THE SYMMETRIC DERIVATIVE ON THE \((k-1)\)-DIMENSIONAL HYPERSPHERE

BY

VICTOR L. SHAPIRO

1. Introduction. Let \(x\) be a point on the unit \((k-1)\)-dimensional hypersphere \(\Omega\) in Euclidean \(k\)-space, \(k \geq 3\), and let \(\mu\) be a completely additive set function of bounded variation defined on the Borel sets of \(\Omega\). Let \(D(x, h)\) represent the spherical cap on \(\Omega\) obtained by intersecting \(\Omega\) with a sphere whose center is \(x\) and radius is \(2 \sin h/2\), and let \(|D(x, h)|\) be the \((k-1)\)-dimensional volume of \(D(x, h)\). Then \(\mu\) will be said to have a symmetric derivative at \(x\), designated by \(\mu_s(x)\), if \(|D(x, h)|^{-1}\mu[D(x, h)]\) tends to \(\mu_s(x)\) as \(h\) tends to zero.

Let \(S[\mu] = \sum_{n=0}^{\infty} Y_n(x)\) be the Stieltjes series of surface harmonics defined by \(\mu\). We shall show in this paper that if \(\mu_s(x_0)\) exists and is finite and \(\mu\) satisfies the global condition \(|\mu| [D(x_0', \epsilon)] = 0\) for some \(\epsilon > 0\), where \(x_0'\) is the point diametrically opposite to \(x_0\) and \(|\mu|\) is the total variation of \(\mu\), then \(S\) is summable \((C, \delta)\), \(\delta > (k-2)/2 + 1\), to \(\mu_s(x_0)\). This result generalizes the well-known result for Fourier-Stieltjes series where \(\delta > 1\), see [8, p. 55]. In case the global condition is not satisfied, we obtain that \(S[\mu]\) is summable \((C, \eta)\) to \(\mu_s(x_0)\) where \(\eta > k-2\) for \(k \geq 4\) and \(\eta > 3/2\) for \(k = 3\).

In the special case when \(\mu\) is absolutely continuous and \(Y_n(x_0) = O(n^{-1})\), we shall show that a necessary and sufficient condition that \(S[\mu]\) converges at \(x_0\) to the finite value \(\beta\) is that \(\mu_s(x_0)\) exists and equals \(\beta\). This fact generalized a result previously obtained by Hardy and Littlewood [5, p. 229] for Fourier series.

2. Definitions and notation. \(\lambda\) will always designate the value \((k-2)/2\), and \(P_n^\lambda\) will designate the Gegenbauer (ultraspherical) polynomials defined by the equation,

\[
(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(\cos \theta).
\]

With the help of these functions, we can associate to every additive set function defined on the Borel sets of \(\Omega\) and of bounded variation there, a sequence of surface harmonics by means of the equation

\[\text{Presented to the Society, December 29, 1955; received by the editors November 4, 1955.}
\]

(\(\dagger\)) National Science Foundation Fellow.
where $\gamma$ is the angle between $x$ and $y$, that is if $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$,
$$
\cos \gamma = (x, y) = x_1y_1 + \cdots + x_ky_k.
$$

As shown in [3, Chap. 11], $r^n Y_n(x)$ gives rise to an homogeneous harmonic polynomial of degree $n$ in Euclidean $k$-space. We define $S[d\mu] = \sum_{n=0}^{n} Y_n(x)$ and call this the Stieltjes-series of surface harmonics associated to $\mu$.

By the terminology $\sum_{n=0}^{n} Y_n(x)$ is $(C, \alpha)$ summable to a given value, we mean the usual Cesaro summability defined for example in [8, Chap. 3].

We note that $D(x_0, h)$ defined in the introduction is the set
$$
\{ x, (x, x_0) \geq \cos h \}.
$$

$[\lambda]$ will mean the integral part of $\lambda$ and $|\mu|(E)$ will stand for the total variation of $\mu$ in $E$.

3. Statement of main results. We shall prove the following theorem for Stieltjes-series of surface harmonics:

**Theorem 1.** Let $\mu$ be a completely additive set function defined on the Borel sets of $\Omega$ and of bounded variation on $\Omega$, and let $S[d\mu] = \sum_{n=0}^{n} Y_n(x)$. Suppose $\mu(x_0)$ exists and is finite. Then $S[d\mu]$ is $(C, \eta)$, $\eta > \max (3/2, k - 2)$, summable to $\mu(x_0)$. If, furthermore, $\mu$ satisfies the condition that $|\mu| [D(x_0', \epsilon)] = 0$ for some $\epsilon > 0$, where $x_0'$ is diametrically opposite to $x$, then $S[d\mu]$ is $(C, \delta)$ summable to $\mu(x_0)$, $\delta > \lambda + 1$.

Concerning the convergence of series of surface harmonics, we shall prove the following theorem:

**Theorem 2.** Let $f$ be an integrable function on $\Omega$ and define $\mu(E)$, for $E$ a Borel set on $\Omega$, by $\mu(E) = \int_{E} f(x) d\Omega(x)$ where $d\Omega(x)$ is the $(k-1)$-dimensional volume element on $\Omega$. Let $S[f] = \sum_{n=0}^{n} Y_n(x)$, and suppose that $Y_n(x_0) = 0(n^{-1})$. Then a necessary and sufficient condition that $S[f]$ converges at $x_0$ to $\beta$ is that $\mu(x_0) = \beta$.

**Remark 1.** By [1, Theorem 2], it is easily seen that the condition $|\mu| [D(x_0', \epsilon)] = 0$ for some $\epsilon > 0$ in Theorem 1 can be replaced by the condition that in $D(x_0', \epsilon)$, $\mu$ is absolutely continuous with $\mu(E) = \int_{E} f(y) d\Omega(y)$ and that
$$
\int_{D(x_0', \epsilon)} \frac{|f(y)|}{[1 - (x_0', y)]^{1/2}} d\Omega(y) < \infty.
$$

4. Fundamental lemmas. Before proceeding with the proof of these theorems we shall prove some lemmas. By $S_{\lambda}^\lambda (\cos \theta)$, we shall designate the sum
\[ S_n^{\alpha,\lambda}(\cos \theta) = \sum_{j=0}^{n} (j + \lambda) P_j^\lambda(\cos \theta) A_{n-j}^{\alpha} \]

where \( \sum_{n=0}^{\alpha} A_n x^n = (1 - x)^{-(\alpha + 1)} \) with \( \alpha > 1 \). By \( [g(\cos \theta)]' \), we shall mean \( dg(\cos \theta)/d\theta \).

**Lemma 1.** \( |[S_n^{\alpha,\lambda}(\cos \theta)]'| \leq K(\epsilon) n^{\lambda+1}(\alpha + \lambda + 1) \) for \( n^{-1} \leq \theta \leq \pi - \epsilon \) and \( [\lambda] + 2 > \alpha > [\lambda] + 1 \), where \( K(\epsilon) \) is a constant depending on \( \epsilon \) but not on \( n \).

We first observe that

\[
\sum_{n=0}^{\infty} (n + \lambda) [P_n^\lambda(\cos \theta)] r^n = -\frac{2\lambda(\lambda + 1)r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+2}}. 
\]

Let us suppose that \( \lambda > 1 \). Then since

\[
\frac{1}{(1 - r)^{\alpha+1}} \frac{(1 - r^2)r \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+2}} = \frac{1}{(1 - r)^{\alpha}} \frac{(1 - r^2) r \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+1}} \frac{1}{(1 - r)} \frac{1}{(1 - 2r \cos \theta + r^2)}
\]

we obtain from (1) that

\[
[S_n^{\alpha,\lambda}(\cos \theta)]' = K_1 \sum_{j=0}^{n} [S_j^{\alpha-1,\lambda-1}(\cos \theta)]' T_{n-j}(\cos \theta)
\]

where \( T_n(\cos \theta) = \sum_{j=0}^{n} P_j^1(\cos \theta) \) and \( K_1 \) is a constant. But

\[ P_j^1(\cos \theta) = \sin (j+1)\theta/\sin \theta; \]

therefore,

\[
T_n(\cos \theta) = \frac{\cos \theta/2 - \cos (2n + 3)\theta/2}{2 \sin \theta \sin \theta/2}.
\]

So if we assume that \( [S_n^{\alpha-1,\lambda-1}(\cos \theta)]' \) satisfies the conclusion of the lemma, we see from (3) and (4) that

\[
| [S_n^{\alpha,\lambda}(\cos \theta)]' | \leq K(\epsilon) \theta^{-(\alpha + \lambda - 1) - 2} \sum_{j=0}^{n} \lambda.
\]

Therefore in order to prove the lemma we need prove it only in the special cases \( \lambda = 1/2 \) and \( \lambda = 1 \).

To do this we introduce

\[ S_n^{\alpha,0}(\cos \theta) = A_n^\alpha/2 + \sum_{j=1}^{n} (\cos j\theta) A_{n-j}^{\alpha} \]
and observe by [8, p. 56] that

\[ |S_{n}^{\alpha,0}(\cos \theta)|' \leq K(\epsilon)n^{(\alpha+1)} \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon, 1 < \alpha < 2. \]

For the case \( \lambda = 1/2 \), we rewrite (2) in the form

\[
\frac{1}{(1 - r^{2})r \sin \theta} \frac{(1 - r^{2})r \sin \theta}{(1 - r^{2})r \sin \theta} = \frac{1}{(1 - r^{2})r \sin \theta} \frac{(1 - r^{2})r \sin \theta}{(1 - r^{2})r \sin \theta} = \frac{1}{(1 - r^{2})r \sin \theta} \frac{(1 - r^{2})r \sin \theta}{(1 - r^{2})r \sin \theta}
\]

and obtain that

\[
[S_{n}^{\alpha,1/2}(\cos \theta)]' = K_{2} \sum_{j=0}^{n} [S_{j}^{\alpha,0}(\cos \theta)]' P_{n-j}^{1/2}(\cos \theta)
\]

where \( K_{2} \) is a constant. From [7, p. 160], \( |P_{n/2}(\cos \theta)| \leq (n \sin \theta)^{-1/2} \). So we conclude from (5) and (6) that

\[ |S_{n}^{\alpha,1/2}(\cos \theta)|' \leq K(\epsilon)^{3/2}(\alpha+3/2) \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon \text{ and } 1 < \alpha < 2 \]

which proves the lemma in the special case \( \lambda = 1/2 \).

For the case \( \lambda = 1 \), let us assume that \( 2 < \alpha < 3 \). Then by (2), we obtain that

\[ [S_{n}^{\alpha,1}(\cos \theta)]' = K_{3} \sum_{j=0}^{n} [S_{j}^{\alpha,0}(\cos \theta)]' T_{n-j}(\cos \theta) \]

and (4) and (5) give us that

\[ |[S_{n}^{\alpha,1}(\cos \theta)]'| \leq K(\epsilon)n^{2}(\alpha+2) \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon, \]

and the proof to the lemma is complete.

**Lemma 2.** \( |S_{n}^{\alpha}(\cos \theta)|' \leq Kn^{\alpha+2}+2 \) for \( 0 \leq \theta \leq n^{-1} \) and \( \alpha \geq 0 \), where \( K \) is a constant independent of \( n \).

To prove this lemma we write

\[
\frac{1}{(1 - r^{2})r \sin \theta} \frac{(1 - r^{2})r \sin \theta}{(1 - r^{2})r \sin \theta} = \frac{1}{(1 - r^{2})r \sin \theta} \frac{(1 - r^{2})r \sin \theta}{(1 - r^{2})r \sin \theta}
\]

and obtain that

\[
[S_{n}^{\alpha,\lambda}(\cos \theta)]' = K_{1} \sum_{i=0}^{n} [S_{i}^{\alpha,\lambda-1/2}(\cos \theta)]' P_{n-i}^{1/2}(\cos \theta),
\]
$S^\alpha_0(\cos \theta)$ being defined as in Lemma 1.

If we assume that the conclusion to the lemma holds for $S^{\alpha-1/2}_n(\cos \theta)$, we can use the fact that $|P^{\alpha/2}_n(\cos \theta)| \leq 1$ and obtain from (7) that

$$| [S^{\alpha}_n(\cos \theta)]' | \leq K_2 \sum_{j=0}^{n} j^{\alpha+2\lambda+1} \leq Kn^{\alpha+2\lambda+2}.$$

So to prove the lemma, we need only show that the conclusion of the lemma holds for $S^{\alpha}_n(\cos \theta)$. But

$$| [S^{\alpha}_n(\cos \theta)]' | = \left| \sum_{j=0}^{n} j \sin j \theta A^{\alpha}_{n-j} \right| \leq Kn^{\alpha+2},$$

and the proof is complete.

We now state a lemma of Kogbetliantz [6, p. 139].

**Lemma 3.** For $-1 < \alpha \leq k - 1$ and for $0 \leq \theta \leq \pi$

$$| S^{\alpha, \lambda}_n(\cos \theta) | \leq K(n + 1)^{\lambda-2} (\sin \theta/2)^{-(\alpha+1)}.$$

We next define the expression $B^\lambda_n(h)$ by

$$B^\lambda_n(h) = \int_{0}^{h} P^\lambda_n(\cos \theta) (\sin \theta)^{2\lambda} d\theta / P^\lambda_n(1) \int_{0}^{h} (\sin \theta)^{2\lambda} d\theta$$

and prove the following lemmas:

**Lemma 4.** $|B^\lambda_n(h)| \leq K(hn)^{-\lambda}$ for $0 \leq h \leq \pi/2$ where $K$ is a constant independent of $n$.

**Lemma 5.** $|B^\lambda_{n+1}(h) - B^\lambda_n(h)| \leq Kh^2n$ for $0 \leq h \leq \pi$ where $K$ is a positive constant independent of $n$.

To prove Lemma 4 we have by [7, p. 80, pp. 166 and 167] that

$$| P^\lambda_n(\cos \theta) | \leq K\theta^{\lambda-n} \frac{\lambda-1}{n!} \quad \text{for } 0 \leq \theta \leq \pi/2,$$

$$P^\lambda_n(1) = \binom{n + 2\lambda - 1}{n},$$

$$| P^\lambda_n(\cos \theta) | \leq P^\lambda_n(1).$$

We conclude that the left side of the inequality in Lemma 4 is majorized by a constant multiple of

$$n^{\lambda-1} \int_{0}^{h} \theta^{-\lambda} \theta^{2\lambda} d\theta / n^{2\lambda-1} \int_{0}^{h} \theta^{2\lambda} d\theta,$$

which gives the right side of the inequality, and the lemma is proved.
To prove Lemma 5, we let $g(n)$ equal the square of the normalizing coefficient of $P_n^\lambda(r)$, that is \cite[p. 174]{3} 

\begin{equation}
\int_{-1}^{1} \left[ P_n^\lambda(r) \right]^2 (1 - r^2)^{\lambda-1/2} dr = \frac{\pi^{1/2} \Gamma(\lambda + 1/2)}{(n + \lambda) \Gamma(\lambda)} P_n^\lambda(1)
\end{equation}

and obtain from the Christoffel-Darboux formula \cite[p. 159]{3} that 

\begin{equation}
\frac{P_n^\lambda(\cos \theta) P_{n+1}^\lambda(1) - P_{n+1}^\lambda(\cos \theta) P_n^\lambda(1)}{2 \sin^2 \theta/2} = 2g(n) n + \lambda \sum_{j=0}^{n} [g(j)]^{-1} P_j^\lambda(\cos \theta) P_j^\lambda(1).
\end{equation}

From (8) and (9), we see that the left side of the inequality in Lemma 5 is majorized by a constant multiple of 

\[ \max_{0 \leq \theta \leq \lambda} \left| P_{n+1}^\lambda(1) P_n^\lambda(\cos \theta) - P_n^\lambda(1) P_{n+1}^\lambda(\cos \theta) \right| \]

But by (9), (10), and (11) this expression in turn is majorized by a constant multiple of 

\[ \frac{h^2}{n^{4\lambda-2}} \sum_{j=0}^{n} j j^{2\lambda-1} = h^2 O(n), \]

which is the right side of the inequality in Lemma 5, and the proof of the lemma is complete.

5. Proof of Theorem 1. Let us first suppose that $\mu(E) = 0$ for $E$ a Borel set contained in $D(x_0', \epsilon)$ where $\epsilon$ is a positive number between 0 and $\pi/2$. Then with $\delta > \lambda + 1$ but less than $[\lambda] + 2$, we have 

\begin{equation}
S^\delta_n(x) = \sum_{j=0}^{n} Y_j(x) A_{n-j}^\delta = \frac{\Gamma(\lambda)}{2\pi^{\lambda+1}} \int_{\Omega_D(x_0', \epsilon)} S_n^{\delta,\lambda}(\cos \gamma) d\mu(y)
\end{equation}

with $\cos \gamma = (x, y)$ and $S_n^{\delta,\lambda}(\cos \gamma)$ defined in the beginning of §4.

Next, we introduce the continuous function $f(x)$ which has the following properties:

(i) $f(x) = 0$ for $x$ in $D(x_0', \epsilon)$,
(ii) $f(x_0) = \mu_\lambda(x_0)$, 
(iii) $\int_{D_\delta}(x) d\Omega(x) = \mu [D(x_0, \pi - \epsilon)]$, 

and set $\mu(E) = \int_{\Omega_D(x_0, \epsilon)} f(x) d\Omega(x)$ for $E$ a Borel set on $\Omega$. By \cite[Theorem 2]{1}, $R_n^{\delta}(x_0)/A_n^\delta - \mu_\lambda(x_0)$ for $\delta > \lambda$ when $R_n$ is defined by the last expression in (12) with $\mu$ replaced by $\mu_\lambda$. Consequently to prove the second part of this theorem it is sufficient to show that
(13) \( \frac{2\pi^{\lambda+1}}{\Gamma(\lambda)} \left[ R_n^\lambda(x_0) - S_n^\lambda(x_0) \right] = \int_{\Omega - D(x_0', \epsilon)} S_n^{\delta, \lambda}\left[ (x_0, y) \right] d\mu_2(y) = o(n^\delta) \)

with \( \lambda + 2 > \delta > \lambda + 1 \) and \( \mu_2 = \mu_1 - \mu \).

To do this, we define a one-dimensional completely additive set function \( \sigma \) of bounded variation on \([0, \pi]\) in the following manner:

To every Borel set \( Z \) on \([0, \pi]\) associate the set \( Z_{k-1} \) on \( \pi \) where \( Z_{k-1} = \{ x, x \in \Omega \text{ and } (x, x_0) = \cos \theta \text{ where } \theta \in Z \} \). Then \( \sigma \) is defined by \( \sigma(Z) = \mu_2(Z_{k-1}) \). So in particular if \( Z = [0, h] \), \( \sigma(Z) = \mu_2[D(x_0, h)] \).

With this definition we obtain that

\[
\int_{\Omega - D(x_0', \epsilon)} S_n^{\delta, \lambda}\left[ (x_0, y) \right] d\mu_2(y) = \int_0^{\pi - \epsilon} S_n^{\delta, \lambda}\left[ \cos \theta \right] d\sigma(\theta) \]

(14)

\[
= - \int_0^{\pi - \epsilon} \mu_2[D(x_0, \theta)] S_n^{\delta, \lambda}\left[ \cos \theta \right]' d\theta \]

since both \( \mu_2[D(x_0, \pi - \epsilon)] \) and \( \mu_2[D(x_0, 0)] \) are zero.

Now by construction \( \mu_2[D(x_0, \theta)] = o\left[ |D(x_0, \theta)| \right] = o(\theta^{2\lambda+1}) \) as \( \theta \to 0 \).

Therefore by Lemma 2,

\[
\int_0^{\pi - \epsilon} \mu_2[D(x_0, \theta)] S_n^{\delta, \lambda}\left[ \cos \theta \right]' d\theta = o\left( \int_0^{1/n} \theta^{2\lambda+1} n^{\lambda+2\lambda+2} d\theta \right) = o(n^\delta),
\]

(15)

and by Lemma 1,

\[
\left| \int_{\pi - \epsilon}^\pi \mu_2[D(x_0, \theta)] S_n^{\delta, \lambda}\left[ \cos \theta \right]' d\theta \right| \leq \psi_1(h)n^{\lambda+1} \int_{\pi - \epsilon}^\pi \theta^{2\lambda+1} \theta^{-(\delta+\lambda+1)} d\theta \leq \psi_2(h)n^\delta,
\]

(16)

where \( \psi_i(h)(i = 1, 2) \) tends to zero with \( h \).

From Lemma 1, we also obtain that

\[
\int_0^{\pi - \epsilon} \mu_2[D(x_0, \theta)] S_n^{\delta, \lambda}\left[ \cos \theta \right]' d\theta = O(n^{\lambda+1-\delta} n^\delta) = o(n^\delta).
\]

(17)

We therefore conclude from (13), (14), (15), (16), and (17) that

\[
\limsup_{n \to \infty} \left| R_n^\delta(x_0) - S_n^\delta(x_0) \right| n^{-\delta} = 0.
\]

Since \( R_n^\delta(x_0)/A_n^\delta \to \mu_*(x_0) \), the second part of Theorem 1 is proved.

To prove the first part of the theorem, we suppose that \( k - 2 < \eta < k - 1 \) for \( k = 4 \) and \( 3/2 < \eta < 2 \) for \( k = 3 \) and set \( \mu = \mu_3 + \mu_4 \), where for Borel sets \( E \),
\( \mu_3(E) = 0 \) for \( E \subseteq \Omega - D(x_0, 3\pi/4) \) and \( \mu_4(E) = 0 \) for \( E \subseteq D(x_0, 3\pi/4) \). Then
\[
\frac{S^n(x_0)}{A^n} = \frac{\Gamma(\lambda)}{2 A^n \pi^{n+1}} \int_{\Omega} S^n(x_0, \cos \gamma) [d\mu_3(y) + d\mu_4(y)] = I^n_\infty + II^n_\infty.
\]

By the part of the theorem proved above, \( I^n_\infty \to \mu_4(x_0) \) for \( \eta > \lambda + 1 \). By Lemma 3 with \( k-2 < \eta < k-1 \),
\[
K(n + 1) \frac{k-2}{\pi^{k-1}} \sum_{j=0}^{n} (j + \lambda)(-1)^j P^\lambda_j(1) A^n_{n-j}.
\]

By \([8, p. 43]\) for \( II^n_{k-2} \) to tend to zero, \( (n+\lambda)P_n(1) \) would have to be \( o(n^{k-2}) \). But \( (n+\lambda)P_n(1)n^{-(k-2)} \to 1/(k-3)! \) as \( n \to \infty \). So in Theorem 1, \( \eta \) must be chosen greater than \( k-2 \).

6. Proof of Theorem 2. The sufficiency condition of Theorem 2 follows immediately from Theorem 1 and the usual Tauberian theorem for Cesàro summability \([4, p. 121]\).

To prove the necessity, we set \( F_h(x) = |D(x, h)|^{-1} \int_{D(x, h)} f(z) d\Omega(z) \) and obtain that
\[
\int_{\Omega} P^\lambda_n[(x, y)] F_h(y) d\Omega(y) = |D(x, h)|^{-1} \int_{\Omega} f(z) \left[ \int_{\Omega} P^\lambda_n[(x, y)] X_{D(x, h)}(z) d\Omega(y) \right] d\Omega(z)
\]
\[
= |D(x, h)|^{-1} \int_{\Omega} f(z) \left[ \int_{D(x, h)} P^\lambda_n[(x, y)] d\Omega(y) \right] d\Omega(z)
\]
where \( X_E(x) \) stands for the characteristic function of the set \( E \).

Then by \([3, p. 243]\),
\[
P^\lambda_n[(x, y)] = P^\lambda_n(1) \left| \Omega \right| [h(n)]^{-1} \sum_{i=1}^{h(n)} S^i_n(x) S^i_d(y)
\]
where \( h(n) \) is the maximum number of linearly independent surface harmon-
ics of degree $n$ on $\Omega$, and $S'_n(x)$, $j = 1, \ldots, h(n)$, are a set of linear independent, orthonormal surface harmonics on $\Omega$.

On the other hand, by [3, p. 240] with $z = (1, 0, \ldots, 0)$ and $x$ given by spherical coordinates in terms of $z$, $S'_n(x)$ can be chosen as a constant multiple of

\begin{equation}
Y(m_q; \theta_q, \pm \phi) = e^{\pm i m_0 \phi} \prod_{q=0}^{2\lambda-1} (\sin \theta_{q+1})^{m_{q+1}+1} P_{m_{q+1}}^{m_{q+1}+1}(\cos \theta_{q+1})
\end{equation}

where $n = m_0 \geq m_1 \geq \ldots \geq m_{2\lambda} \geq 0$. [For the spherical coordinate notation see [3, p. 233] with $p = 2\lambda$.] Let us call $S'_n(x)$ the function obtained in (20) when the sequence $(n, 0, \ldots, 0)$ is used. Then $S'_n(x)$ is the function $P_n^\lambda[(x, z)]$, normalized.

We shall now show that,

\begin{equation}
\int_{D(x,h)} S'_n(y) d\Omega(y) = 0, \text{ for } j \neq 1.
\end{equation}

For there must then exist some $q \neq 0$ such that $m_q \neq 0$. Let $q_1$ be the last such $q$. Recalling that

\begin{equation}
d\Omega(y) = (\sin \theta_1)^{2\lambda}(\sin \theta_2)^{2\lambda-1}\cdots(\sin \theta_{2\lambda})d\theta_1 \cdots d\theta_{2\lambda}d\phi
\end{equation}

where $0 \leq \theta_q \leq \pi$ ($q = 1, \ldots, 2\lambda$), and $0 \leq \phi \leq 2\pi$, we see immediately that (21) holds in case $q_1 = 2\lambda$. Let us suppose then that $q_1 \neq 2\lambda$. Then

\begin{equation} (\sin \theta_{q_1+1})^{m_{q_1+1}+1} P_{m_{q_1+1}}^{m_{q_1+1}+1}(\cos \theta_{q_1+1}) = P_{m_{q_1}}^{\lambda-1/2q_1}(\cos \theta_{q_1+1}). \end{equation}

But by [2, p. 177]

\begin{equation} \int_0^{\pi} P_n^{\lambda-1/2q}(\cos \theta)(\sin \theta)^{2\lambda-q} d\theta = 0 \quad \text{for } n \neq 0 \end{equation}

and consequently (21) holds.

We thus obtain from (18), (19), (20), and (21) that

\begin{equation}
\int_{\Omega} P_n^\lambda[(x, y)] F_h(y) d\Omega(y) = \zeta(n) \int_{D(x,h)} P_n^\lambda[(y, w)] d\Omega(y) \int_{\Omega} f(z) P_n^\lambda[(x, z)] d\Omega(z)
\end{equation}

where

\begin{equation} \zeta(n) = \frac{P_n^\lambda(1)}{h(n)} \left\{ \int_{\Omega} (P_n^\lambda[(x, w)])^2 d\Omega(x) \right\}^{-1}
\end{equation}

\begin{equation} = \frac{1}{D(w, h) P_n^\lambda(1)} \quad \text{by [3, p. 236, formula 29].} \end{equation}
Consequently with $S[f] = \sum_{n=0}^{\infty} Y_n(x)$ and $B_n^\lambda(h)$ as in Lemmas 4 and 5, we conclude from (22) that

$$S[F_h] = \sum_{n=0}^{\infty} Y_n(x) B_n^\lambda(h)$$

and, furthermore, from the continuity of $F_h(x)$, from the fact that $Y_n(x) = O(n^{-1})$, and from Lemma 4, that

$$F_h(x_0) = \sum_{n=0}^{q[h^{-1}]} Y_n(x_0) B_n^\lambda(h) + \sum_{n=q[h^{-1}]+1}^{\infty} Y_n(x_0) B_n^\lambda(h)$$

where $q$ is a fixed, large positive integer.

Since there clearly is no loss of generality in assuming that

$$P_n = \sum_{n=0}^{\infty} Y_n(x_0) \to 0,$$

we shall make this assumption and shall show this implies that $F_h(x_0) \to 0$.

By Lemma 4 the second sum on the right in (23) is majorized by a constant multiple of

$$h^{-\lambda} \sum_{n=0}^{q[h^{-1}]} n^{-1+\lambda} = O(q^{-\lambda}).$$

Using Abel summation by parts on the first sum on the right side in (23), we obtain that this sum is equal to

$$\sum_{n=0}^{q[h^{-1}]-1} R_n[B_n^\lambda(h) - B_{n+1}^\lambda(h)] + R_{q[h^{-1}]} B_{q[h^{-1}]}^\lambda(h).$$

By Lemma 4 the second term in (25) tends to zero as $h \to 0$. By Lemma 5 the first term in (25) is majorized by

$$h^2 \sum_{n=0}^{q[h^{-1}]} o(n)$$

which tends to zero with $h$. We consequently conclude from (23) and (24) that

$$\limsup_{h \to 0} |F_h(x_0)| = O(q^{-\lambda}).$$

But this fact implies that $\lim_{h \to 0} F_h(x_0) = 0$, and the proof of Theorem 2 is complete.

In closing we point out that with no change in the proof of Theorem 2 the condition that $\mu$ be absolutely continuous can be replaced with one requiring only that $\mu [O(x, h)]$ be continuous at $x_0$ for $h$ small.
Bibliography


Rutgers University,
New Brunswick, N. J.
The Institute for Advanced Study,
Princeton, N. J.

Errata, Volume 79


Page 329, lines 20–21. For “those with superscript “Q” partial recursive, uniformly in Q);” read “those with superscript “Q” partial recursive uniformly in Q);”.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use