SUMMATION OF BOUNDED DIVERGENT SEQUENCES, TOPOLOGICAL METHODS

BY

ALBERT WILANSKY AND KARL ZELLER

1. **Introduction.** We treat matrix transformations $t=As$, where $t=\{t_n\}$, $s=\{s_n\}$, $A=(a_{nk})$, and $t_n=\sum_{k=1}^{\infty} a_{nk}s_k$, $n=1, 2, \cdots$; c, m are the spaces of convergent and bounded sequences, respectively; c_A, the *convergence domain* of A, is the space of sequences s (said to be *summed* by A) whose A transform $t=As$ is defined and convergent; m_A, the *boundedness domain* of A, is the space of sequences whose A transform is defined and bounded. We shall always assume that A is conservative, i.e. $c\subset c_A$ and hence $m\subset m_A$. In the usual way, we give m_A a metric which makes it a locally convex complete metric space, i.e. an (F) space, and which is such that c_A is a closed subspace. In the case that A is reversible this is the norm topology given by Mazur [2], the norm being called p_0, below. This suggests an investigation of the topological properties of c as a subspace of c_A, and m as a subspace of m_A. An interesting possibility is that c is dense in c_A. Such matrices have important properties in the theory of generalized limits. See Mazur [2]. We shall, at the opposite extreme, be interested in matrices A such that c is closed in c_A, extending and completing the results of Wilansky [2], and Zeller [1]. We shall show that this condition is equivalent to the condition that m is closed in m_A, and also to the condition that A sums no bounded divergent sequences. The first results in this direction are due to Wilansky [2]. As a special case we obtain a result of Tropper [1].

The second part of this paper is devoted to the construction of matrices A such that c_A contains no bounded divergent sequences, and contains certain preassigned unbounded sequences, being in the case of a finite number of sequences, the smallest linear space including c and containing these sequences. This answers a question of Lorentz [1], and yields, as special cases, results of Agnew [3], Mazur [1], Darevsky [1], Tolba [1], Petersen [1], and Zeller [4].

The facts about linear topological spaces which we shall use are as follows. Two comparable F-metrics are equivalent; see Banach [1, Theorem 6, p. 41]. Given an (F) space with topology defined by semi-norms p_n, a semi-norm q, defined on a subspace, is discontinuous if and only if given $\epsilon>0$ and positive integer b, there exists x such that $q(x)=1$, $p_k(x)<\epsilon$ for $k=1, 2, \cdots, b$; see Zeller [2], Bourbaki [1], compare Banach [1, Theorem 1, p. 54].

2. **Topologies.** Let $\|s\|=\sup_n|s_n|$. With this norm, c and m are Banach

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spaces. We next define semi-norms of three types such that c_A and m_A are (F) spaces with the set of all these semi-norms as shown in Zeller [2]. For convenience, they are arranged as a single sequence.

For $s=\{s_n\}$, and $r=1, 2, \cdots$, set

$$p_0(s) = \sup_n \left| \sum_k a_{nk}s_k \right|,$$

$$p_{2r-1}(s) = \sup_m \left| \sum_{k=1}^m a_{rk}s_k \right|,$$

$$p_{2r}(s) = |s_r|.$$

Define $s^n \to s$ as $p_i(s^n - s) \to 0$ for each i.

Notice that the norm topology on m is stronger (finer) than the F-topology since, for each r, $p_r(s) \leqq M\|s\|$ for some M, since A is conservative.

If A is reversible, or if A is row-finite and 1-1, we may omit all the semi-norms except p_0, when we obtain the classical case in which c_A is a Banach space equivalent with a closed subspace of c with the norm topology.

3. **Closure of c and m.** If A is conservative, we may define $a_k = \lim_n a_{nk}$ for each k, and conclude that $\sum |a_k| < \infty$. It is also true that $\sum_k |a_{nk}|$ is bounded; it is clearly no restriction to assume

$$\|A\| = \sup_n \sum_k |a_{nk}| = 1, \tag{1}$$

and we shall assume this in the present section.

THEOREM 1. *Let A be conservative. The following three conditions are equivalent*:

(a) *c is closed in c_A,*

(b) *m is closed in m_A,*

(c) *A sums no bounded divergent sequences.*

We conjecture that these conditions are also equivalent to

(d) *Every sequence in m_A is $u+v$, where $u \in c_A$, $v \in m$, i.e. $m_A = c_A + m$.*
This condition is easily seen to hold for the matrices given in Theorem 3.

In the proof we shall use the fact that if m is not closed in m_A, the norm topology is strictly stronger than the F-topology.

(b)$\to$(a). If m is closed, the F-topology is the same as the norm topology on m, hence on c. Thus c is closed.

(a)$\to$(b). Suppose that m is not closed. Also not closed, since it is of finite deficiency in m, is the subspace consisting of those bounded sequences s such that $s_i = 0$ for $i < K$, where K is any positive integer. Thus, given $\epsilon > 0$, and integers b, K, there exists $s \in m$ with

$$s_k = 0 \qquad \text{for } k < K, \tag{2}$$

$$\sup |s_n| = 1, \tag{3}$$

$$p_k(s) < \epsilon \qquad \text{for } k < b. \tag{4}$$

The proof will be completed when a convergent sequence is exhibited which has properties (3) and (4), for given ϵ, b. This sequence will be $y = \{\theta_k s_k\}$ where $\{s_k\}$ is a certain bounded sequence, and $\{\theta_k\}$ is a null sequence whose construction we now give.

Let $\epsilon > 0$, and a positive integer b be given. Choose K such that

$$\sum_K^\infty |a_k| < \epsilon. \tag{5}$$

Choose s satisfying (2), (3), (4) with this K and the given ϵ, b. Choose nondecreasing sequences of positive integers $\{u_n\}$, $\{v_n\}$, tending to infinity, with $u_n < v_n$ for all n and such that, for all n,

$$\sum_{k=1}^{u_n - 1} |a_{nk} - a_k| < \epsilon, \tag{6}$$

$$\sum_{k=v_n+1}^{\infty} |a_{nk}| < \epsilon. \tag{7}$$

It follows from (5) and (6) that

$$\sum_{k=K}^{u_n - 1} |a_{nk}| < 2\epsilon. \tag{8}$$

Let $\{\theta_n\}$ be given as follows: $1 \geqq \theta_n \geqq \theta_{n+1} \geqq 0$ for all n; $\theta_1 = \theta_2 = \cdots = 1$, where sufficiently many terms are taken that $\sup_n |\theta_n s_n| > 1/2$; θ_n tends monotonely to 0 in such a way that

$$\theta_{u_n} - \theta_{v_n} < \epsilon. \tag{9}$$

This can be done, for example, by letting $\{w_n\}$ be an increasing sequence of positive integers such that at most one of its terms lies between u_n and v_n for each n, then choosing $\{\theta_n\}$ such that $\theta_{w_n} - \theta_{w_{n+1}} < \epsilon/2$. Compare Agnew [2], Zeller [3, p. 140].

With $y = \{\theta_n s_n\}$, we complete the proof by showing that $p_k(y) < 5\epsilon$ for $k < b$. (Clearly, a trivial modification of y will satisfy (3), (4).)

We have, for each n,

$$\begin{aligned} \sum_k a_{nk} s_k \theta_k = \sum_{k=K}^{\infty} &= \theta_{v_n} \sum_{k=K}^{\infty} a_{nk} s_k + \sum_{k=K}^{u_n - 1} + \sum_{k=u_n}^{v_n} + \sum_{k=v_n+1}^{\infty} a_{nk} s_k (\theta_k - \theta_{v_n}) \\ &= t_1 + t_2 + t_3 + t_4, \text{ say.} \end{aligned}$$

Now $|t_1| \leqq p_0(s) < \epsilon$ by (4); $|t_2| \leqq 2\epsilon$ by (8); $|t_3| \leqq [\theta_{u_n} - \theta_{v_n}] \sum_k |a_{nk}| < \epsilon$ by (1), (3), and (9); $|t_4| < \epsilon$ by (7).

Hence $p_0(y) < 5\epsilon$. Next, for $2n < b$, $p_{2n}(y) = \theta_n p_{2n}(s) < \epsilon$.

Finally, for each m, n, $\left| \sum_{k=1}^{m} a_{nk}s_k\theta_k \right| \leqq \sup_{r\leqq m} \left| \sum_{k=1}^{r} a_{nk}s_k \right|$ by Abel's inequality. Hence $p_{2n-1}(y) \leqq p_{2n-1}(s) < \epsilon$, for $2n-1 < b$.

(a)→(c). Suppose that c is closed. It follows that A is *coregular*, i.e. that $\rho = \lim_n \sum_k a_{nk} - \sum a_k \neq 0$; for if A is *conull* ($\rho = 0$), it follows from Zeller [2, Theorem, 5.2] that $f(1, 1, 1, \cdots = 0$ for any continuous linear functional f such that, for each k, $f(0, 0, \cdots, 0, 1)$ (kth place), $0, 0, \cdots) = 0$. This is false for the norm topology on c. Let s be a bounded sequence, $s \in c_A$. We shall show that s is convergent. Consideration of the sequence (differing from s by a constant sequence) whose nth term is $s_n + \rho^{-1}(\sum a_k s_k - \lim_n \sum_k a_{nk}s_k)$ shows that we may assume that the A-limit of s is $\sum a_k s_k$. Then $s \in c$, since, as we shall now show, $s \in \bar{c}$, where the closure is taken in the F-topology.

As in the proof that $(a) \to (b)$, we can find a sequence $\{\theta_n\}$ decreasing monotonely to zero sufficiently slowly that $y = \{\theta_n s_n\}$ approximates s. More precisely, given $\epsilon > 0$ and a positive integer b, there exists a null sequence $\{\theta_n\}$ with $p_k(y-s) < \epsilon$ for $k < b$. We omit the details which are similar to those of the earlier construction.

Two other proofs that (a)→(c) may be based on the form of the continuous linear functionals on c_A. Such a functional f has the form

$$f(s) = t \lim_n \sum_k a_{nk}s_k + \sum t_k s_k, \qquad \sum |t_k| < \infty$$

for bounded sequences $s \in c_A$. See Zeller [2]. If f vanishes on c, $t = t_k = 0$ for all k, hence $f(s) = 0$ for each bounded s and so $\bar{c} \supset c_A \cap m$ for every coregular A. This proves the result. Also one can easily check that for a sequence s which is summed to $\sum a_k s_k$, the sequence $s^r = \{s_1, s_2, \cdots, s_r, 0, 0, 0, \cdots\}$ tends weakly to s as $r \to \infty$. Since for a linear subspace (indeed, for a convex set) the weak and strong closures coincide, this provides a third proof of the result.

(c)→(a). Assume that c is not closed in c_A.

Case I. A is coregular. We assume that $\rho = 1$, trivial modifications yield the result for general $\rho \neq 0$. Given $\epsilon > 0$, and integers b, K, there exists $s \in c$ satisfying (2), (3), and (4). (See the reasoning given for these.)

The A-limit of s is, in absolute value, less than ϵ since $p_0(s) < \epsilon$. But, since s is convergent, its A-limit is $\lim s_n + \sum_{k=K}^{\infty} a_k s_k$. Hence, for K satisfying (5), $|\lim s_n| < 2\epsilon$. Since $s_n = 0$ for small n it is clear that s has a "hump," i.e. a finite interval of positive integers such that for n not in this interval $|s_n| < 2\epsilon$, and $|s_n| = 1$ for some n in the interval.

We now let ϵ be, successively, 2^{-r-3}, $r = 1, 2, \cdots$, and choose, correspondingly, s^r satisfying (3), (4) with $b = r$. We can further arrange that the humps of these sequences do not overlap, and that there are infinitely many positive integers not in any hump. This can be done inductively by having s^r satisfy

(2) with $K = K_r$ so choosen that (5) holds and also $K_r - 1$ is larger than all the integers in the hump of s^{r-1}.

Let $s = \sum s^r$. This series converges in c_A because of (4). Any positive integer n lies in at most one hump, hence $|s_n| \leqq 1 + 2\sum 2^{-r-3}$, and so s is bounded. Finally s is divergent: first suppose that n belongs to no hump, then $|s_n| \leqq 2\sum 2^{-r-3} = 1/4$; next for each r, we can find an n (in the hump of s^r) such that $|s_n^r| = 1$, then $|s_n| \geqq 1 - 2\sum 2^{-r-3} = 3/4$.

Case II. A is conull. The sequence $\{0, 0, \cdots, 0, s_k, s_{k+1}, \cdots\}$, with s_k increasing slowly to 1 and decreasing to 0, satisfies (2), (3), (4) for sufficiently large K and r. Moreover it has a hump and so the technique of Case I yields the result.

A more precise remark is that all sufficiently slowly oscillating sequences are summable by A. See Zeller [3].

We see that a conull matrix never satisfies (a), (b), (c).

As a corollary we obtain below an extension of a theorem of Tropper [1]. Her remark, pp. 671–672, that we need consider only normal matrices is not clear; for example if A has an inverse with bounded columns, the inverse of the normal matrix which is the same as A for bounded sequences might possibly not have bounded columns.

A *reversible* matrix A is one such that $y = Ax$ holds for one and only one x, for each $y \in c$. We refer in the next result to the (right inverse) matrix B given by Banach [1, p. 50].

Corollary. *Let A be reversible, conservative, and (strictly) stronger than convergence. Suppose that B has bounded columns. Then A sums a bounded divergent sequence.*

We first give a lemma. For some of the terminology see Banach [1, p. 90]. The following ideas are standard: Call $\{t_n\}$ orthogonal to A if $\sum |t_n| < \infty$ and $\sum_n t_n a_{nk} = 0$ for each k. Call A *of type M* if only 0 is orthogonal to A.

Lemma 1. *Let A be reversible, conservative, and suppose that B has bounded columns. Then A is of type M.*

We have, for each r, $0 = \sum_k (\sum_n t_n a_{nk}) b_{kr} = \sum_n t_n \sum_k a_{nk} b_{kr} = t_r$. This proves the lemma. M. S. MacPhail informs us that this lemma appears in a forthcoming article in vol. 6 of the Canadian Journal of Mathematics.

Since a conull matrix sums bounded divergent sequences, we may suppose that A is coregular. Then c is dense in c_A (Banach [1, Lemma 1, p. 91], Wilansky [1, Theorem 3.2.1 (d)]). By Theorem 1, the corollary follows.

4. **Convergence domains without bounded divergent sequences.** The first part of the next result is given in Agnew [3, Theorem 5.3] for regular matrices.

Theorem 2. *Let A be a conservative matrix such that $L = \liminf_{n\to\infty} (|a_{nn}| - \sum_{k \neq n} |a_{nk}|) > 0$. Then A sums no bounded divergent sequence. Suppose,* *further, that* $|a_{nn}|-\sum_{k\neq n}|a_{nk}|>0$ *for each* n. *Then* c_A (*resp.* m_A) *is the smallest linear space including* c (*resp.* m) *and the set of all sequences* s *such that* $As=0$.

We can not omit the additional hypothesis in the second part of the theorem. The restriction is, however, not serious since a matrix obtained by changing a finite number of terms in A has the same convergence domain as A.

Suppose first that s is a bounded sequence which is summable by A. We shall show that s is convergent. We may assume that each s_k is a limit point of s (add a suitable null sequence). We may also assume that the A-limit of s is 0, for we may subtract from s the constant sequence, each of whose terms is $\lim_n \sum_k a_{nk}s_k/\lim_n \sum_k a_{nk}$. (That the denominator of this fraction is not 0 follows from the fact that $L>0$.)

Thus $t=As$ is a null sequence and $\sup|s_k|=\limsup|s_k|=l$, say. But, for each n, $|t_n|\geqq|a_{nn}s_n|-|\sum_{k\neq n}a_{nk}s_k|\geqq|a_{nn}s_n|-l\sum_{k\neq n}|a_{nk}|$. If we now let $n\rightarrow\infty$ through a sequence of values such that $|s_n|\rightarrow l$ we obtain $0\geqq lL$, hence $l=0$, and so $s=0$. This completes the proof that A sums no bounded divergent sequences.

Before proceeding we give a lemma.

LEMMA 2. *Under all the hypotheses of Theorem* 2, *given a bounded sequence* t, *there exists exactly one bounded sequence* s *with* $t=As$. *If* t *is convergent, this* s *will be convergent.*

The second statement is trivial since A sums no bounded divergent sequences. To prove the first, let D be a diagonal matrix with $d_{nn}=a_{nn}$. Let $B=I-D^{-1}A$, where I is the identity matrix. Then $\|B\|\equiv\sup_n\sum_k|b_{nk}|=\sup_n\sum_{k\neq n}|a_{nk}/a_{nn}|<1$. Hence the series $\sum_{k=0}^{\infty}B^k$ converges to a matrix C with $\|C\|<\infty$, in the sense that $\|C-\sum_{k=0}^{m}B^k\|\rightarrow 0$ as $m\rightarrow\infty$, and $CD^{-1}=A^{-1}$. (Compare Hille [1, Theorem 5.2.1, p. 92]; Cooke [1, p. 30]).

Now, given a bounded sequence t, set $s=A^{-1}t$. The result follows, associativity of multiplication being guaranteed by the hypotheses. This concludes the proof of Lemma 2.

Returning to Theorem 2, suppose $y\in m_A$. Set $t=Ay\in m$. Use Lemma 2 to obtain a bounded s such that $t=As$. Then $A(y-s)=0$. Similarly, if $y\in c_A$, we obtain a convergent s with $A(y-s)=0$. This completes the proof of Theorem 2.

The procedure given in the proof of Lemma 2 is familiar in the solution of a finite set of linear equations. Similar considerations have been applied to infinite systems by Agnew [1; 4], Love [1], Rado [1], and especially Parameswaran [1]. Instead of the infinite series, the iteration process $x^{n+1}=s+x^n-Ax^n$, $x^0=s$, has been used. This is an equivalent process.

Theorem 2 has such applications as the following:

(a) Mazur [1, p. 604] defines $a_{nn}=\theta_n$, $a_{n,n-1}=1$, $a_{nk}=0$ otherwise, where

$\theta_n \to 0$, $0 < \theta_n < 1$, and shows that c_A contains only one divergent sequence (modulo c). By removing the first row of A we obtain the same result from Theorem 2.

(b) Petersen [1, p. 75] showed the same for A given by $a_{nn} = (n+1)/(n+2)$, $a_{n,n+1} = 1/(n+2)$, $a_{nk} = 0$ otherwise. Again this follows directly from Theorem 2. See also Wilansky [2, p. 739], where the same result is obtained by consideration of the continuous linear functionals.

(c) A reversible conservative matrix with $L > 0$ sums no divergent sequences. (This generalizes Theorem 1.3 of Agnew [3], in two directions.)

THEOREM 3. *Let the sequences $s^1, s^2, \cdots$ be linearly independent modulo bounded sequences (i.e. no finite nontrivial linear combination is bounded), and let r be a positive integer.*

Then there exist regular row-finite matrices A and B, such that

(i) *c_A (resp. m_A) is the smallest linear sequence space including c (resp. m) and $s^1, s^2, \cdots, s^r$;*

(ii) *c_B contains $s^1, s^2, \cdots$, but no bounded divergent sequence.*

This answers a question of Lorentz [1].

We actually prove a little more. Given ϵ, $0 < \epsilon < 1$, we shall, in each case, choose A so that $a_{nn} = 1$, $\sum_{k \neq n} |a_{nk}| < \epsilon$ for each n. It is this condition which, by Theorem 2, ensures that A sums no bounded divergent sequences.

Proof of (i). Suppose first that $r = 1$. (In this case the result is essentially known; Mazur [1], Darevsky [1], Zeller [4].) We denote s^1 by s. Each row of the matrix A about to be constructed will contain one or two nonzero entries. For each n, let $a_{nn} = 1$; $a_{n,k(n)} = -s_n/s_{k(n)}$ where $\{k(n)\}$ is a strictly increasing sequence of indices so chosen that for each n, $k(n) > n$, $s_{k(n)} \neq 0$, $|s_n/s_{k(n)}| < \epsilon$, and $s_{k(n)}/s_{k(n+1)} \to 0$ as $n \to \infty$; $a_{nk} = 0$ otherwise.

The result now follows from Theorem 2.

For convenience let us denote the above matrix by $A(s; \epsilon)$.

For $r = 2$, the desired matrix is A_2A_1, where A_2, A_1 are chosen as follows. Let $A_1 = A(s^1; \epsilon/4)$. Then t, the A_1 transform of s^2, is unbounded since m_{A_1} is the smallest linear space including m and s^1. Let $A_2 = A(t; \epsilon/4)$. Then $A = A_2A_1$ satisfies all the conditions of the theorem, including the condition mentioned after the statement of the theorem. For example, let $x \in c_A$. Then $A_1x \in c_{A_2}$, hence $A_1x = u + \lambda t = u + \lambda A_1 s^2$, where λ is a number and $u \in c$. Thus $x - \lambda s^2 \in c_{A_1}$, hence $x - \lambda s^2 = v + \mu s^1$ where μ is a number and $v \in c$.

For general r, let $A_1 = A(s^1; 2^{-r}\epsilon)$, $A_2 = A(A_1 s^2; 2^{-r}\epsilon), \cdots$, and $A = A_r \cdots A_1$. This concludes the proof of (i).

Let us denote this matrix by $A(s^1, s^2, \cdots, s^r; \epsilon)$.

Proof of (ii). For each r, let $A_r = A(s^1, s^2, \cdots, s^r; \epsilon/r)$.

Let A be the matrix whose nth row is the nth row of A_n, for each n. Since $a_{nn} = 1$, $\sum_{k \neq n} |a_{nk}| < \epsilon/n$, and $a_{nk} = 0$ for $k < n$, A is regular and sums no bounded divergent sequence. (Here a very simple direct proof, not using

Theorem 2, is available. The matrix A', obtained by omitting the main diagonal of A, sums every bounded sequence. Hence if x is bounded and Ax is convergent, then x is convergent since $x_n = \sum a_{nk}x_k - \sum_{k\neq n} a_{nk}x_k$.)

For $n > r$, the nth term of the A transform of s^r is 0. Hence A sums each s^r.

The construction of reversible matrices with these properties is more troublesome.

In part (ii) we are unable to state exactly what sequences lie in c_A and m_A, except that there are others besides finite linear combinations of the s^r and convergent sequences. Since these spaces are complete they cannot have countable dimension over their closed subspaces c and m, respectively. In fact, even if c is not closed, so that c_A contains bounded divergent sequences, $c_A \cap m$ is not separable in the norm topology so that the same result holds.

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LEHIGH UNIVERSITY,
BETHLEHEM, PA.
UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.