HOMOTOPY RESOLUTIONS OF SEMI-SIMPLICIAL COMPLEXES

BY

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INTRODUCTION

One of the major problems of homotopy theory concerns the classification of topological spaces into homotopy types. This problem has as an obvious generalization that of the classification of spaces with operators into equivariant homotopy types.

We make here a contribution toward the solution of the latter problem. Our treatment stems from those of Eilenberg and MacLane, Postnikov, Olum and Zilber, in that it uses the singular complex of a topological space in order to restate topological problems as combinatorial ones. But as Eilenberg and Zilber have pointed out, singular complexes, considered as algebraic structures, may be subsumed in the more general category of semi-simplicial complexes [3]. Our procedure will be to drop completely all topological structure and to consider the classification as a purely combinatorial problem in the domain of these semi-simplicial complexes.

Thus the theory developed here is completely independent of topology. It is however almost completely parallel to the usual homotopy theory, and many of the same theorems will be found in it, though the proofs are often quite different.

The treatment is divided into four parts. Chapter I develops, after a brief resume of the fundamental facts about semi-simplicial complexes, the notion of semi-simplicial complexes with operators. The operators envisaged are the cognates in the combinatorial domain of topological groups. Thus the complexes with operators correspond to spaces with topological groups of operators. In the most important cases the group operates without fixed points; these complexes are to be thought of as analogous to principal fibre bundles.

Chapter II introduces the notion of homotopy groups of a semi-simplicial complex. These are to be used as coefficient domains for obstruction cochains, obstruction theory being the principal tool in the rest of the paper. The method here is quite unlike that used in the topological case; homotopy groups are characterized, simultaneously with obstructions, by a system of axioms. It is not however true that they can be defined for any semi-simplicial complex. Those complexes which have homotopy groups form a subcategory, and it is in this subcategory (which includes singular complexes) that the classification theory operates.

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In Chapter III there are adduced methods for the construction of complexes with homotopy. This is done with certain elementary structures: the universal bundles of operator groups, and the complexes $K(\Pi, n)$. The former are principal bundles all of whose homotopy groups vanish. The latter, introduced by Eilenberg and MacLane [1], are complexes having exactly one nonvanishing homotopy group. In addition a technique is developed for adding one homotopy group, in dimension n, say, to a complex all of whose homotopy groups vanish in dimensions greater than or equal to n.

In the last chapter the notion of a homotopy-resolution of a complex with homotopy on which a group operates without fixed points is introduced. This is a representation of the complex by means of another constructed, according to the recipes of Chapter III, out of the universal bundle of the group and the complexes $K(\Pi, n)$. In terms of these homotopy-resolutions the principal classification theorem can be stated: complexes are equivalent if and only if they have isomorphic resolutions.

The notion of a homotopy-resolution, which is built up dimension-wise, so to speak out of the homotopy groups of a complex, has a parallel in the theory of chain complexes with operators, in the notion of a homological resolution of such a complex [7]. It would seem that it represents a method of broad utility in topology. In the homotopy form it has been used before, somewhat less explicitly, by Postnikov [9]. There is certainly considerable overlap here with the results of the articles referred to; the exact amount is difficult to estimate since the method of Postnikov is rather different and the presentation, at least in the English version due to Hilton, somewhat cryptic. There is an even closer connection with unpublished work of J. A. Zilber, to whom the author must acknowledge his indebtedness for helpful conversations.

Chapter I

1. **Semi-simplicial complexes.** The semi-simplicial complexes to which we shall refer are always the complete semi-simplicial complexes of Eilenberg and Zilber [3]; we shall consistently omit the word "complete." The definition is that of the reference. We paraphrase it for notational convenience.

Let Δ_r, $r=0, 1, \cdots$, be a sequence of ordered simplices, and let Ω be the category of weakly order-preserving simplicial maps of these simplices into each other. Then a semi-simplicial complex consists of a collection of disjoint sets $K_0, K_1, \cdots$ together with a structure which associates to any $\sigma \in K_q$ and $\omega:\Delta_r \to \Delta_q$ in Ω an element $\sigma\omega$ of K_r in such a way that if ω' is another element of Ω, say $\omega':\Delta_s \to \Delta_r$, then

$$(\sigma\omega)\omega' = \sigma(\omega\omega'). \tag{1.1}$$

The elements of K_q are called q-simplices. If $\sigma \in K_q$ then $q = \dim \sigma$ is called the dimension of σ.

It is proved in [3] that any simplex σ in K can be written uniquely in the form

$$\sigma = \tau\omega$$

where τ is of minimum dimension. The simplex σ is said to be collapsed if $D\sigma = \dim \tau < \dim \sigma$. If for all $\sigma \in K$, $D\sigma = 0$, then K is said to be discrete.

In Ω the maps $e_i^q : \Delta_{q-1} \to \Delta_q$ are distinguished. These are the one-one maps whose images omit the ith vertex of Δ_q. If $\sigma \in K_q$, then $\sigma^i = \sigma e_i^q$ is called the ith face of σ. The maps e_i^q are easily seen to satisfy

$$e_i^q e_j^{q-1} = e_j^q e_{i-1}^{q-1}, \qquad i > j,$$

and thus

$$\sigma^{ij} = \sigma^{j,i-1}, \qquad i > j. \tag{1.2}$$

The integer chain complex $C(K)$ of K is defined by letting $C_q(K)$ be the free abelian group generated by K_q and setting, for $\sigma \in K_q$,

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma^i.$$

It follows from (1.2) that $\partial\partial = 0$. The chain and cochain complexes with any coefficients are defined accordingly, and give rise to the homology and cohomology groups of K.

The subgroups $C_q^d(K)$ generated by collapsed simplices of K_q constitute a subcomplex $C^d(K) \subset C(K)$. The quotient complex $C^n(K) = C(K)/C^d(K)$ is called the complex of normalized chains. For any coefficient group G, the subcochain complex $C_n(K; G)$ consisting of cochains which annihilate $C^d(K)$ is called the complex of normalized cochains. It is proved in [3] that $C^d(K)$ is always acyclic. It follows that for any coefficients, the natural maps $H_q(K; G) \to H_q(C^n(K) \otimes G)$ and $H^q(C_n(K, G)) \to H^q(K; G)$ are isomorphisms onto. That is, the homology and cohomology of K may also be computed by means of normalized chains and cochains.

Two 0-simplices α, β of K are said to be immediate neighbors if they are vertices of the same 1-simplex, i.e. if for some $\sigma \in K_1$, $\sigma^0 = \alpha$ and $\sigma^1 = \beta$ or vice versa. α and β are said to be in the same component of K if there are 0-simplices

$$\alpha = \gamma_0, \gamma_1, \cdots, \gamma_n = \beta$$

such that γ_i, γ_{i+1} are immediate neighbors. Two simplices of any dimension are in the same component of K if their vertices are. This is clearly an equivalence relation such that the components are subcomplexes of K (the definition of subcomplex is the obvious one). K is connected if it has only one component.

If K and L are semi-simplicial complexes, then $K \times L$ is the semi-simplicial complex defined by

$$(K \times L)_q = K_q \times L_q,$$
$$(\sigma, \tau)\omega = (\sigma\omega, \tau\omega).$$

For the homology of these products, the Künneth relations hold [4].

The prime example for semi-simplicial complexes is the singular complex of a topological space. If X is a space the singular complex $S(X)$ is defined by taking for $S(X)_q$ the set of all continuous maps $\sigma:\Delta_q \to X$ and letting $\sigma\omega$ be the map defined by composition.

If a simplicial complex has its vertices partially ordered in such a way that the vertices of any simplex are simply ordered, those simplices of its singular complex which are order-preserving simplicial maps form a subcomplex, the canonical semi-simplicial complex of the simplicial complex. It is well known that this inclusion gives rise to isomorphisms in homology.

We shall denote the canonical semi-simplicial complex of Δ_q by $\mathbf{\Delta}_q$.

If K and L are semi-simplicial complexes a simplicial map $f:K \to L$ is a map such that $f(K_q) \subset L_q$ for all q and such that if $\sigma \in K_q$ and $\omega:\Delta_r \to \Delta_q$ is in Ω then

$$f(\sigma\omega) = (f\sigma)\omega.$$

Since K_q freely generates $C_q(K)$ a simplicial map gives rise to a chain map $f: C_*(K) \to C_*(L)$ and thus to homomorphisms f_* in homology and f^* in cohomology.

If σ is any q-simplex of a semi-simplicial complex K then there is a unique simplicial map

$$\sigma^\Delta:\mathbf{\Delta}_q \to K$$

with $\sigma^\Delta \Delta_q = \sigma$.

We shall rather loosely use I to stand for any semi-simplicial complex arising from a decomposition of the unit interval. Collapsed simplices over the endpoints will be written 0, 1.

If $f_0, f_1: K \to L$ are simplicial, a simplicial homotopy of f_0 and f_1 is a simplicial map $F: K \times I \to L$ such that for all $\sigma \in K$

$$F(\sigma, 0) = f_0\sigma, \qquad F(\sigma, 1) = f_1\sigma.$$

A simplicial homotopy induces a chain homotopy in the following fashion.

Let $a_q \in C_{q+1}(\mathbf{\Delta}_q \times I)$ be that fundamental cycle of the prism modulo its boundary for which $(\Delta_q, 1)$ appears with positive sign in ∂a_q. Then define $D: C_*(K) \to C_*(K \times I)$, of degree $+1$, by $D\sigma = (\sigma^\Delta \times j)a_q$, where j is the identity map of I. We then have

$$\partial D\sigma + D\partial\sigma = (\sigma, 1) - (\sigma, 0)$$

and thus $\tilde{F} = FD$ is a chain homotopy of f_0 and f_1.

The q-skeleton of a semi-simplicial complex K (where $q = 0, 1, \cdots$) con-

sists not of the simplices of dimension less than or equal to q, for these do not make up a subcomplex, but rather of all those simplices σ such that $D\sigma \leqq q$. The notation K^q will be used for the q-skeleton.

If, again, $f_0, f_1: K \to L$, a homotopy through dimension q of f_0 and f_1 is a simplicial map $F: K^q \times I \to L$ which is a homotopy of $f_0 | K^q$ and $f_1 | K^q$.

Two simplicial maps are homotopic if there is a homotopy connecting them; if for every q there is a homotopy through dimension q connecting them, they are weakly homotopic. Clearly homotopy implies weak homotopy.

If $f_0, f_1: K \to L$ are weakly homotopic, then on each skeleton of the chain-complexes of K and L the chain maps induced by f_0 and f_1 are chain-homotopic. It follows that the induced homomorphisms in homology and cohomology are equal:

$$f_{0*} = f_{1*}, \qquad f_0^* = f_1^*,$$

and thus that the homology and cohomology functors are weak homotopy invariants.

2. **Semi-simplicial groups.** A *semi-simplicial group* Γ is a semi-simplicial complex together with a simplicial map $\Gamma \times \Gamma \to \Gamma$ which, restricted to each of the sets Γ_q, is a group composition. This definition is prompted by the behavior of the singular complex of a topological group(1). In fact, if G is a topological group, then $S(G)$ is a semi-simplicial group with the composition

$$(\sigma\tau)y = (\sigma y)(\tau y) \qquad \sigma, \tau \in S(G)_q, \ y \in \Delta_q.$$

A subgroup of Γ is a subcomplex Λ such that for each q, Λ_q is a subgroup of Γ_q. If each Λ_q is a normal subgroup, Λ is a normal subgroup of Γ.

An operation of a semi-simplicial group Γ on a semi-simplicial complex K is a simplicial map $\Gamma \times K \to K$ such that if $\sigma, \sigma' \in \Gamma_q$, $\tau \in K_q$ then

$$(\sigma\sigma')\tau = \sigma(\sigma'\tau).$$

If Γ operates on K, we shall say that K is a Γ-*complex*. If in addition the operation is such that $\sigma\tau = \tau$ implies that σ is the identity, we shall say that Γ operates freely, and that K is a Γ-*bundle*. A semi-simplicial group operates on itself by left translations, the operation being of course free.

If K is a Γ-complex, the orbit-complex K/Γ is defined as follows: $(K/\Gamma)_q$ is the set of equivalence classes in K_q under the relations

$$\tau \sim \tau' \text{ if and only if } \tau = \sigma\tau' \text{ for some } \sigma \in \Gamma_q;$$

while for $\rho \in (K/\Gamma)_q$, if $\omega: \Delta_r \to \Delta_q$ is in Ω, $\rho\omega$ is the class of $\tau\omega$ for any τ in ρ.

(1) The chain complex, with any group of coefficients, of a semi-simplicial complex, is of course an *FD*-complex in the sense of Eilenberg-MacLane [2]. The chain-complex of a semi-simplicial group, when the coefficients are in a ring, forms an *R*-complex in the sense of the above-mentioned paper, when the multiplication in each dimension is defined by regarding the chain group as the group algebra of the group of simplices of that dimension.

If a topological group G operates on a topological space X then $S(X)$ is a $S(G)$-complex under an operation analogous to that given above for $S(G)$. If X is a principal bundle under the operation of G, then $S(X)$ is a $S(G)$ bundle. Moreover, $S(X)/S(G)$ is canonically isomorphic to the singular complex of the base-space.

A subgroup Λ of a semi-simplicial group Γ operates freely on Γ by left translation. Γ/Λ is the "coset space"; if Λ is a normal subgroup, Γ/Λ inherits a group structure from Γ.

If Γ and Γ' are semi-simplicial groups, a homomorphism $\phi:\Gamma\to\Gamma'$ is a simplicial map which is a homomorphism of the group structures in each Γ_q, Γ'_q.

If K and K' are Γ and Γ'-complexes respectively, and $\phi:\Gamma\to\Gamma'$ is a homomorphism, a simplicial map $f:K\to K'$ is ϕ-equivariant if for any simplices $\sigma\in\Gamma$, $\tau\in K$ of the same dimension

$$f(\sigma\tau) = (\phi\sigma)(f\tau). \tag{2.1}$$

If $\Gamma=\Gamma'$ and ϕ is the identity, we may say that f is Γ-equivariant, or just equivariant. The class of ϕ-equivariant maps $f:K\to K'$ is denoted by $\mathcal{E}(\phi; K, K')$ or, if ϕ is the identity, by $\mathcal{E}(\Gamma; K, K')$.

If K is a Γ-complex, the map $\eta:K\to K/\Gamma$ which takes simplices into their equivalence classes is simplicial. In the situation of (2.1), there is a unique simplicial map of K/Γ into K'/Γ' which commutes with these maps:

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ \eta\downarrow & & \downarrow\eta' \\ K/\Gamma & \to & K'/\Gamma' \end{array}$$

We denote it by f/ϕ, or, if ϕ is the identity, by f/Γ.

If K is a Γ-complex and Λ is a subgroup of Γ then K is of course a Λ-complex under the restriction to Λ of the operation of Γ. If Λ is a normal subgroup then K/Λ is a Γ/Λ-complex under the operation given by

$$(\theta\sigma)(\eta\tau) = \eta(\sigma\tau), \qquad \sigma\in\Gamma_q,\ \tau\in K_q$$

where $\theta:\Gamma\to\Gamma/\Lambda$ and $\eta:K\to K/\Lambda$ are the canonical maps. To see this, it is only necessary to verify that if $\theta\sigma'=\theta\sigma$ and $\eta\tau'=\eta\tau$ then $\eta(\sigma'\tau')=\eta(\sigma\tau)$. But these conditions imply that $\sigma'=\lambda\sigma$ and $\tau'=\mu\tau$ for some λ, $\mu\in\Lambda_q$. Since Λ is a normal subgroup, $\sigma\mu=\nu\sigma$ for some $\nu\in\Lambda_q$. Thus

$$\rho(\sigma'\tau') = \eta(\lambda\sigma\mu\tau) = \eta(\lambda\nu\sigma\tau) = \eta(\sigma\tau).$$

If Γ and Γ' are semi-simplicial groups and Λ, Λ' are normal subgroups, and $\phi:\Gamma\to\Gamma'$ is a homomorphism such that $\phi(\Lambda)\subset\Lambda'$, then, as is easily seen,

$$\phi/\phi_1:\Gamma/\Lambda\to\Gamma'/\Lambda'$$

is also a homomorphism, where $\phi_1 = \phi | \Lambda$. If K and K' are Γ and Γ'-complexes, and $f: K \to K'$ is ϕ-equivariant, then $f/\phi_1: K/\Lambda \to K'/\Lambda'$ is ϕ/ϕ_1-equivariant.

If K is a Γ-complex and $f: X \to K/\Gamma$, the *induced* Γ-*complex* $\hat{f}^{-1}(K)$ is defined by letting $\hat{f}^{-1}(K)$ be that subcomplex of $X \times K$ consisting of simplices (σ, τ) such that $f\sigma = \eta\tau$, where again $\eta: K \to K/\Gamma$ is the canonical map, and defining an operation of Γ on $\hat{f}^{-1}(K)$ by $\gamma(\sigma, \tau) = (\sigma, \gamma\tau)$ for γ a simplex of appropriate dimension in Γ. The map $\hat{f}: \hat{f}^{-1}(K) \to K$ defined by $\hat{f}(\sigma, \tau) = \tau$ is clearly equivariant. The complex $\hat{f}^{-1}(K)/\Gamma$ may be identified with X; under this identification, $\hat{f}/\Gamma = f$.

LEMMA 2.2. *If K and L are Γ-complexes and $f: L \to K$ is equivariant, then there exists an equivariant map $F: L \to (f/\Gamma)^{\frown -1} K$ such that F/Γ is the identity on L/Γ.*

It is given, of course, by $F\sigma = (\mu\sigma, f\sigma)$, where $\mu: L \to L/\Gamma$ is the canonical map.

For a Γ-bundle X we define the notion of the *difference-map*. If $\eta: X \to X/\Gamma$ is the canonical map, then

$$Z = \{(\sigma, \tau) \mid \eta\sigma = \eta\tau\} \subset X \times X$$

is a subcomplex, and

$$\Psi(\sigma, \tau)\tau = \sigma$$

defines a simplicial map $\Psi: Z \to \Gamma$, the difference-map of X with respect to Γ. Several properties are worth noting. If σ, τ, ρ are in X and $\eta\sigma = \eta\tau = \eta\rho$ then

$$\Psi(\sigma, \tau)\Psi(\tau, \rho) = \Psi(\sigma, \rho). \tag{2.3}$$

If γ, γ' are simplices of the appropriate dimension in Γ,

$$\Psi(\gamma\sigma, \gamma'\tau) = \gamma\Psi(\sigma, \tau)\gamma'^{-1}. \tag{2.4}$$

The difference map is also natural under equivariant maps: if X' is a Γ'-bundle, $\phi: \Gamma \to \Gamma'$ is a homomorphism and $f: X \to X'$ is ϕ-equivariant then, if Ψ' is the difference-map of X' with respect to Γ',

$$\Psi'(f\sigma, f\tau) = \phi\Psi(\sigma, \tau). \tag{2.5}$$

If Λ is a subgroup of Γ, the difference-map of X with respect to Λ is a restriction of the difference-map Ψ. Denoting this restriction by $\Psi_1: Z_1 \to \Lambda$ we have, if Λ is a normal subgroup and $\gamma \in \Gamma$,

$$\Psi_1(\gamma\sigma, \gamma\tau) = \gamma\Psi_1(\sigma, \tau)\gamma^{-1}. \tag{2.6}$$

But Ψ_1 is in fact defined even if X is not a Γ-bundle, as long as Λ operates freely. It is clear that (2.6) holds in this case also.

To say that a semi-simplicial group Γ operates on a semi-simplicial group Λ means that Γ operates on the underlying complex of Λ in such a way that

for each q, Γ_q operates as a group of automorphisms on Λ_q. In this situation we shall write the operation with an interposed "$\cdot$".

When Γ operates on Λ, the product complex $\Lambda\times\Gamma$ becomes a group, the *splitting extension of* Γ *by* Λ, under the multiplication

$$(\lambda', \gamma')(\lambda, \gamma) = (\lambda'[\gamma'\cdot\lambda], \gamma'\gamma).$$

This group will be denoted by $\Lambda\cdot\Gamma$; if the operation of Γ on Λ is trivial, it is just the product group, i.e. the product group in each dimension, and may be denoted by $\Lambda\times\Gamma$.

The subgroups $\{(1, \gamma)\}$ and $\{(\lambda, 1)\}$ of $\Lambda\cdot\Gamma$, where 1 stands for the appropriate unit element, may be identified with Γ and Λ respectively. The latter is an invariant subgroup. The definition of the group operation in $\Lambda\cdot\Gamma$ is then just that one under which the operation of Γ on Λ is given by inner automorphisms:

$$\gamma\cdot\lambda = \gamma\lambda\gamma^{-1}. \tag{2.7}$$

If another group Γ' operates on a group Λ', if $\Phi:\Gamma\to\Gamma'$ is a homomorphism, and $\phi:\Lambda\to\Lambda'$ is a Φ-equivariant homomorphism, then it is easy to see that $\phi\times\Phi:\Lambda\cdot\Gamma\to\Lambda'\cdot\Gamma'$ is also a homomorphism.

A semi-simplicial complex K is a $\Lambda\cdot\Gamma$-complex if and only if it is both a Λ-complex and a Γ-complex and the operations behave in the following fashion under commutation:

$$\gamma[\lambda\sigma] = (\gamma\cdot\lambda)[\gamma\sigma]. \tag{2.8}$$

If K is a $\Lambda\cdot\Gamma$-complex, K' is a $\Lambda'\cdot\Gamma'$-complex and $f:K\to K'$ is both Φ and ϕ-equivariant, then f is $\phi\times\Phi$-equivariant.

If Γ is a semi-simplicial group and $\alpha, \beta\in\Gamma_0$ are immediate neighbors, then for any $\gamma\in\Gamma_0$, $\gamma\alpha$ and $\gamma\beta$ are also immediate neighbors. For if α and β are vertices of a 1-simplex σ, then $\gamma\alpha$ and $\gamma\beta$ are vertices of $\gamma'\sigma$, where γ' is the collapsed 1-simplex lying over γ. It follows that the component of the identity ${}^0\Gamma\subset\Gamma$ is a normal subgroup. The quotient group $\Gamma/{}^0\Gamma$ is of course discrete.

For any semi-simplicial group Γ the incidence operation $\sigma\to\sigma\omega$, for $\omega:\Delta_r\to\Delta_q$ in Ω is a homomorphism of Γ_q into Γ_r. A discrete group is characterized by the fact that it is an isomorphism, depending only on q and r but not ω. Thus a discrete semi-simplicial group consists of a sequence of groups and a transitive system of isomorphisms and is invariantly determined by, say, the 0-dimensional group. This determination sets up a one-to-one correspondence between ordinary discrete groups and discrete semi-simplicial groups. For any semi-simplicial group Γ, the ordinary discrete group corresponding to $\Gamma/{}^0\Gamma$ will be denoted by $\pi_0(\Gamma)$.

We shall say that a semi-simplicial group Γ operates on a discrete group Π if it operates on the corresponding semi-simplicial group; identifying all the groups of simplices with Π, this simply means that each Γ_q operates on

Π in such a way that for any suitable σ, ω and $p \in \Pi$,

$$(\sigma\omega)p = \sigma p. \tag{2.9}$$

In such a case it is easy to see that the component of the identity operates trivially. Thus the operations of Γ on Π are in natural one-to-one correspondence with the operations of $\pi_0(\Gamma)$ on Π.

3. **Two theorems on Γ-bundles.** If K and L are semi-simplicial complexes, $f: K \to L$, and for some subcomplex L' of L a map $g: L' \to K$ is such that fg is the identity on L', we say that g is a *cross-section of f on L'*. The set of such cross-sections is denoted by $\chi(f; L')$ or, if $L' = L$, by $\chi(f)$.

If X and Y are Γ-complexes we shall in general impose on $X \times Y$ the operation of Γ given by $\gamma(\sigma, \tau) = (\gamma\sigma, \gamma\tau)$. This makes $X \times Y$ a Γ-complex; if either X or Y is a Γ-bundle, so is $X \times Y$.

Now suppose X and Y are Γ-complexes and Λ is a normal subgroup of Γ, and let $p: X \times Y \to X$ be the projection. We shall define a map

$$\mathcal{W}: \mathcal{E}(\Gamma; X, Y) \to \chi(p/\Lambda) \cap \mathcal{E}(\Gamma/\Lambda; X/\Lambda, (X \times Y)/\Lambda).$$

It is convenient to do this in two stages, introducing temporarily the factorization $\mathcal{W} = \mathcal{B}\mathcal{A}$ where

$$\mathcal{A}: \mathcal{E}(\Gamma; X, Y) \to \chi(p) \cap \mathcal{E}(\Gamma; X, X \times Y)$$

and

$$\mathcal{B}: \chi(p) \cap \mathcal{E}(\Gamma; X, X \times Y) \to \chi(p/\Lambda) \cap \mathcal{E}(\Gamma/\Lambda; X/\Lambda, (X \times Y)/\Lambda)$$

are given by $(\mathcal{A}f)\sigma = (\sigma, f\sigma)$ and $\mathcal{B}g = g/\Lambda$. In order to see that $\mathcal{W}f$ is a cross-section of p/Λ it is sufficient to consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathcal{A}f} & X \times Y & \xrightarrow{p} & X \\ \eta \downarrow & & \downarrow \eta' & & \downarrow \eta \\ X/\Lambda & \xrightarrow{\mathcal{W}f} & (X \times Y)/\Lambda & \xrightarrow{p/\Lambda} & X/\Lambda \end{array}$$

where the vertical maps are the canonical projections, for $p(\mathcal{A}f)$ is the identity on X.

THEOREM 3.1. *If Λ is a normal subgroup of the semi-simplicial group Γ, and if X and Y are Γ-complexes such that X is a Λ-bundle, then if $\mathcal{W}$ is the map defined above*

$$\mathcal{W}: \mathcal{E}(\Gamma; X, Y) \approx \chi(p/\Lambda) \cap \mathcal{E}(\Gamma/\Lambda; X/\Lambda, (X \times Y)/\Lambda).$$

It is clear that in any case $\mathcal{A}$ has an inverse. The proof proceeds by exhibiting an inverse for $\mathcal{B}$.

Let Ψ and Ψ' be the difference-maps of X and $X \times Y$ respectively, with respect to Λ. For $h \in \chi(p/\Lambda) \cap \mathcal{E}(\Gamma/\Lambda; X/\Lambda, (X \times Y)/\Lambda)$ and $\sigma \in X$ we define

$$(\mathcal{C}h)\sigma = \Psi(\sigma, p\tau)\tau$$

where τ is any simplex in $X \times Y$ such that $\eta'\tau = h\eta\sigma$. This is in fact independent of τ. If τ' is another such simplex, then

$$\Psi(\sigma, p\,\tau')\tau' = \Psi(\sigma, p\tau)\Psi(p\tau, p\tau')\tau' = \Psi(\sigma, p\tau)\Psi'(\tau, \tau')\tau' = \Psi(\sigma, p\tau)\tau$$

by (2.2, 2.3, 2.5). Now $\mathcal{C}h$ is a simplicial map of X into $X \times Y$: since Ψ and the operation of the group are simplicial, if ω is an appropriate map in Ω then

$$(\mathcal{C}h)(\sigma\omega) = \Psi(\sigma\omega, p\tau\omega)(\tau\omega) = [\Psi(\sigma, p\tau)\tau]\omega.$$

Moreover, $\mathcal{C}h$ is a cross-section of p, for

$$p(\mathcal{C}h)\sigma = p\Psi(\sigma, p\tau)\tau = \Psi(\sigma, p\tau)p\tau = \sigma,$$

since p is equivariant. Furthermore, $(\mathcal{C}h)$ is Γ-equivariant. If we write $\theta: \Gamma \to \Gamma/\Lambda$ for the canonical homomorphism then η and η' are θ-equivariant. Also, if $\eta'\tau = h\eta\sigma$ and $\gamma \in \Gamma$ then

$$\eta'\gamma\tau = (\theta\gamma)(h\eta\sigma) = h[(\theta\gamma)(\eta\sigma)] - h\eta(\gamma\sigma).$$

Thus

$$(\mathcal{C}h)(\gamma\sigma) = \Psi(\gamma\sigma, p\gamma\tau)(\gamma\tau) = \gamma\Psi(\sigma, p\tau)\gamma^{-1}\gamma\tau = \gamma[(\mathcal{C}h)\sigma],$$

by (2.6). Finally,

$$(\mathcal{B}\mathcal{C}h)\eta\sigma = \eta'[(\mathcal{C}h)\sigma] = \eta'[\Psi(\sigma, p\tau)\tau] = [\theta\Psi(\sigma, p\tau)]h\eta\sigma = h\eta\sigma$$

since $\Psi(\sigma, \rho\tau) \in \Lambda$, which is the kernel of θ. On the other hand, for $g \in \chi(p) \cap \mathcal{E}(\Gamma; X, X \times Y)$ and $\sigma \in X$,

$$(\mathcal{C}\mathcal{B}g)\sigma = \Psi(\sigma, p\tau)\tau$$

where $\eta'\tau = (\mathcal{B}g)\eta\sigma = \eta' g\sigma$. Since this is independent of τ, we may take $\tau = g\sigma$. Thus

$$(\mathcal{C}\mathcal{B}g)\sigma = \Psi(\sigma, pg\sigma)g\sigma = \Psi(\sigma, \sigma)g\sigma = g\sigma.$$

This completes the proof of Theorem 3.1. It is convenient to have explicit formulae for $\mathcal{W}$ and $\mathcal{W}^{-1}$. These are the following. Retaining all the above notation,

$$\begin{aligned} (\mathcal{W}f)\eta\sigma &= \eta'(\sigma, f\sigma), \\ (\mathcal{W}^{-1}h)\sigma &= \rho \end{aligned} \tag{3.2}$$

where ρ is the unique simplex in Y such that $\eta'(\sigma, \rho) = h\eta\sigma$.

If K and K' are, respectively, Γ and Γ'-complexes, if $\Phi: \Gamma \to \Gamma'$ is a homomorphism, and $f: K/\Gamma \to K'/\Gamma'$, a lifting of f is a Φ-equivariant map $F: K \to K'$ such that $F/\Phi = f$. The set of liftings is denoted by $\mathcal{L}(f; \Phi)$.

Now suppose that Γ and Γ' are semi-simplicial groups operating, respec-

tively, on semi-simplicial groups Λ and Λ', that $\Phi:\Gamma\to\Gamma'$ is a homomorphism, and $\phi:\Lambda\to\Lambda'$ is a Φ-equivariant homomorphism. Suppose in addition that X and X' are, respectively, $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$-complexes, and that $f:X/\Lambda\to X'/\Lambda'$ is Φ-equivariant. We shall want to investigate the set $\mathcal{L}(f;\phi)\cap\mathcal{E}(\Phi;X,X')$ of $\phi\times\Phi$-equivariant liftings of f.

In order to do this we define first an operation of Γ on $\Lambda\times\Lambda'$ by

$$\gamma\cdot(\lambda,\lambda') = (\gamma\cdot\lambda,\ [\Phi\gamma]\cdot\lambda').$$

Next we define an operation of the splitting extension $(\Lambda\times\Lambda')\cdot\Gamma$ on the complex

$$Y = (f\eta)^{\frown -1}X' = \{(\sigma,\sigma')\mid \eta'\sigma' = f\eta\sigma\}\subset X\times X'$$

(where $\eta:X\to X/\Lambda$ and $\eta':X'\to X'/\Lambda'$ are the canonical maps) by

$$(\lambda,\lambda',\gamma)(\sigma,\sigma') = (\lambda\gamma\sigma,\ \lambda'[\Phi\gamma]\sigma').$$

Finally, we define an operation, which we shall denote by "o," of $(\Lambda\times\Lambda')\cdot\Gamma$ on the complex Λ', an operation in which the operators are not automorphisms:

$$(\lambda,\lambda',\gamma)\circ\lambda_0' = \Phi\gamma\cdot[(\phi\lambda)\lambda_0'\lambda'^{-1}].$$

We are now in a position to define a map $\mathcal{T}:\mathcal{E}((\Lambda\times\Lambda')\cdot\Gamma;\ Y,\Lambda')\to\mathcal{L}(f;\phi)\cap\mathcal{E}(\Phi;X,X')$ by setting, for $G:Y\to\Lambda'$ equivariant, and $\sigma\in X$,

$$(\mathcal{T}G)\sigma = G(\sigma,\sigma')\sigma'$$

where σ' is any simplex in X' such that $(\sigma,\sigma')\in Y$. Let us observe first that this is independent of σ'. For if σ'' is another such simplex in X' then $\sigma''=\lambda'\sigma'$ for some $\lambda'\in\Lambda'$, and thus $(\sigma,\sigma'')=(1,\lambda',1)(\sigma,\sigma')$ and

$$G(\sigma,\sigma'')\sigma'' = [(1,\lambda',1)\circ G(\sigma,\sigma')](\lambda'\sigma') = G(\sigma,\sigma')\lambda'^{-1}\lambda'\sigma' = G(\sigma,\sigma')\sigma'.$$

It is easy to see that $\mathcal{T}G$ is simplicial. To justify the definition it is necessary to show that $\mathcal{T}G$ is $\phi\times\Phi$-equivariant and that $(\mathcal{T}G)/\phi=f$. But since, for $\lambda\in\Lambda$, $\eta\lambda\sigma=\eta\sigma$,

$$(\mathcal{T}G)\lambda\sigma = G(\lambda\sigma,\sigma')\sigma' = [(\lambda,1,1)\circ G(\sigma,\sigma')]\sigma' = (\phi\lambda)G(\sigma,\sigma')\sigma'$$

while for $\gamma\in\Gamma$, since $\eta'(\Phi\gamma)\sigma'=(\Phi\gamma)\eta'\sigma'$ and thus $(\gamma\sigma,(\Phi\gamma)\sigma')=(1,1,\gamma)\cdot(\sigma,\sigma')\in Y$,

$$(\mathcal{T}G)\gamma\sigma = [(1,1,\gamma)\circ G(\sigma,\sigma')]\sigma' = (\Phi\gamma)G(\sigma,\sigma')\sigma',$$

the equivariance with respect to $\phi\times\Phi$ holds. As to $(\mathcal{T}G)/\phi$,

$$[(\mathcal{T}G)/\phi]\eta\sigma = \eta'(\mathcal{T}G)\sigma' = \eta'G(\sigma,\sigma')\sigma' = \eta'\sigma' = f\eta\sigma.$$

THEOREM 3.3. *If $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$ are splitting semi-simplicial group extensions, $\phi\times\Phi$: $\Lambda\cdot\Gamma\to\Lambda'\cdot\Gamma'$ is a homomorphism, X and X' are $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$-*

complexes such that X' is a Λ'-bundle and $f: X/\Lambda \to X'/\Lambda'$ is Φ-equivariant, then, $\mathcal{T}$ being the map defined above,

$$\mathcal{T}: \mathcal{E}((\Lambda \times \Lambda')\cdot\Gamma;\, Y, \Lambda') \approx (\mathcal{L}f;\, \phi) \cap \mathcal{E}(\Phi;\, X, X').$$

The inverse $\mathcal{T}^{-1}$ of $\mathcal{T}$ is defined as follows. If $F \in \mathcal{L}(f;\, \phi) \cap \mathcal{E}(\Phi;\, X, X')$ and $(\sigma, \sigma') \in Y$ then

$$(\mathcal{T}^{-1}F)(\sigma, \sigma') = \Psi(F\sigma, \sigma')$$

where Ψ' is the difference-map of X' with respect to Λ' (cf. 2.4, 2.6). To see that $\mathcal{T}^{-1}$ is in fact inverse to $\mathcal{T}$ we may write

$$(\mathcal{T}^{-1}\mathcal{T}G)(\sigma, \sigma') = \Psi'((\mathcal{T}G)\sigma, \sigma') = \Psi'(G(\sigma, \sigma')\sigma', \sigma') = G(\sigma, \sigma')$$

and

$$(\mathcal{T}\mathcal{T}^{-1}F)\sigma = (\mathcal{T}^{-1}F)(\sigma, \sigma') = \Psi'(F\sigma, \sigma')\sigma' = F\sigma.$$

REMARK. If in the above situation X is also a Λ-bundle, then Y is a $\Lambda \times \Lambda'$-bundle and Theorems 3.1 and 3.3 can be combined to give a one-to-one correspondence between the cross-sections of $p/\Lambda \times \Lambda'$ where $p: Y \times \Lambda' \to Y$ is the projection and the liftings of f, more precisely, between the sets $\chi(p/\Lambda \times \Lambda') \cap \mathcal{E}(\Gamma;\, X/\Lambda,\, (Y \times \Lambda')/\Lambda \times \Lambda')$ and $\mathcal{L}(f;\, \phi) \cap \mathcal{E}((\Lambda \times \Lambda')\cdot\Gamma;\, Y, \Lambda')$, after making the identification $Y/\Lambda \times \Lambda' = X/\Lambda$.

In this form, the topological analogue of the theorem was used by Hu (cf. [8], where it is attributed to Ehresmann) in the special case $\Gamma = \{1\}$, $\Lambda = \Lambda'$, $X = X'$, ϕ and f the identities. The topological analogue of (3.1) was proved by the author [5] in the case $\Gamma = \{1\}$.

4. Equivariant homology and cohomology. If K is a Γ-complex and Γ operates on a discrete abelian group Π the equivariant homology and cohomology groups of K are defined as follows.

In the chain-complex $C_*(K;\, \Pi) = C_*(K) \otimes \Pi$ the subgroup generated by chains of the form

$$(\gamma\sigma) \otimes p - \sigma \otimes (\gamma p)$$

where $\gamma \in \Gamma$, $\sigma \in K$, $p \in \Pi$, form a subcomplex $C_*^{R,\Gamma}(K;\, \Pi)$, the so-called complex of residual chains, for by (2.9),

$$\begin{aligned}\partial[(\gamma\sigma) \otimes p - \sigma \otimes (\gamma p)] &= \sum (-1)^i(\gamma\sigma)^i \otimes p - \sum(-1)^i\sigma^i \otimes (\gamma p)\\ &= \sum (-1)^i[(\gamma^i\sigma^i) \otimes p - \sigma^i \otimes (\gamma^i p)].\end{aligned}$$

The quotient $C_*^{\Gamma}(K;\, \Pi) = C_*(K;\, \Pi)/C_*^{R,\Gamma}(K;\, \Pi)$ is the complex of equivariant chains. Its homology, denoted by $H_*^{\Gamma}(K;\, \Pi)$, is the equivariant homology of K with respect to Γ, with coefficients in Π.

The cochains $c \in C^*(K;\, \Pi)$ such that for $\sigma \in K$, $\gamma \in \Gamma$,

$$c(\gamma\sigma) = \gamma(c\sigma)$$

form a subcomplex $C_*^\Gamma(K;\Pi)$, the subcomplex of equivariant cochains, whose homology $H_\Gamma^*(K;\Pi)$ is the equivariant cohomology of K with respect to Γ, with coefficients in Π.

If K' is a Γ'-complex, $\phi:\Gamma\to\Gamma'$ is a homomorphism and $f:K\to K'$ is ϕ-equivariant then, if Γ' operates also on Π in such a way that $(\phi\gamma)p=\gamma p$ for $\gamma\in\Gamma$, $p\in\Pi$, the chain map induced by f takes residual chains into residual chains:

$$f[(\gamma\sigma)\otimes p-\sigma\otimes(\gamma p)]=(\phi\gamma)(f\sigma)\otimes p-f\sigma\otimes(\phi\gamma)p.$$

Thus f defines a chain map on equivariant chains and a homomorphism $f_*:H_*^\Gamma(K;\Pi)\to H_*^{\Gamma'}(K';\Pi)$.

Similarly, if c is an equivariant cochain in K' then cf is an equivariant cochain in K. Thus f defines also homomorphisms $f^*:H_{\Gamma'}^*(K';\Pi)\to H_\Gamma^*(K;\Pi)$.

If Γ operates also on Π' and $\theta:\Pi\to\Pi'$ is a Γ-equivariant homomorphism, then θ induces a chain map of $C_*(K;\Pi)$ into $C_*(K;\Pi')$ by taking $\sigma\otimes p$ into $\sigma\otimes\theta p$. It is easy to see that this takes residual chains into residual chains and that θ thus induces homomorphisms $\theta_*:H_*^\Gamma(K;\Pi)\to H_*^\Gamma(K;\Pi')$. But θ also induces a cochain map of $C^*(K;\Pi)$ into $C^*(K;\Pi')$ by taking a cochain c into θc. This map takes equivariant cochains into equivariant cochains and thus permits the definition of homomorphisms $\theta_*:H_\Gamma^*(K;\Pi)\to H_\Gamma^*(K;\Pi')$.

Finally, if Γ' operates also on Π' in such a way that θ is Γ'-equivariant and $(\phi\gamma)p'=\gamma p'$ for $\gamma\in\Gamma$, $p'\in\Pi'$, then commutativity holds in the diagrams

$$\begin{array}{ccc} H_*^\Gamma(K;\Pi) & \xrightarrow{f_*} & H_*^{\Gamma'}(K';\Pi) \\ \theta_*\downarrow & & \downarrow\theta_* \\ H_*^\Gamma(K;\Pi') & \xrightarrow[f_*]{} & H_*^{\Gamma'}(K';\Pi') \end{array}$$

$$\begin{array}{ccc} H_\Gamma^*(K;\Pi) & \xleftarrow{f^*} & H_{\Gamma'}^*(K';\Pi) \\ \theta_*\downarrow & & \downarrow\theta_* \\ H_\Gamma^*(K;\Pi') & \xleftarrow[f^*]{} & H_{\Gamma'}^*(K';\Pi'). \end{array}$$

The consideration of homotopies of equivariant maps rests on the following observation. If $\sigma\in K_q$ and $\gamma\in\Gamma_q$ then (cf. §1) $D(\gamma\sigma)=((\gamma\sigma)^\Delta\times j)a_q$ is a chain in $C_{q+1}(K\times I)$ which results from operating on each simplex in $D\sigma$ by a collapsed simplex lying over γ (the operation on $K\times I$ results of course from the prescribed operation on K and the trivial one on I). Since such a simplex in Γ operates on a coefficient group in the same way as γ the chain homotopy FD, where $F:K\times I\to K'$ is ϕ-equivariant, takes residual chains into residual chains. Similarly, the associated cochain homotopy takes equivariant cochains into equivariant cochains.

Thus if $f_0, f_1: K \to K'$ are ϕ-equivariant and are connected by a ϕ-equivariant homotopy the induced homomorphisms on equivariant homology and cohomology groups are equal:

$$f_{0*} = f_{1*}, \qquad f_0^* = f_1^*.$$

We must also talk about partial equivariant homotopies, i.e. equivariant homotopies through dimension q. In order to do this it is necessary to observe first that a subcomplex $L \subset K$ is not necessarily invariant, and that in particular the skeletons need not be invariant. The smallest invariant subcomplex containing L is of course $\Gamma L = \{\gamma\sigma \mid \gamma \in \Gamma, \sigma \in L\}$.

We make then the following definition. An equivariant homotopy through dimension q of f_0 and f_1 is an equivariant homotopy $F: \Gamma K^q \times I \to K'$ (observe that $\Gamma(K^q \times I) = \Gamma K^q \times I$)) of $f_0 | \Gamma K^q$ and $f_1 | \Gamma K^q$.

Just as we did in §1 we may define weak equivariant homotopy of equivariant maps and conclude that equivariant homology and cohomology are weak equivariant homotopy invariants.

If X is a Γ-complex, Γ operates on the discrete abelian group Π, and Λ is a normal subgroup of Γ such that the operation on Π restricted to Λ is trivial, then Γ/Λ operates on Π by $(\phi\gamma)p = \gamma p$, where $\phi: \Gamma \to \Gamma/\Lambda$ is the canonical homomorphism, and the canonical map $\eta: X \to X/\Lambda$ is ϕ-equivariant. Thus η induces homomorphisms $\eta_*: H_*^\Gamma(X; \Pi) \to H_*^{\Gamma/\Lambda}(X/\Lambda; \Pi)$ and $\eta^*: H_{\Gamma/\Lambda}^*(X/\Lambda; \Pi) \to H_\Gamma^*(X; \Pi)$. Such maps, or rather similar ones, can also be defined without the restriction that Λ operate trivially on Π, if homology with local coefficients (Überdeckungen) is introduced; however, it is not necessary to make use of this generalization here.

THEOREM 4.1. *η_* and η^* are isomorphisms onto.*

We omit the proof for the case of homology, which will not be used below. For cohomology, we observe that the cochain map of $C_{\Gamma/\Lambda}^*(X/\Lambda; \Pi)$ into $C_\Gamma^*(X; \Pi)$ by η is, since η is onto, an isomorphism into. It is thus sufficient to show that it is onto as well. But suppose that c is a Γ-equivariant cochain on X. Then

$$\bar{c}(\eta\sigma) = c\sigma$$

defines, since Λ operates trivially on Π, a unique cochain on X/Λ for which, evidently, $\bar{c}\eta = c$. To see that c is Γ/Λ-equivariant, we may write

$$\bar{c}(\phi\gamma)(\eta\sigma) = c(\gamma\sigma) = \gamma(c\sigma) = (\phi\gamma)(\bar{c}\eta\sigma).$$

CHAPTER II

5. **Homotopy groups and obstructions.** Homotopy groups are to serve, for us, as coefficient domains for obstructions to the extension of maps. They will accordingly be introduced axiomatically in terms of an obstruction theory. It will follow from the axioms that when a semi-simplicial complex admits an obstruction theory then that theory is essentially unique.

Extensions of maps are constructed, as in the usual obstruction theory, stepwise over skeletons of increasing dimension. As we have observed, a q-skeleton contains collapsed simplices of dimension greater than q. But of course a map defined and, so far as possible, simplicial, on all simplices of dimension $\leqq q$ has a unique extension to the q-skeleton.

The theory we envisage corresponds to that of an arcwise connected topological space which is simple in all dimensions, i.e. one whose fundamental group is abelian and operates trivially on all the higher homotopy groups. This is not a genuine restriction, since we study complexes with operators, and may always refer to a suitable covering space.

Since we shall be interested in extending simplicial maps, arcwise connectedness must be expressed in a very strong fashion in complexes which are to have a homotopy theory. We say that a complex X is *homotopy-connected* if for any complex K and map $f: K^0 \to X$ there is a simplicial extension of f to K^1. Alternatively, this may be expressed by saying that any two zero-simplices of X are vertices of some 1-simplex. If X is homotopy-connected, K is any complex, $L \subset K$ is a subcomplex, and $f: L \to X$, then f always has an extension to $L \cup K^1$.

If X is a homotopy-connected semi-simplicial complex a *homotopy-system* on X consists of a sequence $\pi_1, \pi_2, \cdots$ of abelian groups and an operation c which to any complex K and simplicial map $f: K^q \to X$ assigns a cochain $cf \in C^{q+1}(K; \pi_q)$ in such a way that the following axioms hold.

(H1) cf is a cocycle.

(H2) $cf = 0$ if and only if f can be extended to K^{q+1}.

(H3) If $L \subset K$ is a subcomplex and $h \in C^q_n(K, L; \pi_q)$ then there is a map $f': K^q \to X$ such that

$$f' \,|\, K^{q-1} \cup L^q = f \,|\, K^{q-1} \cup L^q$$

and $cf' = cf + \delta h$.

(H4) If L is any other semi-simplicial complex and $g: L^{q+1} \to K$, then $c(f[g \,|\, L^q]) = (cf)g$.

LEMMA 5.1. *If $L \subset K$ is a subcomplex and $f: K^q \to X$ is extendible to $K^q \cup L^{q+1}$, then $cf \in Z^{q+1}(K, L; \pi_q)$.*

For, if $j: L \to K$ is the inclusion map then (H4) $(cf)j = c(fj) = 0$.

COROLLARY. *cf is normalized.*

This follows by taking for L the smallest subcomplex containing the collapsed $q+1$ simplices.

LEMMA 5.2. *If $L \subset K$ is a subcomplex, $f: K^q \cup L \to X$ and*

$$H^{q+1}(K, L; \pi_q) = \cdots = H^n(K, L; \pi_{n-1}) = 0,$$

then $f|K^{q-1}\cup L$ *can be extended to* $K^n\cup L$. *If in addition* $\pi_q=0$ *then* f *itself can be so extended.*

Let us consider the lowest dimension. By the preceding lemma, $c(f|K^q)$ is a coboundary, say δh, for some $h\in C^q(K, L; \pi_q)$. Then by H3 there is an $f':K^q\to X$ which agrees with f on $K^{q-1}\cup L^q$ and can be extended to K^{q+1}. This extension may clearly be made to agree with f on L^{q+1}. The rest of the lemma follows by a simple induction, the special case being trivial.

By an oriented simplicial q-sphere we shall mean a pair (S, s), where S is the canonical semi-simplicial complex of a simplicial polyhedron homeomorphic to a q-sphere, and $s\in Z_q(S)$ is a fundamental cycle, i.e. a cycle in one of the generators of $H_q(S)$. If we agree that s is a sum of nondegenerate simplices, then s is uniquely determined, up to sign. By an oriented simplicial $q+1$-cell we shall mean a pair (E, e) where E is the canonical semi-simplicial complex of a simplicial polyhedron homeomorphic to a $q+1$-cell and, if $\dot{E}$ is the subcomplex arising from the boundary of the polyhedron, $e\in Z_{q+1}(E, \dot{E})$ is a fundamental cycle. The boundary $(\dot{E}, \partial e)$ of an oriented simplicial $q+1$-cell (E, e) is of course an oriented simplicial q-sphere. We shall denote by $\mathbf{\Delta}_q$ the canonical semi-simplicial complex of Δ_q. Thus $(\mathbf{\Delta}_q, \Delta_q)$ is an oriented simplicial q-cell, and, if the boundary of the standard simplex Δ_{q+1} has as its semi-simplicial complex $\dot{\mathbf{\Delta}}_{q+1}$, then $(\dot{\mathbf{\Delta}}_{q+1}, \partial\Delta_{q+1})$ is an oriented simplicial q-sphere.

In any semi-simplicial complex K the spherical cycles are the elements of the subgroup of $C_*(K)$ generated by cycles of the form gs, where (S, s) is an oriented simplicial sphere and $g:S\to K$. A cochain on K is a spherical annihilator if it is zero on the spherical cycles in K.

Returning to the notation of the beginning of this section we can state the next lemma.

LEMMA 5.3. *cf is a spherical annihilator.*

For, if (S, s) is an oriented simplicial $q+1$-sphere and $g:S\to K$ we may evaluate $(cf)gs$ as follows. First, we may suppose that $(S, s)=(\dot{E}, \partial e)$ for some oriented simplicial $q+2$-cell (E, e). By Lemma 5.2, $fg:S\to X$ may be extended to a map $F:E^{q+1}\to X$. If $j:S\to E$ is the inclusion map then $Fj=fg$ and

$$(cf)gs = c(fg)s = c(Fj)s = (cF)js = (cF)\partial e = 0 \qquad \text{(by H4).}$$

This fact permits the following definition. If (S, s) is an oriented simplicial q-sphere and $g:S\to X$ then the *homotopy class* $\langle g\rangle$ of g is the element of π_q determined in the following way. Let (E, e) be any oriented simplicial $q+1$-cell with boundary (S, s). By Lemma 5.2, g is extendible to $g_1:E^q\to X$. Then $\langle g\rangle=(cg_1)e$. To see that this is independent of E and the extension, suppose (E', e') is another such $q+1$-cell, of which we may assume without loss of generality that $E'\cap E=S$, and $g_1':E'^q\to X$ another such extension. Then

$(E \cup E', e-e')$ is an oriented simplicial $q+1$-sphere and $g_1 \cup g_1' : (E \cup E')^q \to X$ is a simplicial map. Since $e-e'$ is a spherical cycle,

$$0 = c(g_1 \cup g_1')(e - e') = (cg_1)e - (cg_1')e',$$

using, again, Axiom H4. We have obviously the following lemma.

LEMMA 5.4. *$\langle g \rangle = 0$ if and only if g is extendible to any simplicial $q+1$-cell E with $\dot{E} = S$.*

This notion of the homotopy class of a spherical map may be extended to a more general situation. Suppose S is the canonical semi-simplicial complex of a polyhedron of which we assume only that it is a homology q-sphere with integer coefficients, and suppose that s is a fundamental cycle of S. Then if $g: S \to X$ we may define the homotopy class $\langle g \rangle$ of g in a fashion analogous to the foregoing. For if $E \supset S$ arises from an acyclic polyhedron (e.g. the join with a point) then $H_{q+1}(E, S) \approx H_q(S)$ and there is thus a chain e in E with $\partial e = s$. As before, g may be extended to g_1; $E^q \to X$ and $\langle g \rangle$ defined by $\langle g \rangle = (cg_1)e$. If E' is another such complex, and $\partial e' = s$, then $E \cup E'$ is, if $E \cap E' = S$, a homology $q+1$-sphere by the Vietoris theorem, and $e-e'$ is a spherical cycle. Thus the same proof of the uniqueness of the definition of $\langle g \rangle$ may be used.

When elements of the homotopy groups are represented by spherical maps, addition takes place as in the topological case. Suppose (S, s) and (S', s') are oriented simplicial q-spheres whose underlying polyhedra have in common exactly one q-simplex σ, which appears with opposite signs in s and s'. Then the union of these polyhedra with the interior of σ deleted is a homology q-sphere; let us denote its semi-simplicial complex by $\bar{S}$. $(\bar{S}, s+s')$ is, in the sense of the last paragraph, an oriented simplicial homology q-sphere. If $g: S \to X$ and $g': S' \to X$ agree on σ, then $(g \cup g' \mid \bar{S}): \bar{S} \to X$.

LEMMA 5.5. $\langle g \cup g' \mid \bar{S} \rangle = \langle g \rangle + \langle g' \rangle$.

For suppose that (E, e) and (E', e') are oriented simplicial $q+1$-cells with boundaries (S, s) and (S', s'), and $E \cap E' = S \cap S'$. Then $E \cup E'$ is an acyclic complex containing $\bar{S}$, and $\partial(e+e') = s+s'$. Moreover, by Lemma 5.2, $g \cup g'$ is extendible to $g_1: (E \cup E')^q \to X$. Thus

$$\langle g \cup g' \mid \bar{S} \rangle = (cg_1)(e + e') = c(g_1 \mid E^q)e + c(g_1 \mid E'^q)e' = \langle g \rangle + \langle g' \rangle.$$

We turn now to the problem of the comparison of maps whose domains are distinct. Suppose (S, s) and (S', s') are oriented simplicial homological q-spheres, and that S and S' are contained as disjoint subcomplexes in a simplicial homological q-sphere $\bar{S}$, embedded in such a way that s and s' are both fundamental cycles of $\bar{S}$ contained in the same element of $H_q(\bar{S})$. We may compare $g: S \to X$ and $g': S' \to X$ by means of the following lemma.

LEMMA 5.6. *$g \cup g'$ is extendible to $\bar{S}$ if and only if $\langle g \rangle = \langle g' \rangle$.*

Let us observe first that $s-s'=\partial t$ for some $q+1$-chain t in $\bar{S}$. Now suppose that E and E' are acyclic complexes such that $E\cap(\bar{S}\cup E')=S$ and $E'\cap(\bar{S}\cup E)=S'$, and that e and e' are $q+1$-chains in E and E' such that $\partial e=s$ and $\partial e'=s'$. Then $E\cup\bar{S}\cup E'$ is a simplicial homology $q+1$-sphere and $e'+t-e$ is a fundamental cycle.

In any case, $g\cup g'$ can be extended to $g_1:(E\cup\bar{S}\cup E')^q\to X$. Since $e'+t-e$ is a spherical cycle,

$$0 = (cg_1)(e' + t - e) = c(g_1 \mid E')e' + c(g_1 \mid \bar{S})t - c(g_1 \mid E)e$$

and thus

$$c(g_1 \mid \bar{S})t = \langle g\rangle - \langle g'\rangle.$$

But $H^{q+1}(\bar{S}, S\cup S'; \pi_q)\approx\pi_q$, the isomorphism being given by that map which takes a cocycle into its value on t. Thus, in view of H2 and H3, $g\cup g'$ is extendible if and only if $c(g_1\mid\bar{S})t=0$, which proves the lemma.

The criterion furnished by Lemma 5.6 is in fact quite generally applicable. For given two disjoint simplicial polyhedral homology q-spheres, together with fundamental cycles of each, it is possible to embed them in another in the manner described above. This may be done directly, if there is a simplicial map of either into the other of degree 1, by using the mapping cylinder. In general it may be accomplished by mapping a third sphere into each of the two with degree 1, and taking the union of the mapping cylinders. It should be remarked, however, that this construction conceals the intuitive sense of the lemma, which is that $\bar{S}$ is a cylinder with ends S and S', and that an extension of $g\cup g'$ is just a homotopy of these maps.

We shall use Lemmas 5.5 and 5.6 to define homomorphisms induced by simplicial maps on homotopy groups. It is necessary first to observe that in virtue of axiom H3, a complex which admits a homotopy system is very rich in simplices. We may formulate this by means of the following lemma.

LEMMA 5.7. *If (S, s) is an oriented simplicial q-sphere, L is a proper subcomplex of S, and $g:S\to X$, then for any $p\in\pi_q$ there is a map $g':S\to X$ with $g'\mid L=g\mid L$ and $\langle g'\rangle=p$.*

For suppose σ is a q-simplex in S which is not in L; we may suppose it appears with sign $+1$ in s. Now let (E, e) be an oriented simplicial $q+1$-cell with boundary (S, s). If $h\in C^q(E, L; \pi_q)$ is $p-\langle g\rangle$ on σ and zero elsewhere, and $g_1:E^q\to X$ extends g, then by H3 there is a $g_1':E^q\to X$ such that $g_1'\mid L=g_1\mid L$ and $cg_1'=cg_1+\delta h$. Then $g_1'\mid S$ is the required map. For

$$\langle g_1' \mid \bar{S}\rangle = (cg_1')e = (cg_1)e + h\partial e = \langle g\rangle + p - \langle g\rangle.$$

This is really a very strong restriction on X. In particular it means that, given any q-simplex, there are at least as many q-simplices with the same boundary as there are elements in π_q.

Suppose now that $\{\pi_1', \pi_2', \cdots; c'\}$ is a homotopy system on a semi-simplicial complex X'. If $f:X\to X'$ we may define a homomorphism $f_*:\pi_q \to \pi_q'$, for each q, by

$$f_*\langle g\rangle = \langle fg\rangle,$$

where $g:S\to X$ is any map of an oriented simplicial homology q-sphere. Lemma 5.6 implies that f_* is a well-defined map; that it is a homomorphism follows from Lemma 5.5.

It is clear that this homomorphism is functorial in f: if $\{\pi_1'', \pi_2'', \cdots; c''\}$ is a homotopy system on X'' then, for any $f':X'\to X''$,

$$(f'f)_* = f_*' f_*.$$

It is also evident that if i is the identity map on X then $i_*:\pi_q\to\pi_q$ is the identity.

It follows that if $\bar{\pi}_1, \bar{\pi}_2, \cdots; \bar{c}$ is another homotopy system on X then $i_*:\pi_q\approx\bar{\pi}_q$ for all q.

If we write

$$\dot{\sigma} = \sigma^\Delta \mid \dot{\mathbf{\Delta}}_q$$

then, since $(\mathbf{\Delta}_{q+1}, \partial\Delta_{q+1})$ is an oriented simplicial q-sphere, the obstruction of a map $g:K\to X$ must satisfy

$$(cg)\sigma = \langle g\dot{\sigma}\rangle.$$

Thus if $f:X\to X'$ as above, f_* is characterized by

$$f_*(cg) = c(fg). \tag{5.8}$$

We may refer to the collection of semi-simplicial complexes admitting homotopy systems, together with their simplicial maps, as the category of semi-simplicial complexes with homotopy. The preceding remarks imply, then, the following theorem.

THEOREM 5.9. *On the category of semi-simplicial complexes with homotopy, Axioms* H1–4 *and* (5.8) *define uniquely, up to natural equivalence, a covariant functor.*

In view of this theorem we shall denote the groups of the homotopy system associated with a complex X by $\pi_1(X), \pi_2(X), \cdots$.

Now it is a special case of Lemma 5.6 that maps of oriented simplicial q-spheres which are homotopic have the same homotopy class. It follows immediately that if X and Y have homotopy, if $f, f':X\to Y$, and if f and f' are connected by a homotopy through dimension q, then the induced homomorphisms f_* and f_*' agree on $\pi_1(X), \cdots, \pi_q(X)$.

But this implies that we can state the following amplification of Theorem 5.9:

THEOREM 5.10. *The homotopy functor is a weak homotopy invariant.*

We conclude this section by adducing the obvious example of a complex with homotopy, the singular complex of a topological space with simple homotopy.

If X is such a space, the homotopy groups of $S(X)$ are just those of X. The obstructions are given in the following fashion. If $f:K^q \to S(X)$ and $\sigma \in K_{q+1}$ then $f\dot{\sigma}:\dot{\Delta}_{q+1} \to S(X)$ gives rise to a continuous map g of the boundary of the standard $q+1$-simplex into X. This map is defined as follows: if $y \in \Delta^i_{q+1}$ then

$$gy = [f\dot{\sigma}\Delta^i_{q+1}](e_i^{q+1})^{-1}y$$

(recall that $f\dot{\sigma}\Delta^i_{q+1}$ is a singular q-simplex). The homotopy class of g in the group $\pi_q(X)$ is then $(cf)\sigma$.

With this convention the Axioms H1–4 become standard theorems of obstruction theory; we shall not repeat the proofs here.

6. **Obstructions to extending homotopies.** If X is a complex with homotopy, K is any semi-simplicial complex, $f, f':K^q \to X$, and F is a homotopy through dimension $q-1$ of f and f' we shall define an obstruction $d(f, F, f') \in C^q(K; \pi_q(X))$ to extending F to a homotopy through dimension q of f and f'.

Let us observe first that the inclusion of any complex in its product with the interval gives rise to isomorphisms in homology. In view of this, consideration of the cohomology sequence of the triple

$$(K^q \times I, K^q \times \{0\} \cup K^{q-1} \times I \cup K^q \times \{1\}, K^{q-1} \times I)$$

yields the fact that the pair $(K^q \times I, K^q \times \{0\} \cup K^{q-1} \times I \cup K^q \times \{1\})$ has vanishing homology in dimensions $0, \cdots, q$.

Now F, being defined on $K^{q-1} \times I$, may be extended to F_1 on $K^q \times \{0\} \cup K^{q-1} \times I \cup K^q \times \{1\}$ by setting $F_1(\sigma, 0) = f\sigma$ and $F_1(\sigma, 1) = f'\sigma$. But F_1 may then, by Lemma 5.2, be extended to $(K^q \times I)^q$. The obstruction to extending F is then given by

$$d(f, F, f') = (cF_2)D$$

where F_2 is the extension of F_1 to $(K^q \times I)^q$ and D is the homomorphism of degree 1 of $C_*(K)$ into $C_*(K \times I)$ described in connection with chain homotopies.

F_1 is of course completely determined by F, f, and f'. But the extension F_2 is not. In order to see that the arbitrary character of this extension does not affect the definition of the obstruction we observe that it may also be expressed in the following way.

First, $(\Delta_q \times I, D\Delta_q)$ is an oriented simplicial $q+1$-cell with boundary $((\Delta_q \times I)^{\cdot}, \partial D\Delta_q)$. If $\sigma \in K_q$, then $\sigma^\Delta \times j:\Delta_q \times I \to K^q \times I$ and $(\sigma^\Delta \times j)D\Delta_q = D\sigma$. Moreover, $(\sigma^\Delta \times j)(\Delta_q \times I)^\circ$ is contained in the domain of F_1. Thus

$$(f, F, f')\sigma = (cF_2)D\sigma = (cF_2)(\sigma^\Delta \times j)D\Delta_q = c(F_2(\sigma^\Delta \times j))D\Delta_q.$$

But if we write $\sigma^\square$ for $(\sigma^\Delta \times j)\,|\,(\mathbf{\Delta}_q \times I)^\cdot$ then $F_2(\sigma^\Delta \times j)$ is an extension of $F_1\sigma^\square : (\mathbf{\Delta}_q \times I)^\cdot \to X$. Thus

$$d(f, F, f')\sigma = \langle F_1\sigma^\square \rangle. \tag{6.1}$$

Not only does this representation show the independence of F_2, it indicates also the proof of the following lemma.

LEMMA 6.2. *F is extendible to a homotopy through dimension q of f and f' if and only if $d(f, F, f') = 0$.*

For if F is extendible, it is clear that $F_1\sigma^\square$ is also. Then Lemma 5.4 asserts that $d(f, F, f')$ vanishes. Conversely, if each $[F_1\sigma^\square]$ vanishes, then each $F_1\sigma^\square$ may be extended to a map $F_\sigma : \mathbf{\Delta}_q \times I \to X$. But then we may define an extension F_3 of F_1 to $K^q \times I$ by setting

$$F_3(\sigma, \tau) = F_\sigma(\Delta_q, \tau^\square) \qquad \text{for } \sigma \in K_q,\ \tau \in I.$$

If $f, f', f'' : K^q \to X$ and $F : K^{q-1} \times I \to X$, $F' : K^{q-1} \times I' \to X$ are homotopies through dimension $q-1$ of f with f' and of f' with f'', then F and F' may be added to give a homotopy of f with f''. For we may suppose that I and I' are disjoint, except for the "left end-point" $0'$ of I' and the "right end-point" 1 of I, which are the same. Then $F \cup F' : K^{q-1} \times (I \cup I') \to X$ is such a homotopy. By reversing the ordering in I we may also produce from a homotopy connecting two maps a homotopy in the opposite order. Thus if F is a homotopy of f and f', we may canonically define a homotopy $-F$ of f' with f. Then the representation 6.1 together with Lemma 5.5 gives the following lemma.

LEMMA 6.3. $d(f, F \cup F', f'') = d(f, F, f') + d(f', F', f'')$.

$$d(f', -F, f) = -\,d(f, F, f').$$

The obstruction to the extension of a homotopy is of course related to the obstructions to extending the maps it connects. This relation is expressed by the following lemma.

LEMMA 6.4. $\delta d(f, F, f') = cf' - cf$.

To see this we return to the original expression for $d(f, F, f')$. Let us write k, k' for the injections $k\sigma = (\sigma, 0)$, $k'\sigma = (\sigma, 1)$. Then $F_2k = f$, $F_2k' = f'$ and $\partial D + D\partial = k' - k$. Thus

$$d(f, F, f') = (cF_2)D\partial = (cF_2)(k' - k - \partial D) = c(F_2k') - c(F_2k) = cf' - cf.$$

From this we get the following corollary.

LEMMA 6.5. *If $f, f' : K^q \to X$ are homotopic through dimension $q-1$, then cf and cf' belong to the same class in $H^{q+1}(K; \pi_q(X))$.*

In particular, this conclusion is true if f and f' agree on K^{q-1}, for then the stationary homotopy $f\theta: K^{q-1}\times I\to X$, where θ is the projection, connects them. The obstruction to extending this homotopy we shall denote simply by $d(f, f')$. In terms of this obstruction we may amplify somewhat Axiom H3; by dealing with one simplex at a time we arrive at a proof of the following lemma.

LEMMA 6.6. *If $f: K^q\to X$ and $h\in C_n^q(K, L; \pi_q(X))$, where $L\subset K$ is any subcomplex, then there exists a map $f': K^q\to X$ such that $f'|K^{q-1}\cup L=f|K^{q-1}\cup L$ and $d(f, f')=h$.*

There is also an analogue of H3 for homotopies.

LEMMA 6.7. *If $f, f': K^q\cup L\to X$ and $F: (K^{q-1}\cup L)\times I\to X$ is a homotopy of f and f', and if $h\in C_n^{q-1}(K, L; \pi_q(X))$, then there is a homotopy $F': (K^{q-1}\cup L)\times I\to X$ of f and f' such that $F'|(K^{q-2}\cup L)\times I=F|(K^{q-2}\cup L)\times I$ and $d(f, F', f')=d(f, F, f')+\delta h$.*

For we may certainly find a cochain $h'\in C_n^q((K^{q-1}\cup L)\times I, L\times I; \pi_q(X))$ such that $h'D=h$. Define F' by $d(F, F')=h'$. If we extend h' on $K^q\times\{0\}$ and $K^q\times\{1\}$ by setting it equal to zero there we have also $d(F_2, F_2')=h'$ and

$$d(f, F', f') = c(F_2')D = (cF_2)D + h'\partial D = d(f, F, f') + (h'D)\partial.$$

The naturality of the obstruction to the extension of a homotopy is expressed by three formulae. The first is an analogue of H4. If $f, f': K^q\to X$ and $F: K^{q-1}\times I\to X$ is a homotopy of f and f', and if $g: L^q\to K$, then $\bar{g}=(g|L^{q-1})\times j$ can be composed with F to give a homotopy $F\bar{g}: L^{q-1}\times I\to X$ of fg and $f'g$, where, of course, j is the identity on I. It is an immediate consequence of the definition of the obstruction, and of the obvious identity $\bar{g}D=Dg$ that

$$d(fg, F\bar{g}, f'g) = d(f, F, f')g. \tag{6.8}$$

Suppose instead that $g: X\to Y$, where Y also has homotopy. Then it follows immediately from (6.1) that for the homotopy gF of gf and gf' the equality

$$d(gf, gF, gf') = g_*d(f, F, f') \tag{6.9}$$

holds.

The third identity is slightly different. Suppose $g, g': L^q\to K$ and G is a homotopy through dimension q of g and g'. Suppose also that $f: K^q\to X$, where X has homotopy. Then $\overline{G}=(f\times j)(G|L^{q-1}\times I)$ is a homotopy through dimension $q-1$ of fg and fg'. The obstruction of this homotopy is of course

$$d(fg, \overline{G}, fg') = (cf)\tilde{G} \tag{6.10}$$

where $\tilde{G}=GD$ is the chain homotopy associated with G.

Finally we may state the analogue for homotopies of the extension

Lemma 5.2 which, using 6.2 and 6.7, is proved the same way as those lemmas.

LEMMA 6.11. *If $L \subset K$ is a subcomplex, f, f'; $K^n \cup L \to X$, $F: (K^{q-1} \cup L) \times I \to X$ is a homotopy of f and f', and*

$$H^q(K, L; \pi_q(X)) = \cdots = H^{n-1}(K, L; \pi_{n-1}(X)) = 0,$$

then $F \mid (K^{q-2} \cup L) \times I$ can be extended to $(K^{n-1} \cup L) \times I$ as a homotopy of f and f'. If $\pi_q(X) = 0$, then F itself can be so extended.

7. **Homotopy in complexes with operators.** If Γ is a semi-simplicial group and X is a Γ-complex, then for any semi-simplicial complex K and simplicial maps $f: K \to \Gamma$ and $g: K \to X$ we may define a simplicial map $f \cdot g: K \to X$ by

$$(f \cdot g)\sigma = (f\sigma)(g\sigma)$$

for any $\sigma \in K$. For a map $f: K \to \Gamma$ we may define an inverse, which we shall denote by $f^{-1}: K \to \Gamma$ when there is no possibility of confusion with the set-theoretic inverse, by $f^{-1}\sigma = (f\sigma)^{-1}$. Then $f \cdot f^{-1} = f^{-1} \cdot f$ is of course the constant map whose values are the identities in the groups Γ_q. When X is a complex with homotopy we shall relate this operation to the homotopy of X.

Let us suppose first that $X = \Gamma$ and the operation is just group multiplication. Then we have the following analogue of a well-known lemma.

LEMMA 7.1. *If Γ is a semi-simplicial group with homotopy, (S, s) is an oriented simplicial q-sphere, and f, $g: S \to \Gamma$, then $\langle f \cdot g^{-1} \rangle = \langle f \rangle - \langle g \rangle$.*

It follows immediately from Lemma 5.6 that $\langle f \cdot g \rangle$ depends only on $\langle f \rangle$ and $\langle g \rangle$. Thus we may suppose that S is the union of simplicial q-spheres S_1 and S_2 with just one simplex deleted, and that $s = s_1 + s_2$ where s_i is a fundamental cycle of S_i. If we take $f_i: S_i \to \Gamma$ such that $\langle f_1 \rangle = \langle f \rangle$ and $\langle f_2 \rangle = \langle g \rangle$ and such that both are the identity on the simplex deleted in S, then both may be extended to S with the constant value on the other sphere. But then, if f_i' extends f_i,

$$\langle f \cdot g \rangle = \langle f_1' \cdot f_2' \rangle = \langle f_1 \rangle + \langle f_2 \rangle = \langle f \rangle + \langle g \rangle.$$

The conclusion on the difference follows immediately from the fact that the homotopy class of a constant map is zero.

An immediate corollary is the following.

LEMMA 7.2. *If Γ is a semi-simplicial group with homotopy and f, $g: K^q \to \Gamma$, then $c(f \cdot g^{-1}) = cf - cg$.*

For, if $\sigma \in K_{q+1}$,

$$c(f \cdot g^{-1})\sigma = \langle (f \cdot g^{-1})\dot{\sigma} \rangle = \langle (f\dot{\sigma}) \cdot (g\dot{\sigma})^{-1} \rangle = \langle f\dot{\sigma} \rangle - \langle g\dot{\sigma} \rangle.$$

Now if Γ operates on a complex with homotopy we may, with no restriction on Γ, define an operation of Γ_0 on each of the groups $\pi_q(X)$ by setting

$$\gamma\langle f\rangle = \langle\widehat{\gamma}\cdot f\rangle$$

where f is any map of an oriented simplicial q-sphere into X, and $\widehat{\gamma}$ is the "constant" map into the collapsed simplices over $\gamma\in\Gamma_0$. It follows from Lemma 5.6 that this is independent of the representation of an element of $\pi_q(X)$ and from Lemma 5.5 that it gives rise to an automorphism of $\pi_q(X)$. But suppose that γ is in the component of the identity. Then there is a map $g:I\to\Gamma$, for some decomposition of the unit interval, such that $g(0)=\gamma$ and $g(1)=1$. If we write λ and ρ for the left and right projections of $S\times I$ then $(g\rho)\cdot(f\lambda):S\times I\to X$ is a homotopy of $\widehat{\gamma}\cdot f$ with f. Thus $\langle\widehat{\gamma}\cdot f\rangle=\langle f\rangle$ and we have in fact defined an operation of $\pi_0(\Gamma)$ or, what comes to the same thing, an operation of Γ on $\pi_q(X)$.

One special case is that of a semi-simplicial group Γ such that the component ${}^0\Gamma$ of the identity has homotopy, and which operates on itself by inner automorphisms. This leads to an operation of $\pi_0(\Gamma)$ on $\pi_q({}^0\Gamma)$. We may remark in passing that the equivalence classes under the relation of Lemma 5.6 of maps of spheres into Γ form a group, when composition is defined by the multiplication of maps. It is an easy consequence of Lemma 7.1 that, in dimension q, this group is a splitting extension of $\pi_0(\Gamma)$ by $\pi_q({}^0\Gamma)$.

A more important property of such groups is the following.

LEMMA 7.3. *Suppose X is a Γ-complex, and that X and ${}^0\Gamma$ have homotopy, that (S, s) is an oriented simplicial q-sphere, that $f:S\to X$ and that $g:S\to\Gamma$ is nullhomotopic. Then $\langle g\cdot f\rangle=\tilde{g}\langle f\rangle$ where $\tilde{g}$ is that element of $\pi_0(\Gamma)$ corresponding to the component in which the image of g lies.*

This may be proved by straightforward application of Lemma 5.6 to give a homotopy of $g\cdot f$ with $g_0\cdot f$, where g_0 is a constant map.

Lemma 7.3 is the key to the obstruction theory of equivariant maps and homotopies. This is concerned with the following question. If K is a Γ-complex and f an equivariant map defined on the "invariant q-skeleton" ΓK^q, can f be extended equivariantly to ΓK^{q+1}? Since $\Gamma K^q\supset K^q$, the obstruction $cf=c(f|K^q)$ of such a map is defined. For these obstructions we have the following lemma.

LEMMA 7.4. *If K and X are Γ-complexes, X and ${}^0\Gamma$ have homotopy, and $f:\Gamma K^q\to X$ is Γ-equivariant, then $cf\in Z_\Gamma^{q+1}(K;\pi_q(X))$.*

It is only the equivariance which is in question. But if $\sigma\in K_{q+1}$ and $\gamma\in\Gamma_{q+1}$ then

$$(cf)(\gamma\sigma) = \langle f(\gamma\sigma)^{\cdot}\rangle = \langle f(\dot{\gamma}\cdot\dot{\sigma})\rangle = \langle\dot{\gamma}\cdot(f\dot{\sigma})\rangle = \gamma\langle f\dot{\sigma}\rangle.$$

An equivariant homotopy F of equivariant maps $f, f':\Gamma K^q\to X$, defined through dimension $q-1$, is defined on $\Gamma K^{q-1}\times I\supset K^{q-1}\times I$. Thus $d(f, F, f')$ is defined. It is of course equivariant under the hypotheses of Lemma 7.4.

LEMMA 7.5. *If $f, f':\Gamma K^q \to X$ are Γ-equivariant and $F:\Gamma K^{q-1}\times I\to X$ is an equivariant homotopy of f and f', then $d(f, F, f')\in C^q_\Gamma(K;\pi_q(X))$.*

This is proved in complete analogy with 7.4, using the expression 6.1 for the obstruction.

It is of course impossible to prove extension lemmas like 5.2 and 6.11 in full generality for complexes with operators and equivariant maps. However, in the case that the domain is a bundle such statements do hold. For then equivariant maps may be defined first on one simplex in each class conjugate under the operations of the group and extended to the rest of the class by the requirement of equivariance. Using this technique we get the following lemmas.

LEMMA 7.6. *If K is a Γ-bundle, $L\subset K$ is an invariant subcomplex, X is a Γ-complex, ${}^0\Gamma$ and X have homotopy, and $f:\Gamma K^q\to X$ is equivariant, then f is extendible to an equivariant map on ΓL^{q+1} if and only if cf is zero on L.*

LEMMA 7.7. *If, under the hypothesis of Lemma 7.6, h is a normalized cochain in $C^q_\Gamma(K, L;\pi_q(X))$, then there is an equivariant $f':\Gamma K^q\to X$ such that $f'|\Gamma K^{q-1}\cup L=f|\Gamma K^{q-1}\cup L$ and $cf'=cf+\delta h$.*

From these two we get the extension lemma.

LEMMA 7.8. *If, under the hypothesis of Lemma 7.6,*

$$H^{q+1}_\Gamma(K, L;\pi_q(X)) = \cdots = H^n_\Gamma(K, L, \pi_{n-1}(X)) = 0,$$

then $f|\Gamma K^{q-1}\cup L$ can be extended to an equivariant map on $\Gamma K^n\cup L$. If in addition $\pi_q(X)=0$, then f itself can be extended.

We omit the analogues of 7.6 and 7.7 for equivariant homotopies (cf. Lemmas 6.2 and 6.7) and state only the extension lemma.

LEMMA 7.9. *If under the hypothesis of Lemma 7.6 the maps $f, f':\Gamma K^n\cup L\to X$ are equivariant and $F:(\Gamma K^{q-1}\cup L)\times I\to X$ is an equivariant homotopy of f and f', and in addition,*

$$H^q_\Gamma(K, L;\pi_q(X)) = \cdots = H^{n-1}_\Gamma(K, L;\pi_{n-1}(X)) = 0,$$

then the restriction $F|(\Gamma K^{q-2}\cup L)\times I$ may be extended to $(\Gamma K^{n-1}\cup L)\times I$ as an equivariant homotopy of f and f'. If $\pi_q(X)=0$, then F itself may be so extended.

The analogue of Lemma 6.6 should also be noted. It is of course just a strengthening of 7.7.

LEMMA 7.10. *Under the hypothesis of Lemma 7.7 there is an equivariant map $f':\Gamma K^q\cup L\to X$ such that f and f' agree on $\Gamma K^{q-1}\cup L$ and $d(f, f')=h$.*

If X is a Γ-complex, then for any 0-simplex $\sigma_0\in X$, a map $\phi_0:\Gamma\to X$ is

defined by $\phi_0\gamma=\gamma\hat{\sigma}_0$, where $\hat{\sigma}_0$ is the degenerate simplex of the appropriate dimension over σ_0. If both X and ${}^0\Gamma$ have homotopy, this defines homomorphisms $\phi_{0*}:\pi_q({}^0\Gamma)\rightarrow\pi_q(X)$. It would be easy to see that all the maps defined this way are homotopic, and that the homomorphisms ϕ_{0*} are all equal. This fact follows also from the following lemma.

LEMMA 7.11. *Suppose X is a Γ-complex, X and ${}^0\Gamma$ have homotopy, K is any complex, g_0, $g_1:K^q\rightarrow\Gamma$ agree on K^{q-1} and $f:K^q\rightarrow X$. Then*

$$d(g_1\cdot f,\ g_0\cdot f) = \phi_{0*}d(g_1,\ g_0)$$

where ϕ_0 is any one of the maps defined above.

It is of course necessary to prove this only in the case $K=\Delta_q$. In this case there is a homotopy $F:\Delta_q\times I\rightarrow X$ of f with the constant map onto σ_0. But then we have

$$\begin{aligned} d(g_1\cdot f,\ g_0\cdot f) &= d(\phi_0 g_1,\ \phi_0 g_0) + d(g_1\cdot f,\ g_1'\cdot F,\ \phi_0 g_1) - d(g_0\cdot f,\ g_0'\cdot F,\ \phi_0 g_0) \\ &= \phi_{0*}d(g_1,\ g_0) \end{aligned}$$

where g_0' and g_1' are the compositions of g_0 and g_1 with the projection of $\Delta_q\times I$ onto Δ_q.

It seems proper to refer to the common value of ϕ_{0*} as the *injection* of the homotopy of ${}^0\Gamma$ into that of X.

Finally we need a lemma which permits the construction of extensions with prescribed effects on homotopy groups.

LEMMA 7.12. *Suppose X is a Γ-bundle, A and Y are Γ-complexes, X and Y have homotopy, and $\phi:\pi_n(X)\rightarrow\pi_n(Y)$ is any homomorphism. Suppose also that $\eta:X\rightarrow A$, $\lambda:\Gamma A^n\rightarrow X$, and $f:\Gamma A^n\rightarrow Y$ are equivariant maps such that $\lambda\eta\,|\,\Gamma X^{n-1}$ is the identity and that $cf=\phi_*'(c\lambda)$. Then there is an extension $f':\Gamma X^{n+1}\rightarrow Y$ of $f\eta\,|\,\Gamma X^{n-1}$ such that $f_*':\pi_n(X)\rightarrow\pi_n(Y)$ is just ϕ.*

For by the preceding lemma we may define $f':\Gamma X^n\rightarrow Y$ extending $f\eta\,|\,\Gamma X^{n-1}$ such that $d(f',\ f\eta)=-\phi_*d(\lambda\eta,\ \text{identity})$. Then we have

$$cf' = (cf)\eta - \delta d(f',\ f\eta) = (cf)\eta + \phi_*\delta d(\lambda\eta,\ \text{identity}) = (cf)\eta - \phi_*(c\lambda)\eta = 0.$$

Thus we may suppose f' defined on all of ΓX^{n+1}. To see the effect on $\pi_n(X)$ suppose that K is any semi-simplicial complex and $w:K^n\rightarrow X$. Then

$$\begin{aligned} c(f'w) &= c(f\eta w) - \delta d(f'w,\ f\eta w) = c(f\eta w) + \phi_*\delta d(\lambda\eta w,\ w) \\ &= (cf)\eta w - \phi_*(c\lambda)\eta w + \phi_* cw = \phi_*(cw) \end{aligned}$$

which proves the lemma.

8. Obstructions of cross-sections and liftings. Let us return to the situation of Theorem 3.1, supposing that X is a Γ-complex, that Λ is an invariant subgroup of Γ which operates freely on X, that Γ operates also on Y, and that $p:X\times Y\rightarrow X$ is the projection.

Now it is easy to see that $(\Gamma X^q)/\Lambda$ is just $(\Gamma/\Lambda)(X/\Lambda)^q$, in fact, this is true even if Λ does not operate freely. Thus it follows from Theorem 3.1 that Γ-equivariant maps $\alpha:\Gamma X^q \to Y$ are in one-to-one correspondence with Γ/Λ-equivariant maps in $\chi(p/\Lambda;\ (\Gamma/\Lambda)(X/\Lambda)^q)$.

If we suppose that Y and ${}^0\Gamma$ have homotopy then such maps α determine obstruction cocycles which in turn determine obstruction cohomology classes in $H_\Gamma^{q+1}(X;\pi_q(Y))$. Because of Theorem 3.1, these classes may be interpreted as the obstructions to extending over $(\Gamma/\Lambda)(X/\Lambda)^{q+1}$ a Γ/Λ-equivariant cross-section of p/Λ defined on $(\Gamma/\Lambda)(X/\Lambda)^q$. Finally, if Λ operates trivially on $\pi_q(Y)$, we may consider these obstruction classes in the isomorphic cohomology group $H_{\Gamma/\Lambda}^{q+1}(X/\Lambda;\pi_q(Y))$ (cf. Theorem 4.1).

We shall want to consider particularly the case in which the operator group is a splitting extension. Changing notation slightly, we suppose that Γ and Λ operate on X, and Γ operates on Λ in such a way that X is a Λ-bundle and a $\Lambda\cdot\Gamma$-complex. We suppose further that Λ has homotopy, as well as ${}^0\Gamma$; this will imply that ${}^0(\Lambda\cdot\Gamma)$ has homotopy. For the fibre, i.e. in place of Y, we take Λ with the obvious operation of $\Lambda\cdot\Gamma$:

$$(\lambda,\gamma)\lambda_1 = \lambda(\gamma\cdot\lambda_1).$$

It should be observed that this is not an operation by automorphisms of Λ; we may also observe that with this operation $(X\times\Lambda)/\Lambda$ is isomorphic to X as a Γ-complex, though of course no operation by Λ is defined on it.

Making this identification, p/Λ becomes the canonical map $\eta:X\to X/\Lambda$. Thus if we define $\mathfrak{C}^{q+1}(X,\Gamma,\Lambda)\subset H_\Gamma^{q+1}(X/\Lambda;\pi_q(\Lambda))$ to be the set of all classes arising in the manner described above from obstructions to $\Lambda\cdot\Gamma$-equivariant maps $\alpha:\Gamma X^q\to\Lambda$, we can see that $\mathfrak{C}^{q+1}(X,\Gamma,\Lambda)$ is nonempty if and only if there is a Γ-equivariant cross-section of η on $\Gamma(X/\Lambda)^q$, and that $\mathfrak{C}^{q+1}(X,\Gamma,\Lambda)$ contains 0 if and only if there is a Γ-equivariant cross-section of η on $\Gamma(X/\Lambda)^{q+1}$.

In fact, this class is a coset in the cohomology group (cf. Hu [8]), but we shall make no use of this property.

Suppose now that Γ', Λ', and X' are analogous to Γ, Λ, and X, that $\Phi:\Gamma\to\Gamma'$ is a homomorphism, $\phi:\Lambda\to\Lambda'$ is a Φ-equivariant homomorphism, and that $f:X/\Lambda\to X'/\Lambda'$ is Φ-equivariant. Then writing, as in §3, $Y=(f\eta)^{\frown -1}X'$, Theorem 3.3 allows us to identify $(\Lambda\times\Lambda')\cdot\Gamma$-equivariant maps $F:((\Lambda\times\Lambda')\cdot\Gamma)Y^q\to\Lambda'$ with $\phi\times\Phi$-equivariant liftings of f defined on $(\Lambda\cdot\Gamma)X^q$. Assuming that ${}^0(\Lambda\cdot\Gamma)$ and Λ' have homotopy, the obstructions to such maps F may be looked at as obstructions to extending the lifting of f. Applying Theorem 4.1, we see that the cohomology classes of these obstructions may, since $Y/(\Lambda\times\Lambda')$ may be identified, as a Γ-complex, with X/Λ, be considered to be in $H^{q+1}(X/\Lambda;\pi_q(\Lambda'))$.

Let us define, then, $\mathfrak{L}^{q+1}(f,\Phi,\phi)\subset H_\Gamma^{q+1}(X/\Lambda;\pi_q(\Lambda'))$ to be the set of cohomology classes arising in this manner. Then $\mathfrak{L}^{q+1}(f,\Phi,\phi)$ is nonempty if

and only if there is a $\phi\times\Phi$-equivariant lifting of f on $(\Lambda\cdot\Gamma)X^q$, and $\mathfrak{L}^{q+1}(f, \Phi, \phi)$ contains 0 if and only if there is a $\phi\times\Phi$-equivariant lifting of f on $(\Lambda\cdot\Gamma)X^{q+1}$.

We may remark again that $\mathfrak{L}^{q+1}(f, \Phi, \phi)$ is also a coset.

The following theorem, which in a special case (see the remark after Theorem 3.3) is due to Hu, exhibits the relation between the sets of obstruction classes just defined.

THEOREM 8.1. *Suppose $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$ are splitting semi-simplicial group extensions, and that $\phi\times\Phi:\Lambda\cdot\Gamma\to\Lambda'\cdot\Gamma'$ is a homomorphism. Suppose also that X and X' are $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$-complexes such that X is a Λ-bundle and X' a Λ'-bundle. Then, if $f:X/\Lambda\to X'/\Lambda'$ is Φ-equivariant,*

$$\mathfrak{L}^{q+1}(f, \Phi, \phi) \supset \phi_*\mathfrak{C}^{q+1}(X, \Gamma, \Lambda) - f^*\mathfrak{C}^{q+1}(X'\Gamma', \Lambda').$$

Let us write $\mu: Y\to X/\Lambda$, $\eta: X\to X/\Lambda$, $\eta': X'\to X'/\Lambda'$ for the canonical maps. Then we must show that if $\alpha:(\Lambda\cdot\Gamma)X^q\to\Lambda$ and $\alpha':(\Lambda'\cdot\Gamma')X'^q\to\Lambda'$ are, respectively, $\Lambda\cdot\Gamma$ and $\Lambda'\cdot\Gamma'$-equivariant, there is a $((\Lambda\times\Lambda')\cdot\Gamma)$-equivariant map $F:((\Lambda\times\Lambda')\cdot\Gamma)\,Y^q\to\Lambda'$ such that

$$\mu^{*-1}(cF) \sim \eta^{*-1}\phi_*(c\alpha) - [\eta'^{*-1}(c\alpha')]f$$

in $Z_\Gamma^{q+1}(X/\Lambda;\, \pi_q(\Lambda'))$. Now if we recall that $Y\subset X\times X'$ and write $p_1: Y\to X$ and $p_2: Y\to X'$ for the restrictions of the projections, we may define F by $F=(\phi\alpha p_1)\cdot(\alpha' p_2)^{-1}$; it is easy to see that F is a map of the required type. From Lemma 7.2 and 5.8 we have

$$cF = \phi_*c(\alpha p_1) - c(\alpha' p_2).$$

Thus if $(\sigma, \sigma')\in Y_{q+1}$,

$$\begin{aligned}\mu^{*-1}(cF)\mu(\sigma, \sigma') &= \mu^{*-1}(cF)\eta\sigma = (cF)(\sigma, \sigma') = \phi_*(c\alpha)\sigma - (c\alpha')\sigma' \\ &= \eta^{*-1}\phi_*(c\alpha)\eta\sigma - \eta'^{*-1}(c\alpha')\eta'\sigma'\end{aligned}$$

which proves the theorem, since $\eta'\sigma' = f\eta\sigma$.

We should also mention here the obstruction to lifting in the relative case. Suppose $A\subset X$ is a subcomplex invariant under the operation of $\Lambda\cdot\Gamma$, that $f:X/\Lambda\to X'/\Lambda'$ is Φ-equivariant, that $F_0:A\to X'$ is $\phi\times\Phi$-equivariant, and that $F_0/\phi = f|(A/\Lambda)$. Then the obstructions to extending F_0 to a lifting of f are given by the subsets $\mathfrak{L}^{q+1}(f, F_0, \Phi, \phi)\subset H_\Gamma^{q+1}(X/\Lambda, A/\Lambda;\, \pi_q(\Lambda'))$, with $\mathfrak{L}^{q+1}(f, F_0, \Phi, \phi)$ nonempty if and only if there is an extension on $A\cup X^q$, and containing 0 if and only if there is an extension on $A\cup X^{q+1}$.

With this in mind we may prove the relative covering homotopy theorem. We continue the above assumptions, and suppose in addition that Γ operates trivially on all the homotopy groups of Λ'. It should be observed that the theorem we state here is in fact true under much weaker hypotheses. However it is more convenient to prove the less general theorem, which is sufficient for the applications intended.

THEOREM 8.2. *Suppose* $f:(X/\Lambda)\times I\to X'/\Lambda'$ *is* Φ*-equivariant, and that* $F_0:X\times\{0\}\cup A\times I\to X'$ *is a* $\phi\times\Phi$*-equivariant lifting of* $f|(X/\Lambda)\times\{0\}\cup(A/\Lambda)\times I$. *Then* F_0 *may be extended to a* $\phi\times\Phi$*-equivariant lifting* $F:X\times I\to X'$ *of* f.

For the obstructions will lie in the groups $H_\Gamma^{q+1}((X/\Lambda)\times I,\ (X/\Lambda)\times\{0\}\cup(A/\Lambda)\times I;\ \pi_q(\Lambda'))=\{0\}$.

We shall need also a corollary of the last theorem.

THEOREM 8.3. *If* $f_0, f_1:K\to X/\Lambda$ *are homotopic by a homotopy stationary on* $L\subset K$, *then there exists a* Λ*-equivariant map* $F:\hat{f}_0^{-1}(X)\to\hat{f}_1^{-1}(X)$ *such that* F/Λ *is the identity on* K. F *may be taken to be the identity on* $(f_0|L)^{\frown-1}(X)=(f_1|L)^{\frown-1}(X)$.

9. **Γ-minimal complexes.** In this section we adapt the notion, due to Eilenberg and Zilber [3], of a minimal complex. There are several innovations: the presence of operators, the abstract, i.e. semi-simplicial, framework and, most important, the fact that it is now possible to define minimal complexes intrinsically.

If X is a Γ-complex, and both X and ${}^0\Gamma$ have homotopy, we shall say that a simplex $\gamma\in\Gamma$ is a *null-simplex* with respect to X if $\dot{\gamma}=\dot{1}$ and $\phi_*d(\gamma^\Delta, 1^\Delta)=0$, where ϕ_* is the injection of the homotopy of ${}^0\Gamma$ into that of X, and 1 is the identity simplex. By Lemma 7.11 this means that for any $\sigma\in X$ we have $d((\gamma\sigma)^\Delta, \sigma^\Delta)=0$.

We shall say that X is Γ-*minimal* if

(M1) Γ_0 operates transitively on X_0,

(M2) If $\sigma, \tau\in X$, $\dot{\sigma}=\dot{\tau}$ and $d(\sigma^\Delta, \tau^\Delta)=0$ then $\tau=\gamma\sigma$ for some null-simplex $\gamma\in\Gamma$.

For Γ-minimal bundles we have the following lemma.

LEMMA 9.1. *If* X *and* Y *are* Γ*-minimal* Γ*-bundles,* $f:X\to Y$ *is* Γ*-equivariant and* $f_*:\pi_q(X)\approx\pi_q(Y)$ *for all* q, *then* f *is an isomorphism.*

It is clear from M1 that f is an isomorphism in dimension zero. We suppose inductively that it is an isomorphism through dimension $q-1$. Now if $f\sigma=f\tau$ we have, by the hypothesis of induction, $\dot{\sigma}=\dot{\tau}$. But $d(f\sigma^\Delta, f\tau^\Delta)=0=f_*d(\sigma^\Delta, \tau^\Delta)$ whence by M2 and the equivariance of f we get $\sigma=\tau$.

On the other hand, if $\tau\in Y_q$, then $c(f^{-1}\dot{\tau})=0$ and there is thus a $\sigma\in X_q$ such that $f\dot{\sigma}=\dot{\tau}$. But by Lemma 6.6 there is a $\sigma'\in X_q$ such that $\dot{\sigma}'=\dot{\sigma}$ and $d(\sigma'^\Delta, \sigma^\Delta)=f_*^{-1}d(\tau^\Delta, f\sigma^\Delta)$. It follows from M2 that $\tau=f(\gamma\sigma')$ for some null-simplex $\gamma\in\Gamma$.

If X is a Γ-complex with homotopy, an invariant subcomplex $A\subset X$ will be said to be Γ-minimal in X if it satisfies M1, 2 and in addition,

(M3) If $\sigma\in X$ and $\dot{\sigma}\subset A$ then there is a $\tau\in A$ such that $\dot{\tau}=\dot{\sigma}$ and $d(\tau^\Delta, \sigma^\Delta)=0$.

It will appear that if X is a bundle, a subcomplex satisfying M3 has homotopy; thus a subcomplex minimal in X will be itself minimal.

LEMMA 9.2. *If X is a Γ-complex with homotopy and ${}^0\Gamma$ has homotopy, there exists a subcomplex of X which is Γ-minimal in X.*

For the subset of simplices $\gamma\sigma$ where $\gamma \in \Gamma$ and σ is degenerate over a fixed 0-simplex in X is an invariant subcomplex satisfying M1, 2. There is thus a maximal subcomplex satisfying M1, 2, and this must also satisfy M3.

It should be pointed out that if Γ is trivial and X is a singular complex, then a minimal subcomplex of X is just a minimal complex in the sense of Eilenberg and Zilber [3]. The following theorem establishes the properly generalized "main homotopy" of §5 of that paper.

THEOREM 9.3. *If X is a Γ-bundle with homotopy, ${}^0\Gamma$ has homotopy, and $A \subset X$ is an invariant subcomplex satisfying* M3, *then there is an equivariant deformation retraction of X onto A.*

We shall construct inductively a retraction $r: X \to A$ and a homotopy $R: X \times I \to X$, stationary on A, of the identity with r.

There is of course no difficulty in dimension zero. Suppose $r: X^{q-1} \to A$ and $R: X^{q-1} \times I \to X$ have already been defined(²). Then R may be extended to an equivariant homotopy, stationary on A, say $R_1: X^q \times I \to X$, connecting the identity with some map $r_1: X^q \to X$, which, of course, extends r. But by M3 we may extend r to $r: X^q \to A$ in such a way that $d(r, r_1) = 0$. But then

$$d(\text{identity}, R, r) = d(\text{identity}, R, r_1) + d(r_1, r) = 0$$

and R may be extended to a homotopy of the identity with r.

This implies, together with Lemma 9.1, the Whitehead theorem for bundles with homotopy.

THEOREM 9.4. *If X and Y are Γ-bundles with homotopy, ${}^0\Gamma$ also has homotopy, $f: X \to Y$ is Γ-equivariant, and $f_*: \pi_q(X) \approx \pi_q(Y)$ for all q, then f is an equivariant homotopy equivalence.*

Suppose $A \subset X$ and $B \subset Y$ are Γ-minimal subcomplexes. If $r: Y \to B$ is an equivariant deformation retraction, then $r(f|A): A \to B$ is by Lemma 9.1 an isomorphism. But then $[r(f|A)]^{-1}r$ is a homotopy-inverse of f.

In particular, we have observed that such Γ-bundles have isomorphic minimal subcomplexes if and only if they are homotopy-equivalent. Thus the Γ-minimal Γ-bundles give a classification (by isomorphism-type) of the equivariant homotopy-types of Γ-bundles with homotopy.

(²) At this point we drop the cumbersome notation introduced in §4 for the invariant subcomplex determined by a skeleton. The notation $f: X^n \to Y$, when f is equivariant under the operation of a group Γ, is to be interpreted to mean that f is defined on ΓX^n. Since the values f takes on X^n determine those it takes on all of ΓX^n this notation does not lead to any ambiguities.

CHAPTER III

10. **Universal groups and strong classification.** If Γ is a semi-simplicial group, a universal group for Γ is a semi-simplicial group Υ containing Γ as a subgroup and having homotopy with all groups $\pi_q(\Upsilon)=0$. Since Γ operates on Υ by left translation, such a universal group is also a universal Γ-bundle, i.e. a Γ-bundle with vanishing homotopy.

Just as in the topological theory, universal bundles may be used to effect a strong classification of principal bundles with a fixed base-space: If X and X' are Γ-bundles such that $X/\Gamma=X'/\Gamma$, a strong equivalence of X and X' is a Γ-equivariant map $f:X\rightarrow X'$ such that f/Γ is the identity on X/Γ. It follows, of course, that f is an isomorphism of the bundle structures. The following theorem is proved just as in the topological case (cf. for example [10, II, §19]), supposing ${}^0\Gamma$ has homotopy.

THEOREM 10.1. *If U is a universal Γ-bundle, then the strong equivalence classes of Γ-bundles X such that X/Γ is a fixed complex B are in one to one correspondence with the homotopy classes of maps $\beta:B\rightarrow U/\Gamma$.*

It follows, in particular, that U/Γ is determined up to homotopy type by Γ. Because of this theorem, U/Γ is called a classifying space of Γ.

Our principal observation in this section is that, for any semi-simplicial group Γ, there exists a semi-simplicial group $U(\Gamma)$ which is universal for Γ. This we show by defining $U(\Gamma)$ explicitly, in such a way that it becomes clear that U is a covariant functor on the category of semi-simplicial groups and homomorphisms.

Explicitly, a simplex $\sigma\in U(\Upsilon)_q$ is a map which labels each face of $\mathbf{\Delta}_q$ by a simplex in Γ of the same dimension, i.e. a map $\sigma:\mathbf{\Delta}_q\rightarrow\Gamma$ preserving dimension but not in general incidence. The incidence operations are defined by composition of maps:

$$\mathbf{\Delta}_r \xrightarrow{\omega} \mathbf{\Delta}_q \xrightarrow{\sigma} \Gamma.$$

The group operation in $U(\Gamma)_q$ is defined by that in Γ: if $\tau\in\mathbf{\Delta}_q$ and $\sigma,\sigma'\in U(\Gamma)_q$, then

$$(\sigma\sigma')\tau = (\sigma\tau)(\sigma'\tau).$$

With these definitions it is clear that $U(\Gamma)$ is a semi-simplicial group. Γ may be identified with the subgroup of $U(\Gamma)$ consisting of those simplices which are simplicial maps $\sigma:\mathbf{\Delta}_q\rightarrow\Gamma$; the map giving the identification is just $\gamma\rightarrow\gamma^{\Delta}$.

The assertion about the homotopy of $U(\Gamma)$ is more difficult to demonstrate. It is, of course, equivalent to the statement that for any semi-simplicial complex K, integer q, and map $f:K^q\rightarrow U(\Gamma)$ the map has an extension to K^{q+1}. But it is sufficient to prove this for the special case $K=\mathbf{\Delta}_{q+1}$. For, sup-

posing this known, define for each nondegenerate simplex $\sigma \in K_{q+1}$ an extension $g_\sigma : \mathbf{\Delta}_{q+1} \to U(\Gamma)$ of $f\dot{\sigma} : \dot{\mathbf{\Delta}}_{q+1} \to U(\Gamma)$. Then it is easy to see that

$$f\sigma = g_\sigma \Delta_{q+1}$$

defines an extension of f.

Suppose then that f maps $(\mathbf{\Delta}_{q+1})^q = \dot{\mathbf{\Delta}}_{q+1}$ into $U(\Gamma)$. In order to extend it is necessary to define f only on Δ_{q+1}. This we do as follows. If $\tau \in \mathbf{\Delta}_{q+1}$ we set

$$(f\Delta_{q+1})\tau = \begin{cases} (f\tau)\Delta_{\dim \tau} & \text{if } \mathbf{D}\tau < q+1, \\ \text{any simplex in } \Gamma_{\dim \tau} & \text{if } \mathbf{D}\tau = q+1. \end{cases}$$

To see that this extension is simplicial, suppose $\omega : \mathbf{\Delta}_r \to \mathbf{\Delta}_{q+1}$ is in Ω. We must show that

$$f(\Delta_{q+1}\omega) = (f\Delta_{q+1})\omega.$$

We need consider here only the case in which $\mathbf{D}(\omega\Delta_r) < q+1$, since once f is simplicial on Δ_{q+1} it may certainly be extended to the collapsed simplices.

Now ω may be factored as follows:

$$\omega = \lambda^\Delta \omega'$$

for some $\lambda \in \dot{\mathbf{\Delta}}_{q+1}$ and $\omega' \in \Omega$. But then $\Delta_{q+1}\omega = \lambda\omega'$ and thus

$$f(\Delta_{q+1}\omega) = f(\lambda\omega') = (f\lambda)\omega',$$

while for τ and s-simplex in $\mathbf{\Delta}_r$,

$$(f\Delta_{q+1})(\omega\tau) = f(\omega\tau)\Delta_s = f(\lambda^\Delta\omega'\tau)\Delta_s = [(f\lambda)\omega'\tau^\Delta]\Delta_s = (f\lambda)\omega'\tau.$$

Thus we have proved the following theorem.

THEOREM 10.2. *$U(\Gamma)$ is a universal group for Γ.*

If $\Phi : \Gamma \to \Gamma'$, the homomorphism $U(\Phi) : U(\Gamma) \to U(\Gamma')$ is defined by composition: for $\sigma \in U(\Gamma)$

$$U(\Phi)\sigma = \Phi\sigma.$$

$U(\Phi)$ is of course an extension of Φ under the embedding of Γ in $U(\Gamma)$.

11. **The semi-simplicial groups** $K(\Pi, n)$. The notation $K(\Pi, n)$ has come to be used, in topology, to stand for any space with exactly one nonvanishing homotopy group, which is to be found in dimension n and is equal to Π. However, it was defined originally by Eilenberg and MacLane (cf. [2]) as a specific semi-simplicial complex. It is in fact a semi-simplicial group, under their definition, with homotopy, and having precisely those homotopy groups which should be demanded of it. It is central in the homotopy theory of semi-simplicial complexes with homotopy, since it constitutes a kind of "building block" out of which complexes of any homotopy type can be manufactured.

If Π is a discrete abelian group and $n=1, 2, \cdots$, then $K(\Pi, n)$ is defined by

$$K(\Pi, n)_q = Z_n^n(\Delta_q; \Pi).$$

The incidence operations are given by composition with $\omega \in \Omega$:

$$\Delta_r \xrightarrow{\omega} \Delta_q \xrightarrow{\kappa} \Pi$$

defines, for $\kappa \in K(\Pi, n)_q$, an element $\kappa\omega$ of $K(\Pi, n)_r$; the group operation in each dimension is just the addition in the cocycle group.

It is easy to see that $K(\Pi, n)$, so defined, is a semi-simplicial group. The assertion about the homotopy is less trivial.

THEOREM 11.1. *$K(\Pi, n)$ is a complex with homotopy; $\pi_n(K(\Pi, n)) = \Pi$ and $\pi_r(K(\Pi, n)) = 0$ for $r \neq n$.*

Since $K(\Pi, n)_r = 0$ for $r < n$ it is clear that there is nothing to be proved in low dimensions. It will be necessary only to examine the obstruction in dimension $n+1$, and to prove an extension theorem (compare the proof of 10.2) in higher dimensions.

Suppose, then, that $f: K^n \to K(\Pi, n)$, for some semi-simplicial complex K. To f is associated a cochain $f^c \in C_n^n(K; \Pi)$ given by

$$f^c\sigma = (f\sigma)\Delta_n.$$

We shall observe that $cf = \delta f^c$ obeys Axioms H1–4, and will thus be an obstruction belonging to a homotopy system on $K(\Pi, n)$.

Now as a coboundary, cf is of course a cocycle, so that H1 is clear. Also it is clear that for $g: L^{n+1} \to K$ the cochains $f^c g$ and $(fg)^c$ are equal; H4 follows immediately. Axiom H3 is equally trivial, since every cochain in $C_n^n(K; \Pi)$ is of the form f^c. In fact, if h is such a cochain

$$(f\sigma)\Delta_n = h\sigma$$

defines a map with $f^c = h$, since $K(\Pi, n)^{n-1}$ is trivial.

Now if f is extendible to K^{n+1} we have, for $\sigma \in K_{n+1}$,

$$(cf)\sigma = f^c\partial\sigma = \sum (-1)^i f(\sigma^i)\Delta_n = \sum (-1)^i (f\sigma)^i \Delta_n = \delta(f\sigma)\Delta_n = 0.$$

Conversely, if $cf = 0$, we may define, for $\sigma \in K_{n+1}$, a cochain $\kappa_\sigma \in C_n^n(\Delta_{n+1}; \Pi)$ by

$$\kappa_\sigma \Delta_{n+1}^i = (f\sigma^i)\Delta_n = f^c\sigma^i.$$

This is clearly a cocycle, and thus an $n+1$-simplex of $K(\Pi, n)$. Moreover, it is obvious that $f\sigma = \kappa_\sigma$ defines an extension of f to K^{n+1}. Thus Axiom H2 is also satisfied.

It remains only to prove that if $f: \dot{\Delta}_q \to K(\Pi, n)$, where $q-1 > n$, then f is

extendible to $\mathbf{\Delta}_q$. As we have observed in the proof of Theorem 10.2, it will be necessary to define the extension only on Δ_q. This we do as follows: $\kappa = f\Delta_q \in Z_n^n(\mathbf{\Delta}_q; \Pi)$ is given by

$$\kappa\tau = (f\tau)\Delta_n.$$

To see that κ is a cocycle, we may write, for $\sigma \in (\mathbf{\Delta}_q)_{n+1}$,

$$(\delta\kappa)\sigma = \sum (-1)^i(\kappa\sigma^i) = \sum (-1)^i f(\sigma^i)\Delta_n = \sum (-1)^i (f\sigma)^i \Delta_n = \delta(f\sigma)\Delta_n = 0.$$

To see that f so extended is still simplicial, we must show that for $\omega: \mathbf{\Delta}_r \to \mathbf{\Delta}_q$ in Ω,

$$f(\Delta_q\omega) = \kappa\omega \in Z_n^n(\mathbf{\Delta}_r; \Pi).$$

We may, as in the proof of 10.2, suppose that $\dim(\omega\Delta_r) < q$ and thus that ω has a factorization $\omega = \lambda^\Delta\omega'$, for $\lambda \in \dot{\mathbf{\Delta}}_q$ and $\omega' \in \Omega$. Then if τ is an n-simplex in $\mathbf{\Delta}_r$,

$$f(\Delta_q\omega)\tau = f(\Delta_q\lambda^\Delta\omega')\tau = f(\Delta_q\lambda^\Delta)\omega'\tau = (f\lambda)\omega'\tau,$$

while

$$\kappa\omega\tau = f(\omega\tau)\Delta_n = f(\lambda^\Delta\omega'\tau)\Delta_n = [(f\lambda)\omega'\tau^\Delta]\Delta_n = (f\lambda)\omega'\tau.$$

This completes the proof of the theorem.

Now in fact $K(\Pi, n)$ is a covariant functor, for fixed n, on the category of abelian groups and homomorphisms. If $\phi: \Pi \to \Pi'$ is a homomorphism, it defines a homomorphism $K(\phi, n): K(\Pi, n) \to K(\Pi', n)$ by means of the map on cochains $\kappa \to \phi\kappa$. This homomorphism, being a simplicial map, induces a homomorphism $K(\phi, n)_*: \pi_n(K(\Pi, n)) \to \pi_n(K(\Pi', n))$. But these groups are just Π and Π'. As might be expected,

$$K(\phi, n)_* = \phi. \tag{11.2}$$

For if $f: K^n \to K(\Pi, n)$ then

$$(K(\phi, n)f)^c = \phi(f^c),$$
$$\delta[\phi(f^c)] = \phi(\delta f^c).$$

But (11.2) now follows from (5.8).

The homotopy class of a spherical map into $K(\Pi, n)$ has a particularly simple expression. Suppose (S, s) is an oriented simplicial n-sphere and $f: S \to K(\Pi, n)$. Then we may suppose that $(S, s) = (\dot{E}, \partial e)$ for some oriented simplicial $n+1$-cell (E, e) and that $f_1: E^n \to K(\Pi, n)$ extends f. Then

$$\langle f \rangle = (cf_1)e = \delta f_1^c e.$$

and thus

$$\langle f \rangle = f^c s. \tag{11.3}$$

This may be applied to obstructions of homotopies by using (6.1). The result is especially simple for stationary homotopies, since f^c is normalized. If $f, f': K^n \to K(\Pi, n)$ agree on the $n-1$-skeleton, then

$$d(f, f') = f'^c - f^c. \tag{11.4}$$

12. On adjoining homotopy groups.

THEOREM 12.1. *Suppose X is a Λ-bundle, Λ and X/Λ have homotopy, $\pi_q(\Lambda) = 0$ for $q<n$, and $\pi_r(X/\Lambda) = 0$ for $r \geqq n$. Then X has homotopy and*

$$\eta_*: \pi_q(X) \approx \pi_q(X/\Lambda), \qquad q < n,$$

$$\phi_*: \pi_r(\Lambda) \approx \pi_r(X), \qquad r \geqq n,$$

where $\eta: X \to X/\Lambda$ is the canonical map and ϕ_* is the injection of §7.

The proof proceeds by defining obstructions for maps into X, with coefficients in the appropriate homotopy groups of Λ and X/Λ, and showing that these obstructions satisfy axioms H1–4.

In dimensions smaller than n we define, for $f: K^q \to X$, the obstruction $cf = c(\eta f)$. It is clear that this satisfies H1 and H4, as well as half of H2, for if f is extendible, so, certainly, is ηf.

For the rest, since Λ is $n-1$-connected, we can construct a cross-section $S: (X/\Lambda)^n \to X$ of η. Now if $c(\eta f) = 0$, then ηf is extendible, say to $f_1: K^{q+1} \to X/\Lambda$. If Ψ is the difference-map of X the composition-map $\Psi(f \times S\eta f): K^q \to \Lambda$ is also extendible, say to $\Phi: K^{q+1} \to \Lambda$, since $\pi_q(\Lambda) = 0$. An extension of f is then given by $\Phi \cdot (Sf_1): K^{q+1} \to X$.

There remains only H3. If $f: K^q \to X$ and $h \in C_n^q(K, L; \pi_q(X/\Lambda))$, then there is a map $f': K^q \to X/\Lambda$ such that f' and ηf agree on $K^{q-1} \cup L^q$ and $cf' = c(\eta f) + \delta h$. But then

$$f_1 = [\Psi(f \times S\eta f)] \cdot (Sf')$$

is the map required by H3.

Suppose now that $f: K^r \to X$, where $r \geqq n$. Then there exists an extension $f_1: K \to X/\Lambda$ of ηf. Let $Y = \hat{f}_1^{-1}(X)$ be the induced bundle and define $f': Y^r \to \Lambda$ by

$$f'(\sigma, \tau) = \Psi(\tau, f\sigma).$$

Recalling that $Y = \{(\sigma, \tau) \mid f_1\sigma = \eta\tau\} \subset K \times X$ and that, for $\lambda \in \Lambda$, $\lambda(\sigma, \tau) = (\sigma, \lambda\tau)$, we see that f' is well defined and equivariant. We shall show that $(cf')^\sharp$ satisfies Axioms H1–4, where $(cf')^\sharp \in Z_n^{r+1}(K; \pi_r(\Lambda))$ is defined by

$$(cf')^\sharp \mu = cf',$$

$\mu: Y \to K$ being the canonical map. It will follow then that $cf = (cf')^\sharp$ is the obstruction of the unique homotopy system on X.

The first step is to show that $(cf')^\sharp$ is independent of the extension f_1 of

ηf. If f_2 is another such extension, then the stationary homotopy of f_1 and f_2 on K^r is extendible to a homotopy of the two maps, and thus the identity map $Y^r = Z^r$, where Z is the bundle on K induced by f_2, is extendible to an equivariant map $F: Y \to Z$ with F/Λ the identity on K (cf. Theorem 8.3). If the map corresponding to f_2 is $f'': Z^r \to \Lambda$, then $f''F = f'$ and thus for $(\sigma, \tau) \in Y_{r+1}$

$$(cf'')^\sharp\sigma = (cf'')F(\sigma, \tau) = (cf')(\sigma, \tau) = (cf')^\sharp\sigma.$$

Axiom H1 is of course trivial. For H2 observe first that if f is extendible, say to $\bar{f}: K^{r+1} \to X$, we may assume that f_1 extends $\eta\bar{f}$. But then

$$\bar{f}'(\sigma, \tau) = \psi(\tau, \bar{f}\sigma)$$

gives an extension of f' and thus $cf' = 0$. Conversely if $(cf')^\sharp$ is zero, then so is cf' and f' is extendible, say to $\bar{f}': Y^{r+1} \to \Lambda$. Now $f'^{-1} \cdot \hat{f}_1: Y^r \to X$ is, as is easily seen, just $f\mu$. But $\hat{f}'^{-1} \cdot f_1: Y^{r+1} \to X$ can be factored uniquely, $\bar{f}'^{-1} \cdot \hat{f}_1 = \bar{f}\mu$, where $\bar{f}$ extends f.

For Axiom H3 suppose that $h \in C^r_n(K, L; \pi_r(\Lambda))$. Then there is an equivariant $g': Y^r \to \Lambda$ which agrees with f' on $\mu^{-1}(L) \cup Y^{r-1}$ and satisfies $cg' = cf' + \delta(h\mu)$. But $g'^{-1} \cdot f_1: Y^r \to X$ can be uniquely factored, $g'^{-1} \cdot f_1 = g\mu$, and g must then agree with f on $L \cup K^{r-1}$. The obstruction of g is

$$(cg')^\sharp = (cf' + (\delta h)\mu)^\sharp = (cf')^\sharp + \delta h.$$

To see that H4 holds, suppose $g: L^{r+1} \to K$ for some complex L. Then $f_1 g$ extends $\eta f(g \mid L^r)$. If Z is the bundle $(f_1 g)^{\frown -1} X$, then $G(\sigma, \tau) = (g\sigma, \tau)$ defines an equivariant map $G: Z \to Y$ with $G/\Lambda = g$. But the map $(f(g \mid L^r))'$ whose obstruction defines $c(f(g \mid L^r))$ is clearly just $f'(G \mid Z^r)$. Thus

$$c(f(g \mid L^z)) = (cf')^\sharp(G/\Lambda) = (cf)g.$$

We have now shown that X has homotopy and identified its groups with those of Λ and X/Λ. The proof will be completed if we show that η_* and ϕ_* act as the respective identity maps. For η this follows immediately from $cf = c(\eta f)$ in dimensions less than n. In higher dimensions we may proceed as follows: suppose $w: K^r \to \Lambda$ and $\phi_0: \Lambda \to X$, the latter being given by the operation on a degenerate simplex. If $f = \phi_0 w$ then f_1 may be taken to be a constant map, and Y will be just $K \times \phi_0\Lambda$. Since ϕ_0 is an isomorphism, there is a projection $L: Y \to \Lambda$. But then it is clear that f' is just $L \cdot (w\mu)^{-1}$. Since L is extendible, this gives $cf' = -(cw)\mu$ and thus

$$cf = \phi_*(cw) = -(cf')^\sharp = cw,$$

which shows that ϕ_* is the identity.

An easy consequence of this theorem is the following.

COROLLARY 12.2. *Suppose X and X' are Λ and Λ'-bundles satisfying the hypotheses of Theorem* 12.1. *If $\Phi: \Lambda \to \Lambda'$ is a homomorphism and $f: X \to X'$ is Φ-equivariant, then*

$$f_* = (f/\Phi)_*: \pi_q(X) \to \pi_q(X'), \qquad q < n,$$
$$f_* = \Phi_*: \pi_r(X) \to \pi_r(X'), \qquad r \geqq n,$$

where we have made the identification of the homotopy groups of X given by Theorem 12.1.

This can be seen in the lower dimensions by looking at the commutation $\eta f = (f/\Phi)\eta'$, and in the higher ones by $\phi_0'\Phi = f\phi_0$, where ϕ_0' is the map given by the operation on that degenerate simplex of X' which is the image under f of the one used to define ϕ_0.

Returning to the notation used in the proof of Theorem 12.1, we record an alternate formula for the obstruction of a map $f: K^r \to X$, where $r \geqq n$. Since Λ is $n-1$-connected, there is an equivariant map $\theta: X^n \to \Lambda$. But then f' can be expressed as follows:

$$f' = (\theta f_1)\cdot(\theta f \mu)^{-1}.$$

It follows from Lemma 7.2 that $(cf')^\# = c(\theta f_1)^\# - c(\theta f\mu)^\#$ and

$$cf = (c\theta)^\# f_1 - c(\theta f). \tag{12.3}$$

Finally we observe that with the identification provided by Theorem 12.1, Lemma 7.11 takes the following form. If $g_0, g_1: K^r \to \Lambda$ agree on K^{r-1} and $f: K^r \to X$, then

$$d(g_1\cdot f, g_0\cdot f) = d(g_1, g_0). \tag{12.4}$$

13. **$\Gamma(\Pi, n)$-bundles.** If Γ is a semi-simplicial group operating on the discrete abelian group Π, then Γ operates also on $K(\Pi, n)$ by

$$(\gamma\cdot\kappa)\tau = \gamma(\kappa\tau)$$

where $\gamma \in \Gamma_q$, $\kappa \in K(\Pi, n)_q = Z_n^n(\Delta_q; \Pi)$ and $\tau \in (\Delta_q)_n$. We may thus construct the splitting extension, for which we introduce the notation

$$\Gamma(\Pi, n) = K(\Pi, n)\cdot\Gamma.$$

If $\Phi: \Gamma \to \Gamma'$ is a homomorphism, Γ' operates on Π' and $\phi: \Pi \to \Pi'$ is Φ-equivariant, then $K(\phi, n) \times \Phi: \Gamma(\Pi, n) \to \Gamma'(\Pi', n)$ is a homomorphism which we denote by $\Phi(\phi, n)$.

In all of what follows we shall suppose that the semi-simplicial groups considered are such that the component of the identity has homotopy.

If X is a $\Gamma(\Pi, n)$-bundle, there exist $\Gamma(\Pi, n)$-equivariant maps $\theta: X^n \to K(\Pi, n)$, where $\Gamma(\Pi, n)$ operates on $K(\Pi, n)$ by

$$(\kappa, \gamma)\kappa_1 = \kappa(\gamma\kappa_1), \text{ for } (\kappa, \gamma) \in \Gamma(\pi, n),\ \kappa_1 \in K(\pi, n).$$

These maps give rise to obstructions $c\theta$ lying in a unique class in $H^{n+1}_{\Gamma(\Pi,n)}(X; \Pi)$;

the reduced classes $(c\theta)^\sharp$ thus lie in a unique class

$$k^{n+1}(X) \in H_\Gamma^{n+1}(X/K(\Pi, n); \Pi).$$

Because $K(\Pi, n)$ has only one nonvanishing homotopy group, the lifting theorem takes for bundles of the type just considered a particularly simple form. For, referring to Theorem 8.1, the only relevant obstruction classes for the lifting of a Φ-equivariant map $f: X/K(\Pi, n) \to X'/K(\Pi', n)$ are $\mathfrak{C}^{n+1}(X, \Gamma, K(\Pi, n)) = \{k^{n+1}(X)\}$ and $\mathfrak{C}^{n+1}(X', \Gamma', K(\Pi', n)) = \{k^{n+1}(X')\}$. Thus 8.1 takes the following form.

LEMMA 13.1. *A Φ-equivariant map $f: X/K(\Pi, n) \to X'/K(\Pi', n)$ can be lifted to a $\Phi(\phi, n)$-equivariant map $F: X \to X'$ if and only if*

$$\phi_* k^{n+1}(X) = f^* k^{n+1}(X').$$

This lemma leads to a strong classification (cf. §10) of the $\Gamma(\Pi, n)$-bundles X such that $X/K(\Pi, n)$ is a fixed Γ-bundle B.

LEMMA 13.2. *The strong equivalence classes of $\Gamma(\Pi, n)$-bundles X such that $X/K(\Pi, n)$ is a fixed Γ-bundle B are in one-one correspondence with $H_\Gamma^{n+1}(B; \Pi)$ the correspondence being given by $X \to k^{n+1}(X)$.*

The previous lemma states that the correspondence is one-one onto a subset of $H_\Gamma^{n+1}(B; \Pi)$. We need only show that for any $k \in H_\Gamma^{n+1}(B; \Pi)$ there is an X with $k^{n+1}(X) = k$.

Suppose z is any cocycle in k. Define X by setting

$$X_q = \{(\sigma, u) \mid \sigma \in B_q, u \in C_n^n(\Delta_q; \Pi), \delta u = z\sigma^\Delta\},$$
$$(\sigma, u)\omega = (\sigma\omega, u\omega) \quad \text{for} \quad \omega \in \Omega,$$
$$(\kappa, \gamma)(\sigma, u) = (\gamma\sigma, \kappa + \gamma \cdot u)$$

for $\gamma \in \Gamma_q$, $\kappa \in K(\Pi, n)_q$; the operation "$\cdot$" is the obvious extension of that on $K(\Pi, n)$, and the addition is to take place in the cochain group.

It is easy to see that X so defined is a $\Gamma(\Pi, n)$-bundle and that $X/K(\Pi, n)$ may be identified with B. If we consider the map $\theta: X^n \to K(\Pi, n)$ given by

$$\theta(\sigma, u) = u$$

for $(\sigma, u) \in X_n$, which is simplicial since $K(\Pi, n)^{n-1}$ is trivial and $C_n^n(\Delta_n; \Pi)$ contains only cocycles, and which is clearly equivariant,

$$(c\theta)^\sharp\sigma = (c\theta)(\sigma, u) = \delta\theta^c(\sigma, u) = (\delta u)\Delta_{n+1} = z\sigma^\Delta\Delta_{n+1} = z\sigma.$$

We shall want to talk now about $\Gamma(\Pi, n)$-bundles whose higher homotopy groups vanish. We define a *terminal $\Gamma(\Pi, n)$-bundle* X to be one for which $X/K(\Pi, n)$ has homotopy and $\pi_q(X/K(\Pi, n)) = 0$ for $q \geqq n$. It follows from

Theorem 12.1 that X also has homotopy, which differs from that of $X/K(\Pi, n)$ only in that the group Π appears in dimension n.

If X is a terminal $\Gamma(\Pi, n)$-bundle, the class $k^{n+1}(X)$ can be expressed in a particularly simple fashion. Let us observe that since there exists a $\Gamma(\Pi, n)$-equivariant map $\theta: X^n \to K(\Pi, n)$ there exist also, by Theorem 3.1, Γ-equivariant extensions $j: (X/K(\Pi, n))^n \to X$ of the identity on $(X/K(\Pi, n))^{n-1} = X^{n-1}$, for example, $\mathcal{W}\theta$.

LEMMA 13.3. *If $j: (X/K(\Pi, n))^n \to X$ is a Γ-equivariant extension of the identity, then $cj \in k^{n+1}(X)$.*

For j and $\mathcal{W}\theta$ agree on the $n-1$-skeleton and thus the obstructions cj and $c(\mathcal{W}\theta)$ are in the same class in $H_\Gamma^{n+1}(X/K(\Pi, n); \Pi)$. But by (12.3),

$$c(\mathcal{W}\theta) = (c\theta)^\sharp - c(\theta(\mathcal{W}\theta))$$

since $\eta(\mathcal{W}\theta)$ is the identity. Moreover, $\theta(\mathcal{W}\theta)$ is the constant map onto the identity of $K(\Pi, n)$, so that the second term vanishes. The lemma follows.

We are now in a position to prove the following important lemma. We recall that $\eta: X \to X/K(\Pi, n)$ is the canonical map, and write again $\bar{\eta}: X \times I \to X/K(\Pi, n) \times I$ for the product of η with the identity on I. Let us introduce also a terminal $\Gamma'(\Pi', n)$-bundle X' and a homomorphism $\Phi: \Gamma \to \Gamma'$.

LEMMA 13.4. *Suppose $f: X \to X'$ is Φ-equivariant (only!), $\bar{f}: X/K(\Pi, n) \to X'/K(\Pi', n)$ is Φ-equivariant, and*

$$E: (X/K(\Pi, n))^{n-1} \times I \to X'/K(\Pi', n)$$

is a Φ-equivariant homotopy of $f|X^{n-1}$ and $\bar{f}$. Then if $j': (X'/K(\Pi', n))^n \to X'$ is a Γ'-equivariant cross-section, there exists a $\Phi(f_*, n)$-equivariant lifting $F: X \to X'$ of $\bar{f}$ such that $d(f, j'E\bar{\eta}, F) = 0$.

We begin by showing that there exists some lifting of $\bar{f}$. According to Lemma 13.1, the condition for this is that $f_* k^{n+1}(X) = \bar{f}^* k^{n+1}(X')$. But using Lemma 13.3 and recalling (5.8),

$$f_* k^{n+1}(X) \ni c(fj) \sim c(j'\bar{f}) \in f^* k^{n+1}(X')$$

where j is a Γ-equivariant section in X, since through dimension $n-1$, $fj = j'f$, and fj and $j'\bar{f}$ are thus homotopic through dimension $n-1$.

Now suppose $F_0: X \to X'$ is a $\Phi(f_*, n)$-equivariant lifting of $\bar{f}$. Then $d(f, j'E\bar{\eta}, F_0) \in Z_\Gamma^n(X; \Pi')$, for all the maps and homotopies are Φ-equivariant, and both f and F_0 are extendible. We shall show that this obstruction is also $K(\Pi, n)$-equivariant.

To see this, suppose that $\kappa \in K(\Pi, n)_n$. If we write L_κ for left translation by κ in X_n, and let L_κ be the identity in X^{n-1}, then $L_\kappa: X^n \to X^n$ is simplicial. Moreover, $j'E\bar{\eta}$ is also a homotopy of fL_κ and $F_0 L_\kappa$, and the equivariance of the obstruction is asserted by

$$d(fL_\kappa, j'E\bar{\eta}, F_0L_\kappa) = d(f, j'E\bar{\eta}, F_0)$$

if this equality holds for each n-simplex in $K(\Pi, n)$.

But since composition with L_κ leaves maps unchanged through dimension $n-1$, we have

$$\begin{aligned} d(fL_\kappa, j'E\bar{\eta}, F_0L_\kappa) &= d(fL_\kappa, f) + d(f, j'E\bar{\eta}, F_0) + d(F_0, F_0L_\kappa) \\ &= f_*[d(L_\kappa, \text{identity}) + d(\text{identity}, L_\kappa)] + d(f, j'E\bar{\eta}, F_0) \\ &= d(f, j'E\bar{\eta}, F_0) \end{aligned}$$

using (6.9) and 12.2.

Thus $d(f, j'E\bar{\eta}, F_0) \in Z^n_{\Gamma(\Pi,n)}(X; \Pi')$ and we can find a $\Gamma(\Pi, n)$-equivariant map

$$g: X \to K(\Pi', n)$$

such that $g^c = -d(f, j'E\bar{\eta}, F_0)$. If then we define $F: X \to X'$ by $F = g \cdot F_0$, it is clear that F is also a $\Phi(f_*, n)$-equivariant lifting of $\bar{f}$ and we have

$$d(f, j' E\bar{\eta}, F) = d(f, j'E\bar{\eta}, F_0) + d(F_0, g \cdot F_0) = d(f, j'E\bar{\eta}, F_0) + d(g, \text{identity}) = 0$$

by (12.4) and (11.4).

Chapter IV

14. Homotopically segregated Γ-bundles. In §11 we suggested that the groups $K(\Pi, n)$ formed the essential structural elements of a theory of semi-simplicial complexes. Here we make this notion explicit, in terms of "homotopically segregated Γ-bundles," which, in a sense to be expounded below, are constructed out of $K(\Pi, n)$ put together with invariants. The invariants adduced are in essence just those of Postnikov [9] and the unpublished work of Zilber.

A *homotopically segregated* Γ-*bundle* $\mathfrak{K}$ consists of a sequence $\{{}^0K, {}^1K, \cdots\}$ of Γ-bundles with homotopy, with the additional structure of an operation of $K(\pi_q({}^qK), q)$ on ${}^qK$, for $q = 1, 2, \cdots$, such that

(S1) ${}^0K$ is an aspherical Γ-bundle,

(S2) ${}^qK$ is a $\Gamma(\pi_q({}^qK), q)$-bundle, where the operation of Γ on the homotopy group is that given by its operation on the complex,

(S3) ${}^qK/K(\pi_q({}^qK), q) = {}^{q-1}K$.

It follows that each ${}^qK$ is a terminal $\Gamma(\pi_q({}^qK), q)$-bundle. In fact, if we had listed the groups $\pi_q({}^qK)$ as part of the structure, it would not have been necessary to assert that the ${}^qK$ had homotopy.

Now if $\mathfrak{K}$ is a homotopically segregated Γ-bundle, the equalities ${}^qK^{q-1} = {}^{q-1}K^{q-1}$ hold. This permits us to define a complex $\lim \mathfrak{K}$ by

$$(\lim \mathfrak{K})^q = {}^rK^q, \qquad r \geqq q.$$

If we carry over to $\lim \mathfrak{K}$ the operation of Γ on the ${}^qK$, then $\lim \mathfrak{K}$ is clearly

a Γ-bundle. Moreover, it is also evident that $\lim \mathfrak{K}$ is a complex and that

$$\pi_q(\lim \mathfrak{K}) = \pi_q({}^rK), \qquad r \geqq q.$$

We shall abbreviate this common value by $\pi_q(\mathfrak{K})$.

Compositions of the canonical maps ${}^rK \to {}^{r-1}K$ give rise to equivariant maps $\eta_q : \lim \mathfrak{K} \to {}^qK$. Also, the identity maps ${}^qK^q = (\lim K)^q$ have equivariant extensions $j_q : {}^qK^{q+1} \to \lim \mathfrak{K}$; these are of course not unique.

To a homotopically segregated Γ-bundle $\mathfrak{K}$ corresponds a sequence of invariants

$$k^{q+1}(\mathfrak{K}) = k_\Gamma^{q+1}({}^qK) \in H^{q+1}({}^{q-1}K; \pi_q(\mathfrak{K})),$$

for $q = 1, 2, \cdots$. These may be expressed, for example, by

$$cj_{q-1} \in k^{q+1}(\mathfrak{K}).$$

If $\mathfrak{K}'$ is a homotopically segregated Γ'-bundle and $\Phi : \Gamma \to \Gamma'$ is a homomorphism, a Φ-*map* $\mathfrak{f} : \mathfrak{K} \to \mathfrak{K}'$ is a sequence of maps $\{f_0, f_1, \cdots\}$ such that $f_q : {}^qK \to {}^qK'$ is $\Phi(f_{q*}, q)$-equivariant and such that f_q is a lifting of f_{q-1}, i.e. $f_q/K(f_{q*}, n) = f_{q-1}$.

This means that f_q and f_{q-1} agree on ${}^qK^{q-1} = {}^{q-1}K^{q-1} = (\lim \mathfrak{K})^{q-1}$ and thus serve to define a map $\lim \mathfrak{f} : \lim \mathfrak{K} \to \lim \mathfrak{K}'$. It is clear that $\lim \mathfrak{f}$ is Φ-equivariant and that

$$(\lim \mathfrak{f})_* = f_{q*} : \pi_q(\mathfrak{K}) \to \pi_q(\mathfrak{K}').$$

We shall abbreviate the common value of this homomorphism by $\mathfrak{f}_*$.

The effect of a Φ-map $\mathfrak{f}$ on the invariants is indicated by Lemma 13.1. Since each f_q is in fact liftable, we have

$$\mathfrak{f}_* k^{q+1}(\mathfrak{K}) = f_{q+1*} k^{q+1}(\mathfrak{K}').$$

It is a remarkable fact that any Φ-equivariant map of $\lim \mathfrak{K}$ into $\lim \mathfrak{K}'$ can be approximated to within homotopy by limits of Φ-maps. More precisely, we have the following theorem.

THEOREM 14.1. *If* $g : \lim \mathfrak{K} \to \lim \mathfrak{K}'$ *is* Φ-*equivariant, then there is a* Φ-*map* $\mathfrak{f} : \mathfrak{K} \to \mathfrak{K}'$ *such that* $\lim \mathfrak{f}$ *is* Φ-*equivariantly homotopic to* g.

This is proved by induction, using Lemma 13.4. The process starts by defining $f_0 : {}^0K \to {}^0K'$. Now $\eta_1' g j_1 : {}^1K^2 \to {}^1K'$ has, since the higher homotopy groups of ${}^1K'$ vanish, a Φ-equivariant extension $g_1 : {}^1K \to {}^1K'$. There exists of course an equivariant homotopy $E_0 : {}^0K^0 \times I \to {}^0K'$ of $g_1 | {}^0K^0$ and f_0. It follows then from Lemma 13.4 that there is a $\Phi(g_*, 1)$-equivariant lifting f_1 of f_0 and an equivariant homotopy $E_1 : {}^1K \times I \to {}^1K'$ of g_1 and f_1 which agrees with E_0 on ${}^1K^0 \times I$.

In general we may define Φ-equivariant maps $g_q : {}^qK \to {}^qK'$ which extend

$\eta_q' g j_q$. Suppose that we have defined also maps $f_0, \cdots, f_{q-1}$ which form, in the obvious sense, part of a Φ-map, and equivariant homotopies $E_r: {}^rK \times I \to {}^rK'$ of g_r with f_r such that E_r and E_{r-1} agree on ${}^rK^{r-1} \times I$. Then Lemma 13.4 allows this process to go on one step. The f_q, finally, constitute a Φ-map $\mathfrak{f}$; since the E_q agree in low dimensions they define a homotopy $E: \lim \mathfrak{K} \times I \to \lim \mathfrak{K}'$ of g with $\lim \mathfrak{f}$.

15. **Homotopy resolutions.** The connection between Γ-minimal complexes and the other material we have been discussing is in part indicated by the following lemma.

LEMMA 15.1. *If X is a terminal $\Gamma(\Pi, n)$-bundle and $X/K(\Pi, n)$ is Γ-minimal, then so is X.*

There is of course nothing to prove in dimensions less than n. If σ, τ are simplices of dimension at least n satisfying the hypothesis of M2, then their projections in $X/K(\Pi, n)$ do so also. Thus there is a null-simplex $\gamma \in \Gamma$ and a simplex $\kappa \in K(\Pi, n)$ such that $\tau = \kappa\gamma\sigma$. But κ must be trivial: in dimension n this follows from 12.4 and 11.4, in higher dimensions it is simply a consequence of the fact that the boundary of κ is trivial.

COROLLARY 15.2. *If $\mathfrak{K}$ is a homotopically segregated Γ-bundle and ${}^0K$ is minimal, then so are all the ${}^nK$ and $\lim \mathfrak{K}$.*

We may say then that $\mathfrak{K}$ is a *minimal homotopically segregated* Γ-*bundle.*

We may define now the notion of a homotopy-resolution. If X is a Γ-bundle with homotopy, a *homotopy-resolution* of X is a pair $(\mathfrak{K}, \xi)$ where $\mathfrak{K}$ is a minimal homotopically segregated Γ-bundle and $\xi: \lim \mathfrak{K} \to X$ is Γ-equivariant and satisfies $\xi_*: \pi_q(\mathfrak{K}) \approx \pi_q(X)$ for all q.

From the results of §9 we get quickly the following lemma.

LEMMA 15.3. *If X is a Γ-bundle with homotopy and $\mathfrak{K}$ is a minimal homotopically segregated Γ-bundle, and if $\xi: \lim \mathfrak{K} \to X$ is Γ-equivariant then the following conditions are equivalent:*

I. *$(\mathfrak{K}, \xi)$ is a homotopy-resolution of X.*

II. *ξ is an equivariant homotopy equivalence.*

III. *ξ is an isomorphism of $\lim \mathfrak{K}$ onto a subcomplex A which is minimal in X.*

We may now state and prove the principal theorem of this paper, whose sense is that, to within homotopy equivalence, bundles with homotopy may be replaced by minimal homotopically segregated bundles, and maps by maps of homotopically segregated bundles, i.e. "Φ-maps."

THEOREM 15.4. *If X is a Γ-bundle with homotopy, and ${}^0\Gamma$ also has homotopy, then there exists a homotopy-resolution $(\mathfrak{K}, \xi)$ of X. If $(\mathfrak{K}, \xi)$ and $(\mathfrak{K}', \xi')$ are*

homotopy-resolutions of the Γ*-bundle* X *and the* Γ'*-bundle* X', *if* $\Phi:\Gamma\to\Gamma'$ *is a homomorphism, and if* $f:X\to X'$ *is* Φ*-equivariant, then there is a* Φ*-map* $\mathfrak{f}:\mathfrak{K}\to\mathfrak{K}'$ *such that* $\xi'\lim \mathfrak{f}$ *and* $f\xi$ *are* Γ*-equivariantly homotopic.*

To prove the first statement we shall construct recursively complexes ${}^0K$, ${}^1K$, $\cdots$ and maps ${}^n\xi:{}^nK^{n+1}\to X$ such that the complexes form a minimal homotopically segregated Γ-bundle and in addition, for all n, ${}^{n+1}\xi|{}^nK^n={}^n\xi|{}^nK^n$ and ${}^n\xi_*:\pi_n({}^nK)\approx\pi_n(X)$. For then $\xi:\lim \mathfrak{K}\to X$ may be defined by $\xi|{}^nK^n={}^n\xi|{}^nK^n$ and $(\mathfrak{K}, \xi)$ will be the homotopy-resolution in question.

We begin with a minimal aspherical Γ-bundle ${}^0K$, whose existence is assured by Theorem 10.2 and §9. Since X is homotopy-connected, there is an equivariant map ${}^0\xi:{}^0K^1\to X$ (there is no question of isomorphism of homotopy groups here). Suppose inductively that ${}^{q-1}K$, ${}^{q-1}\xi$ have already been defined. By Lemma 13.2 we may find a $\Gamma(\pi_q(X), q)$-bundle ${}^qK$ with basespace (with respect to $K(\pi_q(X), q)\,{}^{q-1}K$) such that $k^{q+1}({}^qK)$ contains $c({}^{q-1}\xi)$. But according to Lemma 13.3, we may then suppose that $c({}^{q-1}\xi)$ is actually the obstruction of some extension of the identity map ${}^{q-1}K^{q-1}={}^qK^{q-1}$. Thus Lemma 7.11 asserts the existence of the required map ${}^q\xi$.

For the second assertion we need only observe that ξ' is an equivariant homotopy-equivalence and that there is thus a homotopy inverse $\zeta':X'\to\lim \mathfrak{K}'$. By Theorem 14.1 there is a Φ-map $\mathfrak{f}:\mathfrak{K}\to\mathfrak{K}'$ such that $\lim \mathfrak{f}$ and $\zeta'f\xi$ are equivariantly homotopic. But this is obviously the map required.

Finally we state a consequence of Lemma 9.1.

THEOREM 15.5. *If, in the situation of Theorem* 15.4, Φ *is an isomorphism and* $f_*:\pi_q(X)\approx\pi_q(X')$ *for all* q, *then* $\mathfrak{f}$ *is an isomorphism.*

16. **Discrete groups of operators.** In this section we shall consider Γ-complexes where Γ is a discrete group. In this case, the results of §15 take on a more definitive form; specifically, homotopy resolutions may be defined functorially for minimal complexes. Moreover, as long as minimal complexes only are considered, the requirement that Γ operate without fixed simplices may be dropped. In this case only we may generalize the notions of homotopically segregated Γ-bundles and homotopy resolutions by dropping the requirement that the complexes in question be Γ-bundles.

This state of affairs is connected with the fact that a discrete group can contain no nontrivial null-simplices. Thus if Γ is a discrete group and X a minimal Γ-complex, then simplices of X homotopic with fixed boundary are identical. In particular X is simplicial in any dimension in which it is aspherical, in the sense that a simplex is completely determined by its boundary.

With this in mind we may prove the following lemma. We assume here, and throughout this paragraph, that Γ, Γ' etc. are discrete groups.

LEMMA 16.1. *If* X *is a minimal* Γ*-complex and* $\pi_r(X)=0$ *for* $r>q$, *then there*

is an operation of $K(\pi_q(X), q)$ *on* X *such that* X *is a* $K(\pi_q(X), q)$*-bundle and* ϕ_* *is the identity on* $\pi_q(X)$, *where* ϕ_* *is the injection of the homotopy group of the group into that of the complex.*

Let us write $K=K(\pi_q(X), q)$, and let $\lambda: K\times X\to K$ and $\rho: K\times X\to X$ be the projections. Then $\theta:(K\times X)^q\to X$ may be defined by $d(\theta, \rho)=\lambda^c$. The obstruction to extending θ vanishes, and since X is aspherical in higher dimensions, θ may be extended to all of $K\times X$. It should be observed that in consequence of the minimality of X this construction determines θ completely.

To see that θ is an operation, we need only observe that the condition $d(\theta, \rho)=\lambda^c$ implies that K_q operates on the set X_q; for higher dimensions the analogous statement follows from the fact that X is simplicial in those dimensions. Similarly, it is clear that K_q operates without fixed points on X_q, and induction shows that X is a K-bundle.

The assertion on ϕ_*, which needs proof only in dimension q, follows immediately from the definition of θ and Lemma 7.11.

This operation we shall describe as the *regular* operation of $K(\pi_q(X), q)$ on X.

LEMMA 16.2. *Under the regular operation,* X *is a* $\Gamma(\pi_q(X), q)$*-complex.*

This is equivalent to the assertion that $\theta: K\times X\to X$ is Γ-equivariant, which is clear in dimension q, since λ and ρ are, and which follows in higher dimensions from the simpliciality of X.

Now for any semi-simplicial complex X we may define $R_q(X)$ to be the semi-simplicial complex obtained from X by identifying simplices with the same q-skeleton, i.e. σ, $\tau\in X_r$ are equivalent if $\sigma^\Delta|(\Delta_r)^q=\tau^\Delta|(\Delta_r)^q$. The incidence operations in $R_q(X)$ are prescribed by the requirement that the canonical map $\eta_q: X\to R_q(X)$ be simplicial. It is clear that $R_q(X)^q=X^q$ and that $R_q(X)$ is simplicial in dimensions greater than q.

LEMMA 16.3. *If* X *has homotopy so has* $R_q(X)$; $\eta_{q*}:\pi_k(X)\approx\pi_k(R_q(X))$ *for* $k\leqq q$ *and* $\pi_r(R_q(X))=0$ *for* $r>q$.

All that needs proof is the following: if $f:\dot{\Delta}_r\to R_q(X)$ and $r>q+1$, then f is extendible. But $f|(\Delta_r)^{q+1}$ can be lifted to $f':(\Delta_r)^{q+1}\to X$ and thus $f|(\Delta_r)^q$ can be extended to $f'':\Delta_r\to X$. But $\eta_q f''$ then extends f.

If X is a Γ-complex then $R_q(X)$ is too, the operation being prescribed by the requirement that η_q be equivariant. If X is Γ-minimal, it is clear that $R_q(X)$ is also. Thus, by Lemma 16.1, $R_q(X)$ is a $K(\pi_q(X), q)$-bundle under the regular operation.

LEMMA 16.4. *If* X *is a minimal* Γ*-complex, then* $R_q(X)/K(\pi_q(X), q)=R_{q-1}(X)$.

Since $K(\pi_q(X), q)$ is trivial through dimension $q-1$, two simplices of

$R_q(X)$ which are conjugate must have the same $q-1$-skeletons. Let us, for any $\sigma \in R_q(X)_r$, write $\sigma' = \sigma^\Delta | (\Delta_r)^q$. If $\sigma, \tau \in R_q(X)_r$ agree on the $q-1$-skeleton, then

$$d(\sigma', \tau') \in Z_n^q(\Delta_r, \pi_q(X)) = K(\pi_q(X), q)_r$$

is a simplex under operation by which σ and τ are conjugate.

But the preceding lemmas assert that $\{R_q(X)\}$ is a homotopically segregated Γ-complex, whenever X is Γ-minimal. Since $\lim \{R_q(X)\} = X$, we have proved the following theorem.

THEOREM 16.5. *If X is a Γ-minimal Γ-complex, then* $(\{R_q(X)\}$, *identity*) *is a homotopy-resolution of X.*

For maps we have the following remarkable lemma.

LEMMA 16.6. *If Y is a $K(\Pi, q)$-bundle, X satisfies the hypotheses of Lemma* 16.1, *and $f: Y \to X$ is any simplicial map, then f is equivariant with respect to $K(f_*\phi_*, q)$ where $\phi_*: \Pi \to \pi_q(Y)$ is the injection.*

In dimension q we use Lemma 7.11: if $\kappa \in K(\Pi, q)_q$ and $\sigma \in Y_q$ then $d(\kappa\sigma^\Delta, \sigma^\Delta) = \phi_*\kappa\Delta_q$. Thus $d(f\kappa\sigma^\Delta, f\sigma^\Delta) = f_*\phi_*\kappa\Delta_q$. Since X is minimal, this implies that $f(\kappa\sigma) = (f_*\phi_*\kappa)(f\sigma)$. The equivariance then follows from the fact that X is simplicial in higher dimensions.

A special case is worthy of note. If $f: K(\Pi, q) \to K(\Pi', q)$, then f is a homomorphism; in fact, $f = K(f_*, q)$. Together with 16.1 this implies that $K(\Pi, q)$ is characterized, not only as a semi-simplicial complex, but even as a semi-simplicial group by the fact that it is minimal and has as its only nonvanishing homotopy group Π in dimension q.

Now if X and X' are any semi-simplicial complexes and $f: X \to X'$, then $R_q(f): R_q(X) \to R_q(X')$ may clearly be defined by the commutation $\eta_q f = R_q(f)\eta_q$. If X and X' are minimal, it follows from Lemma 16.6 that each map $R_q(f)$ is $K(f_*, q)$ equivariant. But it is clear that $R_q(f)/K(f_*, q) = R_{q-1}(f)$. We have, finally, the following theorem.

THEOREM 16.7. *If X and X' are Γ and Γ'-minimal complexes, $\Phi: \Gamma \to \Gamma'$ is a homomorphism, and $f: X \to X'$ is Φ-equivariant, then $\{R_q(f)\}$ is a Φ-map with* $\lim \{R_q(f)\} = f$.

It may be observed that Theorem 16.5 may be used, in conjunction with Theorem 9.4, to give an alternate proof of the existence of homotopy-resolutions of Γ-bundles with homotopy when Γ is discrete. Since minimal subcomplexes are not defined functorially, neither are the homotopy-resolutions so obtained. It is *not* asserted, and is probably false, that homotopy-resolutions exist for Γ-complexes which are neither bundles nor minimal complexes.

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