NOTE ON THE FOURIER INVERSION FORMULA ON GROUPS(1)

BY

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1. Let G be a locally compact separable topological group with a two-sided invariant Haar-measure. Let $\mathfrak{L}_\alpha(G)$ be the space of equivalence classes of complex valued Haar-Lebesgue measurable functions $f(g)$ which satisfy $\int_G |f(g)|^\alpha dg < \infty$, where dg refers to the Haar-measure on G. Then the following is known [2; 3; 10; 12]: There exists a measure space Y, for each $y \in Y$ a Hilbert space or finite-dimensional vector space H_y over the complex numbers, and a weakly closed self-adjoint algebra $W(y)$ of bounded operators on H_y such that $W(y)$ is for every y a factor of type I or II in the sense of [6]. For every $\gamma \in G$ let the unitary operator $R(\gamma)$ on $\mathfrak{L}_2(G)$ be defined by

$$[R(\gamma)f](g) = f(g\gamma) \text{ for } f \in \mathfrak{L}_2(G). \tag{1.1}$$

Let W denote the weakly closed self-adjoint algebra of bounded operators on $\mathfrak{L}_2(G)$ generated by the operators $R(g)$ and denote by Z the center of W. Then $\mathfrak{L}_2(G)$ is the direct integral of the spaces H_y in the sense of [10] to which the center Z "belongs" in the sense of [10] (compare also [2; 3; 12]). We write as usual

$$\mathfrak{L}_2(G) = \int_{\oplus} H_y. \tag{1.2}$$

Using §1 of [3] we see that there exists for each $y \in Y$ a strongly continuous unitary representation

$$g \to V(g, y) \tag{1.3}$$

of G by unitary operators $V(g, y)$ of the space H_y such that the operators $V(g, y)$ generate for fixed y and g varying over all of G the algebra $W(y)$. For $f(g) \in \mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$ the integral

$$\int_G f(g)V(g, y)dg = F(y) \tag{1.4}$$

defines for each fixed $y \in Y$ a bounded operator $F(y)$ which is an element of $W(y)$. *This operator-valued function $F(y)$ is the generalized Fourier-transform of $f(g)$.* It follows from §1 of [3] that if we denote the bounded operator $\int_G f(g)R(g)dg$ acting on $\mathfrak{L}_2(G)$ by F, then F decomposes into $F(y)$ under the

Presented to the Society, May 4, 1953; received by the editors January 25, 1954.

(1) This paper was written in 1952.

direct integral (1.2). This leads to the following more general definition of the generalized Fourier-transform (cf. also [12] in this connection). Let $f(g)$ be any function $\in \mathfrak{L}_2(G)$ and assume for the moment $f(g)=\bar{f}(g^{-1})$. It follows from [0] and [12] that $f(g)$ determines a hypermaximal self-adjoint operator F acting (by means of convolution by $f(g)$ on a certain dense subspace of $\mathfrak{L}_2(G)$). To this F we apply §4 of [3] and obtain an operator-valued function $F(y)$ such that $F(y)$ is for almost every y a hypermaximal self-adjoint operator on H_y. If $f(g)$ is arbitrary $\in \mathfrak{L}_2(G)$ we can write $f(g)=a(g)+ib(g)$ where $a(g)=\bar{a}(g^{-1})$ and $b(g)=\bar{b}(g^{-1})$. It is clear that the operator F defined by means of convolution by $f(g)$ still has a domain of definition which is dense in $\mathfrak{L}_2(G)$ and that this domain contains $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. Since the adjoint F^* of F is easily seen to be an extension of the operator defined by convolution by $\bar{f}(g^{-1})$, it follows that the domain of F^* also contains $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. Hence the operators $(F+F^*)/2$ and $(F-F^*)/2i$ are defined on a dense subspace of $\mathfrak{L}_2(G)$. And the hypermaximal self-adjoint operators A, B which correspond to $a(g)$, $b(g)$, respectively, are seen to be extensions of $(F+F^*)/2$, $(F-F^*)/2i$ respectively. Thus they have domains whose intersection is still a dense linear subspace of $\mathfrak{L}_2(G)$, since it also contains $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. Similarly it now follows from this that the intersection of the domains of the self-adjoint operators $A(y)$ and $B(y)$ is dense in H_y for almost every y. Put $F(y)=A(y)+iB(y)$. Then $F(y)$ is a densely defined operator on H_y for almost every y. *We take this as the definition of the generalized Fourier-transform* $F(y)$ defined now for any $f(g)\in\mathfrak{L}_2(G)$.

$$\mathfrak{T}f = F(y) = A(y) + iB(y). \tag{1.5}$$

Thus the generalized Fourier-transformation $\mathfrak{T}$ is now defined for all $f\in\mathfrak{L}_2(G)$ and coincides with (1.4) for almost all y, if $f\in\mathfrak{L}_1\cap\mathfrak{L}_2$. And $\mathfrak{T}f$ is a linear combination of two hypermaximal self-adjoint operator-valued functions $A(y)$ and $B(y)$ which have a dense common domain for almost every y.

On the algebra $W(y)$ there exists a relative trace which we denote by t_y; thus t_y is for fixed $y\in Y$ a complex-valued linear functional defined on a certain linear subset of $W(y)$ and satisfying the formal properties of the trace of a matrix; (compare Chapter I of [8], especially Theorem I). The relative trace t_y is unique up to constant multiples. We take a fixed normalization of t_y for each y; then the measure on Y can be taken to be such that the following generalized Peter-Weyl-Plancherel formula [2; 3; 12] holds:

$$\int_G f_1(g)\bar{f}_2(g)dg = \int_Y t_y[F_1(y)F_2(y)^*]dy, \tag{1.6}$$

for $f_j(g)\in\mathfrak{L}_2(G)$, with $F_j(y)$ defined by (1.5); here * denotes the adjoint of an operator and dy refers to the above measure on Y suitably normalized. If both $f_1(g)$ and $f_2(g)$ are such that $F_1=\int f_1(g)R(g)dg$ and $F_2=\int f_2(g)R(g)dg$ are bounded operators on $\mathfrak{L}_2(G)$, then $F_j(y)$ is a bounded operator on H_y for

almost every y. And it is part of the assertion of the generalized Plancherel formula that $t_y[F_1(y)F_2(y)^*]$ has meaning for almost every y. If $f_1(g)$ and $f_2(g)$ are arbitrary elements of $\mathfrak{L}_2(G)$, then $F_j(y)$ as defined by (1.5) need not be a bounded operator for any y. But one can then find elements $f_{n,j}(g)$ of $\mathfrak{L}_2(G)$ which converge to $f_j(g)$ such that $F_{n,j}(y)$ is a bounded operator for each n, j, and y and such that $\lim_n t_y[F_{n,j}(y)F_{n,2}(y)^*]$ exists for almost every y. One way of defining $t_y[F_1(y)F_2(y)^*]$ for such $F_j(y)$ is to put it equal to this limit and to observe that this definition is independent of the particular approximating sequence chosen. Then (1.6) holds for all $f_j(g)\in\mathfrak{L}_2(G)$ (cf. [3; 8; 12]).

Given for the moment an arbitrary direct integral of Hilbert-spaces H_y (not necessarily the central decomposition (1.2) above). Relative to it a notion of measurability has been defined in [10] for arbitrary operator-valued functions $A(y)$ defined for all $y\in Y$ and such that for each y the value $A(y)$ is a bounded operator on H_y. This may be summarized as follows. It is possible to introduce in H_y a complete orthonormal set

$$\phi_1(y),\ \phi_2(y),\ \cdots,\ \phi_n(y),\ \cdots \tag{1.7}$$

in such a manner that $A(y)$ is an operator-valued measurable function of y if and only if the inner products $\langle A(y)\phi_m(y),\ \phi_n(y)\rangle$ taken in each H_y form for each fixed pair m, n a numerical measurable function of y; (if for a given y, m or n are greater than the dimension of H_y this numerical function is defined to be zero).

Now let us return to the central decomposition (1.2) and consider those measurable operator valued functions $A(y)$, $B(y)$, $\cdots$ for which the values $A(y)$, $B(y)$, $\cdots$ are for each y elements of $W(y)$ and for which $t_y[A(y)A(y)^*]$ exists for almost every y in the sense of definition 1.4.5 of [8] and such that

$$\|\mathfrak{A}\|^2 = \int_Y t_y[A(y)A(y)^*]dy \tag{1.8}$$

exists and is finite. It follows that

$$\langle\mathfrak{A},\ \mathfrak{B}\rangle = \int_Y t_y[A(y)B(y)^*]dy \tag{1.9}$$

exists and defines an inner product $\langle\mathfrak{A},\ \mathfrak{B}\rangle$ between the two functions $A(y)$, $B(y)$ which satisfies the usual formal properties of an inner product. In particular if one identifies two such functions if and only if they differ on a subset of Y of measure zero, *then one obtains a possibly incomplete Hilbert space which we denote by* $\mathfrak{X}$. And we obtain from equations (1.5), (1.6), and (1.9) above

$$\langle f_1,\ f_2\rangle = \langle\mathfrak{T}f_1,\ \mathfrak{T}f_2\rangle. \tag{1.10}$$

2. In this section let $U:g\to U(g)$ be an arbitrary weakly measurable (and hence strongly continuous) unitary representation of any locally compact separable group G by means of unitary operators $U(g)$ of a separable Hilbert space H. In this section the Haar-measure need not be two-sided invariant. Let us say then that such expressions as Haar-measure, $\mathfrak{L}_1(G)$, $\int f(g)dg$ refer to the right invariant Haar-measure on G throughout this paragraph. Suppose now that

$$H = \int_{\oplus} H_t \tag{2.1}$$

is a direct integral decomposition of H under which $U(g)$ decomposes for each $g\in G$. It was shown in Theorem 1.1 of [3] that one obtains for almost every t a continuous unitary representation $V_t:g\to V_t(g)$ of G in H_t. We wish to improve this result by showing that the sets N_t of G which occur in Theorem 1.1 of [3] can be taken to be empty.

LEMMA 2.1. *Given a direct integral* (2.1) *under which* $U(g)$ *decomposes for each* $g\in G$ *into an operator-valued function* $U(g, t)$:

$$U(g) \sim U(g, t). \tag{2.2}$$

Then there exists in the space H_t *a continuous unitary representation* $V_t:g\to V_t(g)$ *of* G *such that*

(i) $U(g)\sim V_t(g)$, *i.e. for each fixed* g *we have* $U(g, t) = V_t(g)$ *for almost all* t.

(ii) $V_t(g)$ *is a weakly product measurable operator-valued function.*

(iii) V_t *coincides with the unitary representation* V_t *of Theorem* 1.1 *of* [3].

(iv) *The sets* N_t *of Theorem* 1.1 *of* [3] *can be taken to be empty.*

Proof. In accordance with Theorem 1.1 of [3] there exists a continuous unitary representation $V_t:g\to V_t(g)$ with representation space H_t such that if we define for $f(g)\in\mathfrak{L}_1(G)$ an operator-valued function $F(t)$ by

$$F(t) = \int_G f(g)V_t(g)dg \tag{2.3}$$

then $F(t)$ is measurable in t. Hence if Γ is any Haar-measurable subset of G of finite Haar-measure then

$$F(\Gamma, t) = \int_\Gamma V_t(g)dg \tag{2.4}$$

defines a measurable operator-valued function $F(\Gamma, t)$ of t (for fixed Γ). Let the subscripts m, n denote matrix coefficients with respect to the above complete orthonormal system (1.7). Then $F(\Gamma, t)_{mn}$ is for each fixed pair m, n and fixed set Γ a complex-valued measurable function of t. And $V_t(g)_{mn}$ is complex-valued continuous in g for every t, m, and n. Taking the integral

in (2.4) in the weak sense (which is sufficient for our purpose) we obtain from (2.4)

$$F(\Gamma, t)_{mn} = \int_\Gamma V_t(g)_{mn} dg.$$

Now let Γ_j be a descending sequence of compact open subsets of G which converge to a given group element g. Then we obtain

$$\lim_j \frac{F(\Gamma_j, t)_{m,n}}{\mu(\Gamma_j)} = V_t(g)_{mn}$$

for every fixed t, m, n, and g; here $\mu(\Gamma_j)$ denotes the Haar-measure of Γ_j. Since $F(\Gamma_j, t)_{mn}$ is measurable in t for every j, m, and n, it follows that $V_t(g)_{mn}$ is measurable in t for every fixed g, m, and n. Hence $V_t(g)$ is a measurable unitary operator-valued function of t for every fixed $g \in G$.

Hence we may apply §13 of [10] and obtain for each $g \in G$ an operator $V(g)$ acting on H which decomposes into $V_t(g)$:

$$V(g) \sim V_t(g). \tag{2.5}$$

And for any $a, b \in H$ with $a \sim a(t)$ and $b \sim b(t)$ we have

$$\langle V(g)a, b\rangle = \int \langle V_t(g)a(t), b(t)\rangle dt$$

where on the left the inner product $\langle\ ,\ \rangle$ is taken in the space H, on the right in the various spaces H_t. The integrand $\langle V_t(g)a(t), b(t)\rangle$ is a continuous function of g and of course integrable in t. It follows by a familiar argument that $\langle V(g)a, b\rangle$ is a measurable function of g for fixed a and b. Thus $V(g)$ is weakly measurable in g. Replacing in (2.5) g by $g'g^{-1}$ gives $V(g'g^{-1}) \sim V_t(g'g^{-1}) = V_t(g')\,V_t(g^{-1})$; but $V(g')\,V(g^{-1}) \sim V_t(g')\,V_t(g^{-1})$ for any two elements g and g' of G. Therefore $V(g'g^{-1}) = V(g')\,V(g^{-1}) = V(g')\,V(g)^{-1}$; hence $g \to V(g)$ is a weakly measurable and hence strongly continuous unitary representation of G.

We can now apply the argument on p. 531 of [3] to $V_t(g)_{mn}$ and conclude that $V_t(g)_{mn}$ is the partial derivative of a product measurable function and thus itself product measurable. Moreover in accordance with Theorem 1.1 of [3] there exists for each t a subset N_t of G of Haar-measure zero such that $g \notin N_t$ implies $V_t(g) = U(g, t)$. It follows that if $f(g) \in \mathfrak{L}_1(G)$ then

$$\int_G f(g) V_t(g) dg = \int_G f(g) U(g, t) dg. \tag{2.6}$$

Let us now use the fact (derived in the proofs of Theorem 1.1 and Lemma 1.1 of [3]) that the operator $\int f(g)\,U(g)dg$ decomposes under the given direct integral (2.1) into $\int f(g)\,U(g, t)dg$ and the operator $\int f(g)\,V(g)dg$ into $\int f(g)\,V_t(g)dg$.

Hence it follows from [10] that (2.5) implies $\int f(g)\,U(g)dg = \int f(g)\,V(g)dg$ for each $f(g)\in\mathfrak{L}_1(G)$. But both $U(g)$ and $V(g)$ are strongly continuous in g, therefore $U(g)=V(g)$ for every $g\in G$. Since $V(g)\sim V_t(g)$ by construction of $V(g)$, this proves assertion (i) of Lemma 2.1. The product measurability of $V_t(g)$ follows from p. 531 of [3] as has been observed above. This establishes (ii). Assertion (iii) follows simply from the fact that we began the proof by taking the representation $V_t: g\to V_t(g)$ to be that of Theorem 1.1 of [3]. Finally (iv) follows immediately from (i) and (iii) together with the basic properties of operator-valued functions established in [10]. This completes the proof of Lemma 2.1.

REMARK 1. The above implies a strengthening of Theorem 2.1 of [3] on the integral representation of positive definite functions in terms of elementary positive definite functions: The elementary positive definite functions can be taken to be continuous in g and at the same time product-measurable.

REMARK 2. Lemma 2.1 above contains and generalizes part of the assertion of [4]. But it still seems of interest to have at one's disposal a proof of Lemma 2.1 for the special case of Lie groups which differs completely from the argument given here.

REMARK 3. Results closely related to Lemma 2.1 above have been obtained by different methods by R. Godement [1] and I. E. Segal [13].

REMARK 4. This was written in 1952. In the meantime the following paper has appeared which overlaps this paragraph: *A remark on Mautner's decomposition*, Kodai Math. Sem. Rep. (1952) p. 107.

3. We leave now the general direct integral considered in §2 and return to the central decomposition (1.2) of $\mathfrak{L}_2(G)$. If $A\in W$, then A decomposes under the direct integral (1.2) into an operator-valued measurable function $A(y)$ with values in the $W(y)$. We denote this again by

$$A\sim A(y). \tag{3.1}$$

In §7 of [3] a certain subset J of W was defined by the following condition: If $A\in W$ then $A\in J$ if and only if there is a function $a(g)\in\mathfrak{L}_2(G)$ such that

$$Af = a*f \qquad \text{for all } f\in\mathfrak{L}_2(G) \tag{3.2}$$

where $a*f$ denotes convolution. If was shown in Lemma 7.1 of [3] that J is a two-sided ideal of W.

LEMMA 3.1. *Let $P(y)$ be for each y a projection of H_y which is an element of $W(y)$ such that $t_y[P(y)]$ exists for each y, i.e. is finite. If $P(y)$ is a measurable function of y, then $t_y[P(y)]$ is a (numerical) measurable function of y.*

If moreover $\int t_y[P(y)]dy<\infty$, then there exists a projection P on $\mathfrak{L}_2(G)$ such that $P\in J$ and $P\sim P(y)$.

Proof. Since G is unimodular and separable, $W(y)$ is either a factor of

type I or II as has been shown in [3] and [12]. It follows that the set $\overline{N}$ introduced in §21 of [10] can be taken to be the whole space Y. It then follows from §23 and §24 of [10] that with a suitable normalization of the relative dimension function $\dim_y$ on $W(y)$ the expression $\dim_y [P(y)]$ is a numerical measurable function. Moreover our function $t_y[P(y)]$ is then of the form

$$t_y[P(y) = \mathfrak{a}(y)\cdot\dim_y [P(y)] \tag{3.3}$$

where $\mathfrak{a}(y)$ is a positive real-valued function of y, independent of P, and $\mathfrak{a}(y) < \infty$ (cf. §24 of [10] and Lemma 4.2 of [5]). Since the set $\overline{N}$ of [10] is now the whole measure space Y, §24 of [10] implies that $\mathfrak{a}(y)$ is measurable on Y.

Since $P(y)$ is a projection $\in W(y)$ and depends measurably on y, there exists an element P of W such that $P^2 = P = P^* \sim P(y)$. It was shown in the course of the argument of §7 of [3] and explicitly stated in Theorem 2 of [12] that $P \in W$ implies the existence of an ascending sequence P_n of projections, such that $P_n \in J$ for all $n = 1, 2, 3, \cdots$ and $P_n \uparrow P$. Moreover if we introduce the function $\Delta(P)$ as in Lemma 7.3 of [3], then $\Delta(P_n) \uparrow \Delta(P)$ as was shown in the proof of Lemma 7.3 of [3]. Since P_n is an element of $J \subseteq W$ it decomposes under our direct integral (1.2): $P_n \sim P_n(y)$ such that $P_n(y) \leqq P_{n+1}(y) \leqq P(y)$ for all $y \in Y$ and all n. Therefore

$$t_y[P_n(y)] \leqq t_y[P_{n+1}(y)]$$

and hence

$$\lim_n \int_Y t_y[P_n(y)]dy = \int_Y \lim_n t_y[P_n(y)]dy. \tag{3.4}$$

Moreover by §14 of [10] a subsequence of $P_n(y)$ converges to $P(y)$ for almost every y. Replacing the whole sequences P_n, $P_n(y)$ by these subsequences and changing $P_n(y)$ on one set of measure zero, we see that we can take the P_n and $P_n(y)$ such that in addition to the above properties we have $P_n(y) \uparrow P(y)$.

Let us now use the assumption that $\int t_y[P(y)]dy < \infty$. Then we note that this limit (3.4) exists because $t_y[P_n(y)] \leqq t_y[P(y)]$. Now for fixed y, we know from (3.3) that $t_y[P(y)]$ is a positive constant multiple of $\dim_y [P(y)]$, hence by the properties of $\dim_y$ (see [6]) $P_n(y) \uparrow P(y)$ implies $t_y[P_n(y)] \uparrow t_y[P(y)]$. Therefore

$$\lim_n \int_Y t_y[P_n(y)]dy = \int_Y t_y[P(y)]dy.$$

On the other hand $P_n \in J$ implies by Lemma 7.3 and 7.4 of [3] that $\Delta(P_n) = \int t_y[P_n(y)]dy$. Hence $\lim_n \Delta(P_n)$ is finite and equals $\int t_y[P(y)]dy$. But by Lemma 7.3 of [3], $\lim_n \Delta(P_n) = \Delta(P)$, and by the same lemma $\Delta(P) < \infty$ implies $P \in J$. This proves Lemma 3.1.

COROLLARY. *Let*

$$B(y) = \sum_{j=1}^{r} c_j P_j(y) \tag{3.5}$$

where each $P_j(y)$ satisfies the assumption of Lemma 3.1. *In particular we assume $\int t_y[P_j(y)]dy < \infty$. Then there exists an element B of J such that $B \sim B(y)$ and such that the corresponding function $b(g)$ on G defined by* (3.2) *can be taken to be continuous. And for this continuous function $b(g)$ we have*

$$b(1) = \int_Y t_y[B(y)]dy. \tag{3.6}$$

Proof. By Lemma 7.2 of [3] the function $p_j(g)$ which corresponds to the projection $P_j \in J$ can be taken to be continuous. For it we have, by Lemma 7.3 of [3], $\Delta(P_j) = p_j(1)$. Hence by the above we have $p_j(1) = \int t_y[P_j(y)]dy$. On the other hand $\sum_{j=1}^{r} c_j p_j(g) = b(g)$ is a continuous function on G and corresponds to B in the sense of (3.2). Therefore (3.6) holds.

LEMMA 3.2. *Let $A(y) \in W(y)$ for each $y \in Y$ such that $A(y)$ depends measurably on y. If $A(y)$ is for almost every y positive semi-definite and of finite rank in the sense of Chapter* I *of* [8], *then $t_y[A(y)]$ exists and is a measurable function of y.*

Proof. Since $A(y)$ is semi-definite there exists a unique semi-definite element $A_0(y) = A_0(y)^*$ of $W(y)$ such that $A(y) = A_0(y)^2$. And it follows at once from [10] that $A_0(y)$ depends measurably on y, since $A(y)$ does.

Let $E(y)$ be the projection on the closure of the range of $A_0(y)$. Again [10] implies that $E(y)$ is measurable and $E(y) \in W(y)$. Now $A_0(y)$ has finite rank whenever $A(y)$ has. It follows from the definition of rank that $E(y)$ has finite relative dimensionality for almost all y. Hence $t_y[E(y)] < \infty$ for almost all y. Therefore $t_y[E(y)]$ is a measurable function of y, by Lemma 3.1 above. Hence there exist measurable subsets Y' of Y of finite measure, such that $t_y[E(y)]$ is bounded on Y'. Since $A_0(y)$ is measurable, [10] implies that its bound $|||A_0(y)|||$ is a complex-valued measurable function. Hence we may take Y' to be such that also $|||A_0(y)|||$ is bounded on it. And it is clear that it is sufficient to prove our lemma for operator-functions $A(y)$ which satisfy $A(y) = A_0(y) = E(y) = 0_y$ for $y \notin Y'$. Then we have $\int t_y[E(y)]dy < \infty$. Hence by Lemma 3.1 above there exists an element $E^2 = E^* = E$ of J such that $E \sim E(y)$. Let A_0 be the element of W for which $A_0 \sim A_0(y)$. Note that since $A_0(y)$ is measurable and $|||A_0(y)|||$ now bounded on Y, [10] implies that A_0 exists. Now $A_0(y)E(y) = A_0(y)$ implies $A_0E = A_0$. Since $A_0 \in W$, $E \in J$, and since J is a two-sided ideal of W by Lemma 7.1 of [3] we see that $A_0 \in J$. Hence by the Plancherel formula (1.6), $t_y[A_0(y)A_0(y)^*]$ exists for almost every y and is measurable. Since $A_0(y)^* = A_0(y)$ and $A(y) = A_0(y)^2$ the lemma is proved

when $A(y)=0_y$ for $y\notin Y'$ with Y' suitable as described above But from this the assertion of Lemma 3.2 follows in general.

We next observe that the assumption that $A(y)$ be semi-definite can be omitted in Lemma 3.2.

COROLLARY. *Let $A(y)\in W(y)$ and depend measurably on y. If $A(y)$ has finite rank for almost all y then $t_y[A(y)]$ exists and is a measurable function of y.*

Proof. We can write $A(y)=A_1(y)-A_2(y)+iA_3(y)-iA_4(y)$ where all $A_j(y)$ are easily seen to be measurable, semi-definite, and of finite rank. Also $t_y[A(y)]=t_y[A_1(y)]-t_y[A_2(y)]+it_y[A_3(y)]-it_y[A_4(y)]$, and Chapter I of [8] implies that all these traces exist. Since each term on the right is measurable by Lemma 3.2, so is $t_y[A(y)]$.

LEMMA 3.3. *Let $A(y)\in W(y)$ depend measurably on y. If $A(y)$ is of finite rank for almost every y, if the bound $|||A(y)|||$ of the operator $A(y)$ is a bounded numerical function, and if $\int t_y[A(y)A(y)^*]dy<\infty$, then there exists a function $a(g)\in\mathfrak{L}_2(G)$ whose Fourier-transform* (1.5) *is $A(y)$.*

Proof. Note that since $A(y)$ has finite rank, so has $A(y)A(y)^*$; hence $t_y[A(y)A(y)^*]$ exists and is a measurable function of y by the last corollary. Therefore the statement of the lemma has meaning.

Let $E(y)$ be the projection onto the closure of the range of $A(y)$. Then $t_y[E(y)]$ is finite for almost every y and measurable by Lemma 3.1. Therefore there exists a sequence of measurable subsets Y_n of Y, each Y_n of finite measure, such that $\bigcup_n Y_n=Y$ and $t_y[E(y)]$ is bounded on each Y_n. Put

$$E_n(y)=\begin{cases}E(y) & \text{for } y\in Y_n,\\ 0_y & \text{for } y\notin Y_n.\end{cases}\tag{3.7}$$

Then $\int t_y[E_n(y)]dy<\infty$, hence by Lemma 3.1 there exists an element E_n of J with $E_n\sim E_n(y)$. Clearly $E_n\leqq E_{n+1}\to E$. Put $A_n(y)=E_n(y)A(y)$. Then the element A_n of W which decomposes into $A_n(y)$ under (1.2) satisfies $A_n=E_nA$, hence $E_nA_n=A_n$, since E_n is clearly a projection. Since $A_n\in W$ and $E_n\in J$, we have again $A_n\in J$ by Lemma 7.1 of [3]. Let $a_n(g)$ be the function $\in\mathfrak{L}_2(G)$ which corresponds to A_n in the sense of (3.2). Then by the Plancherel formula (1.6), $\|a_n\|^2=\int_G|a_n(g)|^2dg=\int_Y t_y[A_n(y)A_n(y)^*]dy$.

Since $E(y)A(y)=A(y)$ and $E_n(y)A(y)=A(y)$ for all $y\in Y$, (3.7) implies

$$A_n(y)=\begin{cases}A(y) & \text{for } y\in Y_n,\\ 0_y & \text{for } y\notin Y_n.\end{cases}\tag{3.8}$$

Hence

$$\int_Y t_y[A_n(y)A_n(y)^*]dy=\int_{Y_n} t_y[A(y)A(y)^*]dy,$$

$$\lim_{n} \int_Y t_y[A_n(y)A_n(y)^*]dy = \int_Y t_y[A(y)A(y)^*]dy,$$

and

$$\lim_{m,n} \int_Y t_y[\{A_n(y) - A_m(y)\} \cdot \{A_n(y) - A_m(y)\}^*]dy = 0.$$

Hence using the Plancherel formula (1.6) again, we obtain $\|a_m - a_n\|^2 \to 0$ in $\mathfrak{L}_2(G)$. Hence by the Riesz-Fischer theorem there exists a function $a(g) \in \mathfrak{L}_2(G)$ such that $\|a_n - a\|^2 \to 0$ in the $\mathfrak{L}_2$-norm.

If $n \geqq m$, then $E_m A_n = E_m E_n A = E_m A = A_m$ implies $E_m a_n = a_m$, as is easily seen. On the other hand since E_m is a bounded operator on $\mathfrak{L}_2(G)$, the above l.i.m. $a_n(g) = a(g)$ implies that $E_m a_n$ converges to $E_m a$ in the $\mathfrak{L}_2$-norm, as $n \to \infty$ and m is fixed. Hence together with $E_m a_n = a_m$ for $n \geqq m$ this implies $E_m a = a_m$.

Now use the assumption that $|||A(y)|||$ is a bounded function of y. Then there exists in accordance with [10] an element A of W such that $A \sim A(y)$ and (3.7) and (3.8) imply $AE_m = E_m A = A_m$.

Now consider any element $\phi(g)$ of $\mathfrak{L}_2(G)$ which satisfies $E_m\phi = \phi$ for some m. Then, if $*$ denotes again convolution of two functions on G, we have $A_m\phi = a_m * \phi = (a * e_m) * \phi = a * (e_m * \phi) = a * (E_m\phi) = a * \phi$. Hence the convolution $a_m * \phi$ of a_m and ϕ exists because $A_m \in J$ implies by the definition of J that $A_m\phi$ and $a_m * \phi$ both exist and are equal. On the other hand $E_m\phi = \phi$ implies $A\phi = A_m\phi$. Thus $A\phi = A_m\phi = a_m * \phi = a * \phi$. This proves that $a * \phi$ exists and equals $A\phi$ for all ϕ in a certain dense linear subspace of $E\mathfrak{L}_2(G)$ (namely for those ϕ for which there is an integer m with $E_m\phi = \phi$; and these ϕ form a dense linear subset of $E\mathfrak{L}_2(G)$ because $E_m \uparrow E$). But A is a bounded everywhere defined operator on $\mathfrak{L}_2(G)$. Hence it follows by a familiar argument that $a * \phi$ exists and equals $A\phi$ whenever $\phi \in E\mathfrak{L}_2(G)$.

If on the other hand ϕ is in the orthogonal complement of $E\mathfrak{L}_2(G)$ in $\mathfrak{L}_2(G)$, i.e. $E\phi = 0$, then $A\phi = AE\phi = 0$ and $E_m\phi = 0$, so that $A_m\phi = A_m E_m\phi = 0 = a_m * \phi$. Hence $a * \phi$ exists and equals 0. This completes the proof that if $a(g)$ is the above function $\in \mathfrak{L}_2(G)$ then the convolution $a * \phi$ exists for every $\phi \in \mathfrak{L}_2(G)$ and equals $A\phi$, and hence that $A \in J$. This completes the proof of Lemma 3.2, because if one applies the definition (1.5) to our function $a(g)$ one sees readily that the operator-valued function $A(y)$ which occurs in the hypothesis of Lemma 3.2 coincides (almost everywhere) with the operator-valued function $\mathfrak{T}a$ of (1.5), since in the present case A is a bounded operator on $\mathfrak{L}_2(G)$. This completes the proof of Lemma 3.3.

Using the above results we are now in a position to prove the following theorem.

THEOREM. *Let J be defined as at the beginning of this section and $\mathfrak{X}$ as in* §1 (*cf.* (1.8) *and* (1.9)). *Then the Fourier-transformation $\mathfrak{T}$ defined by* (1.5) *maps*

J onto a dense linear subset of $\mathfrak{X}$. It follows that $\mathfrak{T}$ maps $\mathfrak{L}_2(G)$ onto the completion of $\mathfrak{X}$ in the norm (1.8).

Proof. Let $W_0(y)$ be the subset of those elements of $W(y)$ which are normed in the sense of definition 1.4.5 of [8]. Thus $W_0(y)$ is the set of those elements Q of $W(y)$ for which

$$t_y[QQ^*] \tag{3.9}$$

can be defined to be finite in a consistent manner. Then $\mathfrak{X}$ is the set of those equivalence classes of measurable operator-valued functions $A(y)$, $B(y)$, $\cdots$ with values in the various $W_0(y)$ and with a finite $\|\mathfrak{A}\|$ as defined by (1.8). By Theorem IV of [8] the elements of $W(y)$ which are of finite rank form a dense linear subset $W_1(y)$ of $W_0(y)$ in the norm (3.9). Let $\mathfrak{X}_1$ be the subset of those elements of $\mathfrak{X}$ for which the function-values are in the various $W_1(y)$ (for almost every y). Then $\mathfrak{X}_1$ is clearly a linear subspace of $\mathfrak{X}$. We wish to prove that $\mathfrak{X}_1$ is dense in $\mathfrak{X}$ in the norm (1.8).

Indeed let $A(y)\in\mathfrak{X}$ and let $E(y)$ be again the projection on the closure of the range of $A(y)$. Then $E(y)$ is measurable as above and there exists an element $E\in W$ such that $E\sim E(y)$. And as above there exists an ascending sequence E_n of projections $\in J$ such that $E_n\uparrow E$ and $E_n(y)\uparrow E(y)$ where E_n decomposes into $E_n(y)$, i.e. $E_n\sim E_n(y)$. Put $A_n(y)=E_n(y)A(y)$. Then $E_n(y)$ is of finite rank and hence so is $E_n(y)A(y)$, because $E_n(y)A(y)H_y\subseteq E_n(y)H_y$. It follows that $t_y[\{A_n(y)-A(y)\}\cdot\{A_n(y)-A(y)\}^*]$ tends to zero monotonically, so that it may be integrated with respect to y. This proves that $\mathfrak{X}_1$ is dense in $\mathfrak{X}$.

Now consider the set $\mathfrak{X}_2$ of those elements of $\mathfrak{X}_1$ for which the numerical function $|||A(y)|||$ is bounded. By considering suitable ascending sequences of measurable subsets Y_n of Y and by replacing operator function-values by 0_y on the complements of the Y_n it is easily seen that $\mathfrak{X}_2$ is dense in $\mathfrak{X}_1$ in the norm (1.8).

On the other hand Lemma 3.3 shows that the Fourier-transformation $\mathfrak{T}$ maps a certain linear subset of J onto $\mathfrak{X}_2$. Hence $\mathfrak{T}$ maps J onto a dense linear subset of $\mathfrak{X}$. This proves the first assertion of the theorem.

Now J is dense in $\mathfrak{L}_2(G)$ and $\mathfrak{T}$ is an isometric mapping. Hence $\mathfrak{T}$ maps $\mathfrak{L}_2(G)$ onto the completion of $\mathfrak{X}$. This completes the proof of the theorem.

Remark 1. The question whether $\mathfrak{T}$ is onto was raised by G. W. Mackey in his survey [11].

Remark 2. It now follows that the hypotheses of Lemma 3.3 can be weakened. The assumption that $A(y)$ be of finite rank can be weakened to that $A(y)$ be normed in the sense of [8]. And the condition on the bound $|||A(y)|||$ can now be omitted.

4. We conclude with the following observations.

Lemma 4.1. *Let $a(g)\in\mathfrak{L}_2(G)$ and assume that there exists a function*

$b(g)\in\mathfrak{L}_2(G)$ such that $a(g)=(b*b^*)(g)=\int_G b(\mathfrak{g}^{-1}g)\bar{b}(\mathfrak{g}^{-1})d\mathfrak{g}$. Then $a(g)$ is continuous in g and for every $g\in G$

$$a(g) = \int_Y t_y[V(g, y)^*A(y)]dy \tag{4.1}$$

where $A(y)$ is the generalized Fourier transform of $a(g)$. The integral in (4.1) is absolutely convergent.

Proof. $a(g)=\int_G b(\mathfrak{g}^{-1}g)b(\mathfrak{g}^{-1})dg=\int_G b(\mathfrak{g}g)\bar{b}(\mathfrak{g})d\mathfrak{g}=\langle R(g^{-1})b, b\rangle$ where $R(g)$ denotes right-translation in $\mathfrak{L}_2(G)$ defined by (1.1) and $\langle\ ,\ \rangle$ the inner product in $\mathfrak{L}_2(G)$. Since the mapping $g\to R(g)$ is strongly (and hence weakly continuous), $\langle R(g^{-1})b, b\rangle$ is (for fixed b) a continuous function of g, hence $a(g)$ is continuous.

Now apply the generalized Plancherel formula (1.6) to $\langle R(g^{-1})b, b\rangle$. Let $B(y)$ be the generalized Fourier transform (1.5) of $b\in\mathfrak{L}_2(G)$. Then the Fourier transform of $R(g^{-1})b$ is $V(g^{-1}, y)B(y)$. Hence (1.6) yields for every $g\in G$

$$a(g) = \langle R(g^{-1})b, b\rangle = \int_Y t_y[V(g^{-1}, y)B(y)B(y)^*]dy. \tag{4.2}$$

Now $a=b*b^*$ implies $A(y)=B(y)B(y)^*$ for almost every y. Hence using $V(g^{-1}, y)=V(g, y)^*$ we see that (4.2) implies (4.1) for every $g\in G$. Moreover by the generalized Plancherel formula (1.6) the integral in (4.2) is absolutely convergent and hence so is that in (4.1).

Now the finite linear combinations of the functions $a(g)$ satisfying the hypothesis of Lemma 4.1 are dense in $\mathfrak{L}_2(G)$ (cf. for instance [0; 12]). Hence we obtain

LEMMA 4.2. *Let $f(g)\in\mathfrak{L}_2(G)$. Then there exists a sequence of functions $f_n(g)\in\mathfrak{L}_2(G)$ such that each $f_n(g)$ satisfies the conclusions of Lemma 4.1 and $f(g)=\text{l.i.m. } f_n(g)$. Thus*

$$f(g) = \text{l.i.m.} \int_Y t_y[V(g, y)^*F_n(y)]dy \tag{4.3}$$

where $F_n(y)$ is the generalized Fourier transform of $f_n(g)$.

REMARK. In accordance(2) with [0] each $f_n(g)$ can even be chosen to be a finite linear combination of self-adjoint idempotents $\in\mathfrak{L}_2(G)$.

LEMMA 4.3. *Assume that $F(y)$ is an element of $W(y)$ and depends measurably on y. Assume that $F(y)$ is of finite rank and let $E(y)$ be the projection on the closure of the range of the operator $F(y)$ in the space H_y. If the integral*

(2) (Added November 8, 1954): At this point it is sufficient to know that $\bar{f}(g^{-1})=f(g)$ defines by means of convolution a self-adjoint hypermaximal operator on $\mathfrak{L}_2(G)$ whenever $f\in\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. In this case this operator on $\mathfrak{L}_2(G)$ is bounded, hence there is no difficulty.

$$\int_Y t_y[F(y)F(y)^*]^{1/2} t_y[E(y)]^{1/2} dy \tag{4.4}$$

is absolutely convergent in the sense of Lebesgue, then

$$\int_Y t_y[V(g, y)F(y)]dy \tag{4.5}$$

is also absolutely convergent and is a bounded continuous function of g on G.

Proof. $|t_y[V(g, y)F(y)]| = |t_y[V(g, y)F(y)E(y)]| \leqq t_y[E(y)]^{1/2} \cdot t_y[V(g, y) F(y)\{V(g, y)F(y)\}^*]^{1/2} = t_y[F(y)F(y)^*]^{1/2} t_y[E(y)]^{1/2}$ by Chapter I of [8]. This proves the absolute convergence of (4.5), if one observes that the assumption that $F(y)$ be of finite rank assures the existence of $t_y[E(y)]$ and hence that (4.4) is meaningful.

In accordance with Lemma 4.1 of [5] and its proof there exists (for almost every y) a one-one linear mapping of a linear subspace of $W(y)$ onto a dense linear subspace of H_y. This linear subspace of $W(y)$ is contained in the set $W_0(y)$ of normed elements of $W(y)$ introduced in connection with (3.9) above and is dense in $W_0(y)$ in the norm (3.9). Moreover we know from p. 749 of [5] that if to any $F(y)$ contained in this dense linear subspace of $W_0(y)$ there corresponds under this mapping the element $f(y)$ of H_y then

$$t_y[F(y)F(y)^*] = \mathfrak{a}(y)\|f(y)\|^2 \tag{4.6}$$

in accordance with equation (4.10) of [5], where $\|f(y)\|$ denotes the norm of $f(y)$ in H_y. It follows that the mapping in question can be extended uniquely to the whole of $W_0(y)$ so that (4.6) is preserved. Now since $F(y)$ is assumed to be of finite rank and since we know that $V(g, y)$ is for each $g \in G$ an element of $W(y)$, it follows that $V(g, y)F(y)$ is of finite rank and hence contained in $W_0(y)$. And there corresponds to it under the above mapping the element $V(g, y)f(y)$ of H_y. Similarly $E(y) \in W_0(y)$ so that there corresponds to it an element of H_y which we denote by $e(y)$. And we obtain

$$t_y[V(g, y)F(y)E(y)] = \mathfrak{a}(y)\langle V(g, y)f(y), e(y)\rangle. \tag{4.7}$$

Since $g \to V(g, y)$ is for fixed y a (strongly and hence also weakly) continuous unitary representation of G, the right side of (4.7) is, for fixed y, $f(y)$ and $e(y)$ a continuous function of g. But we have seen at the beginning of this proof that the absolute value of (4.7) is bounded by an integrable function of y which is independent of g, hence (4.5) is a bounded continuous function of g.

COROLLARY. *If $W(y)$ is of finite type (for almost every y) and if $F(y)$ is an element of $W(y)$ depending measurably on y such that*

$$\int_Y t_y[F(y)F(y)^*]^{1/2} dy < \infty, \tag{4.8}$$

then the conclusions of Lemma 4.3 *hold.*

Proof. Indeed since $W(y)$ is of finite type we may take the identity element of $W(y)$ instead of $E(y)$ and observe that $W(y) = W_0(y)$ in this case.

Note that in the special case where (almost) all $W(y)$ are one-dimensional (i.e. G commutative) $F(y)$ is a scalar multiple of the identity of $W(y)$, hence $F(y)$ can be identified with a numerical function which is by (4.8) an integrable function on the character group of G. Thus the above becomes a classical result when G is commutative. And as in the case of classical Fourier analysis one can now combine the above properties to obtain other forms of the Fourier inversion formula on G and related results.

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