THE ENCLOSING OF CELLS IN THREE SPACE
BY SIMPLE CLOSED SURFACES

BY
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1. Introduction. A subset $X$ of Euclidean $n$-space $R^n$ which is homeomorphic to an Euclidean polyhedron is called tame by Fox-Artin [4][(2)] provided there is a homeomorphism of $R^n$ onto itself which carries $X$ onto an Euclidean polyhedron. A question that naturally arises is under what conditions will a homeomorph of an Euclidean polyhedron be tame.

As a step toward the solution of this problem Harrold [6] shows that an arc or simple closed curve in $R^3$ with a certain property $\mathcal{P}$ (defined below) has a complement which is homeomorphic to the complement of its prototype, an evident necessary condition for being tame.

In this paper an extension of the definition of property $\mathcal{P}$ to $k$-cells in $R^3$, $k = 1, 2, or 3$, is made, and an extension of the result of Harrold to $k$-cells is made at the cost of imposing an extra condition which is always fulfilled when $k = 1$. The techniques used are those of Harrold, and depend strongly on the results of Alexander [1]. Much use is made of the concept of a semi-linear map as used by Graeub [5] and Moise [9]. As in Harrold-Moise [7], the set $K$ is called locally polyhedral at a point $p$ provided there is a neighborhood of $p$ which meets $K$ in a finite (or null) polyhedron. The set $K$ is called locally polyhedral modulo $C$ if it is locally polyhedral at each point of the complement of $C$.

2. Definitions and notation. Euclidean $k$-space will be denoted by $R^k$ and a fixed rectangular Cartesian coordinate system will be assumed chosen for $R^k$. The closure of a subset $A$ of a space $R$ will be denoted $\overline{A}$, or simply $\overline{A}$ if it is clear from the context what space $R$ is meant. The boundary of $A$ in $R$ will be denoted by $\partial R(A)$ and is defined to be $[\overline{R(A)}] \cap [\overline{R}(R \setminus A)]$, where $X \setminus Y$ denotes the set of points in $X$ but not in $Y$. The set $E^k$ is defined to be the set of all points $(x_1, \ldots, x_k)$ of $R^k$ such that $0 \leq x_i \leq 1$ for each $i = 1, \ldots, k$, and $C^k$ denotes a topological image of $E^k$ in $R^3$ for $k = 1, 2, or 3$. The image $C^k$ will be called a $k$-cell and a 0-cell is defined to be a point. The symbol $C$ without a superscript is to be interpreted as a $k$-cell for some $k = 1, 2, or 3$, and the term "a cell" as "a $k$-cell for $k = 0, 1, 2, or 3."

(1) Many of the results in this paper were obtained, or are similar to results obtained, in the author's Doctoral Dissertation. This Dissertation was written at the University of Tennessee under the direction of O. G. Harrold, Jr., during the tenure of a National Science Foundation Fellowship.

(2) The numbers in brackets refer to the bibliography.
If \( h \) is any homeomorphism of \( E^k \) onto \( C \) for \( k = 1, 2, \) or \( 3 \), that subset of \( C \) which is the image under \( h \) of \( B_{R^k}(E^k) \) is denoted by \( \partial C \), and is seen not to depend on \( h \). If \( C \) is a 0-cell, \( \partial C \) is defined to be the null set. An open \( k \)-cell is a homeomorph of \( E^k \backslash B_{R^k}(E^k) \), and a \( k \)-sphere is a homeomorph of \( B_{R^{k+1}}(E^{k+1}) \). A 2-cell (an open 2-cell) will occasionally be called a disk (an open disk).

If \( K \) is a 2-sphere in \( R^3 \) then \( \text{Int} \ K \) and \( \text{Ext} \ K \) will be used to denote respectively the bounded and unbounded domains complementary to \( K \) (see [11, Theorem 5.3]).

The null set will be denoted by \( \square \). The symbol \( \delta(A) \) denotes the diameter of the set \( A \), defined as usual by \( \delta(A) = \sup_{a,b \in A} d(a, b) \) where \( d(a, b) \) denotes Euclidean distance. Two sets \( A \) and \( B \) are called separate in \( R \) provided \( A \cap \text{Cl}_R B = B \cap \text{Cl}_R A = \square \).

The topological product of two spaces \( M \) and \( N \) is denoted by \( M \times N \). Two continuous maps \( f_0 \) and \( f_1 \) of \( A \) into \( B \) are said to be homotopic (in \( B \)) if there is a continuous map \( F \) (called a homotopy) of \( A \times I \) into \( B \) which agrees with \( f_0 \) on \( A \times 0 \) and with \( f_1 \) on \( A \times 1 \). When \( f_0 \) and \( f_1 \) are homeomorphisms, then a homotopy \( F \) of \( f_0 \) and \( f_1 \) is called an isotopy provided the restriction of \( F \) to \( A \times t \) is a homeomorphism for each \( t \) in \( I \).

2.11 Definition. Let \( \mathcal{S} \) denote the non-null class of all homeomorphisms of \( E^k = E^1 \times E^{k-1} \) onto \( C \), and let

\[
\mathcal{X}^k = \{ T \mid T = h(x \times E^{k-1}) \text{ for some } x \in E^1 \text{ and } h \in \mathcal{S} \}.
\]

2.12 Definition. For each \( h \) in \( \mathcal{S} \) let

\[
\mathcal{X}^k_h = \{ T \in \mathcal{X}^k \mid T = h(x \times E^{k-1}) \text{ for some } x \in E^1 \}.
\]

It is evident that if \( T \in \mathcal{X}^k \) then \( C \setminus T \) is either connected or consists of exactly two components \( A_0 \) and \( A_1 \) and that \( A_i \cup T = \text{Cl}(A_i) \) is a \( k \)-cell, \( i = 0, 1 \).

It is well known that the collection of all closed subsets of \( C \) forms a metric space under the Hausdorff metric [8] \( \sigma \) which is defined as follows.

2.13 Definition. \( \sigma(A, B) = \max \left[ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right] \).

By the remark following Definition 2.12 if \( T \in \mathcal{X}^k \) then \( C \setminus T \) may be written as \( M' \cup N' \), where \( M' \) is a component of \( C \setminus T \) and \( N' \) is either a component of \( C \setminus T \) or the null set. Letting \( M = M' \cup T \) and \( N = N' \cup T \), \( T \) determines a triple of non-null closed subsets \( (T, M, N) \) of \( C \). For any pair \( T_1, T_2 \) of elements of \( \mathcal{X}^k \), make the following definition.

2.14 Definition.

\[
\rho(T_1, T_2) = \sigma(T_1, T_2) + \min \left[ \sigma(M_1, M_2) + \sigma(N_1, N_2), \sigma(M_1, N_2) + \sigma(N_1, M_2) \right].
\]

Evidently \( \rho(T, T) = 0 \), and if \( \rho(T_1, T_2) = 0 \) then \( \sigma(T_1, T_2) = 0 \) so that \( T_1 = T_2 \). For any triple \( T_1, T_2, T_3 \) of elements of \( \mathcal{X}^k \) and a proper choice of notation
\[ \rho(T_1, T_2) = \sigma(T_1, T_2) + \sigma(M_1, M_2) + \sigma(N_1, N_2), \]

and

\[ \rho(T_2, T_3) = \sigma(T_2, T_3) + \sigma(M_2, M_3) + \sigma(N_2, N_3). \]

Adding these two and using the triangle inequality for \( \sigma \) gives

\[ \rho(T_1, T_3) \geq \rho(T_1, T_2) + \rho(T_2, T_3) \geq \sigma(T_1, T_3) + \sigma(M_1, M_3) + \sigma(N_1, N_3) \geq \rho(T_1, T_3). \]

Thus \( \rho \) is a metric on \( \mathcal{X}^k \).

The superscript of \( \mathcal{X}^k \) will hereafter be omitted when no loss of clarity results.

2.21 Definition. For every \( T \in \mathcal{X} \) and \( \epsilon > 0 \) define \( \mathcal{B}(T, \epsilon) \) as the set of all \( K \subset \mathbb{R}^3 \) satisfying the following conditions:

1. \( K \) is a topological 2-sphere.
2. \( T \subset \text{Int } K \).
3. \( K \) is locally polyhedral modulo \( C \).
4. \( K \cap C = T_1 \cup T_2 \), where \( T_1 \cap T_2 = \emptyset \), where \( T_i = \emptyset \) or \( T_i \in \mathcal{X} \) for \( i = 1, 2 \), and where \( \text{if } T \subset \partial C, \text{ then } T_1 = \emptyset \).
5. \( K \subset \mathcal{S}(T, \epsilon) \).

It is evident that both \( \mathcal{X} \) and \( \mathcal{B}(T, \epsilon) \) depend on the cell \( C \). Since with few exceptions only one cell \( C \) will be under consideration, this dependence is not indicated in the notation. However, when this is not the case, these sets will be denoted by \( \mathcal{X}(C) \) and \( \mathcal{B}(C, T, \epsilon) \) respectively.

2.22 Definition. \( C \) will be said to have property \( \mathcal{P} \) provided that for each \( T \in \mathcal{X} \) and \( \epsilon > 0 \) the set \( \mathcal{B}(T, \epsilon) \) is non-null.

2.23 Definition. \( C \) will be said to have property \( \mathcal{P} \) relative to a subset \( \mathcal{X}_0 \) of \( \mathcal{X} \) provided for every \( \epsilon > 0 \) and \( T \in \mathcal{X}_0 \) the set \( \mathcal{B}(T, \epsilon) \) is non-null.

2.31 Definition. For every \( T \in \mathcal{X} \) and \( \epsilon > 0 \) let \( \mathcal{D}(T, \epsilon) \) denote the set of all \( D \subset \mathbb{R}^3 \) such that

1. \( D \) is a topological 2-cell.
2. \( \partial D \cap C = \emptyset \).
3. \( D \cap C \in \mathcal{X} \).
4. \( D \) is locally polyhedral modulo \( C \).
5. \( \rho(T, D \cap C) < \epsilon \).
6. If \( C \setminus D \) has two components \( C_1 \) and \( C_2 \), then there is an \( \eta > 0 \) such that if \( N \) is a connected set meeting both \( C_1 \) and \( C_2 \) with \( \delta(N) < \eta \), then \( N \) meets \( D \) also.

2.32 Definition. \( C \) will be said to have the disk property relative to a subset \( \mathcal{X}_0 \) of \( \mathcal{X} \) provided \( \mathcal{D}(T, \epsilon) \) is non-null for every \( \epsilon > 0 \) and \( T \in \mathcal{X}_0 \). If \( C \) has the disk property relative to the whole set \( \mathcal{X} \), then it will simply be said that \( C \) has the disk property.

2.33 Definition. \( C \) will be said to have the uniform disk property relative to a subset \( \mathcal{X}_0 \) of \( \mathcal{X} \) provided to each \( \omega > 0 \) there corresponds a \( \delta > 0 \) such that if \( T \in \mathcal{X}_0 \) and \( \epsilon > 0 \) there is a \( D \) in \( \mathcal{D}(T, \epsilon) \) with \( \delta(\partial D, C) > \delta \) and \( D \subset \mathcal{S}(T, \omega) \).
When $C$ has the uniform disk property relative to $\Sigma_h$ for each $h$ in $\mathcal{H}$, $C$ will be said to have the uniform disk property.

2.4 Definition. $C$ will be said to have the enclosure property provided for each $\epsilon > 0$ there is a polyhedral topological 2-sphere $K$ in $S(C, \epsilon)$ with $C \subseteq \text{Int} \ K$.

3. Relations between the metrics. The metric $\rho$ is chosen for $\Sigma$ in preference to the somewhat more simple metric $\sigma$ chiefly because $\sigma$ does not have the property described in this lemma.

3.1 Lemma. If $U_1$, $U_2$, and $T$ are elements of $\Sigma$ such that $T$ separates $U_1$ and $U_2$ on $C$, then there is a $\beta > 0$ such that every element $T'$ of $\Sigma$ with $\rho(T, T') < \beta$ separates $U_1$ and $U_2$ on $C$.

Proof. The notation may be assumed chosen so that $M_0 \supset U_1$ and $N_0 \supset U_2$ where $M_0$ and $N_0$ are the components of $C \setminus T$. If $T' \in \Sigma$ is disjoint from $U_1$ and $U_2$ and does not separate $U_1$ and $U_2$ on $C$, then both $U_1$ and $U_2$ lie in the same component $M_0'$ of $C \setminus T'$. Choose $p_i \in U_i$, $i = 1, 2$ and let $M = M_0 \cup T$, $N = N_0 \cup T'$, $M' = M_0' \cup T'$, and $N' = N_0' \cup T'$ where $N_0'$ is either null or the component of $C \setminus T'$ other than $M_0'$. Then

$$\rho(T, T') = \min \left[ \sigma(M, M') + \sigma(N, N'), \sigma(M, N') + \sigma(N, M') \right],$$

so

$$\rho(T, T') \geq \min \left[ \sigma(M, M'), \sigma(N, M') \right],$$

or

$$\rho(T, T') \geq \min \left[ \sup_{p \in M'} d(M, p), \sup_{p \in M'} d(N, p) \right];$$

or

$$\rho(T, T') \geq \min \left[ d(M, p_2), d(N, p_1) \right].$$

But $p_2 \in U_2 \subset N$ while $p_1 \in U_1 \subset M$, so $\beta_1 = \min \left[ d(M, p_2), d(N, p_1) \right] > 0$. Also $T \cap (U_1 \cup U_2) = \emptyset$, so $\beta_2 = d(T, U_1 \cup U_2) > 0$. Hence $\beta = \min (\beta_1, \beta_2)$ is positive and independent of $T'$. Further, if $\rho(T, T') < \beta$, then the assumption that $T'$ does not separate $U_1$ and $U_2$ on $C$ leads to a contradiction.

3.2. Lemma. If $T \in \Sigma$ and $\delta > 0$ then there is an $\eta > 0$ such that the complement of $S(T, \delta)$ lies in $\text{Ext} \ K$ for every $K$ in $\Psi(T, \eta)$.

Proof. Choose $\rho \in R^3$ and $r$ sufficiently large that $S(T, \delta) \subset S(\rho, r)$. Then $R = R^3 \setminus S(\rho, r)$ is connected and disjoint from $S(T, \delta/n)$ for each $n = 1, 2, \cdots$, and hence determines a component $R_n$ of $R^3 \setminus S(T, \delta/n)$ for each $n$. The sequence $\{R_n\}$ is monotone nondecreasing and has as limit $\cup R_n$, which is clearly in $R^3 \setminus T$. But any point $x$ of $R^3 \setminus T$ can be joined to $R$ by an arc $A_x$ in $R^3 \setminus T$ and if $\delta/n$ is less than $d(T, A_x \cup R)$ then $A_x$ and hence $x$ is in $R_n$. Thus
the reverse inclusion $(R^n \setminus T) \subset \bigcup R_n$ also holds and $\bigcup R_n = R^3 \setminus T$. This means that any closed set disjoint from $T$ lies in $R_n$ for all sufficiently large $n$. In particular, for the assigned $\delta$ there is an $N$ such that the complement of $S(T, \delta)$ lies in $R_N$. Now if $\eta < d(T, R_N)$ and $K \subset P(T, \eta)$, then $R_N$ is an unbounded set in the complement of $K$ and hence $R_N \subset \text{Ext } K$. Since the complement of $S(T, \delta)$ lies in $R_N$, the desired $\eta$ has been found.

3.3 Corollary. A cell has property $P$ (property $P$ relative to $\mathcal{I}_0$) if and only if for each $\varepsilon > 0$ and $T \in \mathcal{I}$ ($T \in \mathcal{I}_0$) there is a $K$ in $\mathcal{Y}(T, \varepsilon)$ with $\text{Int } K \subset S(T, \varepsilon)$.

4. Property $P$ and the disk property.

4.1. Theorem. In order that $C$ have property $P$ (property $P$ relative to $\mathcal{I}_0$) it is necessary and sufficient that for every $\varepsilon > 0$ and $T \in \mathcal{I}$ ($T \in \mathcal{I}_0$) there be a $K$ in $\mathcal{Y}(T, \varepsilon)$ satisfying the following two conditions:

1. $(K \cup \text{Int } K) \subset S(T, \varepsilon)$.

2. If $K \cap C$ has two components, $U_1$ and $U_2$, then $T$ separates $U_1$ and $U_2$ on $C$.

Proof. The sufficiency is obvious, so suppose $C$ has property $P$ (property $P$ relative to $\mathcal{I}_0$) and let $\varepsilon > 0$ and $T \in \mathcal{I}$ ($T \in \mathcal{I}_0$) be assigned. Two cases arise.

Case 1. $T \cap \partial C = \partial T$. Then $T = h(a \times E^{k-1})$ for some $h \in \mathfrak{S}$ and $0 < a < 1$, and $M_1 = \bigcup_{0 \leq x < a} h(x \times E^{k-1})$ and $M_2 = \bigcup_{a < x \leq 1} h(x \times E^{k-1})$ are the components of $C \setminus T$. Let $p_i$ be a point of $M_i$, $i = 1, 2$, and let $\eta = \min \{\varepsilon, d(T, p_1), d(T, p_2)\}$. Then $\mathcal{Y}(T, \eta) \subset \mathcal{Y}(T, \varepsilon)$ since $\eta < \varepsilon$, so it will suffice to show there is a $K$ in $\mathcal{Y}(T, \eta)$ meeting the requirements. Corollary 3.3 guarantees there is a $K$ in $\mathcal{Y}(T, \eta)$ with $\text{Int } K \subset S(T, \eta)$, and this, together with 5 of Definition 2.21 means that this $K$ satisfies condition 1.

Now let $i$ be either 1 or 2. Since $d(p_i, T) \geq \eta$ and $(K \cup \text{Int } K) \subset S(T, \eta)$, $p_i$ is in $\text{Ext } K$. So $\text{Cl } M_i = M_i \cup T$ contains $p_i \in \text{Ext } K$ and $T \subset \text{Int } K$, so $K \cap \text{Cl } M_i = K \cap (M_i \cup T) = K \cap M_i \neq \emptyset$. Thus $K \cap M_i = U_i \in \mathcal{I}$ by 4 of Definition 2.21. But since $T$ separates $M_1$ and $M_2$ on $C$, it must also separate $U_1 = M_1 \cap K$ and $U_2 = M_2 \cap K$ on $C$, so condition 2 is satisfied.

Case 2. $T \subset \partial C$. Then Corollary 3.3 guarantees condition 1 and condition 2 is vacuously fulfilled, for condition 4 of Definition 2.21 requires that $K \cap C$ have but one component.

4.2. Definition. Let $\mathcal{Y}^*(T, \varepsilon)$ denote the set of all spheres of $\mathcal{Y}(T, \varepsilon)$ satisfying the conclusion of Theorem 4.1.

In terms of this notation Theorem 4.1 states that $C$ has property $P$ (relative to $\mathcal{I}_0$) if and only if $\mathcal{Y}^*(T, \varepsilon)$ is non-null for every $T \in \mathcal{I}$ ($T \in \mathcal{I}_0$) and $\varepsilon > 0$.

4.3. Lemma. If $T \in \mathcal{I}$ and $\varepsilon > 0$, then there is a $\beta > 0$ such that whenever $K \subset \mathcal{Y}^*(T, \beta)$ and $T_1$ is a component of $K \cap C$, then $\rho(T, T_1) < \varepsilon$. 
Proof. Let $\epsilon > 0$ and $T \subseteq \mathcal{I}$ be assigned. Then $C$ may be written as $C_1 \cup C_2$ where $T = h(x_0 \times E^{k-1})$ for some $h \subseteq \mathcal{I}$ and $C_1$ and $C_2$ are the images under $h$ of $U_{0 \leq z \leq x_0} h(x \times E^{k-1})$ and $U_{x_0 \leq z \leq 1} h(x \times E^{k-1})$ respectively. Each $C_i$ is a cell (one is of dimension $k$ and the other of dimension $k$ or $k-1$ according as $T \cap \partial C$ is $T$ or $T'$) and is therefore uniformly locally connected. That is, for $i = 1$ and $i = 2$ there is an $\omega_i > 0$ such that if $t \in T$ and $p \in C_i$ with $d(t, p) < \omega_i$ then some connected subset $Q$ of $C_i$ with $\delta(Q) < \epsilon/4$ contains both $t$ and $p$.

If $C_i \neq T$, straightforward application of the uniform continuity of $h$ produces an $x_i \neq x_0$ meeting the following conditions:

4.31 $T'_i = h(x_i \times E^{k-1})$ is a subset of $C_i$.
4.32 $d(t, T'_i) < \omega_i$ for all $t \in T$.
4.33 $d[h(x, y), T] + d[h(x, y), T'_i] < \epsilon/4$ for all $x$ between $x_i$ and $x_0$ and any $y$ in $E^{k-1}$.

For $i = 1, 2$ let $\beta_i = \min \{d(T, T'_i), \epsilon/4\}$ or $\beta_i = 1$ according as $C_i \neq T$ or $C_i = T$, and let $\beta = \min (\beta_1, \beta_2)$.

Suppose now that $K \subseteq \mathcal{I}(T, \beta)$ and that $T_1$ is a component of $K \cap C$. The inclusion $T_1 \subseteq C_1$ is assumed as a notational convenience. If $t$ is any point of $T$ then by 4.32 there is a $t'_i \in T'_i$ with $d(t, t'_i) < \omega_i$ and hence a connected subset $Q$ of $C_i$ with $\delta(Q) < \epsilon/4$ which contains both $t$ and $t'_i$. But $T \subseteq \text{Int} K$ and, since $K \cap \text{Int} K \subseteq S(T, \beta)$ and $d(T, T'_i) > \beta$, $T'_i \subseteq \text{Ext} K$. The set $Q$ therefore meets both $\text{Int} K$ and $\text{Ext} K$, and must meet $K$, so $t \in Q \cap K$ may be chosen. That $K \cap C_1 = T_1$ follows from condition 2 of Theorem 4.1 and the choice of the sets $C_1$ and $C_2$, so since $Q \cap K \subseteq C_1 \cap K$, $t_1$ is a point of $T_1$. Then $d(t, T_1) \leq d(t, t_1) \leq \delta(Q) \leq \epsilon/4$. But $t$ was arbitrary in $T$ so $\sup_{t \in T} d(t, T_1) \leq \epsilon/4$. Since $T_1 \subseteq K \subseteq S(T, \beta)$, it follows that $\sup_{t \in T_1} d(T, t_1) < \beta \leq \epsilon/4$. Combining these yields $\sigma(T, T_1) \leq \epsilon/4$.

Now let $(T, C_1, C_2)$ and $(T_1, M, N)$ be the triples of sets in the expression for $\rho(T, T_1)$ where $N$ contains $T$. Since $C_2$ is connected and $T_1 \subseteq C_1$, this requires $C_2 \subseteq N$ and $M \subseteq C_1$, so that $\sup_{p \in M} d(p, C_1) = \sup_{q \subseteq C_2} d(N, q) = 0$. This means that

$$\rho(T, T_1) \leq \sigma(T, T_1) + \sup_{p \in C_1} d(M, p) + \sup_{q \in N} d(q, C_2).$$

But if $p \in C_1$, then $p = h(x, y)$ where $0 \leq x \leq x_0$ and $y \in E^{k-1}$. If $x < x_1$ then $p \in M$ so $d(M, p) = 0$, and if $x_1 \leq x \leq x_0$, then, by 4.33 together with the fact that the choice of $\beta$ implies $T'_i \subseteq M$, $d(M, p) \leq d(T'_i, p) \leq \epsilon/4$. Similarly, if $q \subseteq N$ then $q = h(x, y)$ with $x_1 \leq x \leq 1$ so either $x_0 \leq x$, in which case $q \subseteq C_2$ and $d(q, C_2) = 0$, or $x_1 \leq x \leq x_0$, so that by 4.33, $d(q, C_2) \leq d(q, T) < \epsilon/4$. Combining, $\rho(T, T_1) \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon$.

4.4. Theorem. If $C$ has property $P$ (relative to $\mathcal{I}$), then $C$ has the disk property (relative to $\mathcal{I}$).

Proof. Let $T \subseteq \mathcal{I}$ ($T \subseteq \mathcal{I}$) and $\epsilon > 0$ be assigned. Then $C \setminus T$ may be written as $N_1 \cup N_2$, where $N_1$ is a component and $N_2$ is either a component or null.
Since $T = h(x \times E^{k-1})$ for some $h \in \mathfrak{H}$ (the given $h$) and some $x \in N_1 \cup T$ is a $k$-cell. Choose $p_1 \in N_1$ and let $\beta_1 = d(p_1, T)$. By Lemma 4.3 there is a $\beta_2 > 0$ such that if $K \subset \mathfrak{Y}^*(T, \beta_2)$ and $T_1$ is a component of $K \cap C$, then $\rho(T, T_1) < \varepsilon$. Let $\beta = \min (\beta_1, \beta_2)$ and choose $K \subset \mathfrak{Y}^*(T, \beta)$.

Now $K \cap C$ has either a single component $T_1$ or two components $T_1$ and $T_2$ according as $N_2$ is null or not, and this together with the choice of $\beta$ and condition 1 of Theorem 4.1 guarantees that $K \cap M_1 \neq \square$ so that $K \cap M_1 = T_1 \in \mathfrak{X}$ is but a notational convenience.

If $T_2 \neq \square$, then $T_2 \in \mathfrak{X}$. $K$ is then a topological 2-sphere with $T_1$ and $T_2$ a pair of disjoint $(k-1)$-cells on $K$. $K$ is, then, the image of the unit spherical surface $W$ under a homeomorphism $g$ which maps $V$ onto $T_1$ where $V$ is the point $(1, 0, 0)$, or the set $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } x \geq 0\}$, or the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } x \neq 0\}$, according as $C$ is a 1-cell, a 2-cell, or a 3-cell. The set $g^{-1}(T_2)$ is disjoint from $V$ so for some $\alpha > 0$, $\{C \cap \mathfrak{W} \cap S(V, \alpha)\} \cap \{g^{-1}(T_2)\} = \square$. If $X = W \cap S(V, \alpha)$, then $\partial X$ is a simple closed curve separating $g^{-1}(T_2)$ and $V = g^{-1}(T_1)$ on $W$. Then $S = g(\partial X)$ is a simple closed curve separating $T_1$ and $T_2$ on $K$.

If $T_2$ is null $S$ may be chosen to be any simple closed curve on $K \cap C$. In either case then, the curve $S$ divides $K$ into two closed disks, one of which meets $C$ at $T_1$ and at no other points. Call this disk $D$. It will now be shown that $D, T,$ and $\varepsilon$ have the six properties of Definition 2.31.

1. $D$ is a topological 2-cell by the Schoenflies theorem.
2. $\partial D = S$ and $S \subset K \cap C$ so $\partial D \cap C = \square$.
3. $D \cap C = T_1 \in \mathfrak{X}$.
4. $D \subset K$ so $D$ is locally polyhedral mod $C$ by 3 of Definition 2.21.
5. $\rho(T, D \cap C) = \rho(T, T_1) < \varepsilon$ by choice of $\beta$ and $K$.
6. Since $K \cap N_1 = T_1$ and $K$ separates $p \in \text{Ext} K$ from $T \cap \text{Int} K$ on $N_1 \cup T$, $(N_1 \cup T) \setminus T_1$ may be written as the union of $C_1 = N_1 \cap \text{Ext} K$ containing $p$ and $C_2 = (N_1 \cup T) \cap \text{Int} K$ containing $T$. Since $(C_1 \cap N_1) \cap (C_2 \cap N_2)$ is null or $T$ according as $N_2$ is null or not, it follows that $C_1$ and $N_2$ are separate so that $T_1 \cap C \cap N_2 = \square$ and the two components of $C \setminus T_1$ must be $C_1$ and $C_2 \cup N_2$. Thus $\omega > 0$ can be taken less than $d(T_1, N_2 \cup (K \setminus D))$ so that $N_2 \cap S(T_1, \omega) = \square$ and $K \cap S(T_1, \omega) = D \cap S(T_1, \omega)$.

It is asserted that there is then an $\eta > 0$ such that if $M$ is a connected set meeting both $C_1$ and $C_2 \cup N_2$ with $\delta(M) < \eta$, then $M \subset S(T_1, \omega)$. For if this is not the case for $i = 1, 2, \cdots$ there is a connected set $M_i$ with $\delta(M_i) < 1/i$ and a triple of points, $p_i \in M_i \cap C_1$, $q_i \in M_i \cap (C_2 \cup N_2)$ and $r_i \in M_i$ with $d(r_i, T_1) > \omega$. By a standard procedure a subsequence of indices $k_1, k_2, \cdots$ can be chosen such that $\{p_{k_i}\}, \{q_{k_i}\}$ and $\{r_{k_i}\}$ all converge, and it is readily seen that they must all have the same limit point $q$. As the limit of $\{p_{k_i}\} \subset C_1$ and $\{q_{k_i}\} \subset C_2 \cup N_2, q$ must lie in $(C_1 \cap C_2) \cap (C_2 \cup N_2) = T_1$, contradicting the fact that as the limit of $\{r_{k_i}\}$, $d(q, T_1) \geq \omega$. Hence the asserted $\eta$ exists, and is the number required to fulfill condition 6. For suppose $M$ is
connected, $\delta(M) < \eta$, and $M$ meets both $C_1$ and $C_2 \cup N_2$. Then $M \subset S(T_1, \omega)$ so $M \cap N_2 \subset S(T_1, \omega) \cap N_2 = \emptyset$ and $M$ meets both $C_1 \subset \text{Ext } K$ and $C_2 \subset \text{Int } K$. This requires that $M$ meet $K$ and $M \cap K \subset K \cap S(T_1, \omega) = D \cap S(T_1, \omega)$.

Thus $D \subset D(T, \epsilon) \neq \emptyset$, and $C$ has the disk property (relative to $\mathcal{X}_b$).

5. The interior radii.

5.1. Theorem. If $\mathcal{X}_0$ is any compact subset of $\mathcal{X}$ and $C$ has property $\mathcal{P}$ relative to $\mathcal{X}_0$, then for each $\epsilon > 0$ there is a $\gamma > 0$ such that for every $T \subset \mathcal{X}_0$ the inequality $\sup_{K \in \mathcal{P}^*(T, \epsilon)} d(T, K) > \gamma$ holds.

Proof. Let $\gamma(T, \epsilon)$ denote the supremum in the statement of the theorem. If the conclusion is false, then for each integer $n$ there is a $T_n$ in $\mathcal{X}_0$ such that $\gamma(T_n, \epsilon) \leq 1/n$. Since $\mathcal{X}_0$ is compact, some subsequence $\{T_{n_k}\}$ converges to $T_0 \subset \mathcal{X}_0$, and to simplify the notation it will be assumed that $\{T_n\}$ converges to $T_0$. Now $\mathcal{P}^*(T_0, \epsilon/2)$ is non-null from the hypothesis that $C$ has property $\mathcal{P}$ relative to $\mathcal{X}_0$ and Theorem 4.1, so choose $K \in \mathcal{P}^*(T_0, \epsilon/2)$.

For any fixed $i$ there is a $p \in T_i$ and a $q \in K$ such that

$$d(p, q) = d(T_i, K).$$

(5.11)

So for any $r \in T_0$, $d(p, q) + d(p, r) \geq d(q, r)$, or

$$d(T_i, K) \geq d(q, r) - d(p, r).$$

(5.12)

Since $T_0 \subset \text{Int } K$, $d(T_0, K) = 2\omega > 0$, and $d(q, r) \geq d(T_0, K)$, so 5.12 yields

$$d(T_i, K) \geq 2\omega - d(p, r).$$

(5.13)

But $r$ was arbitrary in $T_0$, so

$$d(T_i, K) \geq 2\omega - \inf_{r \in T_0} d(p, r) = 2\omega - d(p, T_0),$$

and, since $\rho(T_i, T_0) \geq d(p, T_0)$, $d(T_i, K) \geq 2\omega - \rho(T_i, T_0)$. But $\{T_i\}$ converges to $T_0$ so there is an $M_1$ such that for all $i > M_1$, $\rho(T_i, T_0) < \omega$. Combining these facts,

$$i > M_1 \quad \text{implies} \quad d(T_i, K) \geq 2\omega - \omega = \omega.$$  

(5.14)

Now if $x \in \mathbb{R}^3$ is such that $d(x, T_0) < \omega$, since $d(T_0, K) = 2\omega$, then $x \in \text{Int } K$. But if $i > M_1$, $\rho(T_i, T_0) < \omega$ so $d(x, T_0) < \omega$ for every $x$ in $T_i$, i.e.,

$$i > M_1 \quad \text{implies} \quad T_i \subset \text{Int } K.$$  

(5.15)

Let $q$ be any point of $K \cup \text{Int } K$. Since $K \cup \text{Int } K \subset S(T_0, \epsilon/2)$, there is a point $r$ of $T_0$ such that $d(r, q) < \epsilon/2$. Corresponding to this $r$ there is a $p_i \in T_i$ such that $d(r, p_i) = d(r, T_i) \leq \rho(T_0, T_i)$. Then $d(p_i, q) \leq d(p_i, r) + d(r, q) \leq \rho(T_0, T_i) + \epsilon/2$. But there is an $M_2$ such that if $i > M_2$ then $\rho(T_0, T_i) < \epsilon/2$. For such $i$ the above inequality is $d(r, p_i) \leq \epsilon/2 + \epsilon/2 = \epsilon$. So, given $q$ arbitrary in $K \cup \text{Int } K$ and $i > M_2$ there is a $p_i \in T_i$ such that $d(q, p_i) < \epsilon$. The point $p_i$
depends on \( q \), but the \( M_2 \) depends only on the convergence of \( \{ T_i \} \) to \( T_0 \). So

\[
(5.16) \quad \ i > M_2 \implies K \cup \text{Int } K \subset S(T_i, \epsilon).
\]

Suppose now that \( K \cap C = U_1 \cup U_2 \) where \( U_1 \) and \( U_2 \) are in \( \mathcal{X} \). Then since \( K \in \mathcal{P}^*(\mathcal{X}_0, \epsilon/2) \), condition 2 of Theorem 4.1 requires that \( T_0 \) separate \( U_1 \) and \( U_2 \) on \( C \). By Lemma 3.1, if \( \rho(T_i, T_0) \) is less than some fixed \( \beta > 0 \), \( T_i \) also separates \( U_1 \) and \( U_2 \) on \( C \), so there is an \( M_3 \) such that this is the case for all \( i > M_3 \). If \( K \cap C \) does not have two components take \( M_3 = 1 \). In any case then

\[
(5.17) \quad \ i > M_3 \implies \text{when } K \cap C \text{ has two components}
\]

they are separated on \( C \) by \( T_i \).

Now let \( M = \max (M_1, M_2, M_3) \), and suppose \( i > M \). Then \( K \) satisfies conditions 1, 3, and 4 of Definition 2.21 for the set \( \mathcal{P}(T_i, \epsilon) \) by virtue of being in \( \mathcal{P}^*(T_0, \epsilon/2) \), since these conditions do not involve the \( T \) and \( \epsilon \). But 5.15 and 5.16 are valid since \( i > M \), so conditions 2 and 5 of Definition 2.21 are also satisfied and \( K \in \mathcal{P}(T_i, \epsilon) \). Conditions 1 and 2 of Theorem 4.1 are both fulfilled by \( K, T_i, \) and \( \epsilon \) by virtue of 5.16 and 5.17. Thus \( i > M \) implies \( K \in \mathcal{P}^*(T_i, \epsilon) \).

But then \( \gamma(T_i, \epsilon) \geq d(T_i, K), \) so from 5.14, \( i > M \) implies \( \gamma(T_i, \epsilon) > \omega. \)

Since \( \gamma(T_i, \epsilon) < 1/\epsilon \) by choice, this is a contradiction and the theorem is established.

5.2. Theorem. \( \mathcal{X}_h \) is an arc in \( \mathcal{X} \) for every \( h \) in \( \mathcal{S} \).

\textbf{Proof.} Define \( \phi: E^1 \rightarrow \mathcal{X}_h \) bv

\[
\phi(x) = h(x \times E^{k-1}) \equiv d_{T_z}
\]

It is obvious that \( \phi \) is 1-1 and onto. Let \( x_i \in E^1 \) and \( \omega > 0 \) be assigned. Since \( h \) is uniformly continuous, there is a \( \theta > 0 \) such that whenever \( |x_2 - x_1| < \theta \), then \( d[h(x_2, y), h(x_1, y)] < \omega/3 \) for all \( y \) in \( E^{k-1} \). Let \( M_i \) be the set \{ \( h(x, y) \mid 0 \leq x \leq x_i \} \), and \( N_i \) be \{ \( h(x, y) \mid x_i \leq x \leq 1 \} \), \( i = 1, 2 \). Then \( \rho(T_1, T_2) = \sigma(T_1, T_2) + \sigma(M_1, M_2) + \sigma(N_1, N_2) \), and an elementary calculation shows each of the terms on the right is less than \( \omega/3 \). Thus \( \phi \) is a continuous 1-1 map of a compact space onto a Hausdorff space and must be topological, so \( \mathcal{X}_h \) is an arc.

5.3. Corollary. If \( C \) has property \( \mathcal{P} \) relative to \( \mathcal{X}_h \) and \( T_z \) denotes the \((k-1)\)-cell \( h(x \times E^{k-1}) \), then for each \( \epsilon > 0 \) there is a \( \gamma > 0 \) such that for all \( x \) in \( E^1 \) the inequality \( \gamma(T_z, \epsilon) > \gamma \) holds.

6. The construction lemmas. Let \( h \) be a fixed homeomorphism of \( E^k \) onto \( C \) and let \( a \) be a point of \( T_0 \) \( \partial T_0 \), where \( T_z \) denotes \( h(x \times E^{k-1}), 0 \leq x \leq 1 \). Since \( a \in \partial \mathcal{C}_1 (R^n, C) \), well known results in the theory of accessibility assure that there is an arc \( A' \) from a point \( a_1 \) to \( a \) which meets \( C \) only at \( a \). If \( A'' \) is a topological ray in \( R^n \setminus C \) with initial point \( a_1 \), then \( A' \cup A'' \) contains a topological ray \( A \) which meets \( C \) only at its initial point \( a \).
Similarly a topological ray $B$ meeting $C$ only at its initial point $b \in T_1 \setminus \partial T_1$ can be chosen, and the two rays $A$ and $B$ may be taken disjoint and locally polyhedral modulo $C$.

A partial order on the elements of $\mathcal{X}$ is now defined by $T < U$ provided both the conditions $T$ separates $A$ and $U$ on $A \cup C \cup B$ and $U$ separates $T$ and $B$ on $A \cup C \cup B$ are met. This order is extended to the set consisting of all elements of $T$ and all points of $A \cup B$ by $U < T$ and $T < V$ for every choice of $U \in A$, $V \in B$, and $T$ an element of $\mathcal{X}$ which separates $A$ and $B$ on $A \cup C \cup B$. For each $T$ in the set on which this order is defined it will be convenient to let $\mathcal{A}(T)$ and $\mathcal{B}(T)$ denote the components of $(A \cup C \cup B) \setminus T$ containing an unbounded subset of $A$ and an unbounded subset of $B$ respectively. It is easily verified that $T < U$ holds if and only if both $T \cup \mathcal{A}(T) \subset \mathcal{A}(U)$ and $\mathcal{B}(T) \supset U \cup \mathcal{B}(U)$.

It should also be noted that for elements of $\mathcal{X}$ the order relation depends only on the choice of $a \in T_0 \setminus \partial T_0$ and $b \in T_1 \setminus \partial T_1$, and not on the choice of rays $A$, $B$, from $a$, $b$. In the remainder of this section $h$, $A$, and $B$ will be assumed chosen and fixed.

6.1. Lemma. Let $D$, $K_1$, $T$ and $\varepsilon > 0$ be related as follows:
1. $T \in \mathcal{X}$ and separates $A$ and $B$ on $A \cup C \cup B$.
2. $K_1 \in \mathcal{P}*(T, \varepsilon)$.
3. $K \cap C = U_1 \cup U_2$ where both $U_1$ and $U_2$ are in $\mathcal{X}$.
4. Either $U_1 < T < U_2$ or $U_1 < U < T < U_2$, where $D \cap C = U \in \mathcal{X}$.
5. $D \in \mathcal{Y}(U, \omega)$ for every $\omega > 0$.
6. $(D \cup K_1) \cap (A \cup B) = \emptyset$.
7. $d(\partial D, C) > \varepsilon$.

Then there is a $K_2$ in $\mathcal{Y}*(T, \varepsilon)$ with $K_2 \cap C$ either $U_1 \cup U$ or $U \cup U_2$ according as $T < U$ or $U < T$.

Proof. Suppose $U_1 < T < U < U_2$. Then these four sets are pairwise disjoint and $U$ separates $U_1$ and $U_2$ on $C$. Since $U \in \mathcal{X}$, there is a $g \in \mathcal{S}$ such that $U = g(x \times E^{k-1})$ for some $x$ with $0 < x < 1$, for neither $g(0 \times E^{k-1})$ nor $g(1 \times E^{k-1})$ can separate $U_1$ and $U_2$. Consequently $C \setminus U = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ contains $U_1$ and $T$, $\mathcal{B}$ contains $U_2$, and $\mathcal{A} \cup U$ is a $k$-cell. An arc $E$ in $\mathcal{A} \cup U$ with one endpoint in $U$, the other in $U_1$ and otherwise disjoint from $U \cup U_1$ can be constructed. Since $T$ separates $U_1$ and $U$ on $C$, $E$ meets $T$ so a sub-arc $E'$ of $E$ meeting both $T$ and $U$ and not meeting $U_1$ can be found. Then $T \cup U \cup E'$ is a connected set in the complement of $K_1$ so since $T \subset \text{Int } K_1$, $U$ also is in $\text{Int } K_1$.

Thus $D \cap \text{Int } K_1 \neq \emptyset$, and since $K_1 \cup \text{Int } K_1 \subset S(T, \varepsilon) \subset S(C, \varepsilon)$ while $d(\partial D, C) > \varepsilon$, $D \cap \text{Ext } K_1 \neq \emptyset$ so that $K_1 \cap D \neq \emptyset$. Since $D \cap C \cap K_1 = U \cap K_1$ and $U \subset \text{Int } K_1$, $D \cap C \cap K_1 = \emptyset$ so both $D$ and $K_1$ are locally polyhedral in some neighborhood $V$ of $D \cap K_1$ containing no points of $C \cup \partial D$, and $V$ may be taken disjoint from $A \cup B$ also by condition 6 of the hypothesis. By shifting
the vertices of $D$ which lie in $V$ a distance so small that no point of $D$ outside $V$ is moved, $D$ and $K_1$ may be brought into relative general position. Then $D \cap K_1$ will consist of a finite collection of mutually disjoint simple closed curves $s_1, s_2, \ldots, s_n$.

Each $s_k$ bounds a unique sub-disk $D_k$ of $D$ and a pair of disks $X_k$ and $Y_k$ on $K_1$, where $Y_k$ contains $U_1$. An arc $c$ of $C$ can be chosen so that $c$ meets $\partial C$ only at its two end points $a$ and $b$. Then $A \cup c \cup B$ is a topological line and hence a continuous 1-cycle, so the theory of linkages may be applied to $A \cup c \cup B$ and the curves $s_k$. The arc $c$ may be chosen so that it meets each of the sets $U_1$, $U$, and $U_2$ in a single point, so that for any choice of $k$ either $X_k \cap C = U_2$, $Y_k \cap C = U_1$ and $D_k \cap C = U$ or $X_k \cap C = \Box$, $Y_k \cap C = U_1 \cup U_2$, and $D_k \cap C = \Box$ according as $s_k$ links $A \cup c \cup B$ or not. Thus if any $s_k$ fails to link $A \cup c \cup B$, the corresponding $X_k$ is a sub-disk of $K_1 \setminus (U_1 \cup U_2)$ and an index $j$ can be found so that $X_j \cap D = s_j$. For this $j$ the set $(D \setminus D_j) \setminus X_j$ is a disk, and a neighborhood $V$ of $X_j$ can be chosen so that $V \cap (A \cup c \cup B) = \Box$ and $V \cap D \cap K_1 \subset X_j$. Then $(D \setminus D_j) \setminus X_j$ can be deformed away from $X_j$ semi-linearly so that no point outside $V$ is moved, and the resulting disk $D^*$ has the same boundary and intersection with $A \cup c \cup B$ as $D$ but has at least one less component of intersection with $K_1$ which fails to link $A \cup c \cup B$. This process can be repeated and after doing so a finite number of times a new disk $D_1$ is found such that each $s_k$ in $D \cap K_1$ links $A \cup c \cup B$.

That $\partial C \subset \partial D^*$ follows from the fact that the new $D_1$, like the old, contains $U \subset \text{Int } K_1$ and $\partial D \subset \text{Ext } K_1$, so there is an index $i$ such that $D_i \cap K_1 = s_i$. Let $K_2 = Y_i \cup D_i$. Since $s_i$ links $A \cup c \cup B$, $Y_i \cap C = U_1$ and $D_i \cap C = U$, so $K_2 \cap C = U_1 \cup U$ and $K_2$ is the required set if it can be shown to be in $F(T, \epsilon)$. To show this it must first be shown that $K_2 \in P(T, \epsilon)$, i.e., that $K_2$, $T$, and $\epsilon$ satisfy the five conditions of Definition 2.21.

1. Since $D_i \cap Y_i \subset D_i \subset K_1 = s_i$ and $\partial D_i = \partial Y_i = s_i$, $K_2$ is a topological 2-sphere.

2. In order to prove $T \subset \text{Int } K_2$ the fact that $C \setminus (U_1 \cup U)$ has at most three components is needed. To establish this the following more general statement, which will be useful later, is to be proved.

6.11. If $V_1$ and $V_2$ are disjoint elements of $\Sigma$, then $C \setminus (V_1 \cup V_2)$ has at most three components.

For if $V_1 \subset \Sigma$, then $V_1 = g(v \times E^{k-1})$ for some $g \in \Sigma$ and $0 \leq v \leq 1$. Letting $I_0 = \{(x, y, z) \in E^k : 0 \leq x < v\}$ and $I_1 = \{(x, y, z) \in E^k : v < x \leq 1\}$, it is seen that $C \setminus V_1$ is the union of $M_0 = g(I_0 \times E^{k-1})$ and $M_1 = g(I_1 \times E^{k-1})$. Further, $M_0 \cup V_1$ and $M_1 \cup V_1$ are both $k$-cells if $v$ is neither 0 nor 1, and since the adjustment needed in the following arguments when $v = 0$ or $v = 1$ are easily supplied, the assumption $0 < v < 1$ is made. Since $M_0$ and $M_1$ are topologically the same it is also assumed without loss of generality that $V_2 \subset M_1$. Then $C \setminus (V_1 \cup V_2) = (C \setminus V_1) \setminus V_2 = M_0 \cup (M_1 \setminus V_2)$ and if this set has more than three components $M_1 \setminus V_2$ must have more than two, i.e., $M_1 \setminus V_2 = N_1 \cup N_2 \cup N_3$ where $M_0$, $N_1$, $N_2$, ...
and $N_3$ are pairwise disjoint non-null sets and each $N_i$ is a union of components of $M_1 \setminus V_2$. But since $V_2$ is in $\mathcal{F}$, $C \setminus V_2$ has at most two components so at least two of the sets $N_i$, say $N_1$ and $N_2$, have closures that meet $M_0 \cup V_1$. Since $C_i \subset C \cap M_1 = M_1 \cup V_1$, this requires that neither $V_1 \cap C_i$ nor $V_3 \cap C_i$ be null. But $M_1 \cup V_1$ is a $k$-cell and $V_1 \cap V_2 = \emptyset$, so there is a set $M'_1$ which is topologically the product of $E^{k-1}$ and an open interval which contains all points of $M_1$ in some neighborhood of $V_1$. Thus $N_1$ and $N_2$ both meet a connected subset of $M_1 \setminus V_2$, contradicting the choice of $N_1$ and $N_2$. This proves 6.11.

Applying 6.11 to the present situation yields that $C \setminus (U_1 \cup U)$ has components $M \cup N \cup R$ and that the two components of $C \setminus U$ are $A = M \cup U_1 \cup N$ and $B = R$, where, it will be recalled, $A$ is the component of $C \setminus U$ containing $U_1$ and $T$ while $U_2 \subset B$. Since $K_1 \cap C = U_1 \cup U_2$ and $K_1$ separates $T \subset \text{Int } K_1$ from $A \cap C = a$ in $R^3$, it follows that the assumption that $M \cup A \subset \text{Ext } K_1$ and $T \subset N \subset \text{Int } K_1$ is but a notational convenience. Since $N$ is connected and does not meet $K_2$, if $T \subset \text{Ext } K_2$ then $N \subset \text{Ext } K_2$. Since $M \cup A$ is an unbounded subset of the complement of $K_2$, $M \cup A \subset \text{Ext } K_2$ also. But $\text{Ext } K_2$ is locally connected and since $M$ and $N$ have common limit points (in $U_1$), there are arbitrarily small connected sets in the complement of $K_2$ meeting both $M$ and $N$. If these sets are sufficiently small they must lie in a neighborhood of $U_1$ containing no point of $D$ and hence do not meet $K_1 \setminus K_2 \subset D$. This requires that they not meet $K_1$ and affords a contradiction since $M \subset \text{Ext } K_1$ and $N \subset \text{Int } K_1$.

3. Since both $K_1$ and $D$ are locally polyhedral mod $C$, $K_2 \subset K_1 \cup D$ must be also.

4. By construction $K_2 \cap C = U_1 \cup U$ and both $U_1$ and $U$ are in $\mathcal{F}$.

5. If $p$ is any point of $\text{Ext } K_1$, then there is a topological ray $J$ from $p$ in $\text{Ext } K_1$ and $J \cap C \subset J \cap K_1 = \emptyset$. But the disk $D_i$ is in $K_1 \cap \text{Int } K_1$ so $J \cap D_i = \emptyset$ also. Hence $J \cap K_2 = J \cap (X_1 \setminus D) = \emptyset$ so $J$ is an unbounded subset of the complement of $K_2$ which requires that $J$, and a fortiori $p$, be in $\text{Ext } K_2$. This proves $\text{Ext } K_1 \subset \text{Ext } K_2$ so that by complementation in $R^3$, $K_2 \cup \text{Int } K_2 \subset K_1 \cup \text{Int } K_1 \subset S(T, \epsilon)$.

Thus $K_2$ is in $\mathcal{B}(T, \epsilon)$ and by the proof of 5, $K_2$ satisfies condition 1 of Theorem 4.1 also. That the second condition of that theorem is satisfied follows immediately from the hypothesis $U_1 < T < U$. Thus $K_2 \in \mathcal{B}^*(T, \epsilon)$.

This completes the proof of the lemma for the case $U_1 < T < U < U_2$, and the only other possible case, $U_1 < U < T < U_2$, can be made to depend on the first case in the following way. Consider the effect upon the order relation and the hypotheses and conclusion of the lemma if the names $A$ and $B$ are interchanged and $h^* = hr$ is used instead of $h$, where $r$ is the homeomorphism of $E^k$ onto itself defined by $r(x_1, x_2, \ldots, x_k) = (1 - x_1, x_2, \ldots, x_k)$. Evidently the new $A$ and $B$ are as required since $T_0$ and $T_1$ are interchanged, and since the order relation is reversed as well as the sets $U_1$ and $U_2$, the second case is reduced to the first.
6.2. **Definition.** For every $T \in \mathcal{T}$ and $\varepsilon > 0$ let $\mathcal{Q}(T, \varepsilon)$ denote the collection of all $K \subset R^3$ such that

1. $K$ is a topological 2-sphere.
2. $T \subset \text{Int} \, K$.
3. $K$ is locally polyhedral modulo $C$.
4. $K \cap [A \cup (C \cup B)] = L \cup R$ where each of the sets $L$ and $R$ is either an element of $\mathcal{T}$ or a point of $(A \cup B) \setminus C$.
5. $L < T < R$.
6. $K \subset S(C, \varepsilon)$.

It should be noted that $\mathcal{Q}(T, \varepsilon)$ depends on $h, A$, and $B$. This dependence is not indicated in the notation since in the applications $h, A$, and $B$ will be chosen and fixed.

6.3. **Lemma.** Let $0 \leq x_1 < x_2 \leq 1$, $T_{x_i} = h(x_i \times E^{k-1})$, $K_i \in \mathcal{Q}(T_{x_i}, \varepsilon)$ and $K_i \cap (A \cup (C \cup B)) = L_i \cup R_i$, $i = 1, 2$, be such that $L_1 < L_2 < R_1 < R_2$. Then there is a $K_3$ in $\mathcal{Q}(T_{x_1}, \varepsilon)$ with $K_3 \cap (A \cup (C \cup B)) = L_1 \cup R_2$ and $(A \cup (C \cup B)) \cap \text{Int} \, K_3 = (A \cup (C \cup B)) \cap [(\text{Int} \, K_1) \cup (\text{Int} \, K_2)]$.

**Proof.** As before the adjustments needed for the cases where some or all of the sets $L_1, R_1, L_2, R_2$ are points of $(A \cup B) \setminus C$ are easily made, so only the case where all are elements of $\mathcal{T}$ will be considered. As a preliminary step the following statement is to be proved.

6.31. If $K \in \mathcal{Q}(T, \varepsilon)$ then $(A \cup (C \cup B)) \cap \text{Int} \, K = B(L) \cup A(R)$.

Since $L < T < R$, $L$ separates $C \setminus A = a$ and $T$ on $C$, $R$ separates $T$ and $b = B \setminus C$, and each separates $a$ and $b$ on $C$, so the three components of $C \setminus (L \cup R)$ as guaranteed by 6.11 must be $C_a, C_b$, and $C_T$; that is, the one containing $a$, the one containing $b$, and the one containing $T$ respectively. Thus $A \cup (C \cup B) = A \cup C_a \cup L \cup C_T \cup R \cup C_b \cup B$ and since $A \cup C_a$ and $C_b \cup B$ are unbounded connected sets in the complement of $K$, they lie in Ext $K$. Hence $(A \cup (C \cup B)) \cap \text{Int} \, K \subset C_T$ and, since $C_T$ is connected, does not meet $K$, and contains $T \subset \text{Int} \, K$, the reverse inclusion also holds so

$$C_T = (A \cup C \cup B) \cap \text{Int} \, K.$$

Now the two components $A(L)$ and $B(L)$ of $(A \cup (C \cup B)) \setminus L$ are then $A \cup C_a$ and $C_T \cup R \cup C_b \cup B$, and since $L < T$ implies $T \subset B(L)$ it follows that $A(L) = A \cup C_a$ and $B(L) = C_T \cup R \cup C_b \cup B$. Similarly from $T < R$ it is seen that $A(R) = A \cup C_a \cup L \cup C_T$ while $B(R) = C_b \cup B$. Thus $A(R) \cap B(L) = (A \cup C_a \cup L \cup C_T) \cap (C_T \cup R \cup C_b \cup B) = C_T$ which proves 6.31.

To proceed with the proof of Lemma 6.3, it is noted that since $L_1 < L_2 < R_1$ then $L_2$ is in both $B(L_1)$ and $A(R_1)$ so, by 6.31, $L_2$ is in Int $K_1$ and $K_2 \cap \text{Int} \, K_1 \neq \emptyset$. But $R_1 < R_2$ so $R_2 \subset B(R_1)$ and, by 6.31, $B(R_1)$ is in Ext $K_1$. Thus $K_2$ meets Ext $K_1$ as well as Int $K_1$ and $K_2 \cap K_1 \neq \emptyset$ follows. Since $K_2 \cap K_1 \cap C = (L_2 \cup R_2) \cap (L_1 \cup R_1) = \emptyset$ there is a neighborhood $V$ of $K_1 \cap K_2$ containing no points of $A \cup C \cup B$ and both $K_1$ and $K_2$ are locally polyhedral at each point of $V$. By shifting the vertices of $K_2$ lying in $V$ a distance so small that
no point outside $V$ is moved, $K_1$ and $K_2$ may be brought into relative general position so that $K_1 \cap K_2$ is a finite collection of mutually disjoint simple closed curves $s_1, s_2, \ldots, s_n$. Each $s_j$ bounds a pair of sub-disks $X_{ij}$ and $Y_{ij}$ of $K_i$ where $X_{ij}$ contains $L_i$, $i = 1, 2$.

Let an arc $c$ be chosen in $C$ so that it has only its end points $a = A \cap C$ and $b = C \cap B$ in common with $\partial C$ and meets each of the sets $L_1, L_2, R_1,$ and $R_2$ in a single point. Now suppose $s_k$ links $A \cup C \cup B$. Then none of the four disks $X_{1k}, X_{2k}, Y_{1k}, Y_{2k}$ bounded by $s_k$ can lie in the complement of $A \cup C \cup B$ so $X_{ik} \supset L_i$ and $Y_{ik} \supset R_i$, $i = 1, 2$. If on the other hand $s_k$ does not link $A \cup C \cup B$ then each of the sets $X_{ik} \cap (A \cup C \cup B)$ and $Y_{ik} \cap (A \cup C \cup B)$ must consist of two points or no points for each choice of $i$, and since $L_i \subset X_{ik}$ it follows that $R_i \subset X_{ik}$ also and $Y_{ik} \cap (A \cup C \cup B) = \emptyset$, $i = 1, 2$. No other possibilities exist, since $s_k$ either does or does not link $A \cup C \cup B$, so for any $k$ either $R_i \subset Y_{ik}$ for both $i = 1$ and $i = 2$, or $Y_{ik}$ and $Y_{2k}$ are both disjoint from $A \cup C \cup B$.

If there is any index $k$ such that $Y_{1k}$ is disjoint from $A \cup C \cup B$, then there is evidently one such index $j$ such that $Y_{ij} \cap K_2 = s_j$. Then $Z = Y_{ij} \cup Y_{2j}$ is a topological 2-sphere for $Y_{ij} \cap Y_{2j} = Y_{ij} \cap K_2 = s_j$ and $s_j = \partial Y_{ij} = \partial Y_{2j}$. Further, $Z$ is polyhedral since both $Y_{ij}$ and $Y_{2j}$ are subsets of spheres which are locally polyhedral mod $C$ and neither $Y_{ij}$ nor $Y_{2j}$ meets $A \cup C \cup B$. Beispeil III, §3, [5] can be invoked here to produce a semi-linear map of $R^3$ onto itself throwing $Y_{ij}$ onto $Y_{2j}$ and leaving $A \cup C \cup B$ fixed, but the question of how the interiors of $K_1, K_2$, and the interiors of their images under this map are related would then arise. To avoid this difficulty an isotopy on $R^3$ achieving the above result is to be constructed.

There is a semi-linear homeomorphism $f$ of $R^3$ onto itself throwing $Z$ onto the surface of the unit cube $E$ in $R^3$ and evidently it may be supposed that if $E^+$ and $E^-$ denote the portions of the surface of $E$ on and above and on and below the xy-plane respectively, then $f(Y_{ij}) = E^+$ and $f(Y_{2j}) = E^-$. An isotopy $g_t (0 \leq t \leq 1)$ of $R^3$ onto itself which throws $E^-$ onto $E^+$ and moves no point outside $S(E, \delta)$ with $\delta$ arbitrarily small can be defined with no difficulty, and there is no loss in taking $g_1$ semilinear. Then any point of $f(A \cup C \cup B)$ which is interior (exterior) to $f(K_2)$ at the stage $t = 0$ remains interior (exterior) to $g_t f(K_2)$ for each $t$, so that $f^{-1} g_t f(K_2)$ has all the properties required of the original $K_2$. The $\delta$ may be taken so small that no point of $f(K_2)$ lies interior to $S(E, \delta)$ except $f(Y_{2j})$ and an arbitrarily small neighborhood of $f(Y_{2j})$ relative to $f(K_2)$, and the isotopy may be defined so as to move no point of $f(K_2)$ except $f(Y_{2j})$.

Consequently $f^{-1} g_t f(K_2)$ is $(K_2 \setminus Y_{2j}) \cup Y_{ij}$ and this sphere may be deformed semi-linearly away from $K_1$ in a neighborhood of $Y_{ij}$ so small that no point of $A \cup C \cup B$ is moved. The resulting sphere is a new $K_2$ which has all the properties required of the old and which has at least one less intersection $s_k$ with $K_1$ which does not link $A \cup C \cup B$.

Thus after a finite number of repetitions a new $K_2$ is obtained such that
every component of its intersection with $K_1$ links $A \cup C \cup B$. Then each remaining $X_{1p}$ contains $L_1$ and not $R_1$, and an index $p$ is found such that $X_{1p} \cap K_2 = s_p$. Let $K_3 = X_{1p} \cup Y_{2p}$, and note that $K_3 \cap (A \cup C \cup B) = [X_{1p} \cap (A \cup C \cup B)] \cup [Y_{2p} \cap (A \cup C \cup B)] = L_1 \cup R_2$. Thus in view of 6.31, $K_3$ is the desired sphere provided it can be shown to satisfy the six conditions of Definition 6.2 for being an element of $\mathcal{Q}(T_{z_1}, \epsilon)$.

1. Since $X_{1p} \cap Y_{2p} = s_p = \partial X_{1p} = \partial Y_{2p}$, $K_3$ is a topological 2-sphere.

2. The proof that $T_{z_1} \subset \text{Int} K_3$ is essentially a repetition of the argument given to show $T \subset \text{Int} K_1$ in the proof of Lemma 6.1 and hence need only be outlined here. From $K_1 \in \mathcal{Q}(T_{z_1}, \epsilon)$ and $L_1 < L_2 < R_1 < R_2$ it is seen that $\mathcal{A}(T_{z_1}) \setminus L_1$ has two components, $\mathcal{A}(L_1)$ in Ext $K_3$ and $N$ containing $T_{z_1}$. If $T_{z_1} \subset \text{Ext} K_3$, then $N \subset \text{Ext} K_3$ and connected subsets of Ext $K_3$ meeting both $N$ and $\mathcal{A}(L_1)$ with arbitrarily small diameters exist, for $(\text{Cl } N) \cap \mathcal{A}(L_1) = L_1 \neq \square$. But if the diameters of such sets are sufficiently small they are seen to lie in a neighborhood of $L_1$ in which $K_3$ and $K_1$ are identical, and hence lie in the complement of $K_1$. This contradicts $T_{z_1} \subset \text{Int} K_1$, which cannot be false for $K_1 \in \mathcal{Q}(T_{z_1}, \epsilon)$.

3. $K_3 \subset K_1 \cup K_2$ and hence is locally polyhedral modulo $C$ for both $K_1$ and $K_2$ are.

4. $K_3 \cap (A \cup C \cup B) = L_1 \cup R_3$, and these are as required since they are components of intersection of $A \cup C \cup B$ and $K_1$ and $K_2$ respectively.

5. That $L_1 < T_{z_1} < R_2$ holds was shown above.

6. $K_3 \subset K_1 \cup K_2 \subset S(C, \epsilon)$ as required.

This completes the proof.

7. The enclosure property.

7.1. Theorem. If there is an $h \in \mathcal{S}$ such that $C$ has property $\mathcal{P}$ relative to $\mathcal{S}_h$ and the uniform disk property relative to $\mathcal{S}_h$, then $C$ has the enclosure property.

Proof. Let $\epsilon > 0$ be assigned. Let $A'$ and $B'$ be a pair of topological rays with initial points $a = A' \cap C$ in $T_0 \setminus \partial T_0$ and $b = B' \cap C$ in $T_1 \setminus \partial T_1$ respectively. That such a pair of rays which are mutually disjoint and locally polyhedral mod $C$ exist was shown in the preceding section. Let $\omega_1$ denote $\min [\epsilon, d(T_0, T_1)/2, d(T_0, B'), d(T_1, A')]$ and choose $K_0$ in $\mathcal{B}^*(T_0, \omega_1)$ and $K_1$ in $\mathcal{B}^*(T_1, \omega_1)$. Let $A''$ be the sub-arc of $A'$ which is minimal with respect to containing $K_0 \cap A'$. By the choice of $\omega_1$ and the fact that $T_0 \subset \partial C, K_0 \cap C = R_0 \in \mathcal{T}$ and $K_0 \setminus C$ is an open disk so there is an arc $A'''$ in $K_0 \setminus C$ with the same end points as $A''$. The arc $A'''$ may be taken polyhedral and by further subdividing it and shifting all of its vertices except one end point into Ext $K_0$ a sufficiently small distance so as to introduce no new points of intersection with $B \cup C \cup K_1$ the ray $(A' \setminus A'') \cup A'''$ is deformed into a ray $A$ such that $A \cap K_1 = \square$, $A \cap K_0 = L_0$, a point, and $A \cap C = a$, the initial point of $A$. A ray $B$ is constructed from $B'$ in a similar manner so that $B \cap K_0 = \square$, $B \cap K_1 = R_2$, a point, and $B \cap C = b$, the initial point of $B$. 

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Now $K_0$ and $K_1$ are seen to be in $\mathcal{Q}(T_0, \omega_1)$ and $\mathcal{Q}(T_1, \omega_1)$ respectively, for all of the six conditions of 6.2 are satisfied by virtue of the fact that $K_i$ is in $\mathcal{P}^*(T_i, \omega_i)$, $i=0, 1$, except condition 4: $K_i \cap (A \cup C \cup B) = L_i \cup R_i$ where each of the sets $L_i$ and $R_i$ is either an element of $\mathcal{X}$ or a point of $(A \cup B) \setminus C$. But $K_0 \cap (A \cup C \cup B) = (K_0 \cap A) \cup (K_0 \cap C) \cup (K_0 \cap B) = L_0 \cup R_0 \cup \square$ where $L_0$ is a point of $A$ and $R_0$ is in $\mathcal{X}$ by construction. Similarly $K_1 \cap (A \cup C \cup B) = \square \cup L_1 \cup R_1$ where $L_1$ is in $\mathcal{X}$ and $R_1$ is a point of $B$.

Since $T_0 \subset (\text{Int } K_0) \cap (\text{Ext } K_1)$ and $T_1 \subset (\text{Int } K_1) \cap (\text{Ext } K_0)$ it is seen that two real numbers $p, q$ with $0 < p < 1/2 < q < 1$ can be chosen so that $T_x \subset \text{Int } K_0$ for $0 \leq x \leq p$ and $T_x \subset \text{Int } K_1$ for $q \leq x \leq 1$. Let such a $p$ and $q$ be chosen and let $M$ denote $\bigcup_{p \leq x \leq q} T_x$. The plan of proof will be first to show there is a $K_p$ in $\mathcal{Q}(T_p, \epsilon)$ such that $M \subset \text{Int } K_p$, and then to apply Lemma 6.3 twice, first to $K_0$ and $K_p$ and then to the resulting sphere and $\mathcal{X}$, to obtain the desired sphere. In order to construct $K_p$ some preliminary steps are necessary.

If $\mathcal{X}_m$ denotes the subset of $\mathcal{X}_h$ consisting of all $T_x$ in $\mathcal{X}_h$ with $p \leq x \leq q$, $\mathcal{X}_m$ is evidently a sub-arc of $\mathcal{X}_h$ and hence is compact. If we let $\omega_2$ denote $\text{Min } \{\omega_1, d(M, T_0 \cup T_1 \cup A \cup B)\}$ and $\omega_3 > 0$ be the number corresponding to $\omega_2$ guaranteed by the hypothesis that $C$ has the uniform disk property relative to $\mathcal{X}_h$; that is let $\omega_3$ be such that if $T_x \subset \mathcal{X}_h$ and $\eta > 0$ there is a $D$ in $\mathcal{D}(T_x, \eta)$ with $d(\partial(D, C)) > \omega_3$ and $D \subset S(T_x, \omega_3)$. Now, since $\mathcal{X}_m$ is compact and $C$ has property $\mathcal{P}$ relative to $\mathcal{X}_h$ (and hence relative to $\mathcal{X}_m$), there is a $\gamma > 0$ such that for every $T_x \subset \mathcal{X}_m$, i.e., every $p \leq x \leq q$, $\sup_{K \subset \mathcal{P}^*(T_x, \omega_3)} d(T_x, K) > 2\gamma$. Evidently then there is a $K_x$ in $\mathcal{P}^*(T_x, \omega_3)$ such that $T_y \subset \text{Int } K_x$ for all $y$ such that $\rho(T_x, T_y) < \gamma$. Since $h$ is a homeomorphism, it is easily seen that there is an $\alpha > 0$ corresponding to $\gamma$ such that whenever $|x - y| \leq \alpha$ then $\rho(T_x, T_y) < \gamma$.

Without loss of generality it may be supposed that $p - \alpha > 0$ and $q + \alpha < 1$.

Let $x_1 = p - \alpha$, $x_2 = p$, $x_3 = p + \alpha$, $\ldots$, $x_i = p + (i - 2)\alpha$, $\ldots$, $x_{i+1} = p + (j - 1)\alpha$, where $j$ is chosen so that $x_j < q \leq x_{j+1}$. For $i = 2, 3, \ldots, j$, choose $K_i'$ in $\mathcal{P}^*(T_{x_i}, \omega_3)$ such that $T_y \subset \text{Int } K_i'$ for all $y$ with $x_{i-1} \leq y \leq x_{i+1}$. This is possible for if $x_{i-1} \leq y \leq x_{i+1}$ then $|x_i - y| \leq \alpha$ and $\rho(T_{x_i}, T_y) < \gamma$.

It will now be shown that from $K_i'$ a sphere $K_i$ in $\mathcal{P}^*(T_{x_i}, \omega_3)$ can be formed with the property that $K_i \cap C = L_i \cup R_i$ with $\rho(L_i, T_{x_i-1})$ and $\rho(R_i, T_{x_i+1})$ both arbitrarily small, $i = 2, 3, \ldots, j$. To see this let $\beta > 0$ be assigned. For each $i = 1, \ldots, j+1$ choose $D_i \subset D(T_{x_i}, \beta)$ such that $d(\partial D_i, C) > \omega_3$ and $D_i \subset S(T_{x_i}, \omega_3)$. Then $D_{i+1}$, $K_i'$, $T_{x_i}$, and $\omega_3$ satisfy the seven hypotheses of Lemma 6.1, as follows.

1. $T_{x_i}$ is in $\mathcal{X}$ and separates $A \cap C$ and $B \cap C$ on $A \cup C \cup B$ since each element of $\mathcal{X}_h$ has these properties.
2. $K_i' \subset \mathcal{P}^*(T_{x_i}, \omega_3)$ by choice.
3. Since $(K_i' \cup K_i') \subset S(T_{x_i}, \omega_3)$ and $\omega_3 < d(T_{x_i}, A \cup B)$, both $A$ and $B$ are separated from $T_{x_i}$ by $K_i' \cap C$ on $A \cup C \cup B$. Hence $C \cap K_i'$ has two components and consists of a pair of elements $L_i'$ and $R_i'$ of $\mathcal{X}$.
4. From the above, if the notation is properly chosen, $L_i'$ separates $A$
and $T_x$ while $R_i'$ separates $T_x$ and $B$ on $A \cup C \cup B$. By 2 of Theorem 4.1 and Definition 4.2, $T_x$ separates $L_i'$ and $R_i'$ on $A \cup C \cup B$, so $L_i' < T_x < R_i'$. Since $T_y \subset \text{Int} K_i'$ for all $y$ with $|x_i - y| \leq \alpha$ and since $x_i + 1 + \alpha = x_i = x_{i+1} - \alpha$, $L_i' < T_{x_i-1} < T_{x_i} < T_{x_{i+1}} < R_i'$. Letting $U_i$ denote $D_i \cap C$, $i = 0, 1, \ldots, j+1$, and noting that $\rho(U_i, T_{x_i}) < \beta$, it is seen from Lemma 3.1 that if a sufficiently small number is used in place of $\beta$ then $L_i' < U_{i-1} < T_{x_i} < U_{i+1} < R_i'$.

5. Since $U_{i+1} = D_{i+1} \cap C$, then $\rho(U_{i+1}, C \cap D_{i+1}) < \omega$ for every $\omega > 0$ so $D_{i+1}$ satisfies condition 5 of Definition 2.31 for being in $D(U_{i+1}, \omega)$ for every $\omega > 0$. That $D_{i+1}$ also satisfies all the other conditions of 2.31 follows from the fact that $D_{i+1}$ was chosen in $D(T_{x_{i+1}}, \beta)$.

6. Since $D_{i+1} \subset S(T_{x_{i+1}}, \omega_3)$ and $K_i \in \mathcal{P}^*(T_{x_i}, \omega_3)$, then $D_{i+1} \cup K_i \subset S(M, \omega_2)$ for $D_{i+1} \cap C$, $T_{x_{i+1}}$, and $T_x$, are all in $M$ and $\omega_2 > \omega_3$. But $\omega_2 < d(M, A \cup B)$ so $(D_{i+1} \cup K_i) \cap (A \cup B) = \emptyset$.

7. By choice of $D_{i+1}$, $d(\partial D_{i+1}, C) > \omega_3$.

Thus Lemma 6.1 applies so there is a $K_i''$ in $\mathcal{P}^*(T_{x_i}, \omega_3)$ with $K_i'' \cap C = L_i' \cup U_{i+1}$. In the same way it is verified that $D_{i-1}$, $K_i''$, $T_x$, and $\omega_3$ also satisfy the hypothesis of Lemma 6.1 with $L_i' < U_{i-1} < T_{x_i} < U_{i+1}$, so there is a $K_i$ in $\mathcal{P}^*(T_{x_i}, \omega_3)$ with $K_i \cap C = U_{i-1} \cup U_{i+1}$, $i = 2, 3, \ldots, j$.

Next it must be shown that $K_i$ is in $\mathcal{Q}(T_{x_i}, \omega_3)$, $i = 2, \ldots, j$. The first three conditions of Definition 6.2 are satisfied by $K_i$, $T_x$, and $\omega_3$ by virtue of $K_i \in \mathcal{P}^*(T_{x_i}, \omega_3)$, while conditions 4 and 5 follow from the construction. Condition 6 also holds, since $S(C, \omega_3) \supset S(T_{x_i}, \omega_3) \supset K_i$.

Now Lemma 6.3 is to be applied to $K_2$ and $K_3$. Since $0 < x_2 < x_3 = x_2 + \alpha < 1$, and since each $K_i$ has been shown to be in $\mathcal{Q}(T_{x_i}, \omega_3)$, Lemma 6.3 applies provided $U_1 < U_2 < U_3 < U_4$. But $U_i = D_i \cap C$ and $\rho(T_{x_i}, U_i) < \beta$, so if $\beta$ was taken sufficiently small then $U_1 < U_2 < \cdots < U_{j+1}$ follows from $T_{x_1} < T_{x_2} < \cdots < T_{x_{j+1}}$ and Lemma 3.1. Thus there is a $K_{23}$ in $\mathcal{Q}(T_{x_j}, \omega_3)$ with $K_{23} \cap C = U_1 \cup U_4$ and $(A \cup C \cup B) \cap \text{Int} K_{23} = [A \cup C \cup B] \cap ([\text{Int} K_2] \cup \text{Int} K_3)$. This process is repeated with $K_{23}$ and $K_4$ to obtain $K_{24}$, with $K_{24}$ and $K_5$ to obtain $K_{25}$, and so on until a sphere $K_{2j}$ in $\mathcal{Q}(T_{x_j}, \omega_3)$ is obtained with $K_{2j} \cap C = U_1 \cup U_{j+1}$ and $(A \cup C \cup B) \cap \text{Int} K_{2j} = (A \cup C \cup B) \cap ([\text{Int} K_1] \cup \cdots \cup ([\text{Int} K_j] \cap \text{Int} K_{2j})$. Every point of $M$ lies in a $T_x$ for some $x$ with $p \leq x \leq q$ and hence with $x_1 \leq x \leq x_2$ so $M \subset \text{Int} K_{2j}$.

Thus three spheres have been obtained, first $K_0$ in $\mathcal{Q}(T_0, \omega_1)$ with $K_0 \cap (A \cup C \cup B) = L_0 \cup R_0$ where $L_0$ is a point of $A$ and $R_0$ separates $T_p$ and $T_q$ on $A \cup C \cup B$, second $K_2$ in $\mathcal{Q}(T_{x_1}, \omega_3)$ with $K_2 \cap (A \cup C \cup B) = U_1 \cup U_{j+1}$ where $U_1$ separates $T_0$ and $T_p$ (= $T_{x_2}$) on $(A \cup C \cup B)$ and $U_{j+1}$ separates $T_q$ (for $q \leq x_{j+1}$) and $T_1$ on $(A \cup C \cup B)$, and third $K_1$ in $\mathcal{Q}(T_1, \omega_1)$ with $K_1 \cap (A \cup C \cup B) = L_1 \cup R_1$ where $L_1$ separates $T_p$ and $T_q$ on $(A \cup C \cup B)$ and $R_1$ is a point of $B$. Hence $L_0 < U_1 < R_0 < U_{j+1}$ and $L_0 < L_1 < U_{j+1} < R_1$ so, since $0 < p < 1$, Lemma 6.3 can be applied, first to $K_0$ and $K_2$ to obtain a $K'$ in $\mathcal{Q}(T_0, \omega_1)$ with $K' \cap (A \cup C \cup B) = L_0 \cup U_{j+1}$, and then to $K'$ and $K_1$ to obtain a $K$ in $\mathcal{Q}(T_0, \omega_1)$ with $K \cap (A \cup C \cup B) = L_0 \cup R_1$. 

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Then \((A \cup C \cup B) \cap \text{Int} \ K\) is the union of those portions of \((A \cup C \cup B)\) which were in the interior of any one of the three spheres \(K_0, K_2,\) and \(K_1\). But any point of \(C\) lies in \(T_x\) for some \(x\), and hence is interior to \(K_0, K_2,\) or \(K_1\) according as \(0 \leq x \leq p, \ p \leq x \leq q,\) or \(q \leq x \leq 1\). Thus \(C \subset \text{Int} \ K\). Also, since \(K \subset \Omega(T_0, \omega_1), \ K \subset S(C, \omega_1) \subset S(C, \epsilon)\). That \(K\) is polyhedral follows from the fact that it is locally polyhedral modulo \(C\) and does not meet \(C\). This completes the proof, but a fact that will prove useful later should be noted here; i.e., the sphere \(K\) meets each of the rays \(A, B\) in a single point.

8. The strong enclosure property. For the standard cell \(E^k\) and a given \(\epsilon > 0\) it is evident that not only is there a polyhedral 2-sphere \(M\) in \(S(E^k, \epsilon)\) with \(E^k \subset \text{Int} \ M\), but the sphere \(M\) may be taken to be the boundary of a convex 3-cell so that a straight ray meeting \(E^k\) only at its initial point meets \(M\) in a single point. Consequently any tame cell will have the following property.

8.1. Definition. A cell \(C\) is said to have the strong enclosure property provided to each \(h \in \mathcal{S}\) there corresponds a pair of disjoint topological rays \(A, B\) and a sequence of polyhedral 2-spheres \(\{M_i\}\) which meet the following conditions:

1. \(A(B)\) meets \(C\) only at its initial point \(a \in T_0 \setminus \partial T_0(b \in T_1 \setminus \partial T_1)\).
2. \(A \cup B\) is locally polyhedral modulo \(C\).
3. \(M_i \subset S(C, 1/i)\) and \(C \subset \text{Int} \ M_i, i = 1, 2, \ldots\).
4. \(M_i\) meets each of the rays \(A, B\) in a single point, \(i = 1, 2, \ldots\).

Whether or not a cell with the enclosure property can fail to have the strong enclosure property is an unanswered question, although it has already been shown that the sets \(A, B,\) and \(\{M_i\}\) can be chosen satisfying all conditions except possibly condition 4 whenever \(C\) has the enclosure property.

8.2. Theorem. If \(C\) is a cell with property \(\mathcal{P}\), then a necessary and sufficient condition that \(C\) have the strong enclosure property is that \(C\) have the uniform disk property.

Proof of sufficiency. Let \(h \in \mathcal{S}\) be assigned and choose rays \(A_0, B_0\) satisfying all the conditions of Definition 8.1 except possibly condition 4. Let \(\epsilon_i\) be a null sequence of positive numbers, and for each integer \(i\) let \(T_{0i}\) be a \((k-1)\)-cell such that (1) \(T_{0i} \cap \partial T_{0i}\) contains \(a = A_0 \cap C, (2) T_{0i} \subset T_0,\) and (3) \(T_{0i} \subset S(a, \epsilon_i)\). Also for each \(i\) let \(T_{1i}\) be a \((k-1)\)-cell with similar relationships with \(b = B_0 \cap C\) and \(T_1\). Then for each \(i\) a homeomorphism \(h_i \in \mathcal{S}\) can be chosen so that \(h_i(0 \times E^{k-1}) = T_{0i}\) and \(h_i(1 \times E^{k-1}) = T_{1i}\). (In case \(C\) is a 1-cell each \(h_i\) may be taken to be \(h\) itself, for then \(T_{0i} = T_0\) and \(T_{1i} = T_1\).)

Now for each successive \(i\) the cell \(C\) has property \(\mathcal{P}\) relative to \(\mathcal{S}_{h_i}\) and the uniform disk property relative to \(\mathcal{S}_{h_i}\), so the construction of Theorem 7.1 with \(\epsilon_i\) used as the \(\epsilon\) can be carried out. It is recalled that in this construction the original rays (call them \(A_{i-1}\) and \(B_{i-1}\)) were replaced by a new pair (call them \(A_i\) and \(B_i\)) which had the property that \(A_i \setminus A_{i-1}, (B_i \setminus B_{i-1})\) was an arc obtained by deforming a sub-arc of a sphere \(K_0\) in \(S(T_{0i}, \epsilon_i)[K_1\) in \(S(T_{1i}, \epsilon_i)\).
an arbitrarily small distance, so that \( A_i \setminus A_{i-1} \subset S(T_{0i}, \varepsilon_i) \) and \( B_i \setminus B_{i-1} \subset S(T_{1i}, \varepsilon_i) \) may be assumed. The result of this construction was a polyhedral sphere (say \( M_i \)) in \( S(C, \varepsilon_i) \) with \( C \subset \text{Int } M_i \), which met each of the rays \( A_i, B_i \) in a single point. Evidently the sequence \( \varepsilon_i \) can be chosen so that \( M_i \subset S(C, 1/i) \) and also sufficiently small that any sequence \( \{a_i\} \) with \( a_i \in A_i \setminus A_{i-1} \) must have limit \( a \). To see the latter it is only necessary to note that if \( a_i \in A_i \setminus A_{i-1} \) then \( a_i \in S(T_{0i}, \varepsilon_i) \) so there is an \( a^* \) in \( T_{0i} \) such that \( d(a_i, a) \leq d(a_i, a^*) + d(a^*, a) \leq \varepsilon_i + d(a^*, a) \). But \( d(a^*, a) < \varepsilon_i \), for \( T_{0i} \subset S(a, \varepsilon_i) \) by choice, so \( d(a_i, a) \leq 2\varepsilon_i \). Similarly any sequence \( \{b_i\} \) with \( b_i \in B_i \setminus B_{i-1} \) may be supposed to have limit \( b \).

At the \( i \)th stage of this iterative construction \( A_i (B_i) \) is formed by replacing a sub-arc \( A_i^* (B_i^*) \) of \( A_{i-1} (B_{i-1}) \) by an arc \( A_{i**} = A_i \setminus A_{i-1} (B_{i**} = B_i \setminus B_{i-1}) \). Let \( A = [A_0 \cup A_1^*] \cup [A_i**] \) and \( B = [B_0 \cup B_1^*] \cup [B_i**] \). If the sequence \( \{\varepsilon_i\} \) is properly chosen, then \( A \) and \( A_i \) are identical on a sub-ray of \( A \) which includes all points of \( A \) that are separated from \( a \) by \( A_i^* \), so that \( A \setminus a \) is topologically a ray with the initial point deleted. This implies that \( A \) is a ray with initial point \( a \) provided \( A_i \) is the same exterior to a neighborhood of \( a \) which lies in \( \text{Int } M_i \), and this can be guaranteed by choosing \( \varepsilon_i \) sufficiently small.

**Proof of necessity.** Let \( h \in \mathcal{H} \) and \( \omega > 0 \) be assigned. In order to show that \( C \) has the uniform disk property it is necessary to exhibit a \( \delta > 0 \) such that for every \( T \in \mathcal{I}_h \) and \( \varepsilon > 0 \) there is a \( D \) in \( \mathfrak{D}(T, \varepsilon) \) with \( d(D, C) > \delta \) and \( D \subset S(T, \omega) \). First the following statement must be established.

**8.21.** There is a \( \beta > 0 \) such that if \( U \) and \( T \) are any pair of elements of \( \mathcal{I}_h \) with \( U \subset S(T, \beta) \), then \( \rho(T, U) < \omega/2 \).

If 8.21 is denied, then for each integer \( i \) there is a pair \( U_i, T_i \) of elements of \( \mathcal{I}_h \) with \( U_i \subset S(T, 1/i) \) and \( \rho(T_i, U_i) \geq \omega/2 \). Since \( \mathcal{I}_h \) is an arc under the metric \( \rho \), it may be assumed that \( \{T_i\} \) and \( \{U_i\} \), considered as elements of \( \mathcal{I}_h \), converge to \( T \in \mathcal{I}_h \) and \( U \in \mathcal{I}_h \) respectively. Since \( \rho \) is a metric, \( \rho(T, U) \geq \omega/2 \), and since \( \mathcal{I}_h \) is an arc, \( T \) and \( U \) must be distinct elements of \( \mathcal{I}_h \) and hence disjoint sets in \( \mathbb{R}^8 \). Then \( \alpha = d(T, U) \) is positive, and by definition of \( \rho \), the inclusions \( T_i \subset S(T, \alpha/4) \) and \( U_i \subset S(U, \alpha/4) \) must hold for all \( i \) greater than some fixed \( N \). But if \( 1/i < \alpha/4 \) and \( i > N \), then \( U_i \subset S(T, \alpha/4) \) and hence \( U \subset S(T, \alpha/2) \), contradicting \( U_i \subset S(U, \alpha/4) \) since these two sets are disjoint. From 8.21 the following will be derived.

**8.22.** There is an \( \eta > 0 \) such that if \( U \) and \( T \) are any pair of elements of \( \mathcal{I}_h \) and \( K \in \mathcal{P}^* (T, \eta) \), then \( U \subset \text{Int } K \) implies \( K \subset \mathcal{P}^* (U, \omega) \).

It is evident that no matter what \( \eta > 0 \) is chosen \( K \) must satisfy all of the requirements of Definition 2.21 for being in \( \mathcal{P}^* (U, \omega) \) except possibly \( K \subset S(U, \omega) \), and that \( U \) must separate the components of \( K \cap C \) whenever there
are two, since \( T \) does so and both \( U \) and \( T \) are in \( \mathcal{I}_h \). Hence 8.22 follows if \( \eta \) can be chosen so that \( (K \cup \text{Int} \ K) \subseteq S(U, \omega) \) whenever \( (K \cup \text{Int} \ K) \subseteq S(T, \eta) \). Let \( \eta = \min \left( \frac{\omega}{2}, \beta \right) \), where \( \beta \) is the number required by 8.21, and suppose \( \rho \subseteq (K \cup \text{Int} \ K) \subseteq S(T, \eta) \). Then \( d(\rho, U) \leq d(\rho, q) + d(q, U) \) for any point \( q \) of \( T \), so \( d(\rho, U) \leq \eta + \sup_{q \in T} d(q, U) \leq \omega/2 + \rho(T, U) \leq \omega/2 + \omega/2 = \omega \), and 8.22 follows.

Now since \( C \) has the strong enclosure property, two rays \( A' \) and \( B' \) and a sequence \( \{M_i\} \) of polyhedral spheres satisfying Definition 8.1 can be chosen. Let \( \alpha = \min \left[ \eta, d(T_0, T_1)/2, d(T_0, B'), d(A', T_1) \right] \) and choose \( K_0 \) in \( \mathcal{P}^*(T_0, \alpha) \) and \( K_p \) in \( \mathcal{P}^*(T_1, \alpha) \), where the index \( p \) will be explained below. The rays \( A' \) and \( B' \) are used in the now familiar way to form rays \( A \) and \( B \) such that \( A \cap K_0 \) is a point, \( A \cap K_p = \emptyset \), \( B \cap K_0 = \emptyset \), and \( B \cap K_p \) is a point. Since \( A \) and \( B \) are identical with \( A' \) and \( B' \) in some neighborhood of \( C \), it may be supposed that \( A, B \), and \( \{M_i\} \) satisfy Definition 8.1, for deleting a finite number of the \( M_i \) would make this true.

For each \( U \in \mathcal{I}_h \) except \( T_0 \) and \( T_1 \), \( \alpha(U) = \min \left[ \eta, d(U, A \cup B) \right] \) is positive, and \( K(U) \subseteq \mathcal{P}^*(U, \alpha(U)) \) can be chosen. Let \( M(U) \) denote the set of all \( T \in \mathcal{I}_h \) such that \( T \subseteq \text{Int} \ K(U) \), and take \( K_0 = K(T_0) \) and \( K_p = K(T_1) \). Then the collection of all \( M(U) \) is an open covering of the arc \( \mathcal{I}_h \) and corresponding to the finite sub-covering which must exist is the collection \( K_0, K_1, \ldots, K_p \). \( K_0 \) and \( K_p \) must be present, since \( T_0 \) and \( T_1 \) lie in no \( M(U) \) except \( M(T_0) \) and \( M(T_1) \) respectively, by the choice of \( \alpha(U) \). This collection and the arcs \( A \) and \( B \) are seen to have the following properties.

1. \( A, B \), and \( \{M_i\} \) satisfy Definition 8.1.
2. \( A \cap K_i = \emptyset \) or \( L_0 \), a point, according as \( i > 0 \) or \( i = 0 \), and \( B \cap K_i = \emptyset \) or \( R_p \), a point, according as \( i < p \) or \( i = p \).
3. \( K_0 \cap K_p = \emptyset \) and \( K_i \cap (T_0 \cup T_1) = \emptyset \) for all \( i \).
4. For each \( T \) in \( \mathcal{I}_h \) there is an index \( i \) such that \( T \subseteq \text{Int} \ K_i \), and hence \( K_i \cup \text{Int} \ K_i \subseteq S(T, \omega) \) by the choice of \( \eta \).

Now for each \( i = 0, 1, \ldots, p \) the set \( K_i \cap (A \cup C \cup B) \) is the union of two components \( L_i \) and \( R_i \), and each \( L_i, R_i \) is in \( \mathcal{I} \) except that \( L_0 \) is a point of \( A \) and \( R_p \) is a point of \( B \). A collection \( \sigma_0, \sigma_1, \ldots, \sigma_p \) of polyhedral simple closed curves can therefore be chosen so that \( \sigma_i \) separates \( L_i \) and \( R_i \) on \( K_i \), \( i = 0, 1, \ldots, p \). It will be shown that \( \delta = d(C, \sigma_0 \cup \cdots \cup \sigma_p) \) is the number corresponding to the assigned \( \omega \) required for the uniform disk property relative to \( \mathcal{I}_h \).

For let \( T \in \mathcal{I}_h \) and \( \epsilon > 0 \) be assigned. Then an index \( j \) can be chosen so that \( T \subseteq \text{Int} \ K_j \). Corresponding to \( \epsilon \) there is an \( \epsilon_1 > 0 \) such that if \( D \) is in \( \mathcal{D}(T, \epsilon_1) \) then \( D \) is also in \( \mathcal{D}(T, \epsilon) \) and \( D \cap C \subseteq \text{Int} \ K_j \). It may be assumed that \( D \subseteq \text{Int} \ K_j \) for some sub-disk of the original \( D \) is in \( \mathcal{D}(T, \epsilon) \) and has this property. From the sequence \( \{M_i\} \) choose a sphere \( M \) which separates \( C \) from \( \partial D \cup \sigma_1 \cup \cdots \cup \sigma_p \) and meets \( A \) and \( B \) in the points \( a \) and \( b \) respectively. Then
$M$ and $D$ may be taken in relative general position so that $M \cap D$ (which is non-null) is the union of a finite collection of mutually disjoint simple closed curves $s_1, \ldots, s_n$. Each $s_i$ bounds a disk $D_i$ on $D$ and a pair of disks $X_i$ and $Y_i$ on $M$, where $X_i$ contains the point $a = M \cap A$. If there is an $i$ for which $D_i \cap C = \emptyset$, then the theory of linkages may be used as before to show that $Y_i \cap (A \cup C \cup B) = \emptyset$, and $M$ may be replaced by the result of deforming $(M \setminus Y_i) \cup D_i$ semi-linearly away from $D$ so as to reduce the number of indices $i$ for which $D_i \cap C = \emptyset$. The fact that $(D_i \cup Y_i) \cap (A \cup C \cup B) = \emptyset$ is used to guarantee that the new $M$ retains the property $C \subseteq \text{Int } M$. After a finite number of repetitions, each remaining $D_i$ contains $D \cap C$, so that they are simply ordered by set inclusion and $D$ can be replaced by $D_1$, the minimal remaining $D_i$. Then $D_1 \subseteq \Sigma(T, \epsilon)$, $D_1 \subseteq \text{Int } K_j$, and $D_1 \cap M = s_1 = \partial D_1$.

Now $M$ and $K_j$ are taken in relative general position and application of the theory of linkages yields the result that any simple closed curves on $K_j \cap M$ which bound a sub-disk of $K_j \setminus (A \cup C \cup B)$ also bound a sub-disk of $M \setminus (A \cup C \cup B)$. A new $M$ is formed for which the number of such curves of intersection with $K_j$ is less than before. It must be noted that $\partial D_1$ is a subset of the new $M$ and that $\sigma_j$ does not meet it. Repetition removes all such components of $M \cap K_j$ so that $M \cap K_j$ becomes the union of a finite collection of mutually disjoint simple closed curves $s'_1, s'_2, \ldots, s'_m$, each of which separates $a$ and $b$ on $M$ as well as $L_j$ and $R_j$ on $K_j$. Then $\partial D_1 = s_1$ is interior to $K_j$ and lies on $M$, so that $s_1 \cap s'_i = \emptyset$ for every choice of $i$ and the curves have a natural order $s'_1, s'_2, \ldots, s'_m, s_1, s'_{i+1}, \ldots, s'_m$ induced by set inclusion of the sub-disks of $M \setminus b$ which they bound. By the choice of this order $s'_i$ and $s_1$ together separate $M$ into three components, an open disk containing $a$, an open disk containing $b$, and an open annular ring $R$ such that $R \cap K_j = \emptyset$ and $\text{Cl } R = R \cup s'_1 \cup s_1$. Since $s_1 \subseteq \text{Int } K_j$, $\text{Cl } R$ is contained in $K_j \cup \text{Int } K_j$. Similarly $s'_i$ and $\sigma_j$ bound an annular ring $R^*$ on $K_j$. Evidently $D^* = D_1 \cup R \cup R^*$ is a disk which is contained in $K_j \cup \text{Int } K_j$ and hence in $S(T, \omega)$ and $\partial D^* = \sigma_j$ so that $d(C, D^*) > \delta$. Thus $D^*$ is the required disk provided it can be shown to be in $D(T, \epsilon)$.

To show this it must be verified that $D^*$, $T$, and $\epsilon$ satisfy the six conditions of Definition 2.31. Condition 1 is fulfilled by construction and condition 2 follows from the fact that $\partial D^* = \sigma_j$ was chosen to separate $L_j$ and $R_j$, the components of $K_j \cap C$, on $K_j$. Since $D^* \cap C = D_1 \cap C = D \cap C$, condition 3 is fulfilled. Since $D^* \subseteq (D \cup M \cup K_j)$ and all three of these sets are locally polyhedral modulo $C$, so must $D^*$ be. This is condition 4, and condition 5 follows from $D^* \cap C = D \cap C$ and $D \subseteq \Sigma(T, \epsilon)$. Since there is some neighborhood $U$ of $D^* \cap C$ such that $U \cap D^* = U \cap D$, condition 6 for $D^*$ follows also from $D \subseteq \Sigma(T, \epsilon)$. This completes the proof.

8.3. Theorem. If $C$ has the strong enclosure property and the disk property, then $C$ has property $P$. 

Proof. Let $T \in \mathfrak{T}$ and $\epsilon > 0$ be assigned. Only the case $T \cap \partial C = \partial T$ will be considered since the adjustments to the following argument needed when $T \cap \partial C = T$ are easily made. Therefore there is an $h \in \mathfrak{S}$ and a number $p$ with $0 < p < 1$ such that $T = h(\bar{X} \times E^{k-1})$. For any pair of numbers $m$, $n$ with $0 < m < p < n < 1$, a triple of 1-cells $I_0 = \{x \in E^1 | 0 \leq x \leq m\}$, $I_1 = \{x \in E^1 | n \leq x \leq 1\}$, and $I_2 = \{x \in E^1 | m \leq x \leq n\}$ is defined. This in turn determines a triple of sub-cells of $C$ of dimension $k$, $C_i = h(I_i \times E^{k-1})$, $i = 0$, 1, and 2. Evidently $T_0 \subset C_0$, $T_1 \subset C_1$, and $T \subset C_2$, and if $p - m$ and $n - p$ are taken sufficiently small, $C_0 \subset S(T, \epsilon/2)$. Now choose another pair of numbers $r$, $s$ so that $m < r < p < s < n$, and note that $T_1$ separates $C_0$ and $T_1$ on $C$ while $T$ separates $T$ and $C_1$ on $C$.

Since $C$ has the disk property a pair of disks $D_r \in \mathfrak{D}(T_r, \beta)$ and $D_s \in \mathfrak{D}(T_s, \beta)$ may be chosen, and by Lemma 3.1 if $\beta$ is taken sufficiently small then $D_r (D_s)$ separates $T$ and $T_m (T_n)$ on $C$. This requires that $D_r (D_s)$ separate $T$ and $C_0 (C_1)$ on $C$. Also $R = D_r \cap C$ and $S = D_s \cap C$ are a pair of elements of $\mathfrak{X}$ which lie in $C_2$ and hence it may be supposed that $D_r \cup D_s$ is in $S(T, \epsilon)$, since this can be assured by taking sub-disks of the original ones.

A pair of rays $A$, $B$, and a sequence of spheres $\{M_i\}$ satisfying Definition 8.1 are now chosen, and again taking sub-disks of $D_r$ and $D_s$ as new disks $D_r$, $D_s$ the relation $(D_r \cup D_s) \cap (A \cup B) = \square$ can be made to hold. Now let $\delta$ be $\min \{\epsilon/2, d(C, \partial D_r \cup \partial D_s)\}$ and choose a polyhedral 2-sphere $M$ from the sequence $\{M_i\}$ so that $M$ is in $S(C, \delta)$, $C \subset \text{Int} M$, and $M$ meets $A$ and $B$ in a single point each. Taking $M$ and $D_r \cup D_s$ in relative general position, using the theory of linkages, and replacing sub-disks of $M \setminus (A \cup B)$ by sub-disks of $(D_r \cup D_s) \setminus C$, a new $M$ is formed such that each component of $M \setminus (D_r \cup D_s)$ is a simple closed curve separating $a = M \cap A$ and $b = M \cap B$ on $M$. Since $R$ separates $a$ and $S$ on $C$, if the $\delta$ above was chosen sufficiently small then $D_r$ separates $a$ and $D_s$ on $S(C, \delta)$ and hence on $M$. But this requires that some component $s_1$ of $D_r \cap M$ separate $a$ and $D_s \cap M$ on $M$ and hence a simple closed curve $s_j$ of $D_r \cap M$ can be found such that if $X_j$ is the sub-disc of $M \setminus D_s$ bounded by $s_j$ then $X_j \cap D_r = s_j$. Let $M_0 ^*$ denote the 2-sphere which is the union of $X_j$ and the sub-disc of $D_r$ bounded by $s_j$. That $C_0 \subset \text{Int} M_0 ^*$ is readily established, for $C_0 \subset \text{Ext} M_0$ leads to a contradiction. Similarly a 2-sphere $M_1 ^*$ with $C_1 \subset \text{Int} M_1$ is formed from $M$ and $D_s$.

Now let $\eta > 0$ be chosen less than $\epsilon/2$ and so that $S(C_0, \eta) \subset \text{Int} M_0 ^*$, $S(C_1, \eta) \subset \text{Int} M_1 ^*$, and $S(C, \eta) \cap (M_0 ^* \cup M_1 ^*) = S(C, \eta) \cap (D_r \cup D_s)$. Choose from the sequence $\{M_i\}$ a 2-sphere $N$ with $N \subset S(C, \eta)$ such that $C \subset \text{Int} N$ and $N$ meets each of the sets $A$ and $B$ in a single point. As before, it may be assumed that each component of $N \cap (D_r \cup D_s) = N \cap (M_0 ^* \cup M_1 ^*)$ is a simple closed curve separating $N \cap A$ and $N \cap B$ on $N$. These simple closed curves $s_1, s_2, \cdots, s_n$ bound sub-disks $X_1, X_2, \cdots, X_n$ of $N \setminus B$ and since the $s_i$ are disjoint and each $X_i$ contains $N \cap A$, it may be assumed that $X_1 \subset X_2 \subset \cdots \subset X_n$. Some of the $s_i$ are on $D_r$ and some are on $D_s$, so an index $j$ can be found...
such that $s_j$ lies on one of these two sets (say on $D_1$) while $s_{j+1}$ lies on the other. The pair $s_j, s_{j+1}$ bound an annular ring $R_j$ on $N$ and $s_j$ bounds a disk $D_{r_j}$ on $D$, while $s_{j+1}$ bounds a disk $D_{s_j}$ on $D$. Let $K = D_{r_j} \cup R_j \cup D_{s_j}$. That $K$ is a topological 2-sphere which is locally polyhedral modulo $C$ follows from the construction, as does the fact that $K \cap C = R \cup S$ where both $R$ and $S$ are in $T$.

Thus $R$ is in $\Psi(T, \epsilon)$ provided $T \subset \text{Int } K$ and $K \subset S(T, \epsilon)$.

To see that the latter holds, it is noted first that $D_{r_j} \cup D_{s_j} \subset D_r \cup D_s \subset S(T, \epsilon/2)$, so $K \subset S(T, \epsilon)$ if $R_j \subset S(T, \epsilon)$. Suppose there is a point $p$ of $R_j$ such that $d(p, T) \geq \epsilon$. Since $R_j \subset S(C, \eta)$ and $\eta < \epsilon/2$, there is a point $q$ of $C$ such that $d(p, q) < \epsilon/2$. That $q \in C$ cannot be, for then $d(p, T) \leq d(p, q) + d(q, T) < \epsilon$, so $q \in C_i$ for $i = 0$ or $i = 1$. But $S(C_i, \eta) \subset \text{Int } M_i^*$ so this requires $p \in \text{Int } M_i^*$ for one choice of $i$, say $i = 0$. The choice of $M_0^*$ and $M_1^*$ is seen to assure that $M_0^* \subset \text{Ext } M_1^*$ and $M_1^* \subset \text{Ext } M_0^*$, and since $R_j \cap (M_0^* \cup M_1^*) = s_j \cup s_{j+1} = \partial R_j$, $R_j \setminus (s_j \cup s_{j+1})$ lies in $(\text{Ext } M_0^*) \cap (\text{Ext } M_1^*)$. Hence $R_j \cap \text{Int } M_0^* = \emptyset$ and $p \in \text{Int } M_0^*$ is a contradiction, proving that $K \subset S(T, \epsilon)$.

It follows from the arguments above that $\text{Int } K$ and $\text{Int } M_i^*$ are disjoint for $i = 0$ and $i = 1$. Hence the components $C'_0$ and $C'_1$ of $C \setminus K$ determined by $C_0$ and $C_1$ are in $\text{Ext } K$. If $T \subset \text{Ext } K$ then the third component $C_T$ of $C \setminus K$ is also in $\text{Ext } K$ and a contradiction is reached just as in the proof of Lemma 6.1.

Thus since $\epsilon$ and $T$ were arbitrary and $K \subset \Psi(T, \epsilon)$ has been found, it follows that $C$ has property $\Phi$.

9. **Conclusion.** Theorems 8.2 and 8.3 combine to give the following result.

9.1. **Theorem.** If $C$ is a $k$-cell in $R^3$ for $k = 1, 2, \text{ or } 3$, and has any two of the following three properties, then it also has the third.

1. Property $\Phi$.
2. The uniform disk property.
3. The strong enclosure property.

9.2. **Theorem.** If $C$ is a 1-cell in $R^3$ with property $\Phi$, then $C$ has the strong enclosure property.

If the word strong is deleted here, this becomes a restatement of Theorem 1 of Harrold [6]. The proof given by Harrold, together with the proof of sufficiency in Theorem 8.2, establish 9.2 as stated.

9.3. **Corollary.** If $C$ is a 1-cell with property $\Phi$, then $C$ has the uniform disk property.

Example 1.1 of Fox-Artin [4] is a 1-cell which can be shown to have the uniform disk property but not the enclosure property, and hence of course, not property $\Phi$. Whether or not a 2-cell or a 3-cell with property $\Phi$ can fail to have the enclosure property and/or the uniform disk property is an unanswered question.
Bibliography


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