ON LINEAR, SECOND ORDER DIFFERENTIAL EQUATIONS
IN THE UNIT CIRCLE

BY

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1. Introduction. In the differential equation

\[ W'' + p(z)W' + q(z)W = 0 \quad (\'= d/dz), \]

let \( z \) be a real or complex variable, \( q(z) \) a continuous and \( p(z) \) a continuously differentiable function on the domain under consideration. The function

\[ I(z) = q(z) - (1/4)p(z) - (1/2)p'(z) \]

is called the invariant of (1). If \( W_1(z), W_2(z) \) are linearly independent solutions of (1) and if \( u = W_1/W_2 \), then

\[ 2I = \{u, z\}, \]

where \( \{u, z\} \) is the Schwarzian parameter

\[ \{u, z\} = (u''/u')' - (1/2)(u''/u')^2. \]

The change of dependent variables \( W \rightarrow w = W \exp ((1/2)\int p dz) \) transforms (1) into the normal form

\[ w'' + I(z)w = 0. \]

Hence, in considering zeros of solutions of (1), it can be assumed that (1) has the form (5). The term “solution” will always mean a non-trivial \((\neq 0)\) solution.

This note will be concerned principally with solutions \( w = w(z) \) of the differential equation (5) under the assumption that \( I(z) \) is analytic on the circle \( |z| < 1 \). (Unless the contrary is stated below, it will be supposed that \( I(z) \) satisfies this assumption.)

The following terminology will be used: If no solution of a differential equation has two zeros (on a given \( z \)-set), then the differential equation will be said to be disconjugate (on that set) [11]. Similarly, if no solution has an infinite set of zeros, the differential equation will be called non-oscillatory. (In contrast to the situation on the real field, where Sturm’s separation theorem is valid, it is possible that a solution of (5) can have a finite number of zeros on \( |z| < 1 \) and that another solution has an infinite number of zeros there.)

2. Reduction to a real independent variable. The results on the zeros of solutions of (5) in the case that \( z \) is a complex variable will be deduced from cases where \( I(z) \) is a complex-valued function of a real variable (for example, \( z = x + iy \) for fixed \( y \)). The transfer of these results from the real case will be possible because of the following “comparison” theorem (cf., e.g., [9, p. 319]):

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(*) Let \( I(z) \) be a continuous, complex-valued function of a real variable \( z \) on some interval. If
\[
 v'' + \Re(I(z))v = 0
\]
is disconjugate on the given interval, then (5) is disconjugate on that interval.

In this assertion, (6) can of course be replaced by any Sturm majorant, for example, by
\[
 v'' + |I(z)|v = 0.
\]
The comparison theorem (*) is a particular case of the following trivial fact: If \( A_1 \) and \( A_2 \) are self-adjoint (bounded or unbounded) operators in Hilbert space and \( A_1 \geq \text{const.} > 0 \), then \( \lambda = 0 \) is not in the point spectrum of \( A_1 + iA_2 \). Theorem (*) follows by choosing \( A_1f \) to be the differential operator \( f'' + \Re(I(z))f \) associated with boundary conditions \( f = 0 \) at the end points of the interval and \( A_2f \) to be \( \Im(I(z))f \).

The transformation rule
\[
\{u, Z\} = \{u, z\}(dz/dZ)^2 + \{z, Z\}
\]
for the Schwarzian derivative under the change of (real or complex) independent variables \( z \rightarrow Z \) supplies, by (3), the transformation rule for the invariant of (1) or (5). In particular, (8) reduces to
\[
\{u, Z\} = \{u, z\}(dz/dZ)^2 \quad \text{if} \quad Z = (\alpha z + \beta)/(\gamma z + \delta),
\]
\( \alpha \delta - \beta \gamma \neq 0 \), since \( w = \alpha z + \beta, \gamma z + \delta \) are linearly independent solutions of the equation \( w'' = 0 \) which has the invariant \( I(z) = 0 \).

Assertion (*) immediately implies two results of Nehari [7], which state that if \( u = u(z) \) is an analytic function on the unit circle \( |z| < 1 \), then \( u(z) \) is schlicht on \( |z| < 1 \) whenever \( I(z) \), defined by (3), satisfies either of the inequalities
\[
(10_1) \quad |I(z)| \leq \pi^2/4,
(10_2) \quad |I(z)| \leq 1/(1 - |z|^2)^2
\]
[7, p. 549 and p. 545]. In fact, \( u(z) \) is schlicht on \( |z| < 1 \) whenever (5) is disconjugate there. That either (10_1) or (10_2) implies that (5) is disconjugate on \( |z| < 1 \) can be deduced from (*) as follows:

Ad (10_1). Suppose that some solution \( w = w(z) \) of (5) has two zeros in \( |z| < 1 \), then the remark concerning (9) (in fact, the case \( Z = e^{i\theta}z \)) shows that there is no loss of generality in supposing that these zeros are on the same horizontal segment \( z = x + iy \), with \( y = \text{const.} \) and \( x^2 < 1 - y^2 \). Since the length of the \( x \)-interval joining the two zeros is less than 2, the inequality (10_1) shows that no solution of (7) (hence, by (*), no solution of (5)) can have two zeros on such an interval.
Ad (10a). As verified by direct computation by Nehari [7, p. 547], the
inequality (10a) for the invariant of (5) is unchanged by conformal mappings
$z \mapsto Z$ of the unit circle $|z| < 1$ onto $|Z| < 1$. In fact, if
\begin{equation}
(11)
\begin{aligned}
d_s &= \left| \frac{dz}{dz'} \right| / (1 - |z|^2)
\end{aligned}
\end{equation}
denotes the non-euclidean arc length, which is invariant under the mapping
$z \mapsto Z$, then (9) can be written as
\begin{equation}
(12)
\begin{aligned}
(1 - |Z|^2)^2 \left\{ u, Z \right\} ds = (1 - |z|^2)^2 \left\{ u, z \right\} ds.
\end{aligned}
\end{equation}
Thus the invariance of (10a) follows from (3).

If the assertion concerning (10a) is false, then some solution $w = w(z)$ of
(5) has at least two zeros on $|z| < 1$. In view of the invariance of (10a), it
can be supposed that these zeros are real. Either one of the following two
equivalent arguments shows, by (*), that this leads to a contradiction. First,
for real $z$, (7) has the Sturm majorant
\begin{equation}
(13)
\begin{aligned}
d^2v/dx^2 + v/(1 - x^2)^2 = 0, \quad -1 < x < 1,
\end{aligned}
\end{equation}
which is disconjugate since it possesses the solution $v = (1 - x^2)^{1/2}$ having no
zeros on $-1 < x < 1$; cf. [6]. Second, the change of variables
\begin{equation}
(14)
\begin{aligned}
s = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right)
\end{aligned}
\end{equation}
satisfying (11) for real $z = x$ transforms the invariant $|I(z)|$ of (7), according
to (3) and (8), into the non-positive function $|I(z)| (1 - |z|^2)^2 - 1$ for $z = z(s),$
$-\infty < s < \infty$. Hence, in the case (10a), (5) is disconjugate on $|z| < 1$.

The constants $\pi^2/4$ and 1 are the best possible in (10a) and (10b), respec-
tively; [7, p. 549] and [6, p. 552].

3. A criterion for disconjugateness. The condition (10a) can be replaced
by a somewhat different criterion:

(i) If $C$ is a circular arc in $|z| < 1$ orthogonal to the boundary $|z| = 1$, then the inequality
\begin{equation}
(13)
\begin{aligned}
\int_C (1 - |z|^2) |I(z)| dz \leq 2
\end{aligned}
\end{equation}
implies that (5) is disconjugate on $C$.

If $z = x$ in (7) is a real variable on some interval $a < x < b$ and $I(z)$ is con-
tinuous on this interval, then, according to [4], (7) is disconjugate on this
interval if
\begin{equation}
(14)
\begin{aligned}
\int_a^b (b - x)(x - a) |I(x)| dx \leq b - a
\end{aligned}
\end{equation}
(cf. [8] for a generalization to complex variables). Hence, if $C$ is the real
interval $-1 < x < 1$, (13) and (*) imply that no solution of (5) has two zeros
on the real axis. If $C$ is any circular arc orthogonal to $|z| = 1$, there exists a transformation $z \to Z$ of the type in (9) of the circle $|z| < 1$ onto $|Z| < 1$ such that the image of $C$ is the real segment $-1 < X < 1$, where $Z = X + iY$. It follows from (9) that no solution of (5) has two zeros on $C$ if

$$
\int_{-1}^{1} (1 - |Z|^2) | I(z)(dz/dZ)^2 dZ | \leq 2, \quad Z = X.
$$

Note that (11) and (12) imply that

$$
(1 - |Z|^2) \{u, Z\} dZ = (1 - |z|^2) \{u, z\} dz.
$$

Consequently, the integrals in (13) and (15) are equal, and so (15) follows from (13). Thus (13) assures that no solution of (5) has two zeros on $C$.

4. Disconjugateness and $\mu(1)$. If $C$ is a line segment contained in $|z| < 1$, a sufficient criterion for (7) (hence for (5)) to be disconjugate on $C$ is

$$
\int_C | I(z) dz | \leq 4/L, \quad \text{where } L \text{ is the length of } C
$$

(Liapounoff; cf., e.g., [4]). If $C$ is a chord of $|z| = 1$, this can be improved to

$$
\int_C (1 - |z|^2) | I(z) dz | \leq L
$$

by the criterion (14); see [4].

An inequality of Fejér and Riesz [3] states that

$$
\int_C | I(z) dz | \leq (1/2) \mu(1),
$$

if $C$ is the real line segment $-1 < x < 1$,

$$
\mu(1) = \lim_{r \to 1} \mu(r),
$$

and

$$
\mu(r) = \int_{|z| = r} | I(z) dz |.
$$

According to a remark of Nehari [8, p. 695], (19) is valid if $C$ is any circular arc in $|z| < 1$ orthogonal to $|z| = 1$. Hence, the inequality

$$
\int_C (1 - |z|^2) | I(z) dz | \leq \int_C | I(z) dz |
$$

and (i) give the following:

(ii) The differential equation (5) is disconjugate on $|z| < 1$ whenever
The weakened form \( \mu(1) \leq 2 \) of this condition follows from Nehari’s inequality (21) in [8] and his remark following it. The above use of the inequality (19) is similar to the procedure of Nehari.

The constant 4 in (22) is the ratio of the constants 2 in (13) and 1/2 in (19). Although the inequalities (13) and (19) cannot be improved, it remains undecided whether or not (22) is the “best” possible.

Nehari’s inequality leading to the weakened form \( \mu(1) \leq 2 \) of (22) has the following consequence: The inequality

\[
\mu(r) \leq 4/L
\]

implies that no solution of (5) has two zeros \( z = z_1, z_2 \) in the circle \( |z| < r \) satisfying \( |z_1 - z_2| \leq L \).

5. Solutions satisfying \( w(0) = 0 \). The inequality (23) can be considerably sharpened for \( r \) near 1 in dealing with a particular solution of (5).

(iii) If \( w = w(z) \) is a solution of (5) satisfying \( w(0) = 0 \), then \( w(z) \) has no zero different from \( z = 0 \) in \( |z| < 1 \) if

\[
\mu(r) \leq 1/2r(1 - r) \quad \text{for} \quad 1/2 \leq r < 1.
\]

In order to prove this, grant, for a moment, the fact that no solution of (7) (hence no solution of (5)) can have two zeros on a radius \( z = te^{i\theta} \), where \( 0 \leq t < 1 \), if

\[
\int_0^r t^2 |I(te^{i\theta})| dt \leq r/4(1 - r) \quad \text{for} \quad 0 < r < 1.
\]

The inequality of Fejér and Riesz implies the second of the inequalities

\[
\int_0^r t^2 |I(te^{i\theta})| dt \leq r^2 \int_{-\pi}^{\pi} |I(te^{i\theta})| dt \leq (1/2)r^2\mu(r)
\]

and so (24) implies (25). Thus, in order to prove (iii), it is sufficient to prove the statement concerning (25).

Let \( q_1(s), q_2(s) \) be real-valued, continuous functions on \( 0 < s < \infty \) such that

\[
q_1(s) \geq 0 \quad \text{and} \quad \int_0^\infty q_1(s) ds < \infty.
\]

If the first of the differential equations

\[
d^2v/ds^2 + q_k(s)v = 0 \quad \text{for} \quad k = 1, 2
\]

is disconjugate on \( 0 < s < \infty \) and if

\[
\int_0^\infty |q_k(s)| ds \leq \int_0^\infty q_1(s) ds \quad \text{for} \quad 0 < s < \infty,
\]
then (26) is disconjugate on $0 < s < \infty$ (cf. [5, p. 245] and [11]). The choice $q_1(s) = 1/4s^2$ (Kneser) gives the sufficient condition

$$
\int_0^\infty |q_2(s)| \, ds \leq 1/4s \quad (0 < s < \infty)
$$

for (26) to be disconjugate on $0 < s < \infty$.

If $q(t)$ is continuous in the differential equation

$$
d^2v/dt^2 + q(t)v = 0 \quad (0 < t < 1),
$$

the change of independent variables $s = (1-t)/t$ (which maps $0 < t < 1$ onto $\infty > s > 0$) alters the invariant $q(t)$ of (29) to $q_2(s) = q(t)(dt/ds)^2$, by (9). Since (28) is transformed into

$$
\int_0^t t^2 |q(t)| \, dt \leq 1/4(1 - t) \quad (0 < t < 1),
$$

the statement concerning (25) follows.

By using functions other than $q_1(s) = 1/4s^2$, for example,

$$
q_1(s) = (1/4s^2)(1 + 1/\log^2 s),
$$

it is possible to refine (24) somewhat. It is also possible to refine (iii) in the following direction: Let $0 \leq \alpha < 1$. There exists a constant $K = K$ (independent of $\alpha$ and $I(z)$) such that if

$$
\mu(r) \leq K(1 - \alpha)^2/(1 - r) \quad \text{for} \quad 1/2 \leq r < 1,
$$

then no solution of (5) which has a zero in the circle $|z| \leq \alpha$ has another zero on $|z| < 1$.

6. The solutions in the case $\mu(1) < \infty$. If the condition (22) for (5) to be disconjugate on $|z| < 1$ is weakened to

$$
\mu(1) < \infty,
$$

then, according to Nehari [8], (5) is non-oscillatory on $|z| < 1$. Actually, (31) implies a great deal more about the solutions $w(z)$ of (5) than the fact that $w(z)$ has only a finite number of zeros on $|z| < 1$.

Let (31) hold. Then there exists a function $\psi(\theta)$ of bounded variation on $|\theta| \leq \pi$ such that

$$
\psi(\theta) = \lim_{r \to 1} \int_0^\theta |I(re^{i\phi})| \, d\phi
$$

holds at every continuity point of $\psi(\theta)$.

The properties of the solutions $w(z)$ of (5) under the assumption (31) can be described as follows:
w(z) is uniformly continuous on |z| < 1; in fact, the derivative w'(z) is bounded on |z| < 1. In addition, the radial limits w'(e^{i\theta})=\lim_{r\to 1} w'(re^{i\theta}), as r\to 1, exist for all \theta. The function w'(z) on \{|z| \leq 1\} (defined to be w'(e^{i\theta}) at z=e^{i\theta}) is continuous on \{|z| \leq 1\} except possibly at the points e^{i\theta} where \theta is a discontinuity point of \psi(\theta). The point z=e^{i\theta} is a continuity point of w'(z) if w(e^{i\theta})=0. Finally, there exists one and only one solution w=w(z) of (5) for which w and w' have preassigned radial limits w(e^{i\theta}), w'(e^{i\theta}) at a given point z=e^{i\theta} of |z| =1.

In order to verify these properties, note that if, on a fixed radius, one has

\[ \int_0^1 (1-t) |I(t e^{i\theta})| \, dt < \infty, \]  

then the radial limits w(e^{i\theta}), w'(e^{i\theta}) belonging to a solution w(z) of (5) exist ([1]; cf. [2, pp. 368–370] and [10, pp. 261–268]). Furthermore, there exists one and only one solution having preassigned radial limits w(e^{i\theta}), w'(e^{i\theta}) for a fixed \theta; cf. [10]. Clearly, (19) and (31) imply (33) for every \theta.

Every solution w=w(z) of (5) satisfies

\[ w(z) = w(0) + zw'(0) - \int_0^z (z-\xi)I(\xi)w(\xi)\,d\xi. \]

Hence

\[ |w(re^{i\theta})| \leq A + \int_0^r |I(t e^{i\theta})w(te^{i\theta})| \, dt \quad (A = |w(0)| + |w'(0)|). \]

Consequently, a standard inequality gives

\[ |w(re^{i\theta})| \leq A \exp r \int_0^r |I(te^{i\theta})| \, dt, \]

and so, by the inequality of Fejér and Riesz,

\[ |w(re^{i\theta})| \leq A \exp (1/2)r\mu(r) \leq A \exp (1/2)\mu(1). \]

This proves that w is bounded on |z| < 1. By (34),

\[ w'(z) = w'(0) - \int_0^z I(\xi)w(\xi)\,d\xi. \]

Consequently,

\[ |w'(re^{i\theta})| \leq |w'(0)| + \text{Const.} \int_0^r |I(te^{i\theta})| \, dt \leq |w'(0)| + \text{Const.} \mu(1), \]

so that w'(z) is bounded on |z| < 1.

The relation (35) shows that w'(z) is continuous at z=e^{i\theta} if
\[ \int_{0}^{\theta + h} | I(re^{i\phi})w(re^{i\phi}) | d\phi \to 0, \quad \text{as} \quad (h, r) \to (0, 1). \]

In view of the continuity of \( w(z) \), this is the case when \( \theta \) is a continuity point of \( \psi(\theta) \) or when \( w(e^{i\theta}) = 0 \). This completes the proof of the properties of \( w(z) \) enumerated above.

It is clear that these properties imply that \( w(z) \) has a finite number of zeros on \( |z| \leq 1 \). For otherwise there is a point \( z = e^{i\theta} \) which is a cluster point of zeros of \( w(z) \). Then \( w(e^{i\theta}) = 0 \) and so \( w'(z) \) is continuous at \( z = e^{i\theta} \). Consequently \( w'(e^{i\theta}) = 0 \). But the only solution \( w(z) \) belonging to the (radial) limits \( w(e^{i\theta}) = 0, w'(e^{i\theta}) = 0 \) is the trivial solution \( w(z) = 0 \).

7. An upper estimate for the number of zeros in \( |z| < r \). The inequality (23) furnishes an upper estimate for the number \( N(r) = N(r; w(z)) \) of zeros of a solution \( w(z) \) in the circle \( |z| < r \) \((<1)\):

(iv) Let there exist on \( 0 < r < 1 \) a positive, continuously differentiable, non-decreasing function \( \lambda = \lambda(r) \) satisfying

\[ \mu(r) \leq \lambda(r) \]

and

\[ d\lambda/dr = O(\lambda^2(r)), \quad \text{as} \quad r \to 1. \]

Then

\[ N(r) = O \left( \int_{0}^{r} \lambda^2(r)dr \right), \quad \text{as} \quad r \to 1. \]

(iv) shows that if \( z_1, z_2, \ldots \) are the zeros of a solution \( w(z) \) of (5), then

\[ \sum (1 - |z_k|) = \int (1 - r)dN(r) < \infty \]

is implied by

\[ \int_{1}^{1} (1 - r)\lambda^2(r)dr < \infty. \]

Simple examples seem to indicate that (38) can be improved to the corresponding relation in which \( \lambda^2(r) \) is replaced by \( \lambda(r) \). But this possibility will remain undecided.

In order to prove (iv), note that, according to the inequalities (23) and (36), the distance \( L \) between any pair of zeros of a solution \( w(z) \) on \( |z| \leq r \) satisfies \( L \geq 4/\lambda(r) \). In the proof of (38), it can therefore be supposed that \( \lambda(r) \to \infty \), as \( r \to 1 \).

If \( r \) is sufficiently near 1, the ring \( r - 2/\lambda(r) \leq |z| \leq r \) can be divided into \( 2(2\pi r)/(2/\lambda(r)) = 2\pi r \lambda(r) \) sets (curvilinear quadrilaterals) such that the dis-
tance between any pair of points in any of the sets is less than $4/\lambda(r)$. Consequently, the ring contains at most $2\pi r \lambda(r)$ zeros of a solution $w = w(z)$. The width of the ring is $\Delta r = 2/\lambda(r)$. Hence, the number of zeros is at most $2\pi r \lambda(r) \leq \text{Const.} \lambda^2(r) \Delta r$. On the other hand, the monotony of $\lambda$ implies

$$\int_{\Delta r}^{r} \lambda^2(r) dr \geq \sum \lambda^2(r - \Delta r) \Delta r,$$

if the interval $(0, r)$ is divided into a finite number of pieces. Hence, it is clear that (38) follows if it is verified that

$$\lambda(r) = O(\lambda(r - 2/\lambda(r))), \quad \text{as } r \to 1.$$

But this is a consequence of (37), which implies that

$$\log \left[ \frac{\lambda(r)}{\lambda(r - 2/\lambda(r))} \right] = \int \lambda^{-1} d\lambda = O\left(\int \lambda dr\right) = O(1),$$

as $r \to 1$, where the limits of integration are $r - 2/\lambda(r)$ and $r$.

References


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