EXTENSION OF DERIVATIONS IN CONTINUOUS TRANSFORMATION RINGS(1)

BY

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1. **Introduction.** If A is a finite-dimensional central simple algebra, any derivation of a semisimple subalgebra C into A may be extended to an inner derivation of A [4, p. 102]. Nakayama [7, Theorem 2] has generalized this result to the situation where A and C are simple rings with minimum condition containing the same unit, and the derivation annihilates a simple weakly Galois (cf. definition below) subring C_0 of C. In this paper we extend Nakayama's theorem to the case where A is a continuous transformation ring (i.e. the ring of all continuous linear transformations on a pair of dual spaces), C is a primitive ring with nonzero socle(2) satisfying certain reducibility conditions, and C_0 is an arbitrary weakly Galois subring of C. Under similar hypotheses we prove one further extension theorem where now we do not assume the existence of a weakly Galois subring but instead suppose only that C is a subalgebra over the center of A satisfying a certain finiteness hypothesis.

We are indebted to Professor N. Jacobson for letting us see a manuscript of a book on the theory of rings to be published in the Colloquium series of the American Mathematical Society.

2. **Completely primitive rings.** Throughout this paper we are concerned with an additive abelian group M and certain rings of endomorphisms on it —i.e. our rings are all subrings of the ring E of all endomorphisms of M.

By a derivation δ of a ring R into a ring S containing R we mean an additive mapping of R into S, such that for any a, b in R, $(ab)\delta=(a\delta)b+a(b\delta)$. It is well known that the mapping $a\rightarrow[a, s]=as-sa$ for some s in S is such a derivation. When $R=S$ this is called the inner derivation by s.

In the book on ring theory referred to above Jacobson proves the following lemma by exhibiting the connection between derivations and module extensions. The proof below is a direct one.

LEMMA 1. *Let C be a primitive subring of E with nonzero socle S, such that*(3) *$MS=M$. Then any derivation δ of C into E can be extended to an inner one on E.*

Presented to the Society, September 3, 1954; received by the editors September 11, 1954.

(1) This paper was written while the authors held a grant from the National Science Foundation.

(2) For a discussion of continuous transformation rings and primitive rings with nonzero socles (P.M.I. rings) see [11, §§1, 2].

(3) This condition is equivalent to the complete reducibility of M as a C-module [2, p. 158] and [11, (2.7)].

Proof. Let e be any primitive idempotent of C, i.e. eC is a minimal right ideal of C. As in the proofs of [3, Theorem 4.1] [6, Theorem 9] we subtract an inner derivation of E from δ to obtain a derivation δ_1 of C into E which vanishes on e: since $e^2=e$, $(e\delta)e+e(e\delta)=e\delta$ and so $e(e\delta)e=0$. If $a_1=(e\delta)e-e(e\delta)$ then $[e, a_1]=-e(e\delta)-(e\delta)e=-e\delta$. Thus $c\delta_1=c\delta+[c, a_1]$ is a derivation vanishing on e.

Since $MS=M$, M is a completely reducible C-module(3), $M=\sum_{\oplus}M_\alpha$ where the M_α are faithful irreducible C-modules. Since $M_\alpha eC\neq 0$ we may write $M_\alpha=x_\alpha eC$ for a certain element x_α in M_α. Since eC is a minimal right ideal, $x_\alpha ec=0$ only if $ec=0$. Hence if we let $x_\alpha eca_2=x_\alpha(ec)\delta_1=x_\alpha e(c\delta_1)$ for every α, then a_2 is a well defined element of E. Now for any b in C,

$$x_\alpha ec[b, a_2] = x_\alpha ecba_2 - x_\alpha eca_2 b = x_\alpha e((cb)\delta_1 - (c\delta_1)b) = x_\alpha ec(b\delta_1)$$

so that $b\delta_1=[b, a_2]$ and $b\delta=[b, a_1-a_2]$.

That the assumption $MS=M$ is essential in the above lemma can be seen from the following example (cf. [9, pp. 129–130]): Let M be a vector space of countable dimension over a field Z and let $\{x_i\}$ $(i=0, 1, 2, \cdots)$ be a basis of M over Z. We consider the algebra C of linear transformations of M spanned over Z by the linear transformations e_{ij} $(i, j=1, 2, \cdots)$ defined thus:

$$x_0e_{ij} = x_j,\ i \text{ odd}; \qquad x_0e_{ij} = 0,\ i \text{ even}; \qquad x_ne_{ij} = \delta_{ni}x_j, \qquad n > 0\cdot$$

Clearly C is isomorphic to the algebra of linear transformations it induces on the space spanned by $x_1, x_2, \cdots$ namely, the algebra of all finite matrices, which is known to be primitive and is its own socle S [5, p. 18]. Hence $M=MS\oplus x_0z$. We now consider the linear transformations e'_{ij} given by

$$x_0e'_{ij} = x_j; \qquad x_ne'_{ij} = 0, \qquad n > 0.$$

It is readily verified that $e'_{ij}e_{pq}+e_{ij}e'_{pq}=\delta_{jp}e'_{iq}$ so that the mapping $\sum e_{ij}\alpha_{ij} \to \sum e'_{ij}\alpha_{ij}$, α_{ij} in Z, is a derivation of C into the algebra of linear transformations of M. But for any endomorphism a of M, $x_0a=\sum_0^m x_h\alpha_h$, so that if i is even and $i>m$, $x_0(e_{ij}a-ae_{ij})=0\neq x_0e'_{ij}$. Thus the derivation cannot be extended to an inner one in this case.

In his book Jacobson uses Lemma 1 to prove the next two theorems. Since they have appeared nowhere else and we shall need them in proving Theorems 3 and 4, we reproduce them here.

These theorems concern subrings of the centralizer $\mathfrak{L}(M)$ in E of a division subring D of E. In Theorem 1 the ring of constants contains a ring C_0 which is assumed to be weakly Galois in the sense of Dieudonné [2] and Nakayama [7]: the centralizer of C_0 in E is spanned over D by semilinear transformations.

THEOREM 1. *Let $\mathfrak{L}(M)$ be the ring of all linear transformations on a vector space M over a division ring D. Let C be a primitive subring with nonzero socle*

S such that $MS=M$. Suppose that C contains a weakly Galois subring C_0. Then any derivation δ of C into $\mathfrak{L}(M)$ annihilating C_0 can be extended to an inner one on $\mathfrak{L}(M)$.

Proof. By Lemma 1 there is an endomorphism a of M such that $c\delta=[c, a]$. Since $c_0\delta=0$ for each c_0 in C_0, $a=\sum t_i\alpha_i$, α_i in D, t_i semilinear transformations of M. Hence, for each c in C, $c\delta=[c, a]=\sum[c, t_i]\alpha_i$ is a linear transformation. Now $[c, t_i]$ is a semilinear transformation with the same automorphism of D as t_i so that by Lemme 2b of [2] $c\delta$ is also equal to $\sum[c, t_i]\alpha_i$ summed only over those i for which t_i is associated with an inner automorphism of D. But then by multiplying the t_i by appropriate elements of D they become linear so that we may write $c\delta=\sum[c, t_i]\beta_i$, β_i in D, t_i in $\mathfrak{L}(M)$. If we now write the β_i in terms of a basis $\{\gamma_j\}$ of D over its center Z with $\gamma_1=1$, we obtain $c\delta=\sum[c, s_j]\gamma_j$ with s_j in $\mathfrak{L}(M)$. Since $\mathfrak{L}(M)D$ is isomorphic to $\mathfrak{L}(M)\otimes_Z D$ [2, Théorème 3], $\{\gamma_j\}$ is a basis of $\mathfrak{L}(M)D$ over $\mathfrak{L}(M)$, so that $c\delta=[c, s_1]$.

THEOREM 2. *Let $\mathfrak{L}(M)$ be the ring of all linear transformations on a vector space M over a division ring D with center Z. Let B be a finite-dimensional simple subalgebra of $\mathfrak{L}(M)$ over Z containing the unit of $\mathfrak{L}(M)$. Then any derivation δ of B into $\mathfrak{L}(M)$ which is Z-linear (i.e. $Z\delta=0$) can be extended to an inner derivation of $\mathfrak{L}(M)$.*

Proof. The ring BD of endomorphisms of M is isomorphic to $B\otimes_Z D$ [1, Theorem 8] and so is a simple ring with minimum condition [1, Corollary, p. 98 and Theorem 9]. Extend δ to a derivation of BD into E by setting $D\delta=0$. The result then follows immediately from Lemma 1.

Added in proof (July 22, 1955). We have recently exhibited Lemma 1 and Theorem 2 as special cases of analogous results on higher cohomology groups.

3. **Continuous transformation rings.** We shall next prove an analog (Theorem 3) of Theorem 1 in the case where $\mathfrak{L}(M)$ is replaced by a continuous transformation ring $A=\mathfrak{L}(M, N)$. A corresponding extension of Theorem 2 is already a special case of Hochschild's result [3, Theorem 4.1] except when B is inseparable over Z. We conjecture the extended theorem is true without separability hypotheses but we have no proof.

We first state a lemma which is probably well known.

LEMMA 2. *Let (M, N) be a pair of dual spaces over a division ring D. If p is an endomorphism of M such that there is an adjoint endomorphism p^* of N with $(xp, f)=(x, fp^*)$ for all x in M and all f in N, then p is in $\mathfrak{L}(M, N)$.*

Proof. For every f in N and α in D, $((x\alpha)p, f)=(x\alpha, fp^*)=\alpha(x, fp^*)$ $=\alpha(xp, f)=((xp)\alpha, f)$ so that p is a linear transformation. But every linear transformation with an adjoint is continuous.

THEOREM 3. *Let $A=\mathfrak{L}(M, N)$ be a continuous transformation ring and let*

C be a primitive subring of A with nonzero socle S. Suppose(4) *that $MS=M$, $NS^*=N$, S is contained in the socle of A, and C contains a weakly Galois subring C_0 of A. (Note that the semilinear transformations spanning the centralizer of C_0 need not be continuous.) Then any derivation δ of C into A which vanishes on C_0 can be extended to an inner one on A.*

Proof. We first think of C as a subring of $\mathcal{L}(M)$, the ring of all linear transformations of M over D. Then Theorem 1 insures the existence of a linear transformation t such that $c\delta=[c, t]$. Now let e be any primitive idempotent of C. Just as in the proof of Lemma 1, we find an element a_1 of A such that $[e, t-a_1]=0$ and we let $[c, t-a_1]=c\delta_1$ so that $c\delta_1=[c, s]$ with s a linear transformation of M such that $es=se$. Now for any c in C, $(ece)\delta_1=e(c\delta_1)e=ecse-esce=[ece, ese]$. Since S is in the socle of A, Me is a finite-dimensional space and so [11, (3.20)] assures us that eAe induces all linear transformations on Me. Thus there is an element a_2 in eAe such that $a_2=ese$. We now define a derivation of C into A by $c\delta_2=c\delta_1-[c, a_2]$, so that $(eCe)\delta_2=0$. As in the proof of Lemma 1, we write $M=\sum_\oplus x_\alpha eC$ and $N=\sum_\oplus f_\beta e^*C^*$ for irreducible C-modules $x_\alpha eC$ and irreducible C^*-modules $f_\beta e^*C^*$. As before we define an endomorphism a_3 of M by $x_\alpha eca_3=x_\alpha e(c\delta_2)$, so that, for any b in C, $[b, a_3]=b\delta_2$. If we define a_3^* by $f_\beta e^*d^*a_3^*=-f_\beta e^*(d\delta_2)^*$, we have

$$\begin{aligned}(x_\alpha eca_3, f_\beta e^*d^*) &= (x_\alpha e(c\delta_2), f_\beta e^*d^*) = (x_\alpha e(c\delta_2)de, f_\beta)\\ &= -(x_\alpha ec(d\delta_2)e, f_\beta) = (x_\alpha ec, f_\beta e^*d^*a_3^*).\end{aligned}$$

Hence by Lemma 2, a_3 is in A and $c\delta=[c, a_1+a_2+a_3]$.

If in Theorem 3 we are willing to put further hypotheses on C_0, several of the hypotheses on C can be omitted. For example, if the centralizer of C_0 is spanned by continuous semilinear transformations, then we need only assume that C is primitive and $MS=M$, for here the proof of Theorem 1 applies verbatim. Second, if C_0 is assumed to satisfy all the conditions imposed on C in Theorem 3 as well as being weakly Galois, then by a slight modification of [11, Proposition 3], C also satisfies the hypotheses of Theorem 3 and so Theorem 3 is true, assuming only that C is primitive with nonzero socle. In fact, under these hypotheses the centralizer of C_0 is spanned by continuous semilinear transformations [11, (3.14)]. Third, if we assume C_0 is a Galois subring of A [11, §5], then our second remark applies.

Since our conditions on C are automatic when A and C are simple with minimum condition and with the same unit, Theorems 1 and 3 are generalizations of Nakayama's theorem [7, Theorem 2]. The continuous transformation ring A in Theorem 3 cannot be replaced by an arbitrary primitive ring with nonzero socle; for example, replacing A by its own socle invalidates the theorem, i.e. the derivation can of course still be extended to an inner one of $\mathcal{L}(M, N)$ but this derivation need not be an inner one on $\mathcal{J}(M, N)$.

(4) For the significance of these hypotheses on C, cf. [11, footnote 5].

The following theorem is a sort of dual of Theorem 2; e.g., if $D=Z$ then every centralizer in A of a finite-dimensional simple subalgebra is a C satisfying the hypotheses of Theorem 4 [10, Corollary to Theorem 1].

THEOREM 4. *Let $A=\mathfrak{L}(M, N)$ be a continuous transformation ring with center the field Z. Let C be a primitive subalgebra of A over Z with nonzero socle S. Suppose $MS=M$, $NS^*=N$, and that S is contained in the socle of A. Then if the division ring of C is finite-dimensional*(5) *over Z, any Z-linear derivation δ of C into A can be extended to an inner derivation of A.*

Proof. As before we write $c\delta_1=c\delta+[c, a_1]$ with $e\delta_1=0$ for some primitive idempotent e of C. Thus δ_1 induces a derivation of eCe into eAe which is eZe-linear. As in Theorem 3, eAe is the ring of all linear transformations on Me with center eZe and so applying Theorem 2 to the subalgebra eCe of eAe we find an element a_2 in eAe such that $(ece)\delta_1=[ece, a_2]$. The rest of the proof is identical with that of Theorem 3.

If we restrict ourselves to the case where Z is perfect, Hochschild's result [3, Theorem 4.1] guarantees that any derivation of eCe into eAe can be extended to any inner one of eAe requiring only that eAe be an algebra over eZe containing eCe. Thus the assumption that S is in the socle of A is no longer needed and we have proved the

COROLLARY. *If the center of A is perfect, Theorem* 4 *remains true with the assumption that S be contained in the socle of A deleted.*

Finally we show by an example that the assumption that eCe be finite dimensional over eZe cannot be dropped. Let Z be an arbitrary field of characteristic zero and let $C=Z\{x\}$ be the field of formal power series in one variable over Z. For A we take the ring of formal power series over C in a variable y such that $yx=2xy$. This construction goes back to Hilbert and it is shown in [8, pp. 40–41] that A is a division ring. Direct computation shows the center of A is Z and so all the assumptions of Theorem 4 except the finite-dimensionality of eCe over eZe are fulfilled (note that here $e=1$ and $M=Me$ is a one-dimensional space over A). Now let δ be the derivation of C given by $(\sum\alpha_i x^i)\delta=\sum i\alpha_i x^{i-1}$, α_i in Z. Then δ cannot be extended to an inner derivation of A, for if $x(\sum b_j y^j)-(\sum b_j y^j)x=x\delta=1$ with b_j in C, we get $\sum(1-2^j)b_j xy^j=1$ which is a contradiction.

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(5) By the division ring of C we mean the centralizer of C on a faithful irreducible C-module. It may be concretely represented as eCe where e is any primitive idempotent in C. This eCe is clearly an algebra over the isomorph eZe of Z. The present finite-dimensionality hypothesis can be interpreted as the finite-dimensionality of eCe over eZe.

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