Some Theorems about the Riesz Fractional Integral

By

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I show in this paper that theorems which hold for Riemann-Liouville fractional integrals have analogues holding for the Riesz fractional integral [1]. Theorems 1, 2, and 3 are analogous to well-known results due to Hardy and Littlewood [2]. Theorem 4 is of a different character and is analogous to one recently proved by the author [3].

The Riesz fractional integral \( f_\alpha (P) \) of order \( \alpha \) is given by

\[
f_\alpha (P) = K_m^{-1} \int_E r_{PQ}^{-\alpha m} f(Q) dQ, \quad \text{where} \quad K_m = \pi^{m/2} \Gamma(\alpha/2) [\Gamma((m - \alpha)/2)]^{-1},
\]

\( E \) denotes all of Euclidean \( m \)-space, and \( r_{PQ} \) denotes the distance between \( P \) and \( Q \).

We assume always that \( f(Q) \) is \( L \)-integrable over \( E \).

I prove the following theorems.

**Theorem 1.** If \( f(P) \in \text{Lip} \beta \), \( 0 < \beta < 1 \) then \( f_\alpha (P) \in \text{Lip} (\alpha + \beta) \), \( 0 < \alpha + \beta < 1 \).

**Theorem 2.** If \( f(P) \in L^q \), \( q > 1 \), \( 1 + m/q > \alpha > m/q \), then

\[
f_\alpha (P) \in \text{lip} (\alpha - m/q).
\]

**Theorem 3.** If \( f(P) \in L^q \) and \( 0 < \alpha < m/q \), then

\[
f_\alpha (P) \in L^r, \quad \text{where} \quad \alpha = m(1/q - 1/r).
\]

**Theorem 4.** If \( f(P) \in L^q \) then

(a) for \( 0 < \alpha < m \), \( 2 < q < \infty \), \( f_{\alpha/q} (P) \) is finite everywhere except possibly in a set which is of zero \( \beta \)-capacity for all \( \beta > m - \alpha \);

(b) for \( 0 < \alpha < m \), \( 1 < q < 2 \), \( f_{\alpha/q} (P) \) is finite everywhere except possibly in a set of zero \( (m - \alpha) \)-capacity.

Both (a) and (b) are best possible.

1. Preliminaries. If \( P \) is the point \( (x_1, \ldots, x_m) \) and \( Q \) the point \( (t_1, \ldots, t_m) \) we define the points \( (x_1 + t_1, \ldots, x_m + t_m) \) and \( (x_1 - t_1, \ldots, x_m - t_m) \) to be \( P + Q \) and \( P - Q \) respectively. The distance \( |P| \) of \( P \) from the origin \( 0 = (0, \ldots, 0) \) is given by \( |P|^2 = \sum_{r=1}^m x_r^2 \), and \( |P - Q| \) is the distance \( P \) to \( Q \).

If, for \( 0 \leq \beta \leq 1 \), \( f(P + H) - f(P) = O(|H|^\beta) \) uniformly in \( P \) as \( |H| \to 0 \), we say that \( f(P) \in \text{Lip} \beta \). If, in this, \( O \) is replaced by \( o \) we say that \( f(P) \in \text{lip} \beta \).

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Next, we have
\[ K_m(f_a(P + H) - f_a(P)) = \left( \int_U + \int_{E - U} \right) (|Q - H|^{m-a} - |Q|^{m-a}) f(Q + P) dQ, \]
where \( U \) is the unit hypersphere having the origin as center. For \( |H| < 1/2 \) it is not difficult to establish that
\[ |Q - H|^{m-a} - |Q|^{m-a} = O(|H|) \]
uniformly in \( E - U \). The second integral is thus \( O(H) \) uniformly in \( P \), and so
\[ K_m(f_a(P + H) - f_a(P)) \]
(2)
\[ = \int_U (|Q - H|^{m-a} - |Q|^{m-a}) f(Q + P) dQ + O(|H|). \]

2. Proofs of Theorems 1 and 2. First, Theorem 1. The first term on the right-hand side of (1) of §1 may be rewritten in the form
\[ \int_U (|Q - H|^{m-a} - |Q|^{m-a})(f(Q + P) - f(P)) dQ + f(P) \left\{ \int_{U'} |Q|^{m-a} dQ - \int_U |Q|^{m-a} dQ \right\}, \]
where \( U' \) is the sphere \( U \) transforms into under the transformation \( Q' = Q - H \). The expression in curly brackets is dominated by \( \int_S |Q|^{m-a} dQ \), where \( S = U' + U - U'U \).

Now \( mS < \pi^{m/2} \Gamma((m+2)/2)^{-1}(1 + |H|)^{m-1} = O(|H|) \) and \( |Q|^{m-a} < 2^{m-a} \) in \( S \) for \( |H| < 1/2 \). Consequently, the second term in (2) is \( O(|H|) \).

To deal with the first term we note that it is of order \( H |Q - H|^{m-a} - |Q|^{m-a} |Q|^{a} dQ \) and apply a uniform dilatation transformation of ratio 1:\( |H| \) and then a rotation which takes the transform of \( H \) into the point \( 1 = (1, 0, \ldots, 0) \). The first term is then seen to be less than
\[ |H|^{a+\beta} \int_B |Q - 1|^{m-a} - |Q - 1|^{m-a} |Q|^{\beta} dQ = O(|H|^{a+\beta}), \]
since it is again a simple matter to establish that the integral is finite. This proves Theorem 1.

Next, Theorem 2. Let \( S(r) \) denote the hypersphere of radius \( r \) centered at the origin and write
\[ A(\delta) = S(\delta) - S(|H|), \quad B(\delta) = U - S(\delta), \]
where \( \delta \) will presently be defined. Split the right-hand side of (1) into integrals
\[ I_1 \text{ over } S(|H|), \ I_2 \text{ over } A(\delta), \text{ and } I_3 \text{ over } B(\delta). \] Then, firstly
\[
|I_1| \leq \left\{ \int_{S(|H|)} \left| Q - H \right|^{a-m} - \left| Q \right|^{a-m} |q'dQ| \right\}^{1/q'} \\
\cdot \left\{ \int_{S(|H|)} |f(Q + P)|^{q} dQ \right\}^{1/q}
\]
\[
= |H|^{a-m/q} \left\{ \int_{U} \left| Q - 1 \right|^{a-m} - \left| Q \right|^{a-m} |q'dQ| \right\}^{1/q'} \cdot o(1)
\]
as \[|H| \to 0: \] we use the same transformation on the integral as before. Thus
\[ I = o(|H|^{a-m/q}). \] Further
\[
|I_2| \leq \left\{ \int_{A(\delta)} \left| Q - H \right|^{a-m} - \left| Q \right|^{a-m} |q'dQ| \right\}^{1/q'} \\
\cdot \left\{ \int_{A(\delta)} |f(Q + P)|^{q} dQ \right\}^{1/q}
\]
It is again easy to show that, for \[|H| < \delta/3,\]
\[
\left| Q - H \right|^{a-m} - \left| Q \right|^{a-m} \leq C \left| H \right| \left| Q \right|^{a-m-1}
\]
and thus
\[
|I_2| \leq C \left| H \right| \left\{ \int_{A(\delta)} \left| Q \right|^{(a-m-1)q'} dQ \right\}^{1/q'} \left\{ \int_{A(\delta)} |f(Q + P)|^{q} dQ \right\}^{1/q}
\]
Further
\[
\int_{A(\delta)} \left| Q \right|^{(a-m-1)q'} dQ < \left| H \right|^{(a-1)q'(a-m-1)} \int_{E-U} \left| Q \right|^{(a-m-1)q'} dQ,
\]
so that
\[
|I_2| \leq C \left| H \right|^{a-m/q} \left\{ \int_{A(\delta)} |f(Q + P)|^{q} dQ \right\}^{1/q}
\]
Given any \[\epsilon > 0,\] we can choose \[\delta\] so that \[\int_{A(\delta)} |f(Q + P)|^{q} dQ\] is less than \((\epsilon/C)^q\), and so
\[
|I_2| < \epsilon \left| H \right|^{a-m/q}.
\]
Finally
\[
|I_3| \leq \left\{ \int_{B(\delta)} \left| Q - H \right|^{a-m} - \left| Q \right|^{a-m} |q'dQ| \right\}^{1/q'} \left\{ \int_{\bar{E}} |f(Q + P)|^{q} dQ \right\}^{1/q}
\]
For fixed \(\delta\), \(\left| Q - H \right|^{a-m} - \left| Q \right|^{a-m} = O(\left| H \right|)\) uniformly in \(B(\delta)\), and so
\[I_3 = O(|H|).\]
Thus, finally, \( K_m(f_a(P + H) - f_a(P)) = o(|H|^{a-m/\mu}), \) giving the required result.

3. **Proof of Theorem 3.** We first prove a many-dimensional generalization of a theorem due to Hardy and Littlewood [2, Theorem 3].

**Lemma.** If \( f(P) \in L^q, \ g(Q) \in L^r, \ 1/q + 1/r > 1, \ q > 1, \ r > 1 \) and \( \mu = 2 - 1/q - 1/r \) then

\[
(1) \quad \int_E \int_E |Q - P|^{-\mu} f(P) g(Q) dP dQ \leq KM_q(f) M_r(g),
\]

where \( M_q(f) = \left\{ \int_E |f(P)|^q dP \right\}^{1/q} \) and \( M_r(g) \) is similarly defined.

I prove here the case \( m = 3 \), which is sufficiently typical.

Since an arithmetic mean is greater than the corresponding geometric mean we have

\[
|P - Q|^2 = (x_1 - t_1)^2 + (x_2 - t_2)^2 + (x_3 - t_3)^2 
\geq 3 \left| x_1 - t_1 \right|^{2/3} \left| x_2 - t_2 \right|^{2/3} \left| x_3 - t_3 \right|^{2/3}
\]

and so

\[
|P - Q|^{-\mu} \leq C \left| x_1 - t_1 \right|^{-\mu} \left| x_2 - t_2 \right|^{-\mu} \left| x_3 - t_3 \right|^{-\mu}.
\]

Consequently the left-hand side of (1) is not greater than a constant multiple of

\[
(2) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x_1, x_2, x_3) g(t_1, t_2, t_3)}{|x_1 - t_1|^{\mu} |x_2 - t_2|^{\mu} |x_3 - t_3|^{\mu}} \cdot \, dt_3 dx_3 dt_2 dx_2 dt_1 dx_1.
\]

By the Hardy-Littlewood theorem mentioned, which is the case \( m = 1 \) of the lemma,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| x_3 - t_3 \right|^{-\mu} f(x_1, x_2, x_3) g(t_1, t_2, t_3) \, dt_3 dx_3
\]

is dominated by \( CF(x_1, x_2) G(t_1, t_2) \), where \( F(x_1, x_2) = \left\{ \int_{-\infty}^{\infty} |f(x_1, x_2, x_3)|^q \, dx_3 \right\}^{1/q} \) and \( G(t_1, t_2) \) is defined analogously.

Hence \([2]\) is dominated by

\[
C_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| x_1 - t_1 \right|^{-\mu} \left| x_2 - t_2 \right|^{-\mu} F(x_1, x_2) G(t_1, t_2) \, dt_2 dx_2 dt_1 dx_1.
\]

Applying the case \( m = 1 \) of the lemma again to the inner two integrals we find that (1) is dominated by
where $F(x_1) = \left\{ \int_{-\infty}^{\infty} |F(x_1, x_2)|^q dx_2 \right\}^{1/q}$ and $G(t_1)$ is defined analogously.

A final application of the lemma with $m=1$ shows that (1) is dominated by $C_1 C_2 M_q(F) M_r(G)$. Since $M_q(F)$ equals $\left\{ \int_E |f(P)|^q dP \right\}^{1/q}$ and a similar result holds for $M_r(G)$, we have the required result.

To prove Theorem 3 it is sufficient to prove that, for every $g(P)$ such that $M_q(g) \leq 1$,

$$\int_B f_{a}(P) g(P) dP \leq KM_q(f).$$

The left-hand side of this is equal to

$$(3) \quad K_m^{-1} \int_B \int_B |P - Q|^{a-mf(Q)} g(P) dQ dP$$

and, since $\alpha - m = m(1/q - 1/r) - m = -m(2 - 1/q - 1/r')$, the lemma applies and shows (3) to be, in modulus, not greater than $K'M_q(f) M_r(g) \leq K'M_q(f)$, thus proving the theorem.

4. Preliminaries about Theorem 4. We say, with Frostman [4, p. 26], that a non-negative additive set function $\mu(S)$ defined for all Borel sets in $E$ is a distribution if $\mu(E) = 1$. Further, if $S \subset E$ and $\mu(S) = 1$ we say that the distribution is concentrated on $S$.

Let $S$ be a given set. Suppose that there is a distribution concentrated on $S$ such that

$$V_{\beta} = \sup_{P \in E} \int_B |Q - P|^{-\beta} d\mu(Q)$$

is finite. Then we say that $S$ is of positive $\beta$-capacity. Otherwise $S$ is said to be of zero $\beta$-capacity. Clearly, if $S$ is of positive $\beta$-capacity, it is of positive $\gamma$-capacity for all $\gamma < \beta$. Further, if it is of zero $\beta$-capacity, it is of zero $\gamma$-capacity for all $\gamma > \beta$.

**Lemma.** For $1 < q < 2$, and for every $\epsilon > 0$ for which $q - \epsilon > 1$, we have

$$(1) \quad \int_S \left\{ \int_E |Q - P|^{(a/q') - m} d\mu(Q) \right\}^{q/\epsilon} dP \leq A(\alpha, \epsilon, m, q, S)V_m^{(q-\epsilon)/(q-\epsilon)}$$

where $A(\alpha, \epsilon, m, q, S)$ is a constant depending only on the parameters shown and $S$ is a bounded set.

For $2 \leq q \leq \infty$ we have

$$(2) \quad \int_E \left\{ \int_E |Q - P|^{(a/q') - m} d\mu(Q) \right\}^{q} dP \leq A(\alpha, m)V_m^{q-1}$$
where \( A(\alpha, m) \) is a constant depending only on the parameters shown.

We have
\[
\left\{ \int_E |Q - P|^{(a/q')-m} d\mu(Q) \right\}^q \leq \left\{ \int_E |Q - P|^{-\alpha/q} |Q - P|^{-m} d\mu(Q) \right\}^q \]
\[
\quad \cdot \left\{ \int_E |Q - P|^{-m} d\mu(Q) \right\}^{(q-\epsilon)/(q-\epsilon)'}
\]
by Hölder's inequality. The second factor is not greater than \( V^{(q-\epsilon)/(q-\epsilon)'} \), while the first is \( \int_E |Q - P|^{-\alpha/q-m} d\mu(Q) \). The left-hand side of (1) is therefore not greater than
\[
V_m^{(q-\epsilon)/(q-\epsilon)'} \int_S dP \int_E |Q - P|^{-\alpha/q-m} d\mu(Q).
\]
We invert the order of integration and note that
\[
\int_S |Q - P|^{-\alpha/q-m} dP = A(\alpha, \epsilon, m, q, S), \text{ say.}
\]
Furthermore \( \int_E d\mu(Q) = 1 \). (1) now follows.

To prove (2) I first show the result true for \( q = 2 \) and then that this implies its truth for \( q > 2 \). For this latter part of the proof I am indebted to Professor J. E. Littlewood.

We have first, on inverting the order of integration,
\[
\left\{ \int_E \left( \int_E |Q - P|^{-m} d\mu(Q) \right)^2 dP \right\}^2
\]
\[
= \int_E \int_E \int_E \int_E |Q - P|^{-m} |R - P|^{-m} dP d\mu(Q) dP d\mu(R).
\]
To deal with the inner integral we dilate \( E \) uniformly, taking \( Q \) as the center of dilatation, in the ratio \( 1:|Q - R| \) and then rotate the dilated space so that the transform of \( Q - R \) goes into the point 1. The inner integral then becomes
\[
|Q - R|^{-m} \int_E |U|^{-m} dU + 1 \int_E |U|^{-m} dU = B(\alpha, m) |Q - R|^{-m}.
\]
Consequently, the right-hand side of (3) is dominated by
\[
B(\alpha, m) \int_E \int_E |Q - R|^{-m} d\mu(Q) \leq B(\alpha, m) V_m^{-a}(E).
\]
Since \( \mu(E) = 1 \) this gives the result for \( q = 2 \).

For \( q > 2 \), we have

\[
\int_E \left\{ \int_E |Q - P|^{\alpha/q - m} d\mu(Q) \right\}^q dP
= \int_E \left\{ \int_E |Q - P|^{\left(\frac{(q-2)}{q}\right)(\alpha-m)} |Q - P|^{\left(\frac{\alpha-2m}{2}\right)} d\mu(Q) \right\}^q dP
\]

and this, by Hölder's inequality, does not exceed

\[
J = \int_E \left\{ \int_E |Q - P|^{\alpha-m} d\mu(Q) \right\}^{q-2} \left\{ \int_E |Q - P|^{\left(\frac{\alpha-m}{2}\right)} d\mu(Q) \right\}^2 dP.
\]

The first curly bracket does not exceed \( V_{m-a}^{q-2} \) (by the definition of \( V_{m-a} \)). So

\[
J \leq V_{m-a}^{q-2} \int_E \left\{ \int_E |Q - P|^{\frac{\alpha-m}{2}} d\mu(Q) \right\}^2 dP.
\]

and this, by the result for \( q = 2 \), does not exceed \( V_{m-a}^{q-2} V_{m-a} \). This gives the result for \( q > 2 \).

5. Proof of Theorem 4. Let

\[
S_n(P) = \int_E |Q - P|^{\alpha/q - m} \left[ f(Q) \right]_n dQ,
\]

where

\[
\left[ f(Q) \right]_n = \begin{cases} f(Q) & \text{for } |f(Q)| \leq n \\ n & \text{for } |f(Q)| > n \end{cases}
\]

\( S_n(P) \) is always defined and finite, and to prove the theorem it is sufficient to show that \( S_n(P) \) is bounded everywhere except possibly in a set of zero \( \beta \)-capacity, where \( \beta = m - \alpha \) for \( 1 \leq q \leq 2 \) and \( \beta > m - \alpha \) for \( q > 2 \).

Assume, then, that \( S_n(P) \) is unbounded in a set \( M \) of positive \( \beta \)-capacity. It is then unbounded in a bounded set \( S \) of positive \( \beta \)-capacity. Then, first, there is a distribution concentrated on \( S \) such that \( \int_E |Q - P|^{-\beta} d\mu(Q) \) is bounded for all \( P \). Secondly, there is a function \( n(P) \leq n \), taking only integer values such that \( \int_S S_{n(P)}(P) d\mu(P) \) exists and is unbounded as \( n \to \infty \). This is an adaptation of a known result used by Salem and Zygmund [5, embodied in the proof of Theorem II], but a proof is perhaps not unwelcome.

Let \( \overline{S}(P) = \sup_{0 \leq m \leq n} S_m(P) \) for \( 0 \leq m \leq n \). Then for all \( P \in S \), \( \{ \overline{S}_n(P) \}^{-1} \to 0 \) as \( n \to \infty \). By Egoroff's theorem on uniform convergence it follows that there is a set \( S' \subset S \) such that \( \mu(S - S') \) is as small as we please, and in which \( \{ \overline{S}_n(P) \}^{-1} \to 0 \) uniformly. It follows that, given any large number \( G \), there is an integer \( n_0 = n_0(G) \) such that, for all \( P \in S' \), \( \overline{S}_n(P) > G \) for all \( n > n_0(G) \). Choose \( n(P) \) such that \( S_{n(P)}(P) = \overline{S}_n(P) \). Then
\[ \int_S S_{n(P)}(P) d\mu(P) > G\mu(S') \quad \text{for} \quad n > n_0, \]

and so

\[ \int_S S_{n(P)}(P) d\mu(P) \to +\infty \quad \text{as} \quad n \to \infty. \]

I show this last to be impossible. We have

\[ \left| \int_S S_{n(P)}(P) d\mu(P) \right| = \left| \int_S \int_E |Q - P|^{\alpha/q} \left[ f(Q) \right]_{n(P)} dQ d\mu(P) \right| \]

\[ \leq \int_E |f(Q)| \int_S |Q - P|^{\alpha/q} d\mu(P) dQ \]

and this does not exceed \( M_q(f) M_{q'} \left[ \int_S |Q - P|^{\alpha/q} d\mu(P) \right] \). Now \( M_q(f) < +\infty \) by hypothesis, and we have only to show that

\[ (1) \quad M_{q'} \left[ \int_E |Q - P|^{\alpha/q} d\mu(P) \right] \]

is bounded.

If \( 1 \leq q \leq 2 \) then \( q' \geq 2 \) and (2) of the lemma of §4 immediately gives (1). If \( q > 2 \) we write \( \beta = m - \gamma \). Since \( \gamma < \alpha \) there is an \( r < q \) such that \( \alpha/q = \gamma/r \). We may suppose \( \beta \) so near \( m - \alpha \) that \( 2 < r < q \) since the result, if true for a given \( \beta \), is true for a larger \( \beta \). We may now rewrite (1) in the form

\[ M_{r'} \left[ \int_E |P - Q|^{\gamma/r} d\mu(Q) \right], \]

which, since \( r' > 2 \), is shown to be bounded by invoking (1) of the lemma.

6. **Theorem 4 is best possible.** We show this by constructing a function \( f(P) \in L^q \) and a set \( M \) of positive \( \beta \)-capacity (where \( \beta = m - \alpha \) when \( 1 \leq q \leq 2 \), and \( \beta \) is any number greater than \( m - \alpha \) when \( q > 2 \)) at every point of which \( f_{1/q}(P) \) is infinite. It will avoid unnecessary complication and fully illustrate the general procedure if this is done for the simplest case \( m = 2 \).

\( M \) is constructed as follows. Let \( \{\xi_n\} \) be any sequence such that \( 0 < \xi_n < 1/2 \). Let \( M_0 \) be the unit square \( 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \). From \( M_0 \) remove the rectangle \( \xi_1 < x_1 < 1 - \xi_1, 0 \leq x_2 \leq 1 \) thus leaving the set \( M_1 \). From the left-hand rectangle in \( M_1 \) remove the rectangle \( \xi_1 \xi_2 < x_1 < \xi_1(1 - \xi_2), 0 \leq x_2 \leq 1 \), and make a similar symmetric removal from the right-hand rectangle of \( M_1 \), thus leaving a set consisting of 4 closed rectangles of length 1 and breadth \( \xi_1 \xi_2 \). If we continue in this manner we are left, after the \( n \)th removal, with a set \( M_n \) consisting of \( 2^n \) closed rectangles each of length 1 and breadth \( \xi_1 \xi_2 \cdots \xi_n \). Consequently
\[ m \mathbb{M}_n = 2^n \xi_1 \xi_2 \cdots \xi_n. \]

It is known [5, p. 40] that the projection \( S \) of \( M = \lim M_n \) on the \( x \)-axis will be of positive \( \beta \)-capacity if and only if

\[ \sum_{n=1}^{\infty} 2^{-n} (\xi_1 \xi_2 \cdots \xi_n)^{-\beta} < \infty. \]

If \( S \) is of positive \( \beta \)-capacity there is a distribution \( \nu \) concentrated on \( S \) such that \( \int_0^1 |x_1 - t|^{-\beta} d\nu(t) \) is bounded for all \( x_1 \). Let \( \mu \) be an additive set function defined over \( E \) by

\[ \mu(X) = \int \int_X d\nu(x_1) dx_2. \]

Then

\[ \int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_0^1 \int_0^1 [(x_1 - t_1)^2 + (x_2 - t_2)^2]^{-(\beta+1)/2} dt_2 d\nu(t_1). \]

In the inner integral make the substitution \( x_2 - t_2 = (x_1 - t_1)u \). It is then dominated by

\[ |x_1 - t_1|^{-\beta} \int_{-\infty}^\infty (1 + u^2)^{-(\beta+1)/2} du = A(\beta) |x_1 - t_1|^{-\beta}. \]

Consequently, since \( \mu \) is a distribution concentrated on \( M \),

\[ \int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_E |P - Q|^{-\beta-1} d\mu(Q) \leq A(\beta) \int_0^1 |x_1 - t_1|^{-\beta} d\nu(t_1), \]

which is bounded. \textbf{Thus} \( M \) \textbf{is of positive} \((\beta+1)\)-capacity if \( S \) \textbf{is of positive} \( \beta \)-capacity.

Define \( \{f_n(P)\} \) over \( M_0 \) by

\[ f_0(P) = 0 \text{ in } M_0, \]
\[ f_n(P) = (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/n^{\alpha-1}} \text{ in } M_n, \]
\[ f_n(P) = f_{n-1}(P) \text{ in } M_0 - M_n. \]

Since \( \{f_n(P)\} \) is, eventually, an increasing sequence of measurable functions the function \( f(P) \) given by

\[ f(P) = \lim_{n \to \infty} f_n(P) \text{ in } M_0, \]
\[ f(P) = 0 \text{ in } E - M_0 \]

exists and is measurable over \( E \).

It is easily seen that, for \( n = 1, 2, \cdots \),
\[ f(P) = 0 \text{ in } M_0 - M_1, \]
\[ f(P) = (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q_n - 1} \text{ on } M_n - M_{n+1} \]

so that
\[
\int_E |f(P)|^q \, dP = \int_{M_0} |f(P)|^q \, dP = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} |f(P)|^q \, dP
\]

\[ = \sum_{n=1}^{\infty} (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q - \alpha/(qM_n - qM_{n+1})} \]

\[ = \sum_{n=1}^{\infty} (1 - 2\xi_{n+1}) 2^n (\xi_1 \xi_2 \cdots \xi_n)^{1-\alpha/q - \alpha}. \]

For \( q > 2 \), we may choose \( \delta > 0 \) so that \( 2(1+\delta) < q \), and then put

\[ 2^{\xi_{n+1}^{1-\alpha}} = 1 + (1 + \delta) n^{-1}. \]

Then \( 2^{-n}(\xi_1 \xi_2 \cdots \xi_n)^{q - 1} \sim C n^{-1} \) so that \( (1) \) with \( \beta = 1 - \alpha \) is satisfied, showing \( S \) to be of positive \((1-\alpha)\)-capacity, and hence that \( M \) is of positive \((2-\alpha)\)-capacity.

Further, \( (2) \) is clearly finite, so that \( f \in L^q \) over \( E \).

Let \( P(x_1, x_2) \) be any point of \( M \). Let

\[ M_n(P) = M_n \cdot S[l_2; x_2 - \epsilon_n \leq l_2 \leq x_2 + \epsilon_n], \quad \text{where } \epsilon_n = \xi_1 \xi_2 \cdots \xi_n/2; \]

\[ M_n^*(P) = (M_n - M_{n+1}) \cdot S[l_2; x_2 - \delta_n \leq l_2 \leq x_2 + \delta_n], \quad \text{where } \delta_n = \xi_1 \xi_2 \cdots \xi_n(1 - 2\xi_{n+1})/2. \]

\( M_n(P) \) then consists of \( 2^n \) squares each of side \( \xi_1 \xi_2 \cdots \xi_n \), while \( M_n^*(P) \subset M_n(P) \) and consists of \( 2^n \) squares each of side \( \xi_1 \xi_2 \cdots \xi_n(1 - 2\xi_{n+1}) \). No square in \( M_n^*(P) \) contains \( P \), but one of the squares, \( I_n \) (say), is contained in that one of the squares, \( J_n \) (say), of \( M_n(P) \) which itself contains \( P \). Furthermore, the \( I_n \) \((n = 1, 2, \cdots)\) are disjoint.

Now \( |Q - P| < 2^{1/2} \xi_1 \cdots \xi_n \) for \( Q \) in \( J_n \), and so certainly for \( Q \) in \( I_n \), and thus

\[
K_{2^a/q}(P) = \int_{M_0} |Q - P|^{a/q - 2} f(Q) \, dQ = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} \geq \sum_{n=1}^{\infty} \int_{I_n}.
\]

This last is not less than

\[
\sum_{n=1}^{\infty} (2^{1/2} \xi_1 \cdots \xi_n)^{a/q - 2}(\xi_1 \cdots \xi_n)^{-\alpha/q - \alpha/(qM_n - qM_{n+1})^2} (1 - 2\xi_{n+1})^2
\]

\[ = 2^{a/2q - 1} \sum_{n=1}^{\infty} (1 - 2\xi_{n+1})^2 n^{-1} = + \infty. \]
Consequently, $f_{a/q}(P)$ is infinite at every point of $M$, giving the required example in the case of $q > 2$, thus showing part (a) of Theorem 4 best possible.

For the case $q \leq 2$, let $\beta$ be any positive number less than $1 - \alpha$ and let $\xi$ be such that $2\xi^{1-(1-\alpha+\beta)/2} = 1$. Consider the set $M$ with $\xi_n = \xi$ for all $n$. Since $2\xi^\beta > 1$, $M$ is of positive $(\beta+1)$-capacity. Defining $f(P)$ as before, we use exactly the same argument to show that $f_{a/q}(P) = +\infty$ at every point of $M$. Furthermore, since $2\xi^{1-\alpha} < 1$, (2) is bounded, so that $f \in L^2$.

This shows part (b) of Theorem 4 best possible.

7. The lemma of §4 is best possible. Consider, e.g., (2) of the lemma. Suppose this is not the case, i.e. that there is an $e > 0$ for which, in general,

$$M_{q+\epsilon} \left[ \int_E |Q - P|^{a/q' - m} d\mu(Q) \right] < \infty.$$ 

If, then, $f(P) \in L^{(q+\epsilon)}$, we may say that

$$\left| \int_E S_n(P) d\mu(P) \right| \leq M_{(q+\epsilon)}(f) M_{q+\epsilon} \left[ \int_E |Q - P|^{a/q' - m} d\mu(Q) \right]$$

which is bounded. This would imply that (b) of Theorem 4 is not best possible. Since it is best possible we have shown (2) best possible. A similar argument using (a) would show (1) best possible.

References


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