SOME THEOREMS ON BOUNDARY DISTORTION[1]

BY
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1. We take as our prototype of a result on boundary distortion the well known Löwner's Lemma which states:

Let $\phi(z)$ be a function regular for $|z|<1$ such that $\phi(0)=0$, $|\phi(z)|<1$ for $|z|<1$ and that $\phi(z)$ assumes continuously values of modulus 1 *on an arc A of $|z|=1$. Then the image of A has length at least equal to that of A. Equality of these lengths occurs only for $\phi(z)=e^{i\theta}z$, θ real.*

Extensions of this result and similar results have been treated by various authors. Among these let us mention especially Paatero, Unkelbach and Komatu. However many possibilities of extending this result do not appear to have been recognized. These occur especially when we restrict ourselves to the case of univalent functions. For example, there then exists a direct extension of Löwner's Lemma to domains of finite connectivity greater than one.

2. Our method of treating these results is in all cases what may be called the method of the extremal metric. It is essentially the method used to such advantage by Grötzsch, however we use the more convenient formulation due to Ahlfors and Beurling. Further in some cases it is necessary to combine this method with the method of symmetrization as we have done in earlier works [5; 6; 7; 9].

We shall have to deal with various module problems and in order to avoid repetition we state here the general conditions which we impose. These are to be understood tacitly in each particular case. In each case we deal with a domain D of finite connectivity lying, say, in the w-plane $(w=u+iv)$. On each boundary component of D there is a natural cyclic order among the prime ends, thus we may speak of an arc on a boundary component. With D will be associated certain classes of curves. These may be Jordan curves or open arcs on D tending to a prime end on the boundary of D at either extremity. In any case we will assume them to be locally rectifiable, i.e., every closed subarc on them is rectifiable. By use of local uniformizing parameters this definition extends also to Riemann surfaces. On D we consider the class of functions $\rho(u, v)$ which are non-negative real valued functions of integrable square over D and subject to certain auxiliary conditions of the following sort. For each curve in the various given classes the integral $\int\rho|dw|$ is to exist and satisfy certain subsidiary conditions in the form of possessing lower

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bounds, the value of which may vary from class to class. Then the desired module is given by the minimum of $\iint_D \rho^2 dudv$ where ρ is restricted as indicated. The minimum is attained in all cases with which we deal here and the corresponding metric $\rho|dw|$ will be called the extremal metric. The extremal metric is always unique apart from trivial modifications. In accordance with Teichmüller's Principle it is in each case associated with a quadratic differential. For a quadratic differential $Q(w)dw^2$ we denote the curves on which $Q(w)dw^2>0$ as trajectories, those on which $Q(w)dw^2<0$ as orthogonal trajectories.

3. We begin with a generalization of Löwner's Lemma to multiply-connected domains. Let D be a domain of finite connectivity n, $n\geqq 1$, in the z-plane with boundary components $C_1, C_2, \cdots, C_n$ and a distinguished inner point P. On the boundary component C_1 let there be given an arc A. Let us denote by $F(A)$ the class of functions $f(z)$ regular and univalent in D with $f(P)=0$, satisfying $|f(z)|<1$ for $z\in D$, admitting a continuous extension to A in the sense of natural boundary correspondence and carrying this arc into an arc on $|w|=1$. Let $f_1(z)$ be the function in $F(A)$ under which C_1 corresponds to $|w|=1$, A to the arc α_1 from $e^{-i\theta}$ to $e^{i\theta}$ (in the positive sense) θ real, $0<\theta<\pi$, and under which the boundary components $C_j, j=2, \cdots, n$ correspond to slits on trajectories of the quadratic differential $dw^2/w(w-e^{i\theta})\cdot(w-e^{-i\theta})$. To prove the existence of such a function we first form a two sheeted covering Δ of D branched at the point P and along the curve C_1. This can be mapped into a rectangle in the ζ-plane with sides parallel to the axes with C_1 corresponding to the perimeter of the rectangle so that the boundary arcs covering A go into the vertical sides and the boundary components of Δ covering the boundary components C_j of D, $j=2, \cdots, n$, correspond to horizontal slits [1]. The rectangle can be mapped on the circle $|W|<1$ so that its vertical sides correspond to the arcs from $e^{-i\phi}$ to $e^{i\phi}$ and from $e^{i(\pi-\phi)}$ to $e^{i(\pi+\phi)}$, ϕ real, $0<\phi<\pi/2$. The conformal symmetry of Δ obtained by the correspondence of pairs of points covering the same point in the z-plane appears in its image in the W-plane as a counterclockwise rotation through 180°. Forming $w=W^2$ we have a mapping of Δ into a two sheeted covering surface of $|w|<1$ such that the images of two points of Δ covering the same point of the z-plane cover the same point of the w-plane. Then projecting we obtain the desired mapping $f_1(z)$. Note that $\theta=2\phi$. We now state our extension of Löwner's Lemma.

THEOREM 1. *If $f(z)\in F(A)$ maps A onto an arc on $|w|=1$ of length l then $l\geqq 2\theta$, equality occurring only if $f(z)=e^{-i\chi}f_1(z)$, χ real.*

We note first of all that the metric $|d\zeta|=K|w(w-e^{i\theta})(w-e^{-i\theta})|^{-1/2}|dw|$ is for a suitable constant $K>0$ the extremal metric in the following module problem in the domain $f_1(D)$, i.e., the image of D under f_1. We have one class of curves, Γ, consisting of those open arcs in $f_1(D)$ running from α_1 back to α_1

and separating $w=0$ from the complement of α_1 on $|w|=1$. The subsidiary condition is that $\int_\gamma \rho|dw| \geqq 1$ for $\gamma \in \Gamma$. Suppose that we had $l \leqq 2\theta$. Then for suitable real χ the image of A on $|w|=1$ by $e^{i\chi}f(z)$ would be contained in the arc α_1. Also the image of a curve in Γ by the function $\phi(w)=e^{i\chi}f(f_1^{-1}(w))$ would have length at least one in the metric $|d\zeta|$. Thus the metric $K|\phi'(w)| \cdot |\phi(w)(\phi(w)-e^{i\theta})(\phi(w)-e^{-i\theta})|^{-1/2}|dw|$ in $f_1(D)$ would be admissible in the above module problem. Since the area of the image of D under $e^{i\chi}f$ in the metric $|d\zeta|$ is at most equal to that of $f_1(D)$ in this metric the metric $K|\phi'(w)| \cdot |\phi(w)(\phi(w)-e^{i\theta})(\phi(w)-e^{-i\theta})|^{-1/2}|dw|$ would be extremal. A standard type of argument [3] then shows that $e^{i\chi}f(z)=f_1(z)$ completing the proof of Theorem 1.

Of course Löwner's Lemma says that for a simply-connected domain this result holds without the requirement of univalence on the functions involved. The statement in this case can be proved by an extension of the present method [4] but in view of the simplicity of Löwner's original proof it does not seem worth while to go into details on this.

4. Let now D be a domain of finite connectivity n, $n \geqq 1$, in the z-plane with boundary components $C_1, C_2, \cdots, C_n$. On the boundary component C_1 let there be given disjoint arcs A and B. Let us denote by $F(A, B)$ the class of functions $f(z)$ regular and univalent in D, satisfying $|f(z)|<1$ for $z \in D$, admitting continuous extension to A and B in the sense of natural boundary correspondence and carrying these arcs into arcs on $|w|=1$. Let $f_2(z)$ be the function in $F(A, B)$ under which C_1 corresponds to $|w|=1$, A to the arc α_2 from $e^{-i\theta}$ to $e^{i\theta}$ (in the positive sense), B to the arc β_2 from $e^{i(\pi-\theta)}$ to $e^{i(\pi+\theta)}$, θ real, $0<\theta<\pi/2$, and under which the boundary components C_j, $j=2, \cdots, n$, correspond to slits on trajectories of the quadratic differential

$$dw^2/(w - e^{-i\theta})(w - e^{i\theta})(w - e^{i(\pi-\theta)})(w - e^{i(\pi+\theta)}).$$

To prove the existence of such a function we observe as before that D can be mapped into a rectangle with sides parallel to the axes so that C_1 goes into the perimeter, A and B go into the vertical sides and the boundary components C_j, $j=2, \cdots, n$, go into horizontal slits. Then a standard mapping from the rectangle to the circle $|w|<1$ gives the desired function for some value of θ in the range indicated.

THEOREM 2. *If $f(z) \in F(A, B)$ maps A into the arc from $e^{i\theta_1}$ to $e^{i\theta_2}$ (in the positive sense) on $|w|=1$ and B into the arc from $e^{i\theta_3}$ to $e^{i\theta_4}$ (in the positive sense) then the cross ratio $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$ is greater than or equal to* $\cos^{-2}\theta$, *equality occurring if and only if* $f(z)=e^{i\chi}(f_2(z)-w_0)/(1-\bar{w}_0f_2(z))$, χ real, $|w_0|<1$.

The cross ratio here is understood to be given by

$$(e^{i\theta_1} - e^{i\theta_3})(e^{i\theta_2} - e^{i\theta_4})/(e^{i\theta_1} - e^{i\theta_4})(e^{i\theta_2} - e^{i\theta_3}).$$

The result of this theorem is essentially well known, although apparently not in the present formulation. It is a consequence of the relation of cross ratio to the module of a quadrangle whose boundary is the unit circle and a familiar result for rectangles. It is included here for completeness and we will not give any further details of the proof. In the case of a simply-connected domain the result holds without restricting the functions to be univalent and in this form it is due to Paatero [15].

5. Next let D be a domain of connectivity n, $n \geqq 2$, in the z-plane with boundary components $C_1, C_2, \cdots, C_n$. On the boundary component C_1 let there be given an arc A. Let the complement of this arc on C_1 be denoted by A^*. Let us denote by $F(A; R)$ the class of functions $f(z)$ regular and univalent in D, satisfying $1 < |f(z)| < R$ for $z \in D$ $(1 < R)$, admitting a continuous extension to A in the sense of natural boundary correspondence, carrying A into an arc on $|w| = R$, admitting a continuous extension to C_2 in the same sense and carrying this boundary component onto the circle $|w| = 1$. The boundary components C_j, $j = 3, \cdots, n$, will frequently be referred to as the residual boundary components of D.

It is well known that D can be mapped onto a subdomain D' of the circular ring $1 < |w| < R'$ $(1 < R')$, so that C_2 corresponds to $|w| = 1$ and C_1 to $|w| = R'$, each in the sense of natural boundary correspondence (after this, this qualifying statement will be tacitly understood). Let D'' be the domain obtained by reflecting D' in the circle $|w| = R'$. Let $\mathfrak{D}$ be the union of D', D'' and the arc on $|w| = R'$ corresponding to A less its end points. It is a domain of connectivity $2n-1$. It is well known that $\mathfrak{D}$ can be mapped onto a subset $\mathfrak{D}'$ of the circular ring $1 < |w| < R''$ so that $|w| = 1$ goes into $|w| = 1$, $|w| = (R')^2$ goes into $|w| = R''$ and the other boundaries of $\mathfrak{D}$ go into radial slits. Because of the reflectional symmetry of $\mathfrak{D}$ and the essential uniqueness of the radial slit mapping the portion of $\mathfrak{D}'$ in $1 < |w| < (R'')^{1/2}$ is the conformal image of D. Thus there exists a function mapping D conformally onto a subdomain of the circular ring $1 < |w| < R_0$ $(1 < R_0)$ so that C_2 corresponds to $|w| = 1$, the arc A corresponds to $|w| = R_0$ with its end points going into $w = -R_0$, A^* goes into the radial segment $-R_0 < w \leqq -p$ $(1 < p < R_0)$ described twice and the residual boundary components of D go into radial slits interior to $1 < |w| < R_0$. Let us denote this function by $f(z, R_0)$.

LEMMA 1. *The class $F(A; R)$ is empty if $R > R_0$.*

Consider the following module problem for D. We have one distinguished class Γ of curves consisting of open arcs on D joining A to C_2. The subsidiary condition is that $\int_\gamma \rho |dz| \geqq 1$ for $\gamma \in \Gamma$. Since module problems are conformally invariant we may regard the realization of D obtained by its mapping under $f(z, R_0)$. For this image the extremal metric is $(|w| \log R_0)^{-1} |dw|$, as is well known, and the value of the module is $2\pi/\log R_0$. If now $f \in F(A; R)$ maps D into a subdomain of $1 < |w| < R$ the metric $(|w| \log R)^{-1} |dw|$ is admissible for the same module problem taken in the corresponding image of D and thus the module is not greater than $2\pi/\log R$. Hence

$$2\pi/\log R_0 \leqq 2\pi/\log R$$

and

$$R \leqq R_0.$$

This proves the lemma.

Let now $\Delta(t)$ denote the domain obtained by slitting the circular ring $1<|w|<R_0$ along the segment $-R_0<w\leqq -t$ where $R_0>t\geqq p$. Let $\phi(w, t)$ map $\Delta(t)$ onto the ring $1<|w|<R(t)(1<R(t))$ so that $w=-t$ goes into the point $w=-R(t)$. Then $\phi(f(z, R_0), t)\in F(A; R(t))$. The function $R(t)$ is clearly strictly decreasing as t decreases from R_0 to p. Let $R(p)=R_1$. Then for R with $R_0>R\geqq R_1$ there is a unique function in the above set in $F(A; R)$. Let us denote this function by $f(z, R)$. Under the mapping $w=f(z, R)$, $R_0\geqq R\geqq R_1$, let A correspond to the arc $\alpha(R)$ on $|w|=R$ for whose points $Re^{i\theta}$, $-\theta(R) \leqq\theta\leqq\theta(R)$, $0<\theta(R)<\pi$.

Lemma 2. *If $f(z)\in F(A; R)$, $R_0\geqq R\geqq R_1$, maps A into an arc on $|w|=R$ of length l then $l\geqq 2R\theta(R)$, equality occurring if and only if $f(z)=e^{-i\chi}f(z, R)$, χ real.*

We regard the same module problem as in Lemma 1, considered now in the image $D(R)$ of D by $f(z, R)$. In it the extremal metric is given by $|d\zeta| =(|\Phi(w, R)|\log R_0)^{-1}|\Phi'(w, R)|\,|dw|$ where $\Phi(w, R)$ is the inverse mapping to $\phi(w, t)$ for $R=R(t)$, $p\leqq t<R_0$, and $\Phi(w, R_0)\equiv w$. Suppose that we had $l\leqq 2R\theta(R)$. Then for suitable real χ the image of A on $|w|=R$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha(R)$. As in the proof of Theorem 1 we would have $e^{i\chi}f(z)=f(z, R)$, proving the lemma.

Let us consider now the quadratic differential on the W-sphere

$$Q_g(W)dW^2 = \frac{(g^2 - W^2)dW^2}{(W^2 - 1)^2}, \qquad 0 \leqq g < 1.$$

It has double poles at $W=1$, -1, ∞ and simple zeros at $W=g$, $-g$ except when $g=0$ in which case it has a double zero at $W=0$. The trajectories of $Q_g(W)dW^2$ include the segment T_1: $-g<W<g$ (which becomes void when $g=0$), a trajectory T_2 with both end points at $W=g$ whose closure separates $W=1$ from the other double poles and a trajectory T_3 symmetric to T_2 with respect to reflection in the imaginary W-axis. The trajectories T_1, T_2, T_3 plus their end points divide the W-sphere into three circle domains [11] K_1, K_2, K_3 containing respectively $W=\infty, 1, -1$. All trajectories in each circle domain are simple closed curves and separate the pole in that domain from the other two poles.

Let now E be any domain of finite connectivity at least equal to three and let $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ be three distinguished boundary components of E. In a separate paper [10] we will prove the following result: E can be mapped conformally into the W-sphere so that $\mathfrak{C}_i$, $i=1, 2, 3$, goes into a trajectory in K_i except that $\mathfrak{C}_2(\mathfrak{C}_3)$ may go into T_2 (T_3) together with an arc on the closure of T_1 (which may reduce to a point) plus possible arcs on the closure of T_3 (T_2) (the latter trajectory not covered by these arcs; such arcs may appear only if the image includes all of T_1), finally the remaining boundary components of E go into arcs on orthogonal trajectories of $Q_g(W)dW^2$ or into the union of several arcs on the closure of orthogonal trajectories.

Consider again the image D' of D in the circular ring $1<|w|<R'$ and D'', its reflection in $|w|=R'$. Let $\mathfrak{C}$ be the union of D', D'' and the open arc on $|w|=R'$ corresponding to A^*. In the situation described in the preceding paragraph let the boundary component of $\mathfrak{C}$ arising from A play the role of $\mathfrak{C}_1$, the circle $|w|=1$ the role of $\mathfrak{C}_2$, and the circle $|w|=(R')^2$ the role of $\mathfrak{C}_3$. In the canonical mapping of $\mathfrak{C}$ associated with the quadratic differential $Q_g(W)dW^2$, as a consequence of the associated uniqueness result and the symmetry property of $\mathfrak{C}$, we see that the image of $\mathfrak{C}$ will be symmetric in the imaginary W-axis. Thus if the image of $\mathfrak{C}_2$ includes a segment on T_1 this cannot reach to the origin. In particular, in the case $g=0$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ correspond to trajectories interior to K_2 and K_3 respectively. We will refer to the situation where $\mathfrak{C}_2$ and $\mathfrak{C}_3$ correspond to such interior trajectories for the present as the nonsingular case.

It is readily verified that as the value g, $0 \leqq g<1$, converges to a value g_0, $0 \leqq g_0<1$, the canonical mapping of $\mathfrak{C}$ associated with the value g converges to the canonical mapping of $\mathfrak{C}$ associated with the value g_0. Thus if the canonical domain associated with g_0 belongs to the nonsingular case the same is true for values of g in a neighborhood of g_0 relative to the segment $0 \leqq g<1$ and the set of values of g for which the nonsingular case obtains is an open set relative to this segment. In particular this situation holds in some interval $0 \leqq g<g'$ with $0<g' \leqq 1$. We will now see that $g'<1$. Indeed, in the contrary case, the trajectory corresponding to $\mathfrak{C}_2$ would converge to the point $W=1$ as g approached 1. If then the trajectory corresponding to $\mathfrak{C}_1$ were bounded from the segment $-1 \leqq W \leqq 1$ the metric $B|W-1|^{-1}|dW|$ would for a suitable constant B provide a metric admissible in the standard module problem for the class of curves joining $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathfrak{C}$ in which the area would be arbitrarily small, contradicting the fact that this module has a finite positive value. On the other hand, if the trajectory corresponding to $\mathfrak{C}_1$ tended to the segment $-1 \leqq W \leqq 1$, the metric $B|dW|$ would for a suitable constant B provide a metric admissible in the standard module problem for the class of curves joining $\mathfrak{C}_2$ and $\mathfrak{C}_3$ in $\mathfrak{C}$ in which the area would be arbitrarily small, again a contradiction. Thus $g'<1$. It is readily seen that, in the canonical mapping corresponding to g', $\mathfrak{C}_2$ and $\mathfrak{C}_3$ go respectively into the closures of T_2 and T_3.

Because of the symmetry property of $\mathfrak{C}$ it follows that D' and so D corresponds to the portion of the image of $\mathfrak{C}$ in $\Re W>0$. Moreover A corresponds to the boundary arc on the image of $\mathfrak{C}_1$ in $\Re W\geqq 0$ and A^* to the open segment intercepted on the imaginary axis by the image of $\mathfrak{C}$. Let the doubly-connected domain bounded by these and the image of $\mathfrak{C}_2$ be mapped conformally on the circular ring $1<|w|<R_1(g)$ in the case of the mapping corresponding to the parameter value g, $0\leqq g\leqq g'$. Let this be done so that the combined mapping from D into the w-plane belongs to $F(A; R_1(g))$ and that A goes into the arc $\alpha_1(g)$ for whose points $R_1(g)e^{i\theta}$, $-\theta_1(g)\leqq\theta\leqq\theta_1(g)$, $0<\theta_1(g)<\pi$. Let us denote this function by $\psi_1(z, g)$.

If $\mathfrak{Q}_1(w, g)dw^2$ denotes the quadratic differential in $1<|w|<R_1(g)$ which has a simple zero at the image $\pi(g)$ lying on $-R_1(g)\leqq w\leqq -1$ of the point g (becoming a double boundary zero in the extreme cases) and simple poles at the points $R_1(g)e^{\pm i\theta_1(g)}$, being positive on $|w|=1$ and on the arc $\alpha_1(g)$ and negative on the arc on $|w|=R_1(g)$ complementary to $\alpha_1(g)$, then the residual boundary components of D go into arcs on orthogonal trajectories of $\mathfrak{Q}_1(w, g)\cdot dw^2$ or into the union of several arcs on the closure of such orthogonal trajectories.

From this we readily verify that $R_1(0)=R_1$. Indeed let us map $1<|w|<R_1(0)$ into the circular ring $1<|w|<R^*$ so that $|w|=1$ goes into itself, the arc $\alpha_1(0)$ goes into $|w|=R^*$, the points $R_1(0)e^{\pm i\theta_1(0)}$ go into $w=-R^*$ and the open arc complementary to $\alpha_1(0)$ goes into the radial segment $-R^*<w\leqq -p^*$ described twice $(1<p^*<R^*)$. Then $\mathfrak{Q}_1(w, 0)dw^2$ transforms into the quadratic differential $-Bdw^2/w^2$ for a suitable positive constant B and thus the images of the residual boundary components of D go into radial slits. Hence by a familiar uniqueness result the mapping of D obtained by following $\psi_1(z, 0)$ with the transformation just indicated must coincide with $f(z, R_0)$. Thus $R^*=R_0$, $p^*=p$ and evidently $R_1(0)=R_1$. Moreover the function $\psi_1(z, 0)$ is identical with $f(z, R_1)$.

LEMMA 3. *If $f(z)\in F(A; R_1(g))$, $0\leqq g\leqq g'$, maps A into an arc on $|w|=R_1(g)$ of length l, then $l\geqq 2R_1(g)\theta_1(g)$, equality occurring if and only if $f(z)=e^{-i\chi}\psi_1(z, g)$, χ real.*

The orthogonal trajectories of $Q_g(W)dW^2$, $g>0$, include three with limiting end point at the point $W=g$, two of which tend to $W=\infty$ and the third to $W=1$. At the point $W=-g$ three further orthogonal trajectories have limiting end point, being obtained from the preceding by reflection in the line $\Re W=0$. These special orthogonal trajectories divide the W-sphere into three strip domains [11] $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{S}_3$. The orthogonal trajectories in $\mathfrak{S}_1$ run from $W=\infty$ to $W=\infty$ separating $W=1$ from $W=-1$. The orthogonal trajectories in $\mathfrak{S}_2$ $(\mathfrak{S}_3)$ run from $W=\infty$ to $W=1$ (-1). At each double pole the orthogonal trajectories have limiting tangential directions [11]. In the metric $|Q_g(W)|^{1/2}|dW|$ the intercepts on orthogonal trajectories in $\mathfrak{S}_1$ by the image of $\mathfrak{C}$ all have equal length, say a $(a>0)$, the intercepts on orthogonal trajectories in $\mathfrak{S}_2$ or $\mathfrak{S}_3$ by the image of $\mathfrak{C}$ all have equal length, say b, where $b>(1/2)a$, $0<g<g'$; $b=(1/2)a$, $g=g'$.

The result of Lemma 3 for $g=0$ follows from Lemma 2. Thus we may assume $g>0$. Consider then in D the following module problem. Let Γ_1 denote the class of open arcs on D running from A back to A and separating C_2 from A^*. Let Γ_2 denote the class of open arcs on D running from A to C_2. The subsidiary conditions are that

$$\int_{\gamma_1} \rho\,|dz| \geqq a, \qquad \gamma_1 \in \Gamma_1; \qquad \int_{\gamma_2} \rho\,|dz| \geqq b, \qquad \gamma_2 \in \Gamma_2.$$

Regarding the conformally equivalent problem in the image of D by $\psi_1(z, g)$ we see that the extremal metric is $|\mathfrak{Q}_1(w, g)|^{1/2}|dw|$ where $\mathfrak{Q}_1(w, g)dw^2$ is the quadratic differential obtained by transferring the quadratic differential $Q_g(W)dW^2$ to the w-plane by the mapping given above.

Then if we had $l\leqq 2R_1(g)\theta_1(g)$, for suitable real χ the image of A on $|w| =R_1(g)$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha_1(g)$. In the image of D by this function the metric $|\mathfrak{Q}_1(w, g)|^{1/2}|dw|$ would be admissible for our present module problem and give a value to the area less than or equal to the module. It would thus be the extremal metric. The standard argument [3] for this situation then gives $e^{i\chi}f(z) =\psi_1(z, g)$ completing the proof of the lemma.

Suppose that $R_1(g_1) =R_1(g_2) =\tilde{R}$ for $g_1\neq g_2$. Then the functions $\psi_1(z, g_1)$, $\psi_1(z, g_2)$ are identical by Lemma 3. Since also $\theta_1(g_1) =\theta_1(g_2)$ while the differentials $\mathfrak{Q}_1(w, g_1)dw^2$ and $\mathfrak{Q}_1(w, g_2)dw^2$ are not identical, $\pi(g_1)$ and $\pi(g_2)$ must be different. If $n>2$ two orthogonal trajectories of these respective differentials must have a common arc. It is then readily verified that this situation can occur only if the images of the residual boundary components of D all lie on the real axis in the w-plane. Further they must lie to the right of $\pi(g_1)$ and $\pi(g_2)$. Let the portion of $-\tilde{R}\leqq w<-1$ to the left of such boundary points be $-\tilde{R}\leqq w<-k$ where $k\geqq 1$ and $k=1$ if and only if $n=2$ or all images of residual boundary components of D lie on $1<w<\tilde{R}$. For any point τ on $-\tilde{R}\leqq \tau \leqq -k$ we can form a quadratic differential $\mathfrak{Q}'(w, \tau)dw^2$ with a simple zero at τ, simple poles at $\tilde{R}e^{\pm i\theta_1(g_1)}$, and whose sign on the boundary is the same as for $\mathfrak{Q}_1(w, g_1)dw^2$. By the uniqueness of canonical mappings corresponding to the parameter value g such a quadratic differential must coincide with $\mathfrak{Q}_1(w, g)dw^2$ for some value g and since, for distinct values of g having $R_1(g) =\tilde{R}$, the points $\pi(g)$ are distinct, the values τ on $-\tilde{R}\leqq \tau \leqq -k$ are in (1, 1) ordered correspondence with the values g on an interval $0\leqq g\leqq g''$ where $g''\leqq g'$, equality occurring if and only if $k=1$.

In this case $R_1(g)\equiv R_1$ for $0\leqq g\leqq g''$, this interval containing g_1 and g_2. This situation occurs if and only if D admits an anticonformal mapping leaving invariant each boundary component as a whole and interchanging the end points of A. If then $g''<g'$ we cannot have $R_1(g_1') =R_1(g_2')$ for

$g_1' \neq g_2'$, g_1' or g_2' in the interval $g'' < g \leqq g'$. If the above symmetry does not obtain we do not have $R_1(g_1) = R_1(g_2)$ for any distinct g_1, g_2 in $0 \leqq g \leqq g'$.

In either case, by Lemmas 2 and 3, $R_1(g)$ for $0 \leqq g \leqq g'$ cannot lie in the interval $R_0 \geqq R > R_1$. Since $R_1(g)$ varies continuously with g and $R_1(0) = R_1$, the value $R_1(g)$ decreases monotonically from R_1 to a value R_2, $R_1 \geqq R_2$, as g varies from 0 to g'. This monotonicity is strict and $R_1 > R_2$ apart from the exceptional cases indicated above. To every value R in the interval $R_1 \geqq R \geqq R_2$ there is a value g in $0 \leqq g \leqq g'$ such that $R_1(g) = R$. If $R_1(g_1) = R_1(g_2)$ then $\psi_1(z, g_1) = \psi_1(z, g_2)$. Thus we may denote the corresponding function $\psi_1(z, g)$ by $f(z, R)$ and also $\theta_1(g)$ by $\theta(R)$, $\alpha_1(g)$ by $\alpha(R)$ for these same values of R. Then we have

COROLLARY 1. *If $f(z) \in F(A; R)$, $R_1 \geqq R \geqq R_2$, maps A onto an arc on $|w| = R$ of length l then $l \geqq 2R\theta(R)$, equality occurring if and only if $f(z) = e^{-i\chi} f(z, R)$, χ real.*

This corollary is an immediate consequence of Lemma 3 and the definitions involved.

Next let us take a two-sheeted covering of D branched about C_1 and C_2. This can be mapped conformally on a plane domain D_1 of connectivity $2n-2$. Let its boundary components arising from C_1 and C_2 be denoted by B_1 and B_2. On B_1 there are four distinguished boundary elements P_1, P_2, P_3, P_4 in natural cyclic order arising from the end points of A. Let the sides P_1P_2, P_3P_4 be those arising from A. In the paper [10] mentioned previously we shall prove a result which implies the following: D_1 can be mapped on a subdomain D_2 of the rectangle in the Z-plane

$$-L < \Re Z < L, \qquad -M < \Im Z < M, \qquad L, M > 0$$

where the ratio $L:M$ is fixed by the conformal structure of D_1, B_1 goes into the perimeter of the rectangle with P_1, P_2, P_3, P_4 corresponding to $L-iM$, $L+iM$, $-L+iM$, $-L-iM$ respectively, B_2 goes into a rectangle with sides parallel to the real and imaginary Z-axes, the lengths of these sides having a preassigned ratio r, $0 \leqq r \leqq \infty$, and the remaining boundary components of D_1 go into slits parallel to the real axis. If $r=0$, B_2 goes into a vertical slit, if $r = \infty$, it goes into a horizontal slit.

D_1 has a symmetry property corresponding to the double covering in the z-plane and the corresponding property of D_2 is symmetry with respect to the origin $Z=0$. In particular the image of B_2 is a rectangle (possibly degenerate) with vertices $l-im$, $l+im$, $-l+im$, $-l-im$, $0 \leqq l < L$, $0 \leqq m < M$. Consequently if we take the portion of D_2 in $\Re Z \geqq 0$ and identify points on $\Re Z = 0$ symmetric with respect to the real axis we obtain a conformal object conformally equivalent to D. From the domain bounded by the two rectangles above we similarly obtain a doubly-connected conformal object in which the preceding one is imbedded. Mapping this on a circular ring $1 < |w| < R_2(r)$

$(1<R_2(r))$ we obtain for each r, $0\leqq r\leqq\infty$, a mapping $\psi_2(z, r)$ of D into such a ring, taking this mapping to lie in $F(A; R_2(r))$ and to be normalized so that A goes into an arc $\alpha_2(r)$ for whose points $R_2(r)e^{i\theta}$, $-\theta_2(r)\leqq\theta\leqq\theta_2(r)$, $0<\theta_2(r)<\pi$.

If we denote by $\mathfrak{Q}_2(w, r)dw^2$ the quadratic differential in $1<|w|<R_2(r)$ obtained by transfer under the above mapping of the quadratic differential $-dZ^2$, this will have simple poles at the points $R_2(r)e^{\pm i\theta_2(r)}$ and simple zeros at the points $e^{\pm i\theta^*_2(r)}$, $0\leqq\theta_2^*(r)\leqq\pi$, the latter coinciding to give a double zero at $w=-1$ when $r=0$ and at $w=1$ when $r=\infty$. The quadratic differential $\mathfrak{Q}_2(w, r)dw^2$ is positive on the image of A and on the arc on $|w|=1$ for whose points $e^{i\theta}$, $-\theta_2^*(r)<\theta<\theta_2^*(r)$, negative on the complementary arcs on $|w|=R_2(r)$, $|w|=1$. Finally the remaining boundary components of D go into arcs on orthogonal trajectories of $\mathfrak{Q}_2(w, r)dw^2$.

LEMMA 4. *If $f(z)\in F(A; R_2(r))$, $0\leqq r\leqq\infty$, maps A into an arc on $|w|=R_2(r)$ of length l then $l\geqq 2R_2(r)\theta_2(r)$ equality occurring if and only if $f(z)=e^{-i\chi}\psi_2(z, r)$, χ real.*

Consider in D the following module problem. Let Γ_1 denote the class of open arcs on D running from A back to A and separating C_2 from A^*. Let Γ_2 denote the class of open arcs on D running from A to C_2. The subsidiary conditions are that

$$\int_{\gamma_1}\rho|dz| \geqq 2L, \qquad \gamma_1\in\Gamma_1; \qquad \int_{\gamma_2}\rho|dz| \geqq L-l, \qquad \gamma_2\in\Gamma_2.$$

The quantities L, l are the dimensions of the rectangles occurring in the above canonical mapping corresponding to the parameter value r. In the realization of D as a domain with identifications in the Z-plane the extremal metric is clearly the Euclidean metric. Thus in the image of D by $\psi_2(z, r)$ the extremal metric is $|\mathfrak{Q}_2(w, r)|^{1/2}|dw|$.

Then if we had $l\leqq 2R_2(r)\theta_2(r)$, for suitable real χ the image of A on $|w|=R_2(r)$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha_2(r)$. In the image of D by this function the metric $|\mathfrak{Q}_2(w, r)|^{1/2}|dw|$ would be admissible for the module problem and give a value to the area less than or equal to the module. It would thus be the extremal metric. As before it then follows that $e^{i\chi}f(z)=\psi_2(z, r)$ and the lemma is proved.

It is readily verified that as the value r, $0\leqq r\leqq\infty$, converges to a value r_0 in this range the canonical mapping of D_1, with say L fixed, associated with the value r converges to the canonical mapping of D_1 associated with the value r_0. In particular $R_2(r)$ varies continuously with r. Suppose now that $R_2(r_1)=R_2(r_2)=\widehat{R}$ for $r_1\neq r_2$. Then the functions $\psi_2(z, r_1)$, $\psi_2(z, r_2)$ are identical by Lemma 4. On the other hand the quadratic differentials $\mathfrak{Q}_2(w, r_1)dw^2$, $\mathfrak{Q}_2(w, r_2)dw^2$ are not identical. Since $\theta_2(r_1)=\theta_2(r_2)$ by Lemma 4 we must have $\theta_2^*(r_1)\neq\theta_2^*(r_2)$. If $n>2$, two orthogonal trajectories of these respective differentials must have a common arc. It is readily verified that this can occur only if the images of the residual boundary components of D lie on the segment $1<w<\widehat{R}$ in the w-plane.

In this case we form for $0\leqq\theta\leqq\pi$ the quadratic differential $\mathfrak{Q}''(w, \theta)dw^2$ with simple poles at $\widehat{R}e^{\pm i\theta_2(r_1)}$ and simple zeros at $e^{\pm i\theta}$ (the latter coinciding in a double zero if $\theta=0$ or π) and exhibiting the same boundary behavior as $\mathfrak{Q}_2(w, r)dw^2$. Take on a two-sheeted covering of $1<|w|<\widehat{R}$ branched about $|w|=1$ and $|w|=\widehat{R}$ the function

$$Z = Z(w) = i\int^{w}(\mathfrak{Q}''(w, \theta))^{1/2}dw.$$

For suitable choice of the constant of integration the composition of this with $\psi_2(z, r_1)$ provides a canonical mapping of D_1 corresponding to some ratio $r(\theta)$ and $r(\theta)$ runs from 0 to ∞ as θ runs from π to 0. By the uniqueness property of this canonical mapping we must have $\psi_2(z, r)\equiv\psi_2(z, r_1)$ for all r, $0\leqq r\leqq\infty$. Thus $R_2(r)\equiv\widehat{R}$ for $0\leqq r\leqq\infty$. If $n=2$ this situation automatically occurs.

A similar argument applied to the quadratic differential $\mathfrak{Q}_1(w, g')dw^2$ shows that $R_2(0)=R_2$. In any case by Lemmas 3 and 4, $R_2(r)$ cannot lie in the range $R>R_2$. Thus either $R_2(r)\equiv R_2$ when we set $R_3=R_2$ or $R_2(r)$ decreases strictly monotonically from R_2 to a value R_3, $R_2>R_3$, as r increases from 0 to ∞. In either case there is to each value R with $R_2\geqq R\geqq R_3$ a value r such that $R_2(r)=R$. If $R_2(r)\equiv R_2$ then $\psi_2(z, r)\equiv f(z, R_2)$. Thus for every R with $R_2\geqq R\geqq R_3$ we may denote the function $\psi_2(z, r)$ by $f(z, R)$ where $R_2(r)=R$ and denote $\theta_2(r)$ by $\theta(R)$, $\alpha_2(r)$ by $\alpha(R)$. Then we have

COROLLARY 2. *If $f(z)\in F(A;\ R)$, $R_2\geqq R\geqq R_3$ maps A onto an arc on $|w|=R$ of length l then $l\geqq 2R\theta(R)$, equality occurring if and only if $f(z)=e^{-i\chi}\cdot f(z, R)$, χ real.*

This corollary is an immediate consequence of Lemma 4 and the definitions involved.

The next class of extremal mappings we treat are very much like those occurring for $R_1\geqq R\geqq R_2$ and for this reason we will not go into as great detail in discussing them. We take again the image D' of D in the circular ring $1<|w|<R'$ and D'' its reflection in $|w|=R'$. Let $\mathfrak{E}'$ be the union of D', D'' and the open arc obtained from the image of A by removing its end points. Let the boundary component of $\mathfrak{E}'$ arising from the closure of the image of A^* be $\mathfrak{C}_1'$, the circle $|w|=1$ be $\mathfrak{C}_2'$ and the circle $|w|=(R')^2$ be $\mathfrak{C}_3'$. Then, as is proved in [10], $\mathfrak{E}'$ admits a canonical conformal mapping associated with the quadratic differential $Q_0(W)dW^2$ where $\mathfrak{C}_i'$, $i=1, 2, 3$, goes into a trajectory in K_i, apart from the same exceptions as in the previous case, and the remaining boundary components of $\mathfrak{E}'$ go into arcs on trajectories of $Q_0(W)dW^2$ or into the union of several arcs on the closure of trajectories.

As a consequence of the symmetry of $\mathfrak{C}'$ under reflection in $|w| = R'$ and the uniqueness result for this canonical mapping the image of $\mathfrak{C}'$ will be symmetric in the imaginary axis. Thus if the image of $\mathfrak{C}_2'$ ($\mathfrak{C}_3'$) includes a segment on T_1 this cannot reach to the origin. As before we refer to the situation where $\mathfrak{C}_2'$ and $\mathfrak{C}_3'$ correspond to trajectories interior to K_2 and K_3 as the nonsingular case. It occurs in particular if $g=0$.

It is readily shown that there is a value g^*, $0<g^*<1$, such that for the canonical mappings of $\mathfrak{C}'$ corresponding to g with $0 \leqq g < g^*$ the nonsingular case occurs while for $g = g^*$ the images of $\mathfrak{C}_2'$ and $\mathfrak{C}_3'$ are the closures of T_2 and T_3. Because of the symmetry property of $\mathfrak{C}'$ it follows that D is conformally equivalent to the portion of the image of $\mathfrak{C}'$ in $\Re W>0$. Further A corresponds to the segment intercepted on the imaginary axis by the closure of the image of $\mathfrak{C}'$ and A^* to the open boundary arc on the image of $\mathfrak{C}_1'$ in $\Re W>0$. Let the doubly-connected domain bounded by these and the image of $\mathfrak{C}_2'$ be mapped conformally on the circular ring $1<|w|<R_3(g)$ in the case of the canonical mapping corresponding to the parameter value g. Let this be done so that the combined mapping from D to the w-plane belongs to $F(A;\, R_3(g))$ and that A goes into the arc $\alpha_3(g)$ for whose points $R_3(g)e^{i\theta}$, $-\theta_3(g) \leqq \theta \leqq \theta_3(g)$, $0<\theta_3(g)<\pi$. Let us denote the function giving this mapping by $\psi_3(z, g)$.

If $\mathfrak{Q}_3(w, g)dw^2$ denotes the quadratic differential in $1<|w|<R_3(g)$ obtained by transfer of $Q_g(W)dW^2$ under the mapping from the W-plane to this domain, it has a simple zero at the image $\pi'(g)$, lying on $1 \leqq w \leqq R_3(g)$, of the point g (becoming a double boundary zero in the extreme cases) and simple poles at the points $R_3(g)e^{\pm i\theta_3(g)}$, being positive on $|w|=1$ and on the open arc complementary to $\alpha_3(g)$ on $|w| = R_3(g)$ and negative on $\alpha_3(g)$. The residual boundary components of D go into arcs on trajectories of $\mathfrak{Q}_3(w, g)dw^2$ or into the union of several arcs on the closure of such trajectories.

LEMMA 5. *If $f(z) \in F(A;\, R_3(g))$, $0 \leqq g \leqq g^*$, maps A into an arc on $|w| = R_3(g)$ of length l then $l \geqq 2R_3(g)\theta_3(g)$, equality occurring if and only if $f(z) = e^{-i\chi}\psi_3(z, g)$, χ real.*

The trajectories of $Q_g(W)dW^2$ in the region between T_2 and the image of $\mathfrak{C}_2'$ are simple closed curves all having length $a'>0$ in the metric $|Q_g(W)|^{1/2} \cdot |dW|$ (when $g=g^*$ this set is empty and we may suppose $a'=0$). The remaining portions of trajectories in the part of the image of $\mathfrak{C}'$ in $\Re W>0$ all have length b', $b'>a'$, and run from $\Re W=0$ back to that line. Analogous remarks evidently hold for the trajectories of $\mathfrak{Q}_3(w, g)dw^2$ in $1<|w|<R_3(g)$.

Consider now in D the following module problem. Let Γ_1 denote the class of open arcs on D running from A back to A and separating C_2 from A^*. Let Γ_3 denote the class of simple closed curves on D separating C_1 and C_2. The subsidiary conditions are that

$$\int_{\gamma_1} \rho\,|dz| \geqq b', \qquad \gamma_1 \in \Gamma_1; \qquad \int_{\gamma_3} \rho\,|dz| \geqq a', \qquad \gamma_3 \in \Gamma_3.$$

Regarding the conformally equivalent problem in the image of D by $\psi_3(z, g)$ we see that the extremal metric is $|\mathfrak{Q}_3(w, g)|^{1/2}|dw|$. Then if we had $l \leqq 2R_3(g)\theta_3(g)$, for suitable real χ, the image of A on $|w| = R_3(g)$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha_3(g)$. In the image of D by this function the metric $|\mathfrak{Q}_3(w, g)|^{1/2}|dw|$ would be admissible for the module problem and give a value to the area less than or equal to the module. It would thus be the extremal metric. Again the standard argument gives $e^{i\chi}f(z) = \psi_3(z, g)$, completing the proof.

It follows as before that $R_3(g)$ varies continuously with g and that $R_3(g^*) = R_3$. Equality of $R_3(g)$ for distinct values of g occurs at most if $R_3(g)$ is constant on an interval $g^{**} \leqq g \leqq g^*$ with $g^{**} \geqq 0$. If $n = 2$, $g^{**} = 0$ and in this case only. If $g^{**} > 0$ the residual boundary components of D go into segments on the interval $1 < w < R_3$ to the right of $\pi'(g^{**})$. In these cases $\psi_3(z, g)$ is the same for all g in the above interval. In any case $R_3(g)$ decreases monotonically from R_3 to a value R_4 as g decreases from g^* to 0. The equality $R_3 = R_4$ occurs if and only if $n = 2$. To every value R in the interval $R_3 \geqq R \geqq R_4$ there is a value g in $0 \leqq g \leqq g^*$ such that $R_3(g) = R$. The unique corresponding function $\psi_3(z, g)$ is denoted by $f(z, R)$, $\theta_3(g)$ by $\theta(R)$, $\alpha_3(g)$ by $\alpha(R)$. Then we have as before

Corollary 3. *If $f(z) \in F(A; R)$, $R_3 \geqq R \geqq R_4$, maps A onto an arc on $|w| = R$ of length l then $l \geqq 2R\theta(R)$ equality occurring if and only if $f(z) = e^{-i\chi} \cdot f(z, R)$, χ real.*

The final class of extremal mappings to be treated have considerable resemblance to those occurring for $R_2 \geqq R \geqq R_3$ and we will again omit some details. We return to the domain $\mathfrak{E}$ and use the canonical mapping [10] of $\mathfrak{E}$ which has the following properties: $|w| = 1$ goes into $|\zeta| = 1$, $|w| = (R')^2$ goes into $|\zeta| = \tilde{R}$, the boundary component arising from A goes into a Jordan curve composed of two circular arcs (centre the origin) and two radial segments (a "logarithmic rectangle") such that the ratio of the logarithmic length of the circular arcs to that of the radial segments has a prescribed value s, $0 \leqq s \leqq \infty$, and the remaining boundary components of $\mathfrak{E}$ go into circular arcs with centre the origin.

As a consequence of the symmetry of $\mathfrak{E}$ in the circle $|w| = R'$ the image will be symmetric in the circle $|\zeta| = \tilde{R}^{1/2}$ and the portion of the image in $|\zeta| < \tilde{R}^{1/2}$ will be conformally equivalent to D. The above logarithmic rectangle will also be symmetric in this circle and its portion in $|\zeta| \leqq \tilde{R}^{1/2}$ will correspond to A. The open arc on $|\zeta| = \tilde{R}^{1/2}$ outside the logarithmic rectangle will correspond to A^*. Let the doubly-connected domain bounded by $|\zeta| = 1$ and the union of these arcs be mapped conformally on the circular ring

$1<|w|<R_4(s)$ so that the combined mapping from D to the w-plane belongs to $F(A; R_4(s))$ and so that A goes into the arc $\alpha_4(s)$ for whose points $R_4(s)e^{i\theta}$, $-\theta_4(s)\leqq\theta\leqq\theta_4(s)$, $0<\theta_4(s)<\pi$. We will denote this function by $\psi_4(z, s)$.

If $\mathfrak{Q}_4(w, s)dw^2$ denotes the quadratic differential in $1<|w|<R_4(s)$ obtained by transfer under the mapping from the ζ-plane to this domain of the quadratic differential $-d\zeta^2/\zeta^2$, this will have for $s\neq\infty$ simple poles at the points $R_4(s)e^{\pm i\theta_4(s)}$ and for $s\neq 0$, ∞ simple zeros at the points $R_4(s)e^{\pm i\theta_4^*(s)}$ with $0<\theta_4^*(s)<\theta_4(s)$; when $s=0$ the zeros coincide to give a double zero at $R_4(s)$ and when $s=\infty$ the zeros and poles cancel each other to give just the differential $-dw^2/w^2$. Further this quadratic differential is positive on $|w|=1$ and on the arcs of $|w|=R_4(s)$ where $-\theta_4^*(s)<\theta<\theta_4^*(s)$ and $\theta_4(s)<\theta<2\pi-\theta_4(s)$ and negative on the other arcs for $s\neq\infty$. The residual boundary components of D go into arcs on trajectories of $\mathfrak{Q}_4(w, s)dw^2$.

LEMMA 6. *If $f(z)\in F(A; R_4(s))$, $0\leqq s\leqq\infty$, maps A into an arc on $|w|=R_4(s)$ of length l then $l\geqq 2R_4(s)\theta_4(s)$, equality occurring if and only if $f(z)=e^{-i\chi}\psi_4(z, s)$, χ real.*

Let us treat first the case $s\neq\infty$. The trajectory of $\mathfrak{Q}_4(w, s)dw^2$ joining its zeros in $1<|w|<R_4(s)$ (or running from the double zero back to it for $s=0$) is readily seen to divide the image of D by $\psi_4(z, s)$ into two subdomains. In the subdomain adjacent to $|w|=1$ every trajectory of $\mathfrak{Q}_4(w, s)dw^2$ is a simple closed curve having length 2π in the metric $|\mathfrak{Q}_4(w, s)|^{1/2}|dw|$. In the other subdomain every trajectory is an open arc running from $\alpha_4(s)$ back to $\alpha_4(s)$, separating $|w|=1$ from the complement of $\alpha_4(s)$ on $|w|=R_4(s)$ and having a fixed length $c\leqq 2\pi$ in the metric $|\mathfrak{Q}_4(w, s)|^{1/2}|dw|$. The equality $c=2\pi$ occurs only when $s=0$.

Consider now in D the following module problem. Let Γ_1 denote the class of open arcs on D running from A back to A and separating C_2 from A^*. Let Γ_3 denote the class of simple closed curves on D separating C_1 and C_2. The subsidiary conditions are that

$$\int_{\gamma_1}\rho|dz| \geqq c, \qquad \gamma_1\in\Gamma_1; \qquad \int_{\gamma_3}\rho|dz| \geqq 2\pi, \qquad \gamma_3\in\Gamma_3.$$

Regarding the conformally equivalent problem in the image of D by $\psi_4(z, s)$ we see that the extremal metric is $|\mathfrak{Q}_4(w, s)|^{1/2}|dw|$. Then if we had $l\leqq 2R_4(s)\theta_4(s)$, for suitable real χ, the image of A on $|w|=R_4(s)$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha_4(s)$. In the image of D by this function the metric $|\mathfrak{Q}_4(w, s)|^{1/2}|dw|$ would be admissible for the module problem and give a value to the area less than or equal to the module. It would thus be the extremal metric. As before the standard argument gives $e^{i\chi}f(z)=\psi_4(z, s)$ completing the proof for $s\neq\infty$.

For $s=\infty$ the mapping $\psi_4(z, \infty)$ is seen at once to be the standard canonical mapping of D on a circular slit domain. In this case the class $F(A; R_4(\infty))$ contains only the functions $e^{-i\chi}\psi_4(z, s)$.

It follows as before that $R_4(s)$ varies continuously with s and that $R_4(0) = R_4$. Equality of values of $R_4(s)$ for distinct values of s occurs only if $n=2$ and then $R_4(s) \equiv R_4$ for $0 \leqq s \leqq \infty$, actually then $R_4 = R_1$. In this case $\psi_4(z, s)$ is the same for all s. Otherwise $R_4(s)$ decreases strictly monotonically from R_4 to a value R_5, $R_4 > R_5$, as s increases from 0 to ∞. If $n=2$ we set $R_5 = R_4$. To every value R in the interval $R_4 \geqq R \geqq R_5$ there is a value s in $0 \leqq s \leqq \infty$ such that $R_4(s) = R$. The unique corresponding function $\psi_4(z, s)$ is denoted by $f(z, R)$, $\theta_4(s)$ by $\theta(R)$, $\alpha_4(s)$ by $\alpha(R)$. Then we have as before

COROLLARY 4. *If $f(z) \in F(A; R)$, $R_4 \geqq R \geqq R_5$, maps A onto an arc on $|w| = R$ of length l then $l \geqq 2R\theta(R)$, equality occurring if and only if $f(z) = e^{-i\chi} \cdot f(z, R)$, χ real.*

We summarize the solution of the problem treated in this section in the following statement.

THEOREM 3. *The class $F(A; R)$ is empty for $R > R_0$ and for $R < R_5$. For R such that $R_0 \geqq R \geqq R_5$ there is an extremal function $f(z, R)$ minimizing the length of the image of A among functions in $F(A; R)$ and unique apart from rotations.*

The only part of this statement which may deserve further comment is that $F(A; R)$ is empty for $R < R_5$. This is an immediate consequence of the well known minimal property of the circular slit mapping. Otherwise Theorem 3 follows from Lemmas 1, 2 and Corollaries 1, 2, 3, 4.

Earlier results in this direction were obtained by Komatu [14] who gave in the case $n=2$ the extremal mappings for cases in which the class $F(A; R)$ was not empty but did not determine the range of values of R for which the latter condition held.

6. The results of the preceding section admit an immediate extension to the case where we replace the circle $|w| = 1$ as inner boundary of the image domain by the boundary of an arbitrary circularly symmetric continuum with connected complement. As a normalization we assume that this continuum contains the point $w=1$ and lies in $|w| \leqq 1$. Its characteristic property is then that its intersection with any circle $|w| = R$, $0 < R \leqq 1$, is either void, the entire circle, or an arc on the circle meeting the positive real axis and symmetric with respect to it. Let H denote the boundary of the continuum. As before D will be a domain of connectivity n, $n \geqq 2$, in the z-plane with boundary components $C_1, C_2, \cdots, C_n$ and on C_1 is given an arc A. We denote by $F_1(A; R)$ the class of functions regular and univalent in D, mapping D onto a subdomain of the doubly-connected domain $D(H, R)$ bounded by H and $|w| = R$, carrying A into an arc on $|w| = R$ and C_2 into H.

As R takes all values greater than one the module of $D(H, R)$ (for the class of curves separating its boundary components) takes each positive value once and only once. Thus $D(H, R)$ is conformally equivalent to just one circular ring $1 < |w| < h(R)$ with $h(R)$ a strictly increasing function of R. Let $\mathfrak{H}(w)$ denote the function mapping $1 < |w| < h(R)$ conformally onto

$D(H, R)$ so that $|w|=1$ goes into H and points on the positive real axis in $1<|w|<h(R)$ go into such points in $D(H, R)$. We shall suppose that H is not $|w|=1$ since this case has already been treated in §5.

For fixed β, $0<\beta<\pi$, let $\alpha(\beta, \phi, R)$, $0\leqq\phi<2\pi$, denote the arc on $|w|=h(R)$ whose points are given by $h(R)e^{i\theta}$, $\phi-\beta\leqq\theta\leqq\phi+\beta$. Let $\alpha^*(\beta, \phi, R)$ be the image of $\alpha(\beta, \phi, R)$ under $\mathfrak{H}$ and let $l(\beta, \phi, R)$ be its length. We now prove

LEMMA 7. $l(\beta, \phi, R)>l(\beta, 0, R)$, $\phi\neq 0$.

Indeed, cutting $1<|w|<h(R)$ along the radial segment on which $\theta=\phi$, we obtain a simply-connected domain. This becomes a quadrangle if we take as one pair of opposite sides the two halves of $\alpha(\beta, \phi, R)$ into which the latter is divided by the ray $\theta=\phi$. The module $M(\beta, R)$ of this quadrangle for the class of curves joining this pair of opposite sides is independent of the value of ϕ. Suppose we had $l(\beta, \phi, R)\leqq l(\beta, 0, R)$ for some $\phi\neq 0$. Let $Q(\beta, \phi, R)$ be the quadrangle which is the image under $\mathfrak{H}$ of the quadrangle corresponding to the value ϕ.

Comparing the present situation with that of [8, §3] we see that $Q(\beta, \phi, R)$ lies inside $|w|=R$ rather than outside which evidently is inessential. However this quadrangle has boundary points on $|w|=R$ other than those on the pair of opposite sides in question. Thus we must use a different definition of symmetrization than that given at the place indicated. This we do by taking the reflection of the quadrangle in $|w|=R$, the union of this with the quadrangle and the pair of opposite sides (without their end points) on $|w|=R$, symmetrizing the doubly-connected domain so obtained in the standard manner and taking the portion of the symmetrized domain inside $|w|=R$ as the symmetrization of the quadrangle, the pair of opposite sides of the latter on $|w|=R$ being the intersections of the symmetrized doubly-connected domain with $|w|=R$. With this convention the result of [8, Theorem 2] remains true.

Let then $\tilde{Q}(\beta, \phi, R)$ be the quadrangle obtained by circular symmetrization of $Q(\beta, \phi, R)$. Then by [8, Theorem 2] its module for the analogous problem is actually larger than $M(\beta, R)$, since under the conditions on H this symmetrization cannot just be a rotation. However if $l(\beta, \phi, R)\leqq l(\beta, 0, R)$ the extremal metric for the corresponding problem in $Q(\beta, 0, R)$ would be admissible in $\tilde{Q}(\beta, \phi, R)$. This is impossible and so $l(\beta, \phi, R)>l(\beta, 0, R)$ as stated.

Let us denote the function $\mathfrak{H}(f(z, h(R)))$ by $f_1(z, R)$ for those values of R for which $R_0\geqq h(R)\geqq R_5$. Let us denote this interval by $S_0\geqq R\geqq S_5$. Let the image of A under this function be the arc for whose points $Re^{i\theta}$ we have $-\Theta(R)\leqq\theta\leqq\Theta(R)$. Then we have

THEOREM 4. *The class $F_1(A;\, R)$ is empty for $R>S_0$ and for $R<S_5$. If $f(z)\in F_1(A;\, R)$, $S_0\geqq R\geqq S_5$, maps A into an arc on $|w|=R$ of length l then $l\geqq 2R\Theta(R)$, equality occurring if and only if $f(z)\equiv f_1(z, R)$.*

This follows at once from Theorem 3 and Lemma 7.

7. A problem closely related to that treated in §5 is obtained by relaxing the conditions on the family of functions considered. As before let D be a domain of connectivity n, $n \geqq 2$, in the z-plane with boundary components $C_1, C_2, \cdots, C_n$ and an arc A given on the component C_1, its complement relative to C_1 being denoted by A^*. Let us now denote by $F_2(A; R)$, $R>1$, the class of functions $f(z)$ regular and univalent in D, satisfying $1<|f(z)|<R$ for $z \in D$, carrying A into an arc on $|w|=R$ and such that the image of C_2 separates the image of D from $|w|<1$. It is clear that $F_2(A; R) \supset F(A; R)$. As we shall see, in some cases the function in $F(A; R)$ minimizing the length of the image of A does the same in $F_2(A; R)$.

LEMMA 8. *The class $F_2(A; R)$ is empty for $R<R_5$. If $f(z) \in F_2(A; R)$, $R_5 \leqq R \leqq R_3$, maps A into an arc on $|w|=R$ of length l then $l \geqq 2R\theta(R)$, equality occurring if and only if $f(z)=e^{-i\chi}f(z, R)$, χ real.*

Indeed the first statement of the lemma follows as in Theorem 3. If we had, for $R_5 \leqq R \leqq R_3$, $l \leqq 2R\theta(R)$ then for suitable real χ the image of A on $|w|=R$ under $e^{i\chi}f(z)$ would be contained in the arc $\alpha(R)$. For the appropriate extremal problem regarded in Lemma 5 or Lemma 6 the extremal metric taken in the image of D by the function $f(z, R)$ would then be admissible in the image of D under $e^{i\chi}f(z)$. The proof of the lemma then follows as before.

The function $f(z, R_3)$ maps D onto the circular ring $1<|w|<R_3$ with C_1 going into $|w|=R_3$, C_2 going into $|w|=1$, and the residual boundaries going into slits on trajectories of a quadratic differential with simple poles at $R_3e^{\pm i\theta(R_3)}$, with a double zero at $w=1$, positive on $|w|=1$ and the arc for whose points $R_3e^{i\theta}$, $\theta(R_3)<\theta<2\pi-\theta(R_3)$, negative on the arc for whose points $R_3e^{i\theta}$, $-\theta(R_3)<\theta<\theta(R_3)$. For any value $R>R_3$, the ring $1<|w|<R_3$ can be mapped conformally on the ring $1<|w|<R$ slit along the segment $1<w \leqq k(R)$, where $k(R)$ is greater than one and is a strictly increasing function of R. Let the function performing this mapping so that $|w|=R_3$ goes into $|w|=R$ and points on the positive real axis go into such points be denoted by $\lambda(w, R)$. Let $\lambda(R_3e^{i\theta(R_3)}, R)=Re^{i\Phi(R)}$. Let the arc on $|w|=R$ for whose points $Re^{i\theta}$, $-\Phi(R) \leqq \theta \leqq \Phi(R)$, be denoted by $\mathfrak{A}(R)$.

The function $\lambda(f(z, R_3), R)$ maps D into $1<|w|<R$ with C_1 going into $|w|=R$, the image of C_2 separating the image of D from $|w|<1$ and the residual boundary components going into slits on trajectories of a quadratic differential $\mathfrak{Q}^*(w, R)dw^2$ in $1<|w|<R$ with simple poles at $Re^{\pm i\Phi(R)}$, a double zero at $w=1$, positive on $|w|=1$ and the arc $\mathfrak{B}(R)$ for whose points $Re^{i\theta}$, $\Phi(R)<\theta<2\pi-\Phi(R)$, negative on the arc $\mathfrak{A}(R)$.

LEMMA 9. *If $f(z) \in F_2(A; R)$, $R>R_3$, maps A into an arc on $|w|=R$ of length l, then $l \geqq 2R\Phi(R)$ equality occurring if and only if $f(z)=e^{-i\chi}\lambda(f(z, R_3), R)$, χ real.*

The trajectories of the quadratic differential $\mathfrak{Q}^*(w, R)dw^2$ in $1<|w|<R$, apart from the segment $1<w<R$, are all open arcs running from $\mathfrak{A}(R)$ back to $\mathfrak{A}(R)$, separating $\mathfrak{B}(R)$ from $|w|=1$ and having a constant length which by suitable normalization of $\mathfrak{Q}^*(w, R)dw^2$ we may take to be 1.

Let us consider in D the following module problem. Let Γ be the class of open arcs on D running from A back to A and separating C_2 from A^*. The subsidiary condition is that

$$\int_\gamma \rho|dz| \geqq 1, \qquad \gamma \in \Gamma.$$

In the image of D by $w=\lambda(f(z, R_3), R)$ the extremal metric for this problem is given by $|\mathfrak{Q}^*(w, R)|^{1/2}|dw|$. The proof of Lemma 9 is now completed as in the previous cases.

We define

$$\begin{aligned} f_2(z, R) &= f(z, R), && R_5 \leqq R \leqq R_3, \\ &= \lambda(f(z, R_3), R), && R_3 < R. \end{aligned}$$

Then we summarize the solution of the problem treated in this section in the following statement.

THEOREM 5. *The class $F_2(A; R)$ is empty for $R<R_5$. For R such that $R\geqq R_5$ there is an extremal function $f_2(z, R)$ minimizing the length of the image of A among functions in $F_2(A; R)$ and unique apart from rotations.*

This result does not appear to be known even in the simple case $n=2$. In this case of course $R_3=R_5$.

The results of this section admit an extension to functions mapping D into a domain like $D(H, R)$ of §6 rather than into a circular ring in the same way as the results of §5 were extended in §6. Since this offers no novel considerations we will not give the explicit statements here.

8. In extremal problems the condition that a boundary component go into a circle is very similar to the condition that a distinguished interior point should go into a certain point with the distortion of the mapping having a given value there. Thus we may expect a close parallel between the problem of §5 and the one to be considered now. Let D be a domain of connectivity n, $n\geqq 1$, with boundary components $C_1, C_2, \cdots, C_n$, a distinguished interior point P and an arc A given on the boundary component C_1. Let us denote by $F(A, P; R)$ the class of functions $f(z)$ regular and univalent in D, satisfying $|f(z)|<R$ $(R>0)$ for $z\in D$, carrying A into an arc on $|w|=R$ and such that $f(P)=0$ while $|f'(P)|$ has a prescribed value, say κ. While the family $F(A, P; R)$ depends on the value κ chosen this dependence is essentially of a trivial nature and thus we do not exhibit it explicitly. We shall again seek to find a function in $F(A, P; R)$ for which the length of the image of A is minimal.

The solution of this problem so closely parallels the work of §5 that we shall confine ourselves to a description of the results. The proofs are readily given on the same lines as the earlier ones, the only modification being that we are dealing with the modulus of a derivative being given at a point rather than the image of a boundary component being given. Considerations of this type were already treated by Grötzsch.

First there is a value r_0 such that for R with $R>r_0$ the class $F(A, P; R)$ is empty. The extremal function $f(z, P, r_0)$ maps D on the circle $|w|<r_0$ slit along the segment $-r_0<w\leqq -q$ $(r_0>q>0)$ and along $n-1$ radial segments, the latter being the images of the boundary components C_i for $i>1$. The arc A corresponds to $|w|=r_0$ with its end points going into $w=-r_0$, the complement of A on C_1 goes into the segment $-r_0<w\leqq -q$ described twice. The only functions in $F(A, P; r_0)$ are $e^{i\chi}f(z, P, r_0)$, χ real.

Next there is an interval $r_0>R\geqq r_1$ $(r_0>r_1)$ such that for R with $r_0>R\geqq r_1$ there is an extremal function $f(z, P, R)$ in $F(A, P; R)$ mapping C_1 into $|w|=R$ together with a segment $-R<w\leqq q(R)$, $R\geqq q(R)>0$, described twice, $q(R)=R$ occurring only for $R=r_1$. The arc A goes into the arc on $|w|=R$ for whose points $Re^{i\theta}$, $-\Psi(R)\leqq\theta\leqq\Psi(R)$ with $0<\Psi(R)<\pi$. The residual boundary components of D go into arcs on trajectories of the quadratic differential

$$(w+R)^2dw^2/w^2(w-Re^{i\Psi(R)})(w-Re^{-i\Psi(R)}).$$

This is followed by an interval $r_1\geqq R\geqq r_2$. This interval reduces to a point when $n=1$ and may do so when $n>1$ if D possesses certain special symmetry properties similar to those considered in §5. For R with $r_1\geqq R\geqq r_2$ there is an extremal function $f(z, P, R)$ in $F(A, P; R)$ carrying C_1 into $|w|=R$. The arc A goes into the arc on $|w|=R$ for whose points $Re^{i\theta}$, $-\Psi(R)\leqq\theta\leqq\Psi(R)$ with $0<\Psi(R)<\pi$. The residual boundary components of D go into arcs on trajectories of the quadratic differential

$$(w+\pi(R))(w+R^2/\pi(R))dw^2/w^2(w-Re^{i\Psi(R)})(w-Re^{-i\Psi(R)})$$

or into the union of several arcs on the closure of such trajectories for $r_1\geqq R>r_2$. Here $R\geqq\pi(R)>0$, the equality $\pi(R)=R$ occurs only for $R=r_1$. For $R=r_2$ the residual boundary components of D go into arcs on trajectories of the quadratic differential $dw^2/w(w-r_2e^{i\Psi(r_2)})(w-r_2e^{-i\Psi(r_2)})$. The function $f(z, P, r_2)$ differs from the function $f_1(z)$ in Theorem 1 only by a constant positive factor.

Next comes an interval $r_2\geqq R\geqq r_3$. This reduces to a point if and only if $n=1$. For R with $r_2\geqq R\geqq r_3$ there is an extremal function $f(z, P, R)$ in $F(A, P; R)$ carrying C_1 onto $|w|=R$. The arc A goes into the arc on $|w|=R$ for whose points $Re^{i\theta}$, $-\Psi(R)\leqq\theta\leqq\Psi(R)$ with $0<\Psi(R)<\pi$. For $r_2>R\geqq r_3$ the residual boundary components go into arcs on trajectories of the quadratic differential

$$-(w-\tau(R))(w-R^2/\tau(R))dw^2/w^2(w-Re^{i\Psi(R)})(w-Re^{-i\Psi(R)})$$

or into the union of several arcs on the closure of such trajectories. Here $0<\tau(R)\leqq R$ with $\tau(R)=R$ occurring only for $R=r_3$. Naturally $f(z, P, r_2)$ is the same as before.

Finally there is an interval $r_3\geqq R\geqq r_4$. This reduces to a point if and only if $n=1$. For R with $r_3\geqq R\geqq r_4$ there is an extremal function $f(z, P, R)$ in $F(A, P; R)$ carrying C_1 onto $|w|=R$. The arc A goes into the arc on $|w|=R$ for whose points $Re^{i\theta}$, $-\Psi(R)\leqq\theta\leqq\Psi(R)$ with $0<\Psi(R)<\pi$. For $r_3>R>r_4$ (when $n\neq 1$) the residual boundary components go into arcs on trajectories of the quadratic differential

$$-(w - Re^{i\Psi^*(R)})(w - Re^{-i\Psi^*(R)})dw^2/w^2(w - Re^{i\Psi(R)})(w - Re^{-i\Psi(R)})$$

where $\Psi(R)>\Psi^*(R)>0$. The function $f(z, P, r_3)$ is the same as before while $f(z, P, r_4)$ gives a canonical circular slit mapping for D. The class $F(A, P; R)$ is empty for $R<r_4$.

THEOREM 6. *The class $F(A, P; R)$ is empty for $R>r_0$ and for $R<r_4$. If $f(z)\in F(A, P; R)$, $r_0\geqq R\geqq r_4$, maps A into an arc on $|w|=R$ of length l then $l\geqq 2R\Psi(R)$, equality occurring if and only if $f(z)=e^{-i\chi}f(z, P, R)$, χ real.*

9. We shall give now some applications of the preceding results to the case of a simply-connected domain which we take for simplicity to be the unit circle.

THEOREM 7. *Let $f(z)$ be regular and univalent for $|z|<1$, satisfying $f(0)=0$ and $|f(z)|<1$. Then if the mapping $w=f(z)$ carries an arc of length l on $|z|=1$ into an arc of length L on $|w|=1$ we have*

$$\sin\frac{L}{4}\geqq |f'(0)|^{-1/2}\sin\frac{l}{4}\cdot$$

Indeed, under our hypotheses there will be an extremal function $F(z)$ as given in §8 carrying an arc of length l on $|z|=1$ into an arc on $|w|=1$ (say of length L^*) and having $|F'(0)|=|f'(0)|$. We may suppose the extremal function carries the arc of $|z|=1$ for whose points $e^{i\theta}$, $-l/2\leqq\theta\leqq l/2$ onto the arc of $|w|=1$ for whose points $e^{i\theta}$, $-L^*/2\leqq\theta\leqq L^*/2$. Then this function is defined by

$$|f'(0)|\, z/(1 - z)^2 = w/(1-w)^2.$$

Thus we have

$$|f'(0)|\, e^{il/2}/(1 - e^{il/2})^2 = e^{iL^*/2}/(1 - e^{iL/2})^2$$

so that

$$|f'(0)|\ \sin^2(L^*/4) = \sin^2(l/4)$$

and since $0<l\leqq 2\pi$, $0<L^*\leqq 2\pi$,

$$\sin(L^*/4) = |f'(0)|^{-1/2}\sin(l/4).$$

Since L^* is the minimal length,

$$\sin(L/4) \geqq |f'(0)|^{-1/2}\sin(l/4)$$

as stated in the theorem. Evidently equality occurs only for the functions $e^{-i\chi}F(z)$, χ real.

This result was first proved by Komatu [13] using Löwner's parametric method, improving an earlier result due to Unkelbach [17; 18].

THEOREM 8. *Let $f(z)$ be regular and univalent for $|z|<1$, satisfy $f(0)=0$ and $|f(z)|<1$ and carry an arc of length l on $|z|=1$ into an arc on $|w|=1$. Then*

$$|f'(0)| \geqq \sin^2(l/4).$$

the image of $|z|<1$ covers the circle

$$|w| < 2\sin^{-2}(l/4) - 1 - 2\{\sin^{-4}(l/2) - \sin^{-2}(l/4)\}^{1/2}$$

and for $|z|=r$

$$r \geqq |f(z)| \geqq (1+r)^2\sin^{-2}(l/4)/2r - 1$$
$$- \{(1+r)^4\sin^{-4}(l/4)/4r^2 - (1+r)^2\sin^{-2}(l/4)/r\}^{1/2}.$$

All these bounds are best possible.

Indeed the image arc on $|w|=1$ can have length at most 2π thus by Theorem 7

$$1 \geqq |f'(0)|^{-1/2}\sin(l/4)$$

and

$$|f'(0)| \geqq \sin^2(l/4).$$

Now if $g(z)$ is regular and univalent in $|z|<1$ and satisfies $g(0)=0$, $|g(z)|<1$, $|g'(0)|=\kappa$ (naturally $\kappa\leqq 1$) the image of $|z|<1$ under g covers the circle

$$|w| < 2\kappa^{-1} - 1 - 2(\kappa^{-2} - \kappa^{-1})^{1/2}$$

as is well known. An elementary calculation shows that the radius of this circle increases as κ increases thus the circle covered by the image of $|z|<1$ under f in the theorem will be smallest when $|f'(0)|$ is smallest, i.e. when $|f'(0)|=\sin^2(l/4)$. In this case the precise circle covered is

$$|w| < 2\sin^{-2}(l/4) - 1 - 2\{\sin^{-4}(l/4) - \sin^{-2}(l/4)\}^{1/2}$$

and hence this circle is always covered, as stated.

Finally for g as in the preceding paragraph we have for $|z|=r$,

$$|g(z)| \geqq (1+r)^2\kappa^{-1}/2r - 1 - \{(1+r)^4\kappa^{-2}/4r^2 - (1+r)\kappa^{-1}/r\}^{1/2}.$$

Once again the value of this bound increases as κ increases thus it will be smallest for $|f'(0)| = \sin^2(l/4)$. This gives

$$|f(z)| \geqq (1+r)^2 \sin^{-2}(l/4)/2r - 1 - \left\{(1+r)^4 \sin^{-4}(l/4)/4r^2 - (1+r)^2 \sin^{-2}(l/4)/r\right\}^{1/2}.$$

The statement concerning the upper bound for $|f(z)|$ is evident as is the fact that the bounds are best possible.

10. Finally we consider a problem in which a symmetrization has to be performed in the course of the proof. Consider the class $F(A; r, R)$, $0<r<1$, of functions $f(z)$, regular and univalent in the unit circle $|z|<1$, satisfying $f(0)=0, f(r)=r$, $|f(z)|<R$ $(R>0)$ and carrying the arc A on $|z|=1$ for whose points $e^{i\theta}$, $-a\leqq\theta\leqq a$, $0<a<\pi$, into an arc on $|w|=R$. As before we will determine when $F(A; r, R)$ is empty and, when it is not, we will determine the function in it for which the length of the image of A is smallest. It should be remarked that the analogous problem can be treated for a domain of connectivity n, $n\geqq 1$, with two distinguished interior points, with an arc given on one boundary component and having an appropriate symmetry property. While the more general problem has some interesting novel features we confine ourselves here to the case above for simplicity.

For $1\geqq\tau\geqq\sin^2(a/2)$ let us consider the function $w=\phi(z; r, \tau)$ determined by the following conditions

$$\frac{\tau z}{(1-z)^2} = \frac{W}{(1-W)^2},$$

$$w = r\left\{1 + \frac{(1-r)^2}{2\tau r} + \left(\frac{(1-r)^4}{4\tau^2 r^2} + \frac{(1-r)^2}{\tau r}\right)^{1/2}\right\} W.$$

It is readily verified that this function belongs to the class $F(A; r, R)$ for

$$R = r\left\{1 + \frac{(1-r)^2}{2\tau r} + \left(\frac{(1-r)^4}{4\tau^2 r^2} + \frac{(1-r)^2}{\tau r}\right)^{1/2}\right\}$$

and we denote it by $f(z; r, R)$. As τ decreases from 1 to $\sin^2(a/2)$, the corresponding value for R increases strictly monotonically from 1 to

$$r\left\{1+(1-r)^2 \sin^{-2}(a/2)/2r+((1-r)^4 \sin^{-4}(a/2)/4r^2+(1-r)^2 \sin^{-2}(a/2)/r)^{1/2}\right\}.$$

We denote this latter quantity by $B(r, a)$.

LEMMA 10. *The class $F(A; r, R)$ is empty for $R<1$ and for $R>B(r, a)$.*

That $F(A; r, R)$ is empty for $R<1$ follows at once by Schwarz's Lemma. To prove the other statement we consider the following module problem. Let Γ be the class of open arcs on $|z|<1$ running from A back to A and separating $z=r$ from $z=0$ and the complementary arc to A on $|z|=1$. The subsidiary condition is that

$$\int_\gamma \rho |dz| \geqq 1, \qquad \gamma \in \Gamma.$$

In the image of $|z|<1$ by $w=f(z;r,B(r,a))$ the extremal metric is given by

$$|d\zeta| = K|w(w-r)(w-r^{-1}B^2(r,a))|^{-1/2}|dw|$$

for a suitable constant $K>0$. Suppose there were a function $f(z)$ in $F(A;r,R)$ for $R>B(r,a)$. Then in the image of $|z|<1$ by f the metric $\rho|dw|$ obtained by taking $\rho(w)=K|w(w-r)(w-r^{-1}B^2(r,a))|^{-1/2}$ in $|w|\leqq B(r,a)$, $\rho(w)\equiv 0$ in $B(r,a)<|w|<R$ would be admissible for the transformed problem and give a value to the area at most equal to the module. Evidently $\rho|dw|$ could not be the extremal metric in that image which provides the desired contradiction.

Let the function $f(z;r,R)$ map A into the arc $\alpha(r,R)$ on $|w|=R$ for whose points $Re^{i\theta}$, $-\theta(r,R)\leqq\theta\leqq\theta(r,R)$.

LEMMA 11. *If* $f(z)\in F(A;r,R)$, $1\leqq R\leqq B(r,a)$, *maps* A *into an arc on* $|w|=R$ *of length* l, then $l\geqq 2R\theta(r,R)$, *equality occurring if and only if* $f(z)\equiv f(z;r,R)$.

Slitting $|z|<1$ from 0 to -1 along the negative real axis and from r to 1 along the positive real axis we obtain a simply-connected domain which becomes a quadrangle if we designate as one pair of opposite sides the two halves of A on which $-a\leqq\theta\leqq 0$ and $0\leqq\theta\leqq a$ respectively. The module of this quadrangle for the class of open arcs joining this pair of opposite sides is evidently equal to the module defined by the module problem of Lemma 10. Let the image of this quadrangle by $f(z)$ be denoted by P, its module by $m(\mathrm{P})$. Let the quadrangle obtained from this by symmetrization according to the prescription given in Lemma 7 be denoted by $\tilde{\mathrm{P}}$. Then the module of $\tilde{\mathrm{P}}$ corresponding to $m(\mathrm{P})$ is strictly greater than $m(\mathrm{P})$ unless $f(z)\equiv f(z;r,R)$ since, except in the latter case, $\tilde{\mathrm{P}}$ is not obtained from P by a rigid rotation.

The extremal metric for the module problem of Lemma 10 in the image of $|z|<1$ by $f(z;r,R)$ is given by

$$K|w+1|\,|w(w-r)(w-r^{-1}R^2)(w-Re^{i\theta(r,R)})(w-Re^{-i\theta(r,R)})|^{-1/2}|dw|$$

for a suitable constant $K>0$. Suppose now we had $l\leqq 2R\theta(r,R)$. Then the preceding metric would be admissible in the problem for determining the module of $\tilde{\mathrm{P}}$ which would then be at most equal to $m(\mathrm{P})$. This shows that $l>2R\theta(r,R)$ unless $f(z)\equiv f(z;r,R)$.

THEOREM 9. *The class* $F(A;r,R)$ *is empty for* $R<1$ *and for* $R>B(r,a)$. *For* R *with* $1\leqq R\leqq B(r,a)$ *there is a unique extremal function* $f(z;r,R)$ *minimizing the length of the image of* A *among functions in* $F(A;r,R)$.

In conclusion let us say that there are in addition to the problems solved or mentioned here a wide variety of interesting questions which can be treated by our methods. We have tried only to manifest a few representative problems to indicate the scope of these methods. Let us mention in particular that there are numerous corresponding results in which Steiner symmetrization rather than circular symmetrization intervenes.

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