NORMAL AUTOMORPHISMS AND THEIR FIXED POINTS

BY

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1. Introduction. The elements of the centralizer $T$ of the group of inner automorphisms $J$ of a group $G$ (in the group $A$ of automorphisms of $G$) are called the normal automorphisms of $G$. The center $Z$ of $G$ is the set of all elements of $G$ which are fixed by each mapping from $J$. Likewise, let $B$ be the set of fixed points held in common by the mappings from $T$. $G/B$ is abelian, and the elements of $T$ which induce either the identity or the involution on $G/B$ form a subgroup $W$ of $T$. We shall investigate the ascending central series of $W$. Just as the ascending central series \{\text{Z}_i\} is formed over $Z=Z_1$, so an ascending series is formed over $B$. Elements of $G$ lying in members of this $B$-series turn out to be fixed points for high powers of normal automorphisms. For automorphisms which induce the identity on $G/Z_n$, we show that the common fixed points lie in the centralizer of $Z_n$ in $G$.

The notation is both obvious and conventional. $G$ denotes a group with automorphism group $A$ and inner automorphism group $J$. The members of the ascending central series of $G$ are the $Z_i$, and the higher commutator subgroups are the $G^{(i)}$ [3]. If, say, the inner automorphism group of a group $H$, different from $G$, is to be denoted, we employ the symbols $J(H)$, and similarly for other groups or subgroups, such as $Z_n(H)$, associated with $H$. For a subgroup $H$ of $G$, the centralizer of $H$ in $G$ will be denoted by $C(H; G)$. If $x$, $y\in G$, then $(x, y) = x^{-1}y^{-1}xy$. For normal subgroups $S$ and $T$ of a group $G$, $S \triangleleft T$ (following R. Baer) will be the commutator quotient of $S$ by $T$, the set of all $x \in G$ such that $(x, t) \in S$ for every $t \in T$. $S \triangleleft T$ is a normal subgroup of $G$. The identity map on a group is indicated by $e$, and the identity element of a group is to be $e$. For a homomorphism $f$ on $G$, the kernel will be written $\text{kern } f$. $\oplus$ denotes direct summation of groups. A periodic group is one in which each element is of finite order, and an abelian periodic group will be called a torsion group. If a periodic group $G$ has a uniform order on its elements, then $G$ is said to be uniform torsion (u.t.), and the least positive uniform order will be called [3] the exponent of $G$. A group is said to be torsion-free if it has no nontrivial elements of finite order. A complete group is one in which the $x^n$ form a set of generators of $G$ for each positive integer $n$. The group of integers is to be $I$; the group of rationals, $R$; the multiplicative group of nonzero rationals, $R^*$; and $I_n$ is to be the group of integers modulo $n$.

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Occasionally, we shall give \( I_n \) its usual representation as the group of integral residue classes, modulo \( n \), so that \( j_n \) will be the residue class, modulo \( n \), in which the integer \( j \) lies.

2. Normal automorphisms. Let \( G \) be a group, and let \( H \) be a normal subgroup of \( G \). Let \( \alpha \) be an automorphism of \( G \) such that \( H \) is admissible under both \( \alpha \) and \( \alpha^{-1} \). That is, \( \alpha(H) \subseteq H \) and \( \alpha^{-1}(H) \subseteq H \). Then \( \alpha \) induces an automorphism \( \alpha' \) on \( G/H \) given by \( \alpha'(xH) = \alpha(x)H \). In particular, if \( H \) is a characteristic subgroup of \( G \), then every automorphism \( \alpha \) of \( G \) induces an automorphism \( \alpha' \) on \( G/H \). If \( \alpha \) induces the identity automorphism \( i \) on \( G/Z_1 \), \( \alpha \) is called a normal automorphism (sometimes center \([3]\) or central \([1]\) automorphism) of \( G \), and \( \alpha(x) \equiv x \mod Z_1 \) for every \( x \in G \). It is easy to see that \( \alpha \in A \) is normal if, and only if, \( (x, \alpha(y)) = (x, y) \) for every \( x, y \in G \). Let \( T_1 \) be the set of all normal automorphisms of \( G \). If automorphism composition is interpreted as a multiplication, \( T_1 \) becomes a subgroup of the automorphism group \( A \) of \( G \) with \( i \) as its identity, and \( T_1 \) is normal in \( A \). It is well known that \( T_1 = C(J; A) \) \([3]\). An endomorphism \( \gamma \) of \( G \) for which \( \gamma(G) \subseteq Z_1 \) is called a central endomorphism. If \( \alpha \in T_1 \), \( \alpha(x) = x\gamma(x) \) for every \( x \in G \), where \( \gamma \) is a central endomorphism of \( G \) with the further property (A) that to each \( y \in G \), there exists a unique \( g = g(y; \gamma) \in G \) with \( \gamma(g) = g^{-1}y \). We might write \( \alpha = \iota + \gamma \).

If, conversely, \( \gamma \) is a central endomorphism with (A), then the mapping \( \alpha \), defined by \( \alpha = \iota + \gamma \), is in \( T_1 \).

Let \( G \) be a group for which \( Z_2 \neq Z_1 \). If \( u \in Z_2 \), then the mapping \( \gamma_u \) given by \( \gamma_u(x) = (x, u) \) is readily seen to be a central endomorphism of \( G \). These \( \gamma_u \) will be called the Grün endomorphisms of \( G \). If \( y \in G \), then \( \gamma_u(uyu^{-1}) = (uyu^{-1})^{-1}y \) so that \( \gamma_u(x) = x^{-1}y \) has the solution \( x = uy^{-1}u \). If, conversely, \( x \) is any solution of \( \gamma_u(x) = x^{-1}y \), then \( uy^{-1}u = y \) so that \( x = uy^{-1}u \); and the solution is unique, establishing (A). Hence \( \alpha_u \), a mapping defined by \( \alpha_u(x) = x\gamma_u(x) = u^{-1}uy = \tau_u(x) \), is in \( T_1 \). Suppose, conversely, that \( \alpha \in T_1 \cap J \). Let \( \alpha = \tau_u \).

Then \( u^{-1}uy = x \mod Z_1 \) for every \( x \in G \), so that \( u \in Z_2 \) and \( \alpha = \alpha_u \). We state

**Lemma 1.** \( T_1 \cap J \cong Z_1(J) \), and the elements of the former are in one-to-one correspondence with the Grün endomorphisms of \( G \).

For endomorphisms \( \gamma_a \) and \( \gamma_b \) satisfying (A), note that \( \gamma_a(G) \subseteq \gamma_b(G) \) implies \( \gamma_a(G) \subseteq \gamma_b(G) \), so that \( \gamma_a(G) \subseteq \gamma_b(G) \) \((n = 1, 2, 3 \ldots)\). \( \gamma_i \) is the trivial endomorphism \( (\gamma_i(x) = e \) for every \( x \in G \)), and \( \gamma_i(G) \subseteq \gamma_a(G) \) for every \( \alpha \in T_1 \).

**Lemma 2.** For \( \alpha \in T_1 \), \( \alpha(\gamma_a(G)) = \gamma_a(G) = \alpha(\gamma_a^{-1}(G)) \).

**Proof.** For \( x \in G \), \( \alpha(\alpha(x)) = \alpha(x) \alpha(\alpha(x)) \), so that \( \alpha(\gamma_a(x)) \in \gamma_a(G) \). Hence \( \alpha(\gamma_a(G)) \subseteq \gamma_a(G) \). \( \alpha^{-1}(x) = x\gamma_a^{-1}(x) \) implies that \( x = \alpha(x) \alpha(\gamma_a^{-1}(x)) = \gamma_a(x) \alpha(\gamma_a^{-1}(x)) \), whence \( \gamma_a(x) = \alpha(\gamma_a^{-1}(x^{-1})) \); and \( \gamma_a(G) \subseteq \alpha(\gamma_a^{-1}(G)) \). Replacing \( x \) by \( x^{-1} \), we have \( \gamma_a(x^{-1}) = \alpha(\gamma_a^{-1}(x^{-1})) \). There exists \( y \in G \) such that \( \alpha(y) = \gamma_a(x^{-1}) = y\gamma_a(y) \). Since \( \gamma_a(G) \) is a subgroup of \( G \), \( y \in \gamma_a(G) \). Thus,
\[ \alpha(\gamma^{-1}(x)) = \alpha(y) \text{ where } y \in \gamma_{\alpha}(G), \text{ so that } \alpha(\gamma^{-1}(G)) \subseteq \alpha(\gamma_{\alpha}(G)). \]

3. **The common fixed points.** The subgroup \( Z_1 \), the center of \( G \), is the set of all elements of \( G \) which are fixed by each inner automorphism \( \tau_v \) of \( G \). For \( T_1 = C(J; A) \), the set analogous to \( Z_1 \) is \( B_1 \), where \( x \in B_1 \) if, and only if, \( \alpha(x) = x \) for every \( \alpha \in T_1 \). If \( F(\alpha) \) is the set of the fixed points of \( \alpha \in T_1 \), then \( F(\alpha) \) is a normal subgroup of \( G \). Since \( B_1 = \bigcap F(\alpha) \), where the cross-cut is taken over all \( \alpha \in T_1 \), \( B_1 \) is likewise a normal subgroup of \( G \). Now \( F(\alpha) = \ker \gamma_{\alpha} \), and \( \gamma_{\alpha}(G) \) is abelian. Thus \( F(\alpha) \supseteq G' \), the derivative of \( G \), for every \( \alpha \in T_1 \), and \( B_1 \supseteq G' \). This shows that \( G/B_1 \) is abelian and that if \( B_1 = (e) \), then \( G \) is abelian.

**Lemma 3.** \( G' \subseteq B_1 \subseteq C(Z_2; G) \).

**Proof.** If \( x \in B_1 \), \( \gamma_u(x) = e \) for every Grün endomorphism \( \gamma_u \), \( u \in Z_2 \). Consequently \( x \) commutes with every such \( u \).

**Corollary.** If \( C(Z_2; G) = Z_1 \), then \( G \) is of class 2.

Suppose that \( H \) is a characteristic subgroup of \( G \), that \( J(H; G) \) is the set of all inner automorphisms \( \tau_v \) of \( G \) where \( v \in H \) (where \( \tau_v(x) = v^{-1}xv \)), and that \( Q(H; G) = Q(H) \) is the set of all automorphisms of \( G \) such that \( \alpha \in Q(H) \) induces the identity automorphism on \( G/H \). For instance, \( J(G; G) = J \), and \( Q(Z_1; G) = T_1 \). \( J(H; G) \) is a normal subgroup of \( Q(H; G) \). Let \( F = F(Q(H; G)) \) be the fixed points common to all mappings in \( Q(H; G) \), and let \( F^* = F(J(H; G)) \) be the fixed points common to all mappings in \( J(H; G) \). \( F^* \supseteq F \). But \( F^* = C(H; G) \), so that \( F(Q(H; G)) \subseteq C(H; G) \). This general result will be used later to establish a variation of Lemma 3.

\( G = B_1 \) if, and only if, \( G \) has no proper normal automorphisms. By Lemma 3, \( G = B_1 \) implies \( G = C(Z_2; G) \) so that every element of \( G \) commutes with every element of \( Z_2 \), and \( Z_2 \subseteq Z_1 \). Hence the ascending central series of \( G \) breaks off with \( Z_1 \) if \( G = B_1 \). Likewise, Lemma 3 has the following obvious

**Corollary.** \( G \) is of class 2 if, and only if, \( B_1 \subseteq Z_1 \).

In particular, \( G' \subseteq Z_1 \) if, and only if, \( B_1 \subseteq Z_1 \).

**Lemma 4.** (a) If \( T_1 \) is finite and if \( Z_1 \) is a torsion group, then \( G/B_1 \) is a torsion group. (b) If \( Z_1 \) is u.t., then \( G/B_1 \) is u.t. and \( \exp G/B_1 | \exp Z_1 \). (c) If \( Z_1 \) is torsion-free, then so is \( G/B_1 \).

**Proof.** (a) For \( x \in G \) and \( \alpha \in T_1 \), \( \alpha(x) = xy_{\alpha}(x) \), where \( y_{\alpha}(x) \in Z_1 \). There exists a least positive integer \( n = n(x; \alpha) \) such that \( y_{\alpha}(x^n) = e \), since \( Z_1 \) is a torsion group. Since \( T_1 \) is finite, we can form \( n(x) \), the least common multiple of all such \( n(x; \alpha) \). For \( \alpha \in T_1 \), \( \alpha(x^{n(x)}) = x^{n(x)} \) so that \( x^{n(x)} \in B_1 \), and \( G/B_1 \) is a torsion group. (b) has a proof which is an obvious modification of the proof of (a). (c) Suppose that \( Z_1 \) is torsion-free and that \( x^n \in B_1 \). Then for \( \alpha \in T_1 \), \( \alpha(x^n) = x^n \). But \( \alpha(x) = x_{\alpha}(x) \), so that \( y_{\alpha}(x^n) = e \). Since \( y_{\alpha}(x) \) is not a periodic
element, \( \gamma_a(x) = e \) and \( x \in B_1 \). Hence \( G/B_1 \) is torsion-free.

**Lemma 5.** If \( G/B_1 \) is complete, then each \( \gamma_a(G) \) is complete; and if, in addition, \( Z_1 \) is torsion-free, \( G/B_1 \) and each \( \gamma_a(G) \) are direct sums of copies of \( R \), the additive group of the rationals.

**Proof.** For \( z \in \gamma_a(G) \), there exist \( x \in G \) and \( \alpha \in T_1 \) with \( \alpha(x) = xz \). Since \( G/B_1 \) is complete, for each positive integer \( n \) there exists \( y \in G \) with \( x \equiv y^n \) mod \( B_1 \). \( \alpha(y^n x^{-1}) = y^n x^{-1} = y^n \gamma_a(y^n) x^{-1} = y^n x^{-1} \gamma_a(y^n) z^{-1} \). Hence \( \gamma_a(y^n) = z \), and \( [\gamma_a(y)]^n = z \). Since \( \gamma_a(G) \) is abelian, this is enough to show that it is complete. If, in addition, \( Z_1 \) is torsion-free, then Lemma 4(c) shows that \( G/B_1 \) is torsion-free. Also each \( \gamma_a(G) \) is torsion-free. But torsion-free, complete abelian groups are direct sums of copies of \( R \).

**4. Automorphisms induced on \( G/B_1 \).** Since \( B_1 \) is admissible under each normal automorphism of \( G \), each such automorphism induces an automorphism on \( G/B_1 \). (See, however, [1] where \( G/G' \) for finite \( G \) is discussed instead.) If \( \alpha \in T_1 \) induces the identity on \( G/B_1 \), then \( \alpha(x) = x \gamma_a(x) \equiv x \mod B_1 \) so that \( \gamma_a(G) \subseteq B_1 \). Conversely, if \( \alpha \in T_1 \) and if \( \gamma_a(G) \subseteq B_1 \), then the induced automorphism \( \alpha' \) has the property \( \alpha'(xB_1) = \alpha(x)B_1 = x \gamma_a(x)B_1 = xB_1 \) for every \( xB_1 \in G/B_1 \). Thus, a necessary and sufficient condition that \( \alpha \in T_1 \) induce \( \iota \) on \( G/B_1 \) is that \( \gamma_a(G) \subseteq B_1 \). Let the set of all such \( \alpha \in T_1 \) be denoted by \( V_1 \). By a well known result [3, p. 78] on automorphisms which leave a normal subgroup \( H \) and the factor group \( G/H \) point-wise fixed, \( V_1 \) is an abelian group under automorphism composition. \( V_1 \) is a normal subgroup of \( T_1 \). For, if \( \alpha \in V_1 \), \( \beta \in T_1 \), then \( \beta^{-1} \alpha \beta(x) = \beta^{-1} \alpha(x \gamma_a(x)) \equiv \beta^{-1} (x \gamma_\beta(x) bc) = xbc \) where \( b, c \in B_1 \), and \( \alpha(x) = xb \), \( \alpha \gamma_a(x) = \gamma_\beta(x) c \). This makes \( V_1 \) normal in \( T_1 \). Moreover, \( \alpha^{-1} \beta^{-1} \alpha \beta(x) = xc \), and we have

**Lemma 6.** If \( \alpha \in V_1 \) and if \( \beta \in T_1 \), then \( \gamma_a(\alpha, \beta) = \gamma_a \gamma_\beta \).

Since \( G/B_1 \) is an abelian group, it has the automorphism \( \omega \) given by \( \omega(y) = y^{-1} \) for every \( y \in G/B_1 \). \( \omega^2 = \iota \), and \( \omega \) is called the **involution automorphism**.

**Lemma 7.** (a) If \( \alpha \in T_1 \) induces the involution automorphism on \( G/B_1 \), then \( \alpha \) induces the involution automorphism on \( \gamma_a(G) \). (b) \( \alpha \in T_1 \) induces \( \omega \) on \( G/B_1 \) if, and only if, \( \gamma_a(\alpha) = \gamma_a(\alpha^{-1}) \) for every \( x \in G \) and for every \( \beta \in T_1 \).

**Proof.** (a) For \( x \in G \), \( \alpha(x) = i \gamma_a(x) \equiv x^{-1} \mod B_1 \) so that \( x^2 \gamma_a(x) \equiv B_1 \) and \( \gamma_a(x^2) \gamma_a(\alpha(x)) = e \). Thus \( \gamma_a(x) \alpha(\gamma_a(x)) = e \), so that \( \alpha(\gamma_a(x)) = \gamma_a(x^{-1}) \), and \( \alpha \) induces \( \omega \) on \( \gamma_a(G) \). (b) \( \alpha \) induces \( \omega \) on \( G/B_1 \) if, and only if, \( x^2 \gamma_a(x) \equiv B_1 \) for every \( x \in G \). For \( \beta \in T_1 \), \( \gamma_\beta(x^2 \gamma_a(x)) = e \) so that \( \gamma_\beta \gamma_a(x) = \gamma_\beta(x^{-2}) \). Conversely, if \( \gamma_\beta \gamma_a(x) = \gamma_\beta(x^{-2}) \) for every \( \beta \in T_1 \), then \( x^2 \gamma_a(x) \equiv B_1 \).

If we let \( W_1 \) be the set of all \( \alpha \in T_1 \) which induce either \( \iota \) or \( \omega \) on \( G/B_1 \), then \( W_1 \) is a group under automorphism composition. Let the set of those elements of \( W_1 \) which are not in \( V_1 \) be denoted by \( W_1^* \). Assume, for the pres-
ent, that this set is nonvoid. It is easy to verify that the elements of $W_1^*$ are carried into elements of $W_1^*$ by the inner automorphisms of the group $T_i$, so that $W_1$ is a normal subgroup of $T_i$. The index $[W_1: V_1] = 2$, and $W_1/V_1 \cong I_2$; for, if $\alpha, \beta \in W_1^*$, then $\alpha^{-1}\beta(x) = \alpha^{-1}(x^{-1}b) = \alpha^{-1}(x^{-1})$, where $b \in B_1$. Since $\alpha(x) = x^{-1}c$ (where $c \in B_1$), $\alpha^{-1}(x^{-1}) = xc^{-1}$, and $\alpha^{-1}\beta(x) \equiv x \mod B_1$ so that $\alpha^{-1}\beta \in V_1$.

**Theorem 1.** (a) For a group $G$, $W_1$ is $j$-nilpotent for a given positive integer $j$, or $Z_j(W_1)$ is included properly in $V_1$. (b) If $W_1^*$ is nonvoid, then $\alpha \in Z_j(W_1) \cap V_1$ if, and only if, $\gamma_\alpha(x^{2j}) = e$ for every $x \in G$ and $\alpha \in V_1$.

**Proof.** If $W_1^*$ is void, then $V_1 = W_1$ and $W_1$ is abelian. Let us therefore assume that $W_1^*$ is nonvoid. First suppose that $j = 1$, and consider $\alpha \in V_1 \cap Z_1(W_1)$. Choose $\beta \in W_1^*$. Then since $\alpha \in Z_1(W_1)$, $(\alpha, \beta) = 1$. Now $\gamma_{(\alpha, \beta)}(x) = \gamma_\alpha \gamma_\beta(x) = \gamma_\alpha(x^{-2})$, by Lemmas 6 and 7(b). Since $\gamma_\alpha(x) = e$ for every $x \in G$, $\gamma_\alpha(x^{-2}) = e$ for every $x \in G$. Conversely, if $\gamma_\alpha(x^{2j}) = e$ for every $x \in G$, then $\gamma_{(\alpha, \beta)}(x) = e$ for every $x \in G$ and for every $\beta \in W_1^*$, by Lemmas 6 and 7(b). But $\gamma_{(\alpha, \beta)}(x) = e$ for every $x \in G$ implies that $(\alpha, \beta) = 1$ for every $\beta \in W_1^*$. Since $V_1$ is abelian, and since $\alpha \in V_1$, $\alpha$ is in $Z_1(W_1)$. We have verified (b) in the case $j = 1$.

Suppose that there exists $\beta \in Z_1(W_1) \cap W_1^*$. Since $[W_1: V_1] = 2$, $\beta \in Z_1(W_1)$ if, and only if, $W_1^* \subset Z_1(W_1)$. If $\alpha \in V_1$, $\beta \in Z_1(W_1) \cap W_1^*$, then $(\alpha, \beta) = 1$ and $\gamma_{(\alpha, \beta)}(x) = e$ for every $x \in G$, by Lemma 6. By Lemma 7(b), $\beta \in W_1^*$ implies $\gamma_\alpha \gamma_\beta(x) = e$ for every $x \in G$. Conversely, if $\gamma_\alpha(x^{2j}) = e$ for every $x \in G$, then $\gamma_{(\alpha, \beta)}(x) = e$ for every $x \in G$ and for every $\beta \in W_1^*$, by Lemmas 6 and 7(b). But $\gamma_{(\alpha, \beta)}(x) = e$ for every $x \in G$ implies that $(\alpha, \beta) = 1$ for every $\beta \in W_1^*$. Since $V_1$ is abelian, and since $\alpha \in V_1$, $\alpha$ is in $Z_1(W_1)$. We have now established (a) for the case $j = 1$.

Now suppose that the theorem holds for the case $j - 1$. If $\beta \in W_1^*$ and if $\alpha \in V_1 \cap Z_{j-1}(W_1)$, $(\alpha, \beta) \in Z_{j-1}(W_1)$. If $x \in G$, $(\alpha, \beta)(x) = x\gamma_\alpha(\gamma_\beta(x)) = x\gamma_\alpha(x^{-2})$. Noting that $(\alpha, \beta) \in V_1$ since $\gamma_\alpha(x^{-2}) \in B_1$, (b) can be applied for the case $j - 1$, and $\gamma_\alpha[(x^{2^{j-1}})^{-2}] = e$, whence $\gamma_\alpha(x^{2^j}) = e$ for every $x \in G$. Conversely, suppose that $\alpha \in V_1$ and that $\gamma_\alpha(x^{2^j}) = e$ for every $x \in G$. Choose $\beta \in W_1^*$. $\gamma_{(\alpha, \beta)}(y) = \gamma_\alpha \gamma_\beta(y) = \gamma_\alpha(y^{-2})$ for every $y \in G$. Let $y = x^{2^{j-1}}$. Then $\gamma_\alpha(y^{-2}) = e$ by assumption, and $\gamma_{(\alpha, \beta)}(x^{2^{j-1}}) = e$ for every $x \in G$. Since $\alpha \in V_1$ implies $\gamma_\alpha(y^{-2}) \in B_1$, $\gamma_{(\alpha, \beta)}(y) \in B_1$ and $(\alpha, \beta) \in V_1$. By (b) for the case $j - 1$, $(\alpha, \beta) \in Z_{j-1}(W_1)$ for every $\beta \in W_1^*$. If $\beta \in V_1$, then the fact that $V_1$ is abelian allows one to conclude that $(\alpha, \beta) = 1 \in Z_{j-1}(W_1)$. Hence $(\alpha, \beta) \in Z_{j-1}(W_1)$ for every $\beta \in W_1$, and $\alpha \in Z_j(W_1)$. This establishes (b) for the case $j$.

Since $[W_1: V_1] = 2$, the elements of $W_1^*$ all have the form $\beta \alpha$ where $\alpha \in V_1$. Suppose now that $\beta \in Z_j(W_1) \cap W_1^*$ and that $\alpha$ and $\delta$ are elements of $V_1$. $\beta \alpha \delta \equiv \beta \alpha \delta \equiv \beta \delta \alpha \equiv \beta \delta \alpha \equiv \beta \delta \alpha \mod Z_{j-1}(W_1)$. Likewise, $\beta \alpha \delta \equiv \delta \alpha \beta \mod Z_{j-1}(W_1)$. Hence if $Z_j(W_1) \cap W_1^*$ is nonvoid, then $W_1^* \subset Z_j(W_1)$. If $\alpha \in V_1$, $\beta \in Z_j(W_1) \cap W_1^*$, then $(\alpha, \beta) \in Z_{j-1}(W_1)$ and $\gamma_{(\alpha, \beta)}(x) = \gamma_\alpha \gamma_\beta(x) = \gamma_\alpha(x^{-2})$. As above,
\[ \alpha \in V_1 \text{ implies } (\alpha, \beta) \in V_1 \text{ so that } (b) \text{ for the case } j - 1 \text{ applies, and } \gamma_\alpha [x^{(2^j - 1)}]^{-2} = e. \] This (b) for the case \( j \) established above, places \( \alpha \in Z_j(W_1) \). \( W_1 = W_1^* \cup V_1 \subset Z_j(W_1) \), and \( W_1 = Z_j(W_1) \).

If \( W_1 \neq Z_j(W_1) \), the above shows that \( Z_j(W_1) \subset V_1 \). If the inclusion is not strict, then \( Z_j(W_1) = V_1 \) and \( I_2 \cong W_1 / V_1 \cong J(W_1 / Z_{j-1}(W_1)) \), an impossibility [2]. This completes the proof of the theorem.

**Corollary 1.** Let \( G \) be a group for which \( W_1^* \) is nonvoid. (a) If \( Z_1 \) is u.t. with exponent dividing \( 2^j \), then \( W_1 \) is nilpotent of class \( \leq j \). (b) If \( Z_1 \) is torsion-free and if \( V_1 \) is nontrivial, then \( W_1 \) is non-nilpotent.

**Proof.** (a) For \( \alpha, \xi \in V_1 \), \( (\alpha, \xi) = 1 \) since \( V_1 \) is abelian. Choose \( \beta \in W_1^* \). Since \([W_1 : V_1] = 2\), \( W_1^* \) is the coset of \( V_1 \) in \( W_1 \) which contains \( \beta \alpha \). For \( x \in G \), \( \alpha(x) = xb \), \( \xi(x) = xd \), and \( \beta(x) = x^{-1}c \), where \( b, d \in Z_1 \cap B_1 \) and \( c \in B_1 \). \( (\beta \alpha, \xi)(x) = \alpha^{-1} \beta^{-1} \xi^{-1} \beta \alpha \xi(x) = \alpha^{-1} \beta^{-1} \xi^{-1} \beta(xbd) = \alpha^{-1} \beta^{-1} \xi^{-1}(x^{-1} c bd) = \alpha^{-1} \beta^{-1} \xi^{-1}(dx^{-1} c bd) \)

\[ \delta_1 \in V_1 \text{ if } \exp Z_1 2^j \text{ then } \delta_1(x^{2^j}) = x^{2^j - 1} (b^{-1} d^{2^j}) = x^{2^j - 1}, \text{ and }\gamma_\xi (x^{2^j}) = e \text{ for every } x \in G. \] By (b) of the theorem, \( \delta_1 \in Z_{j-1}(W_1) \), so that \( W_1 \subset Z_{j-1}(W_1) \), and \( W_1 \) is nilpotent of class \( \leq j \). This establishes (a) of the corollary.

(b) Now suppose that \( Z_1 \) is torsion-free. By hypothesis, we can find \( \alpha \) and \( \xi \in V_1 \) and \( y \in G \) with \( \alpha(y) \neq \xi(y) \), where \( \alpha(y) = yb \) and \( \xi(y) = yd \). Construct \( \delta_1 \) as in part (a) of the corollary. \( \delta_1(x^{2^j}) = x^{2^j - 1} (b^{-1} d^{2^j}) = x^{2^j - 1} \).

**Corollary 2.** Let \( Z_1 \) be torsion-free, \( V_1 \) be nontrivial, \( W_1^* \) be nonvoid and let \( G \) be complete. Then \( W_1 \) has a trivial center.

**Proof.** By Corollary 1(b), \( W_1 \) is not nilpotent. By (a) of the theorem, \( Z_1(W_1) \) is a proper subgroup of \( V_1 \). By (b) of the theorem, \( \alpha \in Z_1(W_1) \) implies \( x^2 \in F(\alpha) \), the set of all fixed points of \( \alpha \), for every \( x \in G \). Since \( G \) is complete, \( G = F(\alpha) \), and \( \alpha = 1 \).

It is fairly obvious that \( \alpha \) and \( \beta \in T_1 \) induce the same automorphism on \( G / B_1 \) if, and only if, \( \alpha \equiv \beta \mod V_1 \); and an equivalent condition is that \( \gamma_\alpha (x) \equiv \gamma_\beta (x) \mod B_1 \) for every \( x \in G \). It follows that if \( \alpha \equiv \beta \mod V_1 \), then there exists an endomorphism \( \lambda_{\alpha, \beta} \) on \( G \) into \( B_1 \cap Z_1 \) such that (1) the kernel of \( \lambda_{\alpha, \beta} \) is just \( F(\alpha^{-1} \beta) = F(\beta^{-1} \alpha) \); (2) \( \gamma_\alpha (x) = \gamma_\beta (x) \lambda_{\alpha, \beta}(x) \); and (3) for \( g \in G \), \( \lambda_{\alpha, \beta}(x) = \beta(x^{-1})g \) has a unique solution \( x = x(g) \in G \). Conversely, if \( \lambda \) is an endomorphism of \( G \) into \( B_1 \cap Z_1 \), if \( \beta \in T_1 \) and if \( \lambda(x) = \beta(x^{-1})g \) has a unique solution \( x = x(g) \) for every \( g \in G \), then the mapping \( \alpha \) defined by \( \alpha(x) = \beta(x) \lambda(x) \) is a normal automorphism of \( G \) such that \( \alpha \equiv \beta \mod V_1 \) and such that \( \lambda = \lambda_{\alpha, \beta} \). We restate as follows:
Lemma 8. If $\beta \in T_1$ and if $\lambda$ is an endomorphism of $G$ into $B_1 \cap Z_1$, then $\beta + \lambda \in T_1$ with $\beta + \lambda \equiv \beta \mod V_1$, if, and only if, $i + \beta - 1 \lambda \in T_1$.

Recall that $\tau_x(y) = x^{-1}yx$.

Lemma 9. (a) $\alpha \in W_1^*$ implies that $\tau_x \alpha^{-1}(x) = \alpha(x)$ for every $x \in G$. (b) $\alpha \in T_1$ and $\alpha^2 = i$ imply that $\alpha$ induces $\omega$ on $\gamma_\alpha(G)$. If, in addition, $\alpha \in Z_1(W_1) \cap V_1$ and if $W_1^*$ is nonvoid, then $\gamma_\alpha(G) \subseteq \ker \gamma_\alpha$.

Proof. (a) is immediate. As for (b), $\alpha \in T_1$ implies that $\alpha(x) = x\gamma_\alpha(x)$ and $\alpha^{-1}(x) = x\alpha^{-1}(\gamma_\alpha(x)) = x\gamma_\alpha^{-1}(x)$. Hence $\alpha^{-1}(\gamma_\alpha(x^{-1})) = \gamma_\alpha^{-1}(x)$, or $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(x^{-1})$, since $\alpha^{-1} = \alpha$, and $\alpha$ induces $\omega$ on $\gamma_\alpha(G)$. From this, $\gamma_\alpha(x)\gamma_\alpha^2(x) = \gamma_\alpha(x^{-1})$, and $\gamma_\alpha(x^2) = \gamma_\alpha^2(x)$. By Theorem 1(b), $\gamma_\alpha(x^2) = e$, so that $\gamma_\alpha^2(x) = e$ and $\gamma_\alpha(G) \subseteq \ker \gamma_\alpha$.

5. The $B$-series. $G/B_1(G)$ is an abelian group so that all of its automorphisms are normal. Define $B_2(G)$ as the complete inverse image in $G$ of $B_1(G/B_1(G))$ under the natural homomorphism of $G$ onto $G/B_1(G)$. In general, suppose that $B_j(G)$ is defined. Then $B_{j+1}/B_j \cong B_1(G/B_j)$. We let $B_0(G) = (e)$. Each $B_j$ is a normal subgroup of $G$, and $i \leq j$ implies that $B_i \subseteq B_j$, so that the $B$-series ascends monotonically in its index. Each $G/B_j$ is abelian ($j > 0$), and $B_{j+1}/B_j$ is the set of elements of $G/B_j$ which are each fixed by all automorphisms of $G/B_j$ ($j > 0$). If $B_{j+1} = B_j$, then for all $k \geq j$, $B_k = B_j$.

Lemma 10. The $B$-series breaks off at $B_1$ if any one of the following holds: (a) $G/B_1$ has no elements of order 2. (b) $Z_1$ is torsion-free, or $Z_1$ has no elements of order 2 or no $\gamma_\alpha(G)$, for $\alpha \in T_1$, has elements of order 2. (c) To each $xB_1$ in $G/B_1$, there exists an automorphism $\theta = \theta_x$, such that $\theta(xB_1) \neq xB_1$. (d) To each $x \in G$, there exists $\alpha = \alpha_x \in A$ such that $\alpha$ induces an automorphism on $G/B_1$, and $\alpha(x) \neq x \mod B_1$. (e) The equation $\xi^2 = \alpha$, for $\alpha \in T_1$, always has a solution in $T_1$.

Proof. (a) Since $G/B_1$ is abelian, it has the involution automorphism $\omega$. If $gB_1 \in B_2/B_1$, then $\omega(gB_1) = gB_1 = g^{-1}B_1$, and $g^2 \in B_1$. Since $G/B_1$ has no elements of order 2, $g \in B_1$ and $B_2 \subseteq B_1$. (b) For $x^2 \in B_1$ and $\alpha \in T_1$, $\alpha(x^2) = x^2 = \gamma_\alpha(x^2)$, and $\gamma_\alpha(x^2) = e$. Since $\gamma_\alpha(G)$ has no elements of order 2, $\gamma_\alpha(x) = e$ and $x \in B_1$. Hence $G/B_1$ has no elements of order 2, and (a) applies. (c) There is no fixed point common to all automorphisms of $G/B_1$, so that $B_2/B_1$ is trivial, and $B_2 = B_1$. (d) $\alpha$ induces $\alpha'$, an automorphism on $G/B_1$. $\alpha'(xB_1) \neq xB_1$ so that (c) can now be applied. (e) If $g \in B_2$, then, as we saw in the proof of (a), $g^2 \in B_1$. For $\beta \in T_1$ there exists an induced automorphism $\beta'$ on $G/B_1$. Since $g \in B_2$, $\beta'(gB_1) = gB_1 = gB_1$. Hence $\beta(g) \equiv g \mod (Z_1 \cap B_1)$. $\beta'(g) = \beta(g\gamma_\beta(g)) = \beta(g)\gamma_\beta(g) = \gamma_\beta(g^2)$, since $\gamma_\beta(g) \in B_1 \cap Z_1$. But $g^2 \in B_1$ implies that $\gamma_\beta(g^2) = e$, so that $\beta^2(g) = g$. Since every $\alpha \in T_1$ is, by hypothesis, a square, $g \in B_1$, and $B_2 \subseteq B_1$.
6. The case $G = B_2$. It is obvious that $G = B_2$ if, and only if, $B_1(G/B_1) = G/B_1$; that is, if, and only if, the identity is the only normal automorphism of $G/B_1$. Since $G/B_1$ is abelian, we see that $G = B_2$ if, and only if, $G/B_1$ has no proper automorphism. But this is equivalent [2; p. 101] to

**Lemma 11.** $G = B_2$ if, and only if $G/B_1 \cong I_2$.

Since $G/B_1 \cong I_2$ in this case, choose $u \in B_2$, $u \in B_1$. Then to each $x \in G$, $x \in B_1$, there exists $b_x \in B_1$ with $x = ub_x$. For $\alpha \in T_1$, $\alpha(x) = x \alpha(u) = x \gamma_\alpha(u)$. Hence $\alpha(x) = x \gamma_\alpha(u)$ if $x \in B_1$, $x \neq x$ if $x \in B_1$. We note that $\gamma_\alpha(u) \in B_1 \cap Z_1$, by the proof of Lemma 10(e). Since $u^2 \in B_1$ (by the proof of Lemma 10(a)), $\gamma_\alpha(u^2) = e$. It is clear that if $\alpha$, $\beta \in T_1$ then $\gamma_\alpha(u) = \gamma_\beta(u) \gamma_\alpha(u)$. $\gamma_\alpha(u) = e$, if, and only if, $\alpha = e$. Moreover, suppose that $c \in Z_1 \cap B_1$ and that $c^2 = e$. Define $\alpha$ by $\alpha(x) = xc$ if $x \in B_1$, $x = x$ if $x \in B_1$, $\alpha(x) = e$, if, and only if, $x = e$. If $y \in B_1$, $\alpha(y) = y$; and if $y \in G$, $y \in B_1$, then $\alpha(y^{-1}) = y^{-1}c = y$. For $x$, $y \in B_1$, $\alpha(xy) = \alpha(x) \alpha(y)$. If $x \in B_1$, $x = ub_x$. $\alpha(x) = ub_x c = xc$. For $y \in B_1$, $\alpha(xy) = xy = \alpha(x) \alpha(y)$. $\alpha(y) = \alpha(y)c$. $\alpha(xy) = \alpha(x) \alpha(y)$, since $c^2 = e$. It is thus seen that $\alpha$ is an automorphism of $G$ and that $\alpha \in T_1 \cap V_1$ (since $\alpha$ induces the identity on $G/Z_1$ and on $G/B_1$). Let $K_1$ be the subgroup of $T_1 \cap V_1$ generated by the elements of order 2 of that group. We have proved

**Theorem 2.** If $G = B_2$, then $T_1$ is an elementary abelian group with exponent 2, and $T_1 = V_1 \cong K_1$.

**Corollary.** If $G = B_2$ and if $\alpha \in T_1$, $\alpha \neq e$, then $\gamma_\alpha(G) \cong I_2$.

**Proof.** By the proof of the theorem, $\ker \gamma_\alpha = B_1$. Apply Lemma 11.

7. Some properties of the $B$-series.

**Lemma 12.** $B_{n+1}(G)/B_1(G) \cong B_n(G/B_1(G))$.

**Proof.** The lemma is valid for $n = 1$. Suppose that it is true for the case $j-1$. Then $B_1((G/B_1)/B_1) \cong (B_1(G/B_1))/B_1(G/B_1)$ since $B_1(B_1(B_1(G/B_1)) = B_2(G/B_1)$, by the induction hypothesis. But $B_1((G/B_1)/B_1) = B_1(G/B_1) \cong B_{j+1}(G/B_1)/B_1(G/B_1)$. Hence $B_{j+1}(G/B_1) \cong B_{j+1}/B_1$.

We say that $G$ is $B$-nilpotent of $B$-class $n$ (or $n$-$B$-nilpotent) if $G = B_n$.

**Corollary.** Suppose that $G$ is not $n$-$B$-nilpotent. The following are equivalent: (a) $G$ is $(n+1)$-$B$-nilpotent. (b) $G/B_1$ is $n$-$B$-nilpotent. (c) $G/B_n \cong I_2$.

**Proof.** The equivalence of (a) and (b) follows from the lemma. $G/B_n \cong I_2$ if, and only if, $B_1(G/B_n) = G/B_n$. But $B_1(G/B_n) \cong B_{n+1}/B_n \cong G/B_n$ if, and only if, $G = B_{n+1}$.

**Lemma 13.** Let $G$ be a group which is $(n+1)$-$B$-nilpotent but not $n$-$B$-nilpotent, and suppose that $Z_{k} \subseteq B_n$ and that $Z_{k-1} \nsubseteq B_n$. Then $G$ is $k$-nilpotent.
Proof. $G = B_{n+1}$ implies that $G/B_n \cong I_2$, by Lemma 12, Corollary. Since $G/Z_k \cong (G/B_n)/(Z_k/B_n)$, $G/Z_k$ must be isomorphic to $I_2$ or $(e)$, the only possible homomorphic images of $G/B_n \cong I_2$. But $G/Z_k \cong J(G/Z_{k-1})$. Since the group of inner automorphisms of a group cannot be a nontrivial cyclic group, $G/Z_k \cong (e)$, and $G = Z_k$.

**Lemma 14.** If $G$ is $n$-$B$-nilpotent where $n \geq 2$, then $\alpha(\gamma_a(x)) = \gamma_a(x^{-1}) \mod B_{n-2}$ for every $\alpha \in T_1$ and for every $x \in G$.

**Proof.** First consider the case $n = 2$. If $G = B_2$, we see, from the discussion before Theorem 2, that $\alpha \in T_1$ implies that $\gamma_a(x) = \gamma_a(u)$ if $x \in B_1$, = e if $x \in B_1$. Here, $u$ is a representative of the non-unity coset of $B_1$ in $G$. Since $\gamma_a(u^2) = e$ and $\gamma_a(u) \in B_1$, we obtain $\gamma_a(x) \alpha(\gamma_a(x)) = \gamma_a(u) \alpha(\gamma_a(u)) = [\gamma_a(u)]^2 = e$ if $x \in B_1$. If $x \in B_1$, the calculation still gives $e$. Recalling that $B_0 = (e)$, we see that the lemma is established for $n = 2$.

Suppose that the lemma holds for the case $n - 1$. Since $G = B_n$, $G/B_1$ is $(n - 1)$-$B$-nilpotent, by Lemma 12. For $\alpha \in T_1$, consider the induced automorphism $\alpha'$ on the abelian group $G/B_1$. By the induction assumption, if $\alpha'(xB_1) = (xB_1)(zB_1)$, then $zB_1 \alpha'(zB_1) \in B_{n-2}(G/B_1)$. Now $\alpha(x) = x \gamma_a(x)$, so that $\alpha'(xB_1) = x \gamma_a(x) B_1$, and $xz \equiv x \gamma_a(x) \mod B_1$. Hence $z \equiv \gamma_a(x) \mod B_1$ so that $zB_1 = \gamma_a(x) B_1$. A substitution shows that $\gamma_a(x) B_1 \alpha'(\gamma_a(x) B_1) = \gamma_a(x) \cdot \alpha(\gamma_a(x)) B_1 \in B_{n-2}(G/B_1)$. But $B_{n-2}(G/B_1) \cong B_{n-2}/B_1$, by Lemma 12. From this we can conclude that $\gamma_a(x) \alpha(\gamma_a(x)) \in B_{n-2}$ for every $\alpha \in T_1$ and for every $x \in G$. The lemma is established.

**Corollary 1.** If $G = B_n$, $n \geq 2$, and if $\alpha \in T_1$, then $\alpha^2(x) \equiv x \mod (Z_1 \cap B_{n-2})$ for every $x \in G$.

**Proof.** $\alpha(x) = x \gamma_a(x)$ implies that $\alpha^2(x) = x \gamma_a(x) \alpha(\gamma_a(x))$. By the lemma, $\gamma_a(x) \alpha(\gamma_a(x)) \in B_{n-2}$.

**Corollary 2.** If $G = B_n$, $n \geq 2$, and if $\alpha \in T_1$ induces $\omega$ on $\gamma_a(G)$ or on $G/B_1$, then $\alpha^2 = \iota$.

**Proof.** If $\alpha$ induces $\omega$ on $\gamma_a(G)$, then $\alpha(\gamma_a(x)) = \gamma_a(x^{-1})$, so that, by the proof of Corollary 1, $\alpha^2(x) = x$ for every $x \in G$. By Lemma 7(a), if $\alpha$ induces $\omega$ on $G/B_1$, then $\alpha$ induces $\omega$ on $\gamma_a(G)$.

**Corollary 3.** Let $M(\alpha)$ be the largest subgroup of $\gamma_a(G)$ on which $\alpha$ induces the involution automorphism. If $G = B_n$, $n \geq 2$, then $\gamma_a \ker \gamma_a^2 = M(\alpha)$, and $\gamma_a^2(G)$ is an $\alpha$-admissible subgroup of $\gamma_a(G) \cap B_{n-2}$.

**Proof.** $x \in \ker \gamma_a^2$ if, and only if, $\gamma_a^2(x) = \gamma_a(x) \alpha(\gamma_a(x)) = e$, that is, equivalently, $\gamma_a(x) = \gamma_a(x^{-1})$. But the latter is equivalent to $\gamma_a(x) \in M(\alpha)$. By Lemma 2, $\gamma_a(G)$ is $\alpha$-admissible, so that, for given $x \in G$, $\alpha(\gamma_a(x)) = \gamma_a(y)$ for a suitable $y = y(x); \alpha$. Then $\gamma_a^2(x) = \gamma_a(x) \gamma_a(y)$, and $\alpha(\gamma_a^2(x)) = \alpha(\gamma_a(x)) \cdot \alpha(\gamma_a(y)) = \gamma_a(y) \alpha \gamma_a(y) = \gamma_a^2(y)$. This shows that $\gamma_a^2(G)$ is $\alpha$-admissible.
Lemma 15. If \( x \in B_n, \ n \geq 2 \), then \( x^{2n-1} \in B_1 \).

**Proof.** The case \( n = 2 \) was treated in the proof of Lemma 10(a). Suppose that the lemma is valid for \( n = j \). If \( x \in B_{j+1}, \ xB_1 \subseteq B_{j+1}/B_1 = B_j(G/B_1) \). By the induction assumption, \( x^{2^{j-1}}B_1 \subseteq B_j(G/B_1) = B_2/B_1 \), and \( x^{2^{j-1}} \in B_2 \). The case \( n = 2 \) now shows that \( [x^{2^{j-1}}]^2 = x^{2^j} \in B_1 \).

**Corollary.** \( B_n/B_1 \) is u.t. abelian with \( \exp (B_n/B_1) | 2^{n-1} \), so that, for an \( n \)-B-nilpotent group \( G \), \( G/B_1 \) is u.t. abelian, and an \( n \)-B-nilpotent group with periodic \( B_1 \) is itself periodic.

Theorem 3. If \( G \) is \( n \)-B-nilpotent, \( n \geq 2 \), then \( W_1 \) is \((n - 1)\)-nilpotent.

**Proof.** If \( W_1^* \) is void, then \( W_1 = V_1 \), an abelian group. Suppose that \( W_1^* \) is nonvoid. \( x \in B_n \) implies \( x^{2n-1} \in B_1 \), by Lemma 15. If \( \alpha \in V_1 \), then \( \gamma_\alpha(x^{2n-1}) = e \) for every \( x \in G \), since \( G = B_n \). By Theorem 1 (b), \( V_1 \subseteq Z_{n-1}(W_1) \). By Theorem 1(a), \( W_1 \) is \((n - 1)\)-nilpotent.

**Corollary.** If \( G \) with torsion-free \( Z_1 \) is \( n \)-B-nilpotent, then \( W_1 = V_1 \), or \( V_1 \) is trivial, and \( W_1 \) is an elementary abelian group with exponent 2.

**Proof.** If \( W_1^* \) is nonvoid, then Theorem 1, Corollary 1(b), and the present theorem show that \( V_1 \) is trivial. Since \( \alpha \in W_1^* \) implies that \( \alpha^2 \in V_1 \), \( W_1 \) is elementary abelian with exponent 2.

Lemma 16. Each \( B_n \) is \( T_1 \)-admissible, and, if \( n \geq 1 \), \( \gamma_\alpha(B_n) \subseteq B_{n-1} \) for every \( \alpha \in T_1 \).

**Proof.** \( B_1 \) is \( T_1 \)-admissible. Suppose that \( B_{n-1}(G) \) is \( T_1 \)-admissible for every group \( G \). \( g \in B_n \) implies that \( gB_{n-1} \subseteq B_1(G/B_{n-1}) \). For \( \alpha \in T_1 \), \( B_{n-1} \) is both \( \alpha \)- and \( \alpha^{-1} \)-admissible (by the induction assumption), and \( \alpha \) induces an automorphism \( \alpha' \) on the abelian group \( G/B_{n-1} \). Since \( gB_{n-1} \subseteq B_1(G/B_{n-1}) \), \( \alpha'(gB_{n-1}) = gB_{n-1} = \alpha(g)B_{n-1} \), and \( \alpha(g) \equiv g \mod B_{n-1} \). Hence \( \gamma_\alpha(B_n) \subseteq B_{n-1} \). Since \( B_{n-1} \subseteq B_n \) and \( g \in B_n, \gamma_\alpha(g) \in B_n \) so that \( B_n \) is \( \alpha \)-admissible.

8. Orbital elements. An element \( x \in G \) is said to be \( n \)-orbital if \( \alpha^n(x) = x \) for every \( \alpha \in T_1 \). Collecting these \( n \)-orbital elements together in a set \( L_n = L_n(G) \), we see that \( L_n \) is a subgroup of \( G \). Since \( T_1 = C(J; A) \), \( L_n \) is normal in \( G \). More generally, \( L_n \) is \( C(T_1; A) \)-admissible. (We shall discuss \( C(T_1; A) \) below.) \( L_1 = B_1 \), and \( m \mid n \) implies \( L_m \subseteq L_n \). Thus \( G' \subseteq B_1 = L_1 \subseteq L_n \) for every positive integer \( n \), and \( G/L_n \) is abelian. From the proof of Lemma 10(e), we see that \( x \in B_2 \) implies \( \alpha^2(x) = x \) for every \( \alpha \in T_1 \), so that \( B_2 \subseteq L_2 \).

For positive integers \( s \geq t \), let \( C(s, t) = s!/t!(s-t)! \). Consider \( C = C(2^{n-1}, r) \), where \( n \geq 2 \) and \( r \leq n - 1 \). If \( r \) is odd,

\[
C = 2^{n-1}s \prod_{k=1}^{(r-1)/2} \left( \frac{2^{n-1} - 2k}{2k} \right)
\]

where \( s \in \mathbb{R} \) is a quotient of odd integers. Let \( k = 2^sd_k \), where \( d_k \) is a non-
negative integer, and $d_k$ is an odd integer. Since $r$ is odd and $\leq 2^{n-1}, r \leq 2^{n-1} - 1$ and $k \leq (r-1)/2 \leq 2^{n-2} - 1$. Thus we have $c_k \leq n-3$, and $(2^{n-1} - 2k)/2k = (2^{n-2} - c_k - d_k)/d_k$, a quotient of odd integers. We have proved that $r$ odd implies that $2^{n-1} \mid C(2^{n-1}, r)$. $C(2^{n-1}, r+1) = [(2^{n-1} - r)/(r+1)]C(2^{n-1}, r)$. For odd $r \geq 5$, the exponent of the highest power of 2 dividing into $r+1$ is $\leq r-2$, so that $2^{n-r+1} \mid C(2^{n-1}, r+1)$, and $2^{n-(r+1)} \mid C(2^{n-1}, r+1)$. If $r = 1$, then $r+1 = 2$, and $2^{n-2} = 2^{n-(r+1)} \mid C(2^{n-1}, r+1)$. If $r = 3$, then $r+1 = 4$, and $2^{n-3} = 2^n \mid C(2^{n-1}, r+1)$. We summarize in

**Lemma 17.** For $n \geq \max (2, r+1), 2^n \mid C(2^{n-1}, r)$.

**Theorem 4.** $B_n \subseteq L_m$, where $m = 2^{n-1}$.

**Proof.** Since the earlier cases have been treated, we assume that $n \geq 3$. Suppose that $g \in B_n$ and that $\alpha \in T_1$. $\alpha^m(g) = \alpha^{m-1}(gb_{n-1})$ where $b_{n-1} \in B_{n-1}$, by Lemma 16. Assume, inductively, that $\alpha^m(g) = \alpha^{m-k}[g \prod_{r=1}^k b_{n-r}^{C(k, r)}]$ where $b_{n-r} \in B_{n-r}$, and $\gamma_r(a) = b_{n-1}$, $\gamma_r(b_r) = b_{r-1}$ ($t = n-k+1, n-k+2, \ldots, n-1$). When $r > n-1$, we take $b_{n-r} = e$. Then

$$\alpha^m(g) = \alpha^{m-(k+1)} \left[ \frac{\sum_{\gamma_r(a) = b_{n-1}}}{gb_{n-1}} \prod_{r=2}^k b_{n-r}^{C(k, r)+C(k, r-1)} \right]_{b_{n-(k+1)}}^{(k+1)}.$$

But $C(k, 1) + 1 = C(k+1, 1), C(k, r) + C(k, r-1) = C(k+1, r)$, and $C(k, k) = 1 = C(k+1, k+1)$. Thus,

$$\alpha^m(g) = \alpha^{m-(k+1)} \left[ g \prod_{r=1}^{k+1} b_{n-r}^{C(k+1, r)} \right],$$

and the induction is complete. Now take $k = m = 2^{n-1}$ and note that $\alpha^0 = e$. Since $b$'s with nonpositive subscript are $e$, we can write $\alpha^m(g) = g \prod_{r=1}^m b_{n-r}^{C(m, r)}$. By Lemma 15, $b_{n-r} \in \gamma_r(B_{n-r+1})$ implies that $b_{n-r} = e$. By Lemma 17, however, $2^n \mid C(m, r)$, so that $\alpha^m(g) = g$.

**Corollary.** If $g \in B_n$, $n \geq 2$, if $\alpha \in T_1$, and if $m = 2^{n-1}$, then $\alpha^{m/2}(g) \equiv g \mod B_1$, and $\gamma^{m/2}(g^2) = e$. In particular, if $G = B_n$, then $\alpha^{m/2} \in V_1$, and $T_1/V_1$ is u.t. with exponent dividing $2^n - 2$.

**Proof.** By Lemma 12, $gb_1 \in B_{n-1}(G/B_1)$. Let $\alpha$ induce $\alpha'$ on $G/B_1$. By the theorem, $\alpha'^{m/2}(gb_1) = gb_1$, and $\alpha^{m/2}(g) \equiv g \mod B_1$; that is, $\alpha^{m/2}(g) = gb$ where $b = \gamma^{m/2}(g) \in B_1$. Also by the theorem, $\alpha^m(g) = g$. But $\alpha(m)(g) = \alpha^{m/2}(\alpha^{m/2}(g)) = \alpha^{m/2}(gb) = gb^2$, and $b^2 = e$.

**Lemma 18.** Let $n$ be an integer $\geq 1$, and let $G$ be a group for which each automorphism of $G/B_n$ can be extended to a normal automorphism of $G$. Then if $\gamma_\alpha(g) \in B_n$ for every $\alpha \in T_1$, $g \in B_{n+1}$.

**Proof.** By hypothesis, $\alpha(g) \equiv g \mod B_n$ for every $\alpha \in T_1$. Let $\alpha$ induce $\alpha'$ on $G/B_n$. $\alpha'(g_{B_n}) = \alpha(g)B_n = gB_n$. Since the set of induced $\alpha'$ coincides with $A(G/B_n) = T_1(G/B_n)$, $gB_n \in B_1(G/B_n) = B_{n+1}/B_n$, and $g \in B_{n+1}$.
9. The centralizer of $T_1$. Since $T_1$ is the centralizer of $J$ in $A$, $U_1 = C(T_1; A) \supseteq J$, where $U_1$ is a normal subgroup of $A$.

Lemma 19. (a) $B_1(G)$ is $U_1$-admissible, and if each automorphism of each $G/B_i$ ($i = 1, 2, 3, \cdots$) can be extended to a normal automorphism of $G$, then each $B_n$, $n \geq 2$, is likewise $U_1$-admissible. (b) $\gamma_a(F(U_i)) \subseteq F(U_i) \subseteq Z_1$ for every $\alpha \in T_1$. (c) Each $\theta \in U_1$ induces an automorphism on each $F(\gamma_\alpha)$, $\alpha \in T_1$.

Proof. (a) For $\theta \in U_1$ and $\alpha \in T_1$, $\theta(a(x)) = \theta(x) \theta(\gamma_\alpha(x)) = \alpha \theta(x) \gamma_\alpha \theta(x)$, so that $\theta \gamma_\alpha = \gamma_\alpha \theta$ for every $\alpha \in T_1$. If $g \in B_1$, then $\gamma_\alpha \theta(g) = \theta \gamma_\alpha(g) = \theta(e) = e$ for $\alpha \in T_1$, and $\theta(g) \in B_1$. Now suppose that $B_n$ is $U_1$-admissible. For $g \in B_{n+1}$, $\gamma_\alpha \theta(g) = \gamma_\alpha \gamma_\alpha(g)$. $\gamma_\alpha(g) \in B_n$, by Lemma 16. By the induction assumption, $\theta \gamma_\alpha(g) \in B_n$. Applying Lemma 18, $\theta(g) \in B_{n+1}$. (b) If $\theta(g) = g$ for every $\theta \in U_1$, then $\gamma_\alpha \theta(g) = \gamma_\alpha(g) = \theta(g)$. Hence $\gamma_\alpha(F(U_i)) \subseteq F(U_i)$ for every $\alpha \in T_1$. If $\theta(g) = g$ for every $\theta \in U_1$, then $\tau_x(g) = g$ for every $x \in G$, since $J \subseteq U_1$. But $\tau_x(g) = g$ for every $x \in G$ implies that $g \in Z_1$. (c) If $g \in F(\gamma_\alpha)$, $\gamma_\alpha(g) = g$, and $\theta \gamma_\alpha(g) = \gamma_\alpha \theta(g) = \theta(g)$, so that $\theta(g) \in F(\gamma_\alpha)$. Conversely, if $\theta(g) \in F(\gamma_\alpha)$, then $\theta \gamma_\alpha(g) = \gamma_\alpha \theta(g) = \theta(g)$. Since $\theta$ is an automorphism, $\gamma_\alpha(g) = g$, and $g \in F(\gamma_\alpha)$.

Theorem 5. Each element of $C(T_1; A)$ induces a normal automorphism on $Z_2$, and there exists a homomorphism on $C(T_1; A)$ into $T_1(Z_2)$ with kernel consisting of all those mappings in $C(T_1; A)$ which reduce to the identity on $Z_2$.

Proof. $\theta \in U_1$ implies that $\theta$ commutes with every Grün automorphism of $G$. If, therefore, $u \in Z_2$, then $\theta(x^{-1} u^{-1} x u) = \theta(x^{-1}) \theta(u^{-1}) \theta(x) \theta(u) = \theta(x^{-1}) u^{-1} \theta(x) u$ for every $x \in G$. $u \theta(u^{-1})$ is, consequently, in the centralizer of every $\theta(x)$, $x \in G$. Since $\theta$ is an automorphism, $u \theta(u^{-1}) \in Z_1(G) \subseteq Z_1(Z_2(G))$, and $\theta(u) \equiv u \mod Z_1(Z_2(G))$, so that $\theta$ restricted to $Z_2$ is normal thereon.

Corollary. If $G$ is of class 2, then $J \subseteq C(T_1; A) \subseteq T_1$, and $C(T_1; A) = Z_1(T_1)$.

10. The higher normal automorphisms. If $\alpha \in A$ has the property $\alpha(x) \equiv x \mod Z_n$ for every $x \in G$, we say that $\alpha$ is an $n$-normal automorphism, and we have described the higher normal automorphisms of $G$. Let $T_n$ be the set of $n$-normal automorphisms of $G$. Under automorphism composition, $T_n$ is a normal subgroup of $G$, and $m \leq n$ implies that $T_m \subseteq T_n$.

Theorem 6. (a) $T_n/T_{n-1}$ is isomorphic to a subgroup of $T_1(G/Z_{n-1})$. (b) $T_n/T_1$ is isomorphic to a subgroup of $T_n(J)$.

Proof. (a) $\alpha \in T_n$ induces an automorphism $\alpha'$ on $G/Z_{n-1}$. For every $x \in G$, $xZ_{n-1} \subseteq G/Z_{n-1}$, and $\alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xzZ_{n-1}$ where $z \in Z_n$. Then $zZ_{n-1} \subseteq Z_1(G/Z_{n-1})$, so that $\alpha'$ is normal on $G/Z_{n-1}$. It is not difficult to see that if $\alpha, \beta \in T_n$, then $(\alpha \beta)' = \alpha' \beta'$, so that $(\cdot)'$ is a homomorphism on $T_n$ into $T_1(G/Z_{n-1})$. Suppose that $\alpha' = \iota$. Then $\alpha'(xZ_{n-1}) = xZ_{n-1}$ for every $x \in G$, and
\[ \alpha(x) \equiv x \mod Z_{n-1}. \] Hence \( \alpha \) induces the identity on \( G/Z_{n-1} \), and \( \alpha \in T_{n-1} \). Conversely, if \( \alpha \in T_{n-1}, \ \alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xZ_{n-1}, \) and then \( \alpha' = \iota \) on \( G/Z_{n-1} \). Therefore, \( \text{ker} \ (') = T_{n-1} \). (b) \( \alpha \in T_{n+1} \) induces an automorphism \( \alpha'' \) on \( G/Z_i \cong J \), given by \( \alpha''(xZ_i) = \alpha(x)Z_i = x\gamma(x)Z_i \), where \( \gamma(x) \in Z_{n+1} \). Hence \( \alpha''(xZ_i) = xZ_1 \mod (Z_{n+1}/Z_1) \). Now \( Z_n(J) \cong Z_n(G/Z_1) \cong Z_{n+1}/Z_1 \), as an induction will show. Hence \( \alpha'' \) is, effectively, in \( T_n(J) \). \( \alpha \) induces \( \iota \) if, and only if, \( \alpha(x) \equiv x \mod Z_1 \) for every \( x \in G \), and \( \text{ker} \ (''') = P_i \).

**Corollary 1.** Let \( \alpha' \) be a nontrivial normal automorphism of \( G/Z_n \) which can be extended to a higher normal automorphism \( \alpha \) of \( G \). Then \( \alpha \in T_n \).

**Corollary 2.** Each \( \alpha \in T_n \) induces a homomorphism of \( G \) and an endomorphism of \( G/Z_{n-1} \) into \( Z_n/Z_{n-1} = Z_1(G/Z_{n-1}) \).

**Proof.** The endomorphism is \( \gamma \gamma' \), and the homomorphism is obtained by following the natural mapping \( \phi_{n-1} \) of \( G \) onto \( G/Z_{n-1} \) by \( \gamma \gamma' \). Moreover, \( \gamma \phi_{n-1}(x) = x^{-1} \gamma(x)Z_{n-1} \) for every \( x \in G \).

Let \( \mathcal{S} \) be a set of automorphisms of \( G \) and let \( N(\mathcal{S}) \) be the set of all \( g \in G \) such that \( \alpha(g) \equiv g \mod Z_1 \) for every \( \alpha \in \mathcal{S} \).

**Lemma 20.** If \( K \) is a subgroup of \( A \), then \( C(K; A) \cap J = J(N(K); G) \). In particular, \( Z_1(A) \cap J = J(N(A); G) \).

**Proof.** If \( \tau \alpha = \alpha \tau \) for every \( \alpha \in K \), then \( g^{-1} \alpha(x)g = \alpha(g^{-1})\alpha(x)\alpha(g) \) for every \( x \in G \), so that \( g\alpha(g^{-1}) \) is in the centralizer of every \( \alpha(x) \). Since \( \alpha \) is an automorphism, \( g\alpha(g^{-1}) \in Z_1 \), and \( \alpha(g) \equiv g \mod Z_1 \), so that \( g \in N(K) \) and \( \tau \in J(N(K); G) \). The proof can be read in reverse to obtain the converse.

**Lemma 21.** The following are equivalent: (a) \( J \subset Z_1(A) \). (b) \( A = T_1 \). Either of these conditions implies that \( G \) is of class 2.

**Proof.** \( A = T_1 \) if, and only if, \( G = N(A) \). But if the latter holds, \( J(N(A); G) = J \); and conversely, if \( J(N(A); G) = J \), \( x \in G \) implies the existence of \( y \in N(A) \) with \( \tau_x = \tau_y \). Then \( x \equiv y \mod Z_1 \), so that, if \( \alpha \in A, \ \alpha(x) \equiv \alpha(y) \equiv y \equiv x \mod Z_1 \). This shows that \( x \in N(A) \) and that \( N(A) = G \). By Lemma 20, \( J \cap Z_1(A) = J(N(A); G) = J \), so that \( J \subset Z_1(A) \), and (a) implies (b). A slight rearrangement of the above argument shows that (b) implies (a). Now if every automorphism of \( G \) is a normal automorphism, \( x^{-1}yx \equiv y \mod Z_1 \) for every \( x, y \in G \). This implies that \( G' \subset Z_1 \), and \( G \) is of class 2.

A similar result is contained in

**Theorem 7.** Let \( G \) be a group with the properties (1) \( J \subset T_n \) and (2) each \( \alpha \in A \) induces \( \iota \) on each \( Z_{j+1}/Z_j \) \( (j = 1, 2, \ldots, n) \). Then \( J \subset Z_n(A) \).

**Proof.** First, we establish three lemmas:

(R) For a group \( G \), \( J \subset T_n \) if, and only if, \( G \) is of class \( n+1 \).

(S) For a group \( G \), \( J \cap Z_n(A) \subset J(Z_{n+1}; G) \).
A group $G$ with property (2) has the further property that $J \cap Z_n(A) = J(Z_{n+1}; G)$ (for the $n$ of property (2)).

To prove (R), use the proof of the last statement of Lemma 21 as a model. As for (S), take $n = 0$. Then $J \cap Z_n(A)$ consists of a single element, and the inclusion is trivially valid. Suppose that it holds for $n = k$. $\tau_\varphi \in J \cap Z_{k+1}(A)$ implies that $(\alpha, \tau_\varphi) \in Z_k(A)$ for every $\alpha \in A$. A brief computation shows that $(\alpha, \tau_\varphi) = \tau_h$, where $h = g \alpha^{-1}(g^{-1})$. By the induction assumption, $g \alpha^{-1}(g^{-1}) \in Z_{k+1}$, and this is to be valid for every $\alpha \in A$. If we take $\alpha = \tau_x$, $x \in G$, then $g x g^{-1} x^{-1} \in Z_{k+1}$ for every $x \in G$, and $g \in Z_{k+2}$. But this means that $J \cap Z_{k+1}(A) \subseteq J(Z_{k+2}; G)$.

To prove (T), let $\{\alpha_i\}$ ($i = 1, 2, \ldots, n$) be any finite set of elements of $A$. For a fixed $g \in G$, define $g_1 = g \alpha_1^{-1}(g^{-1})$. If $g_k$ is defined, let $g_{k+1} = g \alpha_{k+1}^{-1}(g_k^{-1})$. A different finite set of elements of $A$, or even the same set in a different order, may very well lead to a different finite sequence $\{g_i\}$ on $g$. Let $G_i(g) = G_i$ be the set of all $g_i$ obtained in this fashion for fixed $g$ and fixed positive integer $i$. By Lemma 20, $\tau_\varphi \in Z_k(A)$ if, and only if, $g \in N(A)$. But $g \in N(A)$ if, and only if, $\alpha(g) \equiv g \mod Z_1$ for every $\alpha \in A$. The latter condition is equivalent to $G_1 \subseteq Z_1$. Now suppose that $\tau_\varphi \in Z_k(A)$ if, and only if, $g_k \subseteq Z_1$. $\tau_\varphi \in Z_{k+1}(A)$ if, and only if, $J(G_i(g); G) \subseteq Z_k(A)$. By the induction assumption, this is equivalent to $G_k(h) \subseteq Z_1$ for every $h \in G_i(g)$. Since $\cup G_i(h) = G_{k+1}(g)$, where the set union is taken over all $h \in G_i(g)$, $\tau_\varphi \in Z_{k+1}(A)$ if, and only if, $G_{k+1}(g) \subseteq Z_1$.

Now suppose that $\tau_\varphi \in J(Z_{n+1}; G)$. Then $g \in Z_{n+1}$ and $G_1(g) \subseteq Z_n$, since each $\alpha \in A$ induces the identity on $Z_{n+1}/Z_n$. Assume, inductively, that $G_k(g) \subseteq Z_{k+1}/Z_k$. Since each member of $A$ induces the identity on $Z_{n-k+1}/Z_{n-k}$, $G_{k+1}(g) \subseteq Z_{n-k}$. In particular, $G_n(g) \subseteq Z_1$. By the above, $J(Z_{n+1}; G) \subseteq Z_n(A)$. Along with (S), this is enough to establish (T).

To prove the theorem, note that $J \subseteq T_n$ implies, by (R), that $G$ is of class $n+1$. Therefore, in (T), replace $Z_{n+1}$ by $G$. Since $J(G; G) = J$, the theorem is proved.

For a subgroup $K$ of $A$, it is clear that $F(K) \subseteq N(K)$, that $Z_1 \subseteq N(K)$, and that $N(K)$ is $K$-admissible. We prove a preliminary result on $Q(H; G)$ for a characteristic subgroup $H$ of $G$.

**Lemma 22.** Let $H$ be a characteristic subgroup of $G$. (a) $Q(H; G) \cap J = J(H \div G; G)$. (b) $\alpha$ and $\beta$ induce the same automorphism on $G/H$ if, and only if, $\alpha \equiv \beta \mod Q(H; G)$. In particular, $\tau_x \equiv \tau_y$ if, and only if, $x \equiv y \mod H \div G$.

**Proof.** (a) $\tau_\varphi \in Q(H; G)$ if, and only if, $\tau_\varphi(x) \equiv x \mod H$ for every $x \in G$. But this latter condition is equivalent to $(g, x^{-1}) \in H$ for every $x \in G$, and this is true if, and only if, $g \in H \div G$. (b) is obvious.

**Lemma 23.** (a) $T_n \div J = T_{n+1}$. (b) $(T_n, J) \subseteq J(Z_n; G)$. (c) $T_n \cap J = J(Z_{n+1}; G)$.

**Proof.** (a) and (b) can be established by routine arguments. To verify (c),
replace $H$ by $Z_n$ in Lemma 22(a), and note that $Q(Z_n; G) = T_n$ and that $Z_n \div G = Z_{n+1}$.

**Corollary.** (a) For a subgroup $K$ of $A$ and for a positive integer $n$, $N(K) = Z_n$ if, and only if, $C(K; A) \cap J = T_{n-1} \cap J$. (b) $N(K) = Z_1$ if, and only if, $C(K; A) \cap J$ is trivial. (c) If $G$ is $n$-nilpotent, and if $K$ is a subgroup of $A$, then $K \subset T_1$ if, and only if, $C(K; A) \cap J = T_{n-1} \cap J$.

**Proof.** (a) If $N(K) = Z_n(G)$, then $J(Z_n; G) = J(N(K); G) = C(K; A) \cap J$, by Lemma 20. Since $J(Z_n; G) = T_{n-1} \cap J$ (by (c) of the lemma), half the statement is established. Conversely, suppose that $C(K; A) \cap J = T_{n-1} \cap J$. One can readily check the equivalence of the following statements: (1) $x \in Z_n$. (2) $\tau_x \alpha = \alpha \tau_x$ for every $\alpha \in K$. (3) $\tau_x \alpha \tau_x^{-1}(y) = \alpha \tau_x \alpha^{-1}(y)$ for every $y \in G$, $\alpha \in K$. (4) $x^{-1}yx = \alpha(x^{-1})y\alpha(x)$ for every $y \in G$ and every $\alpha \in K$. (5) $\alpha(x) x^{-1} \in Z_1$ for every $\alpha \in K$. (6) $x \in N(K)$. (b) follows from (a) by taking $n = 1$. (c) $K \subset T_1$ if, and only if, $N(K) = G$. Since $G = Z_n$, (a) is applicable.

**Theorem 8.** $G^{(n)} \subset F(T_n) \subset C(Z_n; G)$.

**Proof.** If $\alpha \in T_n$, then $\alpha(x^{-1}y^{-1}xy) = t^{-1}x^{-1}u^{-1}y^{-1}xtyu$ where $t, u \in Z_n$. Hence $\alpha(x^{-1}y^{-1}xy) \equiv x^{-1}y^{-1}xy \mod Z_{n-1}$, and $\alpha$ induces the identity on $G'(Z_{n-1} \cap G')$. Suppose, inductively, that $\alpha \in T_n$ induces the identity on $G^{(1)}/(Z_{n-1} \cap G^{(1)})$. A set of generators of $G^{(k+1)}$ is all $(x, y)$ where $x, y \in G^{(k)}$. $\alpha(x^{-1}y^{-1}xy) = t^{-1}x^{-1}u^{-1}y^{-1}xtyu$, where $t, u \in Z_{n-k}$. Hence $\alpha(x^{-1}y^{-1}xy) \equiv x^{-1}y^{-1}xy \mod Z_{n-k}$; and our induction shows that $\alpha \in T_n$ induces the identity on each $G^{(k)}/(Z_{n-k} \cap G^{(k)})$. Now take $k = n$ so that $Z_{n-k} = (e)$. That is, each $\alpha \in T_n$ induces the identity on $G^{(n)}$, whence $G^{(n)} \subset F(T_n)$. By the discussion after Lemma 3, $F(T_n) \subset C(Z_n; G)$.

**Corollary 1.** $G^{(n)} \subset F(J(Z_{n+1}; G))$.

**Proof.** $T_n \supset J(Z_{n+1}; G)$, by Lemma 23(c).

**Corollary 2.** If $F(T_n) = (e)$ or if $F(J(Z_{n+1}; G)) = (e)$ for some positive integer $n$, then $G$ is solvable [3].

**Corollary 3.** $J(G^{(n)}; G) \subset C(T_{n+1}; A)$.

**Proof.** $\alpha \in T_{n+1}$, and $g \in G^{(n)}$ imply that $\alpha(g) \equiv g \mod Z_1$, by the proof of the theorem. Hence $J(G^{(n)}; G) \subset J(N(T_{n+1}); G) = C(T_{n+1}; A) \cap J$, by Lemma 20.

If we let $U_n = C(T_n; A)$, then, by Lemma 20, $J(N(T_n); G) \subset U_n$. As in Lemma 19(a), $F(T_n)$ is $U_n$-admissible.

11. **Examples.** (A) For positive integers $n > 2$, let $D_n$ denote the $n$th dihedral group, the group of isometries of a regular $n$-gon. $D_n$ is the semi-direct product of $I_n$ and of $I_2$ with the multiplication rules $(x_n, 0_2)(y_n, z_2) = (x_n + y_n, z_2)$ and $(x_n, 1_2)(y_n, z_2) = (x_n - y_n, 1_2 + z_2)$. For $n > 2$, there is a non-trivial element in the center if, and only if, $n$ is even; and in this case, the
center consists of two elements, \((0_n, 0_2)\) and \((h_n, 0_2)\), where \(h\) is an integer such that \(2h = n\). Since \(D_n\) is a group with two generators, there are three non-trivial possibilities for central endomorphisms. The verification of the following results is easy: \(T_1(D_{4k})\) is isomorphic to the Klein four group, \(I_2 \oplus I_2\). Let us denote the four group by \(\mathcal{B}\). \(B_1(D_{4k})\) consists of all \((x_{4k}, 0_2)\) where \(x\) is even, so that \(B_1(D_{4k}) \cong I_{2k}\). Likewise, \(B_1(D_{4k}) = D_{4k}'\), and, in fact, \(D_{4k}/B_1(D_{4k}) \cong \mathcal{B}\). It follows that the \(B\)-series breaks off at \(B_1(D_{4k})\). \(T_1(D_{4k+2}) \cong I_2\), \(B_1(D_{4k+2})\) consists of all \((x_{4k+2}, 0_2)\), so that \(B_1(D_{4k+2}) \cong I_{2k}\). \(D_{4k+2}/B_1(D_{4k+2}) \cong I_2\) so that, by Lemma 11, \(D_{4k+2}\) is 2-B-nilpotent. If \(n = 4\), then \(D_4/Z_1(D_4)\) is isomorphic to \(\mathcal{B}\) whence \(D_4\) is of class 2. Then \(T_2(D_4) = A(D_4)\), and it can be readily verified that \(A(D_4) \cong D_4\) and that \(F(T_2(D_4)) = B_1(D_4) = Z_1(D_4)\).

(B) Let \(G\) be a group of type \((2^n)\). \(G\) is isomorphic to the additive group, modulo 1, of the rationals \(k/2^n\), where \(k\) is an odd integer or 0. Since \(G\) is abelian, \(T_1(G) = A(G)\). \(G\) has a nontrivial automorphism \(\alpha(k/2^n) = 1 - (k/2^n)\) corresponding to the conjugation automorphism on the representation of \(G\) on the unit circle. The only fixed points are 1 = 0 and 1/2. Conversely, if \(\beta\) is any automorphism of \(G\), \(2\beta(1/2) - \beta(1) = 1 = 0\), so that \(\beta(1/2) = 1/2\) or 0. Thus \(B_1 \cong I_2\). Since \(G/B_1 \cong G\), \(B_2\) consists of 0, 1/4, 1/2, and 3/4, and \(B_2 \cong I_4\). In general, \(B_n \cong I_{2^n}\). \(G = UB_n\) where the union is taken over all positive integral values of \(n\).

(C) Let \(G\) be the multiplicative group of all nonsingular 2 by 2 matrices over the field of rationals, \(R\). It is well known that \(Z_i\) consists of all
\[
\begin{pmatrix}
u & 0 \\
0 & \nu
\end{pmatrix},
\]
and that \(Z_2 = Z_1\). By Lemma 1, we have an example of a group for which \(T_1 \cap J\) is trivial. Let \(\mu\) be an endomorphism of the multiplicative group of nonzero rationals \(R^*\) where \(x\mu(x^2a) = 1\) has a unique solution \(x = \mu(a; \mu)\) for every \(a \in R^*\). Let \(d_1\) and \(d_2\) be integers with the restriction \(|d_i| = 1\). Define a mapping \(\alpha = \alpha(\mu; d_1, d_2)\) on \(G\) by
\[
\begin{align*}
\alpha \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} r\mu(r) & 0 \\ 0 & \mu(r) \end{pmatrix} \quad \text{for every } r \in R^*; \\
\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & d_1 \\ d_1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_2 & d_2 \\ 0 & d_2 \end{pmatrix}.
\end{align*}
\]
Then it is possible to prove that \(\alpha\) is a normal automorphism of \(G\), and each normal automorphism of \(G\) is such an \(\alpha(\mu; d_1, d_2)\). A matrix \(M \in G\) is in \(B_1\) if, and only if, it can be factored (without regard to the order of the factors) into a product of an even number of factors
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
an even number of factors

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and a set of factors

$$\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \quad (i = 1, 2, \ldots, n),$$

where $a_1a_2 \cdots a_n = 1$. $V_1$ turns out to be the set of all $\alpha(\mu; d_1, d_2)$ with $\mu(x) = \pm 1$ for every $x \in R^*$. $W_1^*$ consists of all $\alpha$ with $\mu(x) = \pm (1/x)$, and $[T_1: V_1]$ is equal to the number of normal subgroups of index 2 in $G$ which contain

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$W_1$ for this group is abelian. Now $W_1^*$ is nonvoid, $V_1$ is nontrivial and $Z_1$ has the periodic element

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, this example shows that we cannot drop the hypothesis of aperiodicity for $Z_1$ in Theorem 1, Corollary 1(b). $G/B_1 \cong G \oplus R^*$ whence $B_2 = B_1$. Let $\mu$ be an endomorphism of $R^*$ such that, to each positive prime $p$, there exists a positive prime $q$ with $q|\mu(p)| = 1 = q|\mu(q)|$. Then $\alpha(\mu; d_1, d_2) \in T_1$, so that, for this group, $T_1$ is far from trivial and $T_1 \neq W_1$. Negatively, one can show, for instance, that if $\mu$ is an endomorphism of $R^*$ for which $|\mu(p)|$ is always a product of $k$ positive primes (or a product of the reciprocals of $k+1$ positive primes) for every positive prime $p$, then the corresponding $\alpha$ is not an automorphism.

(D) Let $G$ be a group with generators $a$, $b$, and $c$, where $a^2 = e$, $ab = ba$, $ac = ca$, and $bc = cba$. Then every element of $G$ can be written uniquely as a product $a^ib^jc^k$ where $i$ is 0 or 1, and $j$ and $k$ range over the integers. $Z_1 \cong I_2$ and $G/Z_1 \cong R \oplus R$ so that $G$ is nilpotent of class 2. One can verify that $T_1 \cong G$. An element is in $B_1$ if, and only if, $j$ and $k$ are both even. Under any automorphism $\alpha$ each center element, $a^i$, is fixed. There is an automorphism $\beta$ which changes the sign of $j$ in each term. Its set of fixed points is precisely all elements with $j = 0$. There is an automorphism $\delta$ which changes the sign of $k$ in each term, and the corresponding fixed points are all elements with $k = 0$. The cross-cut of these two sets of fixed points is $Z_1$, so that $Z_1 = F(A) = F(T_2)$, and this latter set is included in $F(T_1) = B_1$ properly. (In the example of $D_4$ above, $F(T_1) = F(T_2)$ for the class 2 group $D_4$.)
Presumably, by extending the group of this example or by considering $n$ by $n$ triangular matrices with a diagonal of unities, one could exhibit groups with significant $T_n$, for $n > 2$.

**BIBLIOGRAPHY**


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