $p_n(-x)$ where $x \geq 1/r$. Thus (9) and (10) yield

$$p_n(-x) = 1 + \sum_{k=0}^{n/2-3} (-p_{n,2k+1}x^{2k+1} + p_{n,2k+2}x^{2k+2}) + R_n(x) > 0 \quad (n \text{ even})$$

and

$$p_n(-x) = \sum_{k=0}^{(n-1)/2-2} (p_{n,2k}x^{2k} - p_{n,2k+1}x^{2k+1}) - R_n(x) < 0 \quad (n \text{ odd}).$$

Hence all the real roots of $p_n(z)$ lie in $-1/r < x < 0$.

Now write $q_n(z) = z^n p_n(1/z)$ and consider the polynomial $q_n(rz)$. Since we shall always be concerned with this particular polynomial we may suppress the suffix $n$ and write
\[ g_n(rz) = g(z) = \sum_{k=0}^{n} a_k z^k, \]

does that \( a_k = \rho_{n,n-k} r^k \). Inserting the values of the coefficients \( \rho_{n,k} \) from (1) we can easily observe that

\[ 0 < a_1 = r a_0, \]
\[ a_k > r a_{k+1} > 0, \quad 1 \leq k \leq n - 1. \] (11)

We form, with M. Marden [2, p. 148], the successively derived coefficients \( a_k^{(j)} \) given by

\[ a_k^{(j)} = a_k, \quad a_k^{(j+1)} = a_0^{(j)} a_k^{(j)} - a_{n-j}^{(j)} a_{n-k-j}, \quad 0 \leq k < n - j; \quad 0 \leq j \leq n - 1. \] (12)

As for the first derived coefficients \( a_k^{(1)} \), the relations (11) yield the following:

for \( k = n-1, \)
\[ a_{n-1}^{(1)} = a_0 a_{n-1} - a_n a_1 > 0, \]
and for \( k > 0, \)
\[ a_k^{(1)} - a_{k+1}^{(1)} = a_0 (a_k - a_{k+1}) + a_n (a_{n-k-1} - a_{n-k}) > (1 - 1/r) a_k^{(1)}, \]
so that

\[ a_k^{(1)} > r a_{k+1}^{(1)} > r^{n-k-1} a_{n-1}^{(1)} > 0, \quad 1 \leq k \leq n - 2. \] (13)

Finally, for \( k = 0, \)
\[ a_1^{(1)} - a_0^{(1)} = a_0 (a_1 - a_0) - a_n (a_{n-1} - a_n) < (r - 1) a_0^{(1)}, \]
so that

\[ 0 < a_1^{(1)} < r a_0^{(1)}. \] (14)

A comparison between the relations (13) and (14) for the case \( j = 1 \), on the one hand, and the relations (11) for \( j = 0 \), on the other hand, suggests that for the \( j \)th derived coefficients we should have

\[ 0 < a_1^{(j)} < r a_0^{(j)}, \]
\[ a_k^{(j)} > r a_{k+1}^{(j)} > 0, \quad 1 \leq k \leq n - j - 1. \] (15)

In fact (15) is valid for \( j = 1 \), in view of (13) and (14). Moreover supposing (15) to be true for some \( j < n - 2 \), then the method used in deriving (13) and (14) from (11) can be similarly applied to (15) to derive easily the following relations

\[ 0 < a_1^{(j+1)} < r a_0^{(j+1)}, \]
\[ a_k^{(j+1)} > r a_{k+1}^{(j+1)} > 0, \quad 1 \leq k \leq n - j - 1, \]
so that (15) is true for $1 \leq j \leq n - 2$. We may apply (15) with $j = n - 2$ to deduce that $a_0^{(n-1)} > 0$, so that

$$a_0^{(i)} > 0, \quad 1 \leq j \leq n - 1.$$

Applying Marden's theorem concerning the number of zeros of a polynomial inside the unit circle in terms of the coefficients $a_0^{(i)}$ [2, Theorem (42.1), p. 150] we infer that the polynomial $g(z)$ will have at most one zero inside the unit circle. Consequently the polynomial $p_n(z)$ will have at least $n - 1$ of its zeros within or on the circle $|z| = 1/r$. Suppose that the number of zeros of $p_n(z)$ within or on the circle $|z| = 1/r$ is exactly $n - 1$ and that the remaining zero $\beta$ is outside the circle. By the first part of the proof $\beta$ cannot be real and hence its conjugate $\bar{\beta}$, which is also a zero of $p_n(z)$, lies outside the circle $|z| = 1/r$. We conclude from this contradiction that all the zeros of $p_n(z)$ lie within or on the circle $|z| = 1/r$, and this completes the proof of Theorem B.

REFERENCES


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