MODULAR LIE ALGEBRAS. I

BY

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1. Introduction. A Lie algebra $I$ over a field $E$ of characteristic $p > 0$ is called separable if its Killing form $B(X, Y) = \text{trace}(\text{ad} X \text{ ad} Y)$ is nondegenerate. Separable algebras over algebraically closed fields enjoy many of the properties of complex semi-simple algebras; in particular the Cartan subalgebras are commutative, and they possess root systems which determine the algebras up to isomorphism. These results, and a complete classification of the simple separable algebras over an algebraically closed field of characteristic $p > 7$, have been obtained recently by Seligman [12](2).

In this paper we develop the basic properties of modular Lie algebras; these are separable algebras which are obtained from semi-simple algebras of characteristic zero in the following way. Let $\mathfrak{g}$ be a semi-simple algebra over an algebraically closed field $C$ of characteristic zero. Then $\mathfrak{g}$ possesses a Cartan subalgebra $\mathfrak{h}$, and a Cartan basis $(X_i)$ relative to $\mathfrak{h}$, such that the constants of structure determined by the basis $(X_i)$ belong to an algebraic number field $K$. A finite set of exceptional prime ideals in $K$, which is independent of the choice of the Cartan subalgebra $\mathfrak{h}$ and the basis $(X_i)$, is then defined; this set includes the prime ideals which divide the rational primes 2 and 3, and the primes which divide the determinant of the Killing matrix $(B(X_i, X_j))$. Now let $p$ be a fixed nonexceptional prime, and let $o$ be the ring of $p$-integers in $K$. Then $\Sigma = \sum oX_i$ forms a Lie subring of $\mathfrak{g}$, and $I = \Sigma/p\Sigma$ is a separable Lie algebra over the residue field $\bar{K} = o/p$, with the further properties that the natural homomorphism of $\Sigma \rightarrow I$ maps $\mathfrak{h}$ onto a Cartan subalgebra $\mathfrak{h}$ of $I$, and defines a (1-1) mapping of the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ onto the set of roots of $I$ relative to $\mathfrak{h}$. We call a Lie algebra $I$ defined in this way a modular Lie algebra.

As Seligman has observed, a $p$-power operation can be defined in every separable algebra $I$ of characteristic $p$ in such a way that $I$ becomes a restricted Lie algebra in the sense of Jacobson [9]. The study of restricted representations of $I$, which preserve the $p$-power operation in addition to the usual properties of a representation, is equivalent to the study of right $a$-modules, where $a$ is a finite dimensional enveloping algebra of $I$ called the $u$-algebra. The

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main results of this paper deal with the theory of \( \alpha \)-modules, where \( \alpha \) is the \( \mu \)-algebra of a separable modular algebra \( I \). In §7, it is proved that every irreducible \( \alpha \)-module is absolutely irreducible. The Cartan-Weyl theory of weights of irreducible \( \alpha \)-modules is developed, and in particular it is proved that two irreducible \( \alpha \)-modules \( m \) and \( m' \) which possess leading weights \( \lambda \) and \( \lambda' \), respectively, are \( \alpha \)-isomorphic if and only if \( \lambda = \lambda' \). Not every irreducible \( \alpha \)-module need possess a priori a leading weight; it is proved in §8, however, that if an irreducible \( \alpha \)-module possesses an extreme weight, then it has a leading weight, and in this case the Weyl group of \( I \) acts transitively upon the set of extreme weights of \( m \). In §9 it is proved that every irreducible \( \alpha \)-module which possesses a leading weight is \( \alpha \)-isomorphic to a composition factor of an \( \alpha \)-module obtained by a reduction process from an irreducible representation space of \( I \).

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2. A preliminary remark on the rank of a Lie algebra. We begin with some definitions. Let \( \mathfrak{g} \) be a Lie algebra over an arbitrary field \( E \), having a basis \( X_1, \cdots, X_n \) over \( E \), and let \( X^* = \sum \tau_i X_i \) be the general element of \( \mathfrak{g} \); then \( X^* \in \mathfrak{g}^R(\tau) \) where \( R = \mathbb{E}(\tau_1, \cdots, \tau_n) \) is the field of rational functions in \( n \) variables \( \tau_i \). Let

\[
f(\lambda; \tau) = \lambda^n + \mu_1(\tau)\lambda^{n-1} + \cdots + \mu_n(\tau)
\]

be the characteristic polynomial of the l.t. (linear transformation) \( \text{ad} X^* : Y \to [YX^*] \) acting in \( \mathfrak{g}^R \). Then the \( \mu_i(\tau) \) are homogeneous polynomials in the \( \tau \)'s with coefficients in the ring generated over the subring of \( E \) consisting of the rational integral multiples of 1 by the constants of structure \( c_{ijk} \) of \( \mathfrak{g} \), given by the equations

\[
[X_i X_j] = \sum c_{ijk} X_k.
\]

There exists a unique integer \( l > 0 \), called the rank of \( \mathfrak{g} \), such that \( \mu_{n-l}(\tau) \neq 0 \), and \( \mu_{n-l+1}(\tau) = \cdots = \mu_n(\tau) = 0 \). It is known that the rank of \( \mathfrak{g} \) is independent of the choice of the basis \( X_1, \cdots, X_n \) of \( \mathfrak{g} \). For reference we state the following result, which is an immediate consequence of our definitions.

**Lemma 1.** Let \( \mathfrak{g} \) be a Lie algebra over a field \( E \), and let \( \Omega \) be an extension field of \( E \). Then the rank of \( \mathfrak{g}^\Omega \) is equal to the rank of \( \mathfrak{g} \).

A subalgebra \( \mathfrak{h} \) of a Lie algebra \( \mathfrak{g} \) is called a Cartan subalgebra if (1) \( \mathfrak{h} \) is nilpotent, and (2) \( [\mathfrak{h} , X] \subseteq \mathfrak{h} \) implies \( X \in \mathfrak{h} \) for all \( X \) in \( \mathfrak{g} \).

If the base field is infinite, then Cartan subalgebras are known to exist, and the dimension of a Cartan subalgebra is equal to the rank of \( \mathfrak{g} \).

\(^{(p)}\) By \( \mathfrak{g}^R \) we mean the Lie algebra obtained by extension of the field \( E \) to \( R \).
3. Preliminary results on semi-simple algebras of characteristic zero\(^{(4)}\).

Let \( \mathfrak{g} \) be a semi-simple Lie algebra over an algebraically closed field \( C \) of characteristic zero. We write \( Q \) for the prime field in \( C \), and \( Z \) for the subring of \( Q \) consisting of the multiples of 1; then \( Q \) and \( Z \) are isomorphic to the field of rational numbers and the ring of rational integers, respectively. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \), and let \( \alpha, \beta, \cdots \) be the roots\(^{(5)}\) of \( \mathfrak{g} \) relative to \( \mathfrak{h} \). Let \( B(X, Y) = \text{trace} (\text{ad} X \text{ ad} Y) \) be the Killing form on \( \mathfrak{g} \); then \( B \) is non-degenerate on \( \mathfrak{g} \), and its restriction to \( \mathfrak{h} \) is nondegenerate. For each root \( \alpha \), there exists a unique element \( H_\alpha \) in \( \mathfrak{h} \) such that \( B(H_\alpha, H) = \alpha(H) \) for all \( H \) in \( \mathfrak{h} \). Then \( \alpha(H_\alpha) \) is a nonzero element of \( Q \), and if we put \( H_\alpha = 2\alpha(H_\alpha)^{-1}H_\alpha \), then \( \alpha(H_\alpha) = 2 \). Let \( l = \dim \mathfrak{h} = \text{rank} \mathfrak{g} \). It is known that there exists a set of \( l \) linearly independent roots \( \alpha_1, \cdot \cdot \cdot, \alpha_l \), called a fundamental system of roots, such that every root of \( \mathfrak{g} \) relative to \( \mathfrak{h} \) is an image of one of the \( \alpha_i \), \( 1 \leq i \leq l \), by an element of the group \( W \) generated by the Weyl reflections \( \Lambda \rightarrow \Lambda - \Lambda(H_{\alpha_i})\alpha_i, \) \( 1 \leq i \leq l \). Every root \( \alpha \) can be expressed in the form \( \alpha = \sum_{i=1}^{l} d_i \alpha_i \), where the \( d_i \) are elements of \( Z \). If we write \( H_{\alpha_i} = H_i, 1 \leq i \leq l \), then \( H_1, \cdots, H_l \) form a basis of \( \mathfrak{h} \). The linear functions \( \sum q_i \alpha_i \) with coefficients \( q_i \) in \( Q \) are called rational linear functions on \( \mathfrak{h} \), and may be ordered lexicographically with respect to the ordered set \( (\alpha_1, \cdots, \alpha_l) \). In particular the roots are linearly ordered in this way, and when we write \( \alpha < \beta, \alpha < 0 \), etc. it is to be understood that "<" is the lexicographic order relation. If \( \alpha \) is a root and \( m \) an integer, then \( m\alpha \) is a root if and only if \( m = \pm 1 \). For any pair of roots \( \alpha \) and \( \beta \), \( \alpha(H_\beta) \in Z \).

If the dimension of \( \mathfrak{g} \) is \( n \), then there exist \( n - l \) distinct roots, and root vectors \( E_\alpha, E_\beta, \cdots \) in (1-1) correspondence with them which can be chosen in such a way that the following relations hold.

\[
(3) \quad \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \geq 0} CE_\alpha + \sum_{\alpha > 0} CE_{-\alpha} \quad \text{(direct sum)}; \\
(4) \quad [HH'] = 0, \quad [E_\alpha H] = \alpha(H)E_\alpha, \quad H, H' \in \mathfrak{h}; \\
(5) \quad \begin{cases} 
N_{\alpha\beta}E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root,} \\
0 & \text{if } \alpha + \beta \text{ is not a root,} \\
H_\beta & \text{if } \alpha = -\beta,
\end{cases}
\]

where for each pair of roots \( (\alpha, \beta) \), \( N_{\alpha\beta}^2 \in Z \); and

\[
(6) \quad \begin{cases} 
B(E_\alpha, E_\beta) = 0 & \text{if } \alpha + \beta \neq 0, \\
B(E_{-\alpha}, E_\alpha) = 2B(H_{\alpha'}, H_{\alpha'})^{-1}, \\
B(E_\alpha, H) = 0, & H \in \mathfrak{h}.
\end{cases}
\]

\(^{(4)}\) For proofs of these results we refer to Weyl [14]; for the terminology we use and for a discussion of some of these questions see also Gantmacher [5] and Harish-Chandra [7].

\(^{(5)}\) We do not call the zero linear function a root.
A basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ is called an admissible basis relative to a Cartan subalgebra $\mathfrak{h}$ if $X_1, \ldots, X_m$ are the root vectors $E_{\alpha}$ belonging to the positive roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, $X_{m+1}, \ldots, X_n$ are the root vectors corresponding to the negative roots, and $(X_{m+1}, \ldots, X_{m+i}) = (H_1, \ldots, H_i)$ is the basis of $\mathfrak{h}$ determined by the fundamental system of roots $\alpha_1, \ldots, \alpha_l$, all normalized so that the relations (3), (4), (5), and (6) hold. It is known that if $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$, then there exists an admissible basis of $\mathfrak{g}$ relative to $\mathfrak{h}$. We shall write $(X_i)$ for an admissible basis, and call the field $K$ generated by the constants of structure $N_{ab}$ in (5) the coefficient field determined by the basis. Then $K$ is an algebraic number field, which contains all the constants of structure $c_{ij}$ determined by the basis $(X_i)$.

4. Arithmetical preparations. Let $(X_i)$ be an admissible basis of $\mathfrak{g}$ with coefficient field $K$. Let $a_{ij} = -\alpha_i(H_j) = -\alpha_i(X_{m+j})$, $1 \leq i, j \leq l$; then the $a_{ij}$ are rational integers with certain properties, and the matrix $(a_{ij})$ is called the Weyl matrix of $\mathfrak{g}$ (see [7, p. 29]).

We shall write $\mathfrak{o}_p$ for a discrete valuation ring in $K$, $\mathfrak{p}$ for the corresponding prime ideal, $K$ for the residue class field $\mathfrak{o}_p/\mathfrak{p}$, and $\phi$ for the homomorphism (or place) mapping $\mathfrak{o}_p$ upon $K$.

We note that for any prime ideal $\mathfrak{p}$ in $K$ such that $2 \mathfrak{E} \mathfrak{p}$, the elements $B(E_{-\alpha}, E_{\alpha}) \subset \mathfrak{o}_p$ for all roots $\alpha$. This fact is a consequence of (6), and the formula

$$B(H'_\alpha, H'_\alpha) = \alpha(H'_\alpha) = 4 \left( \sum_\beta \left( p_{\beta\alpha} + q_{\beta\alpha} \right)^2 \right)^{-1},$$

where the sum is over all roots $\beta$, and where $p_{\beta\alpha}$ and $q_{\beta\alpha}$ are the uniquely determined rational integers, $p_{\beta\alpha} \leq 0 \leq q_{\beta\alpha}$, such that $\beta + k\alpha$ is a root if and only if $p_{\beta\alpha} \leq k \leq q_{\beta\alpha}$.

**Definition.** A non-archimedean prime ideal $\mathfrak{p}$ in $K$ is called non-exceptional (relative to the basis $(X_i)$) if the following conditions are satisfied.

(i) $2 \mathfrak{E} \mathfrak{p}$, $3 \mathfrak{E} \mathfrak{p}$;

(ii) det $(a_{ij}) \in \mathfrak{p}$; and

(iii) $B(E_{-\alpha}, E_{\alpha}) \in \mathfrak{p}$ for all roots $\alpha$;

otherwise $\mathfrak{p}$ is called exceptional. Evidently the number of exceptional prime ideals is finite.

Let $\mathfrak{p}$ be a non-archimedean prime ideal in $K$ such that $2 \mathfrak{E} \mathfrak{p}$. Let $\Sigma_\mathfrak{p}$ be the set of linear combinations with coefficients in $\mathfrak{o}_\mathfrak{p}$ of the elements $H_\alpha, E_\alpha$, where $\alpha$ ranges through the set of all roots of $\mathfrak{g}$. Inspection of the table (3)-(5) reveals that $\Sigma_\mathfrak{p}$ is closed under the bracket operation. (The $N_{ab} \subset \mathfrak{o}_\mathfrak{p}$ because their squares are in $Z$.)

Next suppose that $2 \mathfrak{E} \mathfrak{p}$, and $B(E_{-\alpha}, E_{\alpha}) \in \mathfrak{p}$ for all roots $\alpha$. We prove that the admissible basis $(X_i)$ is an $\mathfrak{o}_\mathfrak{p}$-basis for the ring $\Sigma_\mathfrak{p}$. It is sufficient to prove that for all roots $\alpha$, $H_\alpha$ is an $\mathfrak{o}_\mathfrak{p}$-linear combination of the $H_i$, $1 \leq i \leq l$. Let $\alpha = \sum d_i \alpha_i$, where the $d_i \in Z$, $1 \leq i \leq l$. Then
\[ H_a = 2B(H_a', H_a' - 1)H_a' = B(E_{-a}, E_a)H_a' \]
\[ = B(E_{-a}, E_a)(\sum d_i H_a') = 2^{-1}B(E_{-a}, E_a)(\sum B(H_a', H_a')H_i) \]
\[ = \sum_{i=1}^l B(E_{-a}, E_a)B(E_{-a_i}, E_{a_i})^{-1}d_iH_i \]

where the \( B(E_{-a}, E_a)B(E_{-a_i}, E_{a_i})^{-1}d_i \subseteq \mathfrak{p}, 1 \leq i \leq l. \)

Finally, let \( \mathfrak{p} \) be a nonexceptional prime ideal in \( K. \) We prove that if we set \( B_{ij} = B(X_i, X_j), 1 \leq i, j \leq n, \) then the \( B_{ij} \in \mathfrak{p}, \) and \( \det (B_{ij}) \in \mathfrak{p}. \) Since \( \Sigma_p \) is closed under the bracket operation, it follows that \( \Sigma_p \) is mapped into itself by \( \text{ad } X_i, 1 \leq i \leq n. \) But \( B_{ij} = \text{trace (ad } X_i \text{ ad } X_j), \) hence the \( B_{ij} \in \mathfrak{p}. \) From (6) we see that the matrix \( (B_{ij}) \) can be expressed in the form

\[ (B_{ij}) = (B(X_i, X_j)) = \begin{bmatrix} B_0 & B_a \\ B_a^T & B_\beta \end{bmatrix}, \]

where \( B_0 = (B(X_i, X_j)), m + 1 \leq i, j \leq m + l, \) where exactly one block

\[ B_a = \begin{pmatrix} 0 & B(E_{-a}, E_a) \\ B(E_{-a}, E_a) & 0 \end{pmatrix} \]

appears for each pair of roots \( (\alpha, -\alpha), \) and where the matrix \( (B_{ij}) \) has zeros except for the blocks we have listed along the main diagonal. Then we have

\[ \det (B_{ij}) = (\det B_0) \left( \prod_{\alpha > 0} \det B_{\alpha} \right). \]

By (iii) in the definition of an exceptional prime ideal, the second factor on the right does not belong to \( \mathfrak{p}. \) We have for all \( i \) and \( j, \) \( B(H_i, H_j) = -2a_{ij}B(H_i', H_j')^{-1} = -a_{ij}B(E_{-a_i}, E_{a_i}) \) by (6). Therefore

\[ \det B_0 = \det (B(H_i, H_j)) = (-1)^l(\det (a_{ij})) \left( \prod_{1 \leq i \leq l} B(E_{-a_i}, E_{a_i}) \right), \]

so that \( \det B_0 \neq \mathfrak{p}. \) Thus \( (B_{ij}) \in \mathfrak{p} \) if \( \mathfrak{p} \) is nonexceptional.

We shall discuss the uniqueness of the coefficient fields and of the sets of exceptional prime ideals determined by two admissible bases \( (X_i) \) and \( (X'_i) \) relative to Cartan subalgebras \( \mathfrak{g} \) and \( \mathfrak{g}', \) respectively. By a result of Chevalley [4] there exists an automorphism \( \sigma = \exp (\text{ad } Z) \) of the adjoint group of \( \mathfrak{g} \) such that \( \mathfrak{g}' = \mathfrak{g}'. \) An examination of the table (3)–(6) shows that \( (X'_i) \) is an admissible basis of \( \mathfrak{g} \) relative to \( \mathfrak{g}'. \) The coefficient field relative to the basis \( (X_i) \) is identical with the coefficient field of the basis \( (X'_i). \) By a result

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of Weyl [14, p. 372], we have the formula \( N_{\alpha \beta} = -(i+1)k \), where \( \beta = i\alpha, \\cdots, \beta - \alpha, \beta + \alpha, \\cdots, \beta + k\alpha \) is the string of roots of the form \( \beta + j\alpha, j \in \mathbb{Z} \), and consequently the constants \( N_{\alpha \beta} \) and \( N_{\alpha \delta} \) determined by the bases \((X_i')\) and \((X'_\delta)\) of \( \mathfrak{g} \) relative to \( \mathfrak{g}' \) differ at most by a permutation of the roots. Therefore the coefficient field \( K \) is independent of the choice of an admissible basis.

In order to compare the exceptional prime ideals relative to the admissible bases \((X_i)\) and \((X'_i)\), we may assume that the Cartan subalgebras which give rise to these bases are identical. If we let \( E_\alpha \) and \( E'_\alpha \) be the root vectors in the two bases, then since the elements \( H_\alpha' \) are uniquely determined, (6) shows that the primes which divide some \( B(E_\alpha, E'_\alpha) \) are identical with the primes which divide some \( B(E_\alpha', E'_\alpha) \). We use finally the known result that the determinants of the Weyl matrices relative to two fundamental systems of roots of \( \mathfrak{g} \) relative to a Cartan subalgebra \( \mathfrak{g} \) differ at most by a unit factor. Therefore the set of exceptional primes is an invariant of the algebra \( \mathfrak{g} \).

5. Modular Lie algebras. A Lie algebra \( I \) over a field \( E \) of characteristic \( p > 0 \) is called separable if the Killing form \( B(x, y) \) of \( I \) is nondegenerate. It is immediate that if \( I \) is a separable algebra, then so is \( I^F \), where \( F \) is an extension field of \( E \). In any separable algebra \( I \), a \( p \)-power operation can be defined so that \( I \) becomes a restricted Lie algebra in the sense of Jacobson [9]. The definition of \( x^p \) is based on the fact that \((\text{ad } x)^p \) is a derivation of \( I \); hence by [16, p. 53], \((\text{ad } x)^p \) is an inner derivation \( \text{ad } y \). The element \( y \) is uniquely determined because of the separability (see [12]), and hence if we set \( y = x^p \), so that \([xy] = [x^p] = \epsilon(\text{ad } x)^p, \epsilon \in \mathbb{I}, I \) becomes a restricted Lie algebra. In the sequel, "separable algebra" means "restricted separable algebra" under this definition of the \( p \)-power operation. This paper is devoted to a study of certain separable algebras which are constructed as follows.

Let \( \mathfrak{g}, (X_i), K, \mathfrak{g} \) be as in §4, and let \( p \) be a fixed nonexceptional prime, containing the rational prime \( p \). We shall adhere to these notations for the rest of the paper, and we shall write \( a \) for \( a \), \( \bar{K} \) for the residue field \( a/p \), \( \Sigma \) for \( \Sigma_p \), and \( \phi: \phi(a) = \bar{a} \) for the homomorphism of \( a \) onto \( \bar{K} \). Then \( \Sigma = \sum aX_i \) is closed under the bracket operation, and may be regarded as a Lie algebra over the ring \( a \). \( p\Sigma = \sum pX_i \) is an ideal in \( \Sigma \), and \( \Sigma/p\Sigma \) is a Lie algebra over \( a \). Let \( T \) be the natural mapping of \( \Sigma \) onto \( \Sigma/p\Sigma \); we shall write \( x = XT \) for \( X \in \Sigma \), and \( I = \Sigma/p\Sigma \). If we define \( \bar{a}x = \bar{a}(XT) = (aX)T \), for \( a \in a \), then \( I \) becomes a Lie algebra of dimension \( n \) over \( \bar{K} \) which will be called a modular Lie algebra.

Theorem 1. The modular Lie algebra \( I = \Sigma/p\Sigma \) defined at a nonexceptional prime ideal \( p \) is a separable algebra over \( \bar{K} \) whose rank is equal to the rank of \( \mathfrak{g} \). The natural mapping \( T: \Sigma \to I \) maps \( \mathfrak{g} \cap \Sigma \) onto a Cartan subalgebra \( \mathfrak{g} \) of \( I \). The restriction of the Killing form of \( I \) to \( \mathfrak{g} \) is nondegenerate.

Proof. The cosets \( (X_iT) = (x_i) \) form a basis of \( I \) over \( \bar{K} \), and from (2) we have \( [x_i, x_j] = \sum \hat{c}_{ijk}a_k \). If \( B^*(x, y) \) is the Killing form on \( I \), then
\[ B^*(x_i, x_j) = \text{trace} (\text{ad} x_i \text{ ad} x_j) = \sum_{\mu, v = 1}^{n} \epsilon_{\mu v} \epsilon_{x, j} \mu \]

\[ = \phi \left( \sum_{\mu, v = 1}^{n} \epsilon_{\mu v} \epsilon_{x, j} \mu \right) = \phi(B(x_i, x_j)), \]

and by linearity we have, for all \( X, Y \) in \( \Sigma \),

\[ \phi(B(X, Y)) = B^*(XT, YT). \tag{9} \]

In particular, \( \det B^*(x_i, x_j) = \det \phi(B(X_i, X_j)) = \phi(\det(B(X_i, X_j))) \neq 0 \) by the remarks in §4, and hence \( I \) is separable.

For each root \( \alpha \) of \( \mathfrak{g} \) relative to \( \mathfrak{h} \), we have \( \alpha(H_i) \in \mathbb{Z} \leq 0 \) for \( 1 \leq i \leq l \). Therefore, if we set \( \mathfrak{h} = (\mathfrak{g} \cap \Sigma) T \), we can define a unique linear function \( \tilde{\alpha} \) on \( \mathfrak{h} \) whose value on \( H_i T \) is given by \( \tilde{\alpha}(H_i T) = \phi(\alpha(H_i)) \), \( 1 \leq i \leq l \). It follows that for all \( H \in \mathfrak{g} \cap \Sigma \) we have

\[ \phi(\alpha(H)) = \tilde{\alpha}(HT), \tag{10} \]

and in particular, since \( H_i \in \mathfrak{g} \cap \Sigma \), \( \tilde{\alpha}(H_i T) = \phi(\alpha(H_i)) = \phi(2) \neq 0 \) in \( K \) so that the linear functions \( \tilde{\alpha} \) on \( \mathfrak{h} \) are all different from zero. For any root vector \( X_i = E_{\alpha} \), the relation \( [X_i H] = \alpha(H) X_i \), \( H \in \mathfrak{g} \cap \Sigma \), implies that

\[ [x_i h] = \tilde{\alpha}(h)x_i, \quad h \in \mathfrak{h}. \tag{11} \]

From (11) and the fact that all \( \tilde{\alpha} \neq 0 \) we see that \([hx] \subseteq \mathfrak{h} \) implies \( x \in \mathfrak{h} \), and since \( \mathfrak{h} \) is commutative, we conclude that \( \mathfrak{h} = (\mathfrak{g} \cap \Sigma) T \) is a Cartan subalgebra of \( I \).

Let \( f(\lambda; \tau) \) be the characteristic polynomial of \( \text{ad} X^* \), where \( X^* = \sum \tau_i x_i \) is the general element of \( \mathfrak{g} \). If we let \( x^* = \sum \tau'_i x_i \) be the general element of \( I \) with respect to the basis \( (X_i T) = (x_i) \), then it can be verified that the coefficients \( \mu_k(\tau') \) of the characteristic polynomial \( f'(\lambda; \tau') \) of \( \text{ad} x^* \) are obtained from the \( \mu_k(\tau) \) of (1) by replacing the \( \tau_i \) by \( \tau'_i \), \( 1 \leq i \leq n \), and the coefficients of the \( \tau'_i \)'s, which are in \( o \), by their images under \( \phi \). If \( l \) is the rank of \( \mathfrak{g} \), then \( \mu^*_n - 1(\tau') = \cdots = \mu^*_n(\tau') = 0 \), so that rank \( I \geq l \).

In order to prove that the rank of \( I \) does not exceed \( l \), we form \( I^L \), where \( L \) is any infinite field containing \( K \). Then the linear functions \( \tilde{\alpha} \), extended by linearity to \( \mathfrak{h}^L \), are all different from zero. Since \( L \) is an infinite field, there exists an element \( h = \sum \xi_i x_i \), \( \xi_i \in L \), in \( \mathfrak{h}^L \) such that \( \tilde{\alpha}(h) \neq 0 \) for all \( \tilde{\alpha} \). It follows from (11) that the characteristic polynomial of \( \text{ad} h \) has exactly \( l \) roots equal to zero. Now \( x^* = \sum \tau'_i x_i \) also may be viewed as the general element(\( ^{(*)} \)) of \( I^L \). Upon substituting \( \xi_i \) for \( \tau'_i \) in \( \mu_{n - 1}(\tau') \) we obtain \( \mu_{n - 1}(\xi_1, \cdots, \xi_n) \neq 0 \), otherwise the characteristic polynomial of \( \text{ad} h \) would have more than \( l \) roots equal to zero. Thus the rank of \( I^L \) does not exceed \( l \), and by Lemma 1 and what has already been proved, the rank of \( I \) is equal to \( l \).

\( ^{(*)} \) We may assume that the \( \tau'_i \) are transcendental over \( L \).
Finally, by (9) and (6) it follows that $B^*(h, x_i) = 0$ for $1 \leq i \leq m$ and $m + 1 + 1 \leq i \leq n$ and all $h$ in $\mathfrak{h}$. Therefore the restriction of $B^*$ to $\mathfrak{h}$ is nondegenerate, because $I$ is separable. This completes the proof of the theorem.

The linear functions $\bar{\alpha}$ on $\mathfrak{h}$ which were defined in the proof of Theorem 1 are roots of $I$ with respect to $\mathfrak{h}$ (see [12] or [16] for a discussion of roots of Lie algebras of characteristic $p > 0$.) The following result describes the properties of the roots $\bar{\alpha}$ which we shall need.

**Theorem 2.** The mapping $\alpha \mapsto \bar{\alpha}$ is a (1-1) mapping of the set $R$ of roots of $\mathfrak{g}$ relative to $\mathfrak{s}$ onto the set $\bar{R}$ of roots of $I$ relative to $\mathfrak{h}$, such that the following statements are valid.

(a) $\alpha, \beta, \alpha + \beta \in R$ implies $\bar{\alpha} + \bar{\beta} \in \bar{R}$ and $\alpha + \beta \mapsto \bar{\alpha} + \bar{\beta};$

(b) $\alpha \in R$ implies $-\bar{\alpha} \in \bar{R}$ and $-\alpha \mapsto -\bar{\alpha};$

(c) $\bar{\alpha}_i, \ldots, \bar{\alpha}_l$ are linearly independent, and form a fundamental system(7) of roots of $I$ relative to $\mathfrak{h}$.

(d) if $m = 0, 1, \ldots, p - 1$, then $m\bar{\alpha} \in \bar{R}$ if and only if $m = 1$ or $m = p - 1$.

**Proof.** In order to prove that that mapping $\alpha \mapsto \bar{\alpha}$ is (1-1) and onto, it is convenient to extend the base field $K$ to its algebraic closure $\Omega$. The functions $\bar{\alpha}$, extended to $\mathfrak{h}\Omega$ by linearity, are all different from zero, and this fact, together with the multiplication table of the basis $(x_i)$ of $I$, implies that $\mathfrak{h}\Omega$ is a Cartan subalgebra of $I\Omega$, and that the functions $\bar{\alpha}$ are roots of $I\Omega$ with respect to $\mathfrak{h}\Omega$. Suppose that $\bar{\alpha}_i = \bar{\beta}_i$, where $\alpha \neq \beta$; then the dimension of the root space of $\bar{\alpha}$ is not less than two, and since the characteristic of $K$ exceeds 3, this contradicts a result of Jacobson [12, Theorems 5.1 and 5.2]. The mapping $\alpha \mapsto \bar{\alpha}$ is onto, since a root of $I$ relative to $\mathfrak{h}$ distinct from the $\bar{\alpha}$ would define a root of $I\Omega$ distinct from the $\bar{\alpha}$, and this is impossible, since the root vectors belonging to the roots $\bar{\alpha}$, together with the elements of $\mathfrak{h}\Omega$, span $I\Omega$. Over the infinite field $\Omega$, elements of $I\Omega$ belonging to distinct roots are linearly independent.

Statements (a) and (b) are obvious. We have $\bar{\alpha}_i(H_jT) = -\phi(a_{ij})$ for $1 \leq i, j \leq l$. Since $\det \phi(a_{ij}) \neq 0$, the roots $\bar{\alpha}_1, \ldots, \bar{\alpha}_l$ are linearly independent.

Now let $\Lambda$ be any $p$-integral linear function on $\mathfrak{s}$: $\Lambda(H_i) \in \Omega, 1 \leq i \leq l$. We write $\phi\Lambda = \widetilde{\Lambda}$ for the unique linear function on $\mathfrak{h}$ such that $\lambda(H_iT) = \phi(\lambda(H_i)), 1 \leq i \leq l$, in agreement with our definition of the roots $\bar{\alpha}$. Let $S_i$ be the reflection determined by $\alpha_i$ on the dual space of $\mathfrak{s}$, and let $\bar{S}_i$ be the reflection determined by $\bar{\alpha}_i$ on the dual space of $\mathfrak{h}$. Then for all $p$-integral linear functions $\Lambda, \Lambda S_i$ is $p$-integral, and

$$\phi(\Lambda S_i) = \widetilde{\Lambda S_i}, \hspace{1cm} 1 \leq i \leq l.$$

From (12) and the fact that $\alpha_1, \ldots, \alpha_l$ is a fundamental system, (c) follows.

(7) By analogy with Cartan's definition in the characteristic zero theory, we call $\bar{\alpha}_1, \ldots, \bar{\alpha}_l$ a fundamental system if every root $\bar{\alpha}$ is an image of some $\bar{\alpha}_i$ by an element of the group of 1.t. on the dual space of $\mathfrak{h}$ generated by the reflections $\lambda \mapsto \lambda - \lambda(h)$, $1 \leq i \leq l$, where $h_i = H_iT$. 

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directly. Finally, since the characteristic \( p > 3 \), statement (d) holds for \( I^n \), and hence for \( I \), by [12, Theorem 5.3].

We show next that the \( p \)-power operation on the basis elements \( x_i \) of \( I \) is given by the following formulas.

\[
\begin{align*}
    x_i^p &= 0, & 1 \leq i \leq m, m + l + 1 \leq i \leq n; \\
    x_i^p &= x_i, & m + 1 \leq i \leq m + l.
\end{align*}
\]

That the \( x^p = 0 \), \( 1 \leq i \leq m, m + l + 1 \leq i \leq n \), follows from a result of Jacobson [12, Theorem 4.1]. For \( i = 1, \cdots, l \) we have \( x_{m+i} = H_i T \), and by the definition of the \( p \)-power operation it is sufficient to prove that for all \( X \in \Sigma \), \( X (\text{ad} \, H_i)^p \equiv [XH_i] \pmod{p^2} \). We need check it only for a root vector \( X = E_a \), and in this case the statement is obvious, since \( \alpha(H_i) \in Z \).

6. Preliminary results on representation theory. We show in this section that each irreducible representation of \( \mathfrak{g} \) gives rise to a restricted representation of \( I \). Let \( \mathfrak{A} \) be the universal associative algebra of \( \mathfrak{g} \) (see [2, 6, 15]). Let \( P = (i_1, \cdots, i_m) \), \( Q = (j_1, \cdots, j_l) \), \( R = (k_1, \cdots, k_m) \) be row vectors whose coefficients are non-negative rational integers. We write \( |P| = \sum_{i=1}^{m} i_i \), and similarly define \( |Q|, |R| \); we write \( 0 = (0, \cdots, 0) \) so that \( |0| = 0 \). If \( (X_i) \) is an admissible basis of \( \mathfrak{g} \) (see §3), then the elements\(^8\)

\[
Z(P, Q, R) = X_{i_1}^{j_1} \cdots X_m^{j_l} X_{m+1}^{j_1} \cdots X_m^{j_l} X_{m+l+1} \cdots X_{n}^{k_m},
\]

where \( |P|, |Q|, |R| \geq 0 \), form a basis of \( \mathfrak{A} \) over \( \mathbb{C} \). The number \( |P| + |Q| + |R| \) is called the degree of \( Z(P, Q, R) \). The degree of \( F = \sum a(P, Q, R) Z(P, Q, R) \), \( a(P, Q, R) \in \mathbb{C} \), is defined to be the largest \( |P| + |Q| + |R| \) for which \( a(P, Q, R) \neq 0 \), and we write \( \deg F \) for this number, with the convention that \( \deg 0 = -\infty \). Then it is known that

\[
\begin{align*}
    \deg (F + G) &\leq \max (\deg F + \deg G), \\
    \deg (FG) &= \deg F + \deg G, \\
    \deg [FG] &= \deg (FG - GF) < \deg F + \deg G.
\end{align*}
\]

From (14) it follows in particular that \( \deg (X_{i_1} \cdots X_{i_r}) = r, 1 \leq i_j \leq n \), and

\[
\deg \left( \sum_{0 \leq r, 1 \leq i_j \leq n} a_{i_1, \cdots, i_r} X_{i_1} \cdots X_{i_r} \right) = \max_{a_{i_1, \cdots, i_r} \neq 0} (r), \quad a_{i_1, \cdots, i_r} \in \mathbb{C}.
\]

Let \( \mathfrak{B} \) be the set of all linear combinations of the \( Z(P, Q, R) \) with coefficients in \( \mathfrak{o} \). Then \( \mathfrak{B} \) is contained in the subring of \( \mathfrak{A} \) generated by the elements of \( \Sigma = \sum \mathfrak{o} X_i \); and \( \mathfrak{o} \); in fact, \( \mathfrak{B} \) is identical with this subring. All that has to be proved is that \( Z(P, Q, R) Z(P', Q', R') \) is an \( \mathfrak{o} \)-linear combination of the

\(^8\) We shall identify \( \mathfrak{g} \) with the linear part of \( \mathfrak{A} \).
$Z(P, Q, R)$, and this, in turn, follows if we can show that an arbitrary product $X_{i_1} \cdots X_{i_r}$ of the $X_j$ is an $\alpha$-linear combination of the $Z(P, Q, R)$. This fact is established by induction on the degree of $X_{i_1} \cdots X_{i_r}$, and on the minimum number of transpositions of the $X_j$ required to put $X_{i_1} \cdots X_{i_r}$ in the form $Z(P, Q, R)$, i.e. in the form $X_{j_1} \cdots X_{j_r}$, where $j_1 \leq j_2 \leq \cdots \leq j_r$.

The reduction is achieved by the usual straightening process, using the fact that $[X_i X_j] = \sum c_{ijk} X_k$, where the $c_{ijk} \in \omega$.

Let $\alpha$ be the $\omega$-algebra (see [9]) of the modular Lie algebra $I = \Sigma/p\Sigma$ over $\overline{\mathbb{K}}$; we shall identify $I$ with its image in $\alpha$ under the natural imbedding of $I$ into $\alpha$. Then $\alpha$ has a basis over $\omega$ consisting of the standard monomials

\[(15)\quad Z(P, Q, R) = x_1 \cdots x_m x_{m+1} \cdots x_{m+l} x_{m+l+1} \cdots x_n,\]

where $0 \leq i_t, j_t, k_t < p$.

Now $\alpha$, as well as $I$, can be regarded as a Lie algebra over $\omega$, having the elements (15) as a set of generators, and scalar multiplication defined by $\alpha f = \alpha f$ for $a \in \alpha, f \in \alpha$. The mapping $T: \Sigma \rightarrow I$ may be regarded as an $\omega$-linear homomorphism of the Lie algebra $\Sigma$ over $\omega$ into the associative algebra $\alpha$ over $\omega$. Thus $T$ is an $\omega$-linear mapping of $\Sigma \rightarrow \alpha$ such that

\[(16)\quad [X_T]T = [XY_T] = (XT)(YT) - (YT)(XT)\]

for all $X, Y \in \Sigma$. From the properties of the algebra $\mathfrak{B}$ which we have derived, it follows that $T$ can be extended uniquely to an $\omega$-linear associative homomorphism $\hat{T}$ of $\mathfrak{B}$ onto $\alpha$ such that $1 \hat{T} = 1, X \hat{T} = XT$ for all $X \in \Sigma$, and such that $(aX) \hat{T} = a(XT)$ for $X \in \Sigma, a \in \omega$. We shall determine the kernel of the homomorphism $\hat{T}$.

From the definition of the $p$-power operation in $I$, it follows that for each $X \in \Sigma$ there exists an element $X^{[p]}$, which is uniquely determined mod $p\Sigma$, such that $X^{[p]} = (XT)^p$, and such that for all $Y \in \Sigma$,

\[(17)\quad Y(ad X)^p = [YX^{[p]}] \pmod{p\Sigma}.\]

From this formula we obtain the following congruence in $\mathfrak{B}$:

\[(18)\quad [F, X^p - X^{[p]}] \equiv 0 \pmod{p\mathfrak{B}}\]

for all $X \in \Sigma, F \in \mathfrak{B}$. From (16), since $F \rightarrow [FX^{[p]}]$ and $(ad X)^p$ are derivations in $\mathfrak{B}$ modulo $p\mathfrak{B}$, we have

\[(19)\quad F(\text{ad } X)^p \equiv [FX^{[p]}] \pmod{p\mathfrak{B}}.\]

On the other hand we have by a well known identity [10, p. 102], $F(\text{ad } X)^p \equiv [FX^p] \pmod{p\mathfrak{B}}$. From these formulas we obtain (17).

**Lemma 2.** The $\omega$-linear homomorphism $T: \Sigma \rightarrow I$ can be extended to a unique homomorphism $\hat{T}$ of $\mathfrak{B}$ onto the $\omega$-algebra $\alpha$ of $I$, such that the kernel $\mathfrak{K}$ of $\hat{T}$ is the ideal in $\mathfrak{B}$ generated by the elements $X^{[p]} - X^p, 1 \leq i \leq n$, and the elements of $p$.  

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Proof. Obviously the ideal $\mathfrak{I}'$ generated by the $X_{i}^{p} - X_{i}$ and $p$ is contained in $\mathfrak{I}$. Now let $F$ be an element of $\mathfrak{I}$ such that $F \in \mathfrak{I}'$; we prove that $F \mathfrak{I}' \neq 0$. Let $F = \sum a(P, Q, R)Z(P, Q, R)$, $a(P, Q, R) \in \mathfrak{a}$. Since all the $X_{i}^{p}$ have degree not greater than one, it follows that $F$ is congruent mod $\mathfrak{I}'$ to an expression $F' = \sum a(P, Q, R)Z(P, Q, R)$ with coefficients in $\mathfrak{a}$ in which all the components of $P, Q, R$ in a term whose coefficient is not zero are less than $p$, and such that no nonzero coefficient is in $p$. Since $F \in \mathfrak{I}'$, $F' \in \mathfrak{I}'$, and hence $F' \neq 0$. Moreover $F \mathfrak{I}' = F' \mathfrak{I}'$ since $\mathfrak{I}' \subseteq \mathfrak{I}$, and $F' \mathfrak{I}' = 0$ because the elements $a(P, Q, R)$ in which the components of $P, Q, R$ are less than $p$ are linearly independent in $\mathfrak{a}$. Thus $\mathfrak{I}' = \mathfrak{I}$, and the lemma is proved.

Let $\rho$ be a representation of $\mathfrak{I}$ by l.t. in a finite dimensional space $\mathfrak{M}$ over $C$. Then $\mathfrak{M}$ becomes a right $\mathfrak{I}$-module if we define $UF = U\rho(F)$ for $U \in \mathfrak{M}$, $F \in \mathfrak{I}$. Conversely every right $\mathfrak{I}$-module defines a representation of $\mathfrak{I}$. When we speak of $\mathfrak{I}$-modules, it is assumed that the vector spaces involved are finite dimensional. Similar remarks apply to the $\mathfrak{u}$-algebra $\mathfrak{a}$ of $I$. If $\mathfrak{M}$ (resp. $\mathfrak{m}$) is an $\mathfrak{I}$-module (resp. an $\mathfrak{a}$-module), then a linear function $\Lambda$ (resp. $\lambda$) on $\mathfrak{B}$ (resp. $\mathfrak{b}$) is called a weight of $\mathfrak{M}$ (resp. $\mathfrak{m}$) if $\mathfrak{M}_\Lambda = \{U | UH = \Lambda(H)U \text{ for all } H \in \mathfrak{B} \neq 0\}$ (resp. $\mathfrak{m}_\lambda = \{u | uh = \lambda(h)u \text{ for all } h \in \mathfrak{b} \neq 0\}$). A linear function $\Lambda$ on $\mathfrak{B}$ is called an integral linear function if $\Lambda(H_i) \in \mathfrak{Z}$ for $1 \leq i \leq l$; $\Lambda$ is a dominant integral function if $\Lambda(H_i) \geq 0$ for $1 \leq i \leq l$. It is known that every weight of an $\mathfrak{I}$-module $\mathfrak{M}$ is an integral function, and that every $\mathfrak{I}$-module has a highest weight $\Lambda$, with respect to the lexicographic order in the set of rational functions on $\mathfrak{B}$, such that $\Lambda$ is a dominant integral function. The weights of an $\mathfrak{I}$-module $\mathfrak{M}$ have the property that $\Lambda(H_\alpha)$ is an integer for every root $\alpha$, and that if $\Lambda$ is the highest weight, and $\alpha$ any positive root, then $\Lambda(H_\alpha) \geq 0$. If $\mathfrak{M}$ is an irreducible $\mathfrak{I}$-module, then the dimension of the space $\mathfrak{M}_\Lambda$ belonging to the highest weight $\Lambda$ is equal to one. Two irreducible $\mathfrak{I}$-modules having the same highest weight are $\mathfrak{I}$-isomorphic. If $\Lambda$ is any dominant integral function on $\mathfrak{B}$, then there exists a (finite dimensional) irreducible $\mathfrak{I}$-module whose highest weight is $\Lambda$. For proofs of these results, see [7; 14].

Lemma 3. Let $\mathfrak{M}$ be an irreducible $\mathfrak{I}$-module. Then there exists a finitely generated $\mathfrak{a}$-submodule $\mathfrak{M}_0$ of $\mathfrak{M}$ such that $\mathfrak{M}_0$ spans $\mathfrak{M}$, and such that $\mathfrak{M}_0 \mathfrak{B} \subseteq \mathfrak{M}_0$.

Proof. Let $\Lambda$ be the highest weight of $\mathfrak{M}$, and let $U \neq 0$ be an element of $\mathfrak{M}_\Lambda$. By our remark above, for every positive root $\alpha$, $\Lambda(H_\alpha) \geq 0$. By [7, pp. 51, 52], the elements $UZ(0, 0, R)$, $R = (k_1, \cdots, k_m)$, $0 \leq k_j \leq \Lambda(H_{\alpha_j})$, where $\alpha_j$ is the positive root such that $X_{m+1+j}$ belongs to $-\alpha_j$, form a set of $C$-generators of $\mathfrak{M}$. A close inspection of the argument shows that these elements form a set of $\mathfrak{a}$-generators for the $\mathfrak{a}$-module $\mathfrak{M}_0$ consisting of all $\mathfrak{a}$-linear combinations of the elements $UZ(P, Q, R)$, $(P, Q, R)$ arbitrary. Since the elements $Z(P, Q, R)$ form an $\mathfrak{a}$-basis for $\mathfrak{B}$, we have $\mathfrak{M}_0 \mathfrak{B} \subseteq \mathfrak{M}_0$, and the lemma is proved.
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Corollary. The module $W_0/W_0x$ is an $a$-module, finite dimensional over the field $K$.

Proof. The assertion follows from the facts that $W_0$ is a finitely generated $a$-module such that $W_0b \subseteq W_0$, and that $x$ is the kernel of the homomorphism $T$ of $b$ onto $a$.

We shall call the $a$-module $W_0/W_0x$ associated with an irreducible $a$-module $W$ a $\phi$-module belonging to $M$.

Let $n$ be the radical of $a$ (if $I$ is not commutative then $n \neq 0$ by a result of Hochschild [8]). Let $f \rightarrow f^*$ be the natural mapping of $a \rightarrow a/n = a^*$, and let $a^* = \sum e_i^*a^*$ be a decomposition of $a^*$ into minimal right ideals, where the $e_i^*$ are mutually orthogonal idempotents. Since $a$ has an identity element, Theorem 9.3C of [1] implies that there exist mutually orthogonal idempotents $e_i$ such that $e_i+n = e_i^*$ for each $i$, and such that $a = \sum e_i a$ is a decomposition of $a$ into indecomposable right ideals. By Theorem 9.2G of [1], $e_i a$ is $a$-isomorphic to $e_i a$ if and only if $e_i^* a^*$ and $e_i^* a^*$ are $a$-isomorphic. We can partition the $e_i$ appearing in the decomposition $a = \sum e_i a$ into equivalence classes, $e_i$ being equivalent to $e_j$ if $e_i a$ and $e_j a$ are $a$-isomorphic. Let $e^{(1)}, \ldots, e^{(k)}$ be the idempotents in $a$ which are the sums of the elements in the distinct equivalence classes. It follows that $(e^{(1)})^*, \ldots, (e^{(k)})^*$ are central, mutually orthogonal idempotents in $a^*$, which are the identity elements of the distinct simple ideals in the two sided Wedderburn decomposition of $a^*$. If $m$ is any irreducible $a$-module, then since $m$ is $a$-isomorphic to one of the $e_i^* a^*$, there exists one of the $e^{(j)}$ such that $ue^{(j)} = u$ for all $u \in m$. Since the $e^{(j)}$ are mutually orthogonal by construction, $me^{(k)} = 0$ if $k \neq j$, and we shall say that $m$ belongs to the $j$th class. It is easily proved that an irreducible $a$-module $m_i$ belongs to the $j$th class if and only if $m_i e^{(j)} \neq 0$; and that an arbitrary $a$-module $m$ has a composition factor of the $j$th class if and only if $me^{(j)} \neq 0$.

Now let $M$ be an arbitrary irreducible $a$-module, and let $M_0$ be the $a$-submodule of $M$ constructed in Lemma 3. Let $F_1$ be any element of $B$ such that $F_1 T = e^{(1)}$. Then if $u = U + M_0x$ is an element of the $\phi$-module $M_0/M_0x$, we have $ue^{(j)} = UF_1 + M_0x$. We have proved the following lemma.

Lemma 4. A $\phi$-module $M_0/M_0x$ has a composition factor in the $j$th class if and only if $M_0 F_1 \subseteq M_0x$, where $F_1$ is any element of $B$ such that $F_1 T = e^{(1)}$.

7. The Cartan-Weyl theory of weights of representations of a modular Lie algebra. We begin with some general remarks on the representations of associative algebras (see [10] for a complete discussion). Let $a$ be a finite dimensional algebra over a field $E$. An extension field $F \supseteq E$ is called a splitting field if every irreducible $a^p$-module is absolutely irreducible, that is, it remains irreducible for arbitrary extensions of the field $F$. Let $n$ be the radical of $a^p$. Then $F$ is a splitting field if and only if $a^p/n$ is a direct sum of full matrix algebras over $F$. If $F$ is a splitting field and $\Omega \supseteq F$, then $\Omega$ is a splitting field. If $F \subseteq \Omega$, $F$ a splitting field, then every irreducible $a^p$-module $q$ is equal to $m^q$,
where \( m \) is an \( a-F \) submodule of \( q \). This property is characteristic of splitting fields.

**Lemma 5.** Let \( E \) be a perfect field, and let \( \Omega \supseteq E \) be a splitting field. Then \( E \) is a splitting field if every irreducible \( a^0 \)-module \( q \) is equal to \( m^a \), where \( m \) is an \( a-E \)-submodule of \( q \).

**Proof.** Let \( n \) be the radical of \( a \). Then \( a/n \) is a separable algebra, and it follows that \( n^0 \) is the radical of \( a^0 \), so that

\[
(a/n)^0 \cong a^0/n^0 \cong \Omega_{h_1} \oplus \cdots \oplus \Omega_{h_r}
\]

where \( \Omega_h \) is the algebra of \( h \times h \) matrices over \( \Omega \). For each \( i, 1 \leq i \leq r \), there exists a minimal right ideal \( q_i \) in \( a^0/n^0 \) of dimension \( h_i \) over \( \Omega \) (corresponding to the \( i \)th direct summand \( \Omega_{h_i} \)), which, by the hypothesis, has the form \( q_i = (m_i)^0 \), where the \( m_i \) are irreducible \( a-E \)-modules of dimension \( h_i \) over \( E \). Clearly no two of the \( m_i \) can be \( a \)-isomorphic. Therefore, if \( t_i \) is the dimension over \( E \) of the centralizer of \( m_i \), it follows that the dimension \( d \) over \( E \) of \( a/n \) is not less than \( \sum h_i^2 t_i \). On the other hand \( d = \sum h_i^2 \) by (18). Therefore \( \sum h_i^2 \geq \sum h_i^2 t_i \), and hence \( t_i = 1 \) for all \( i \). It now follows from Burnside's Theorem that \( a/n \) is a direct sum of full matrix algebras over \( E \), and that \( E \) is a splitting field.

**Theorem 3.** Let \( I \) be a separable modular algebra \( \Sigma/p \Sigma \), where \( p \) is a non-exceptional prime ideal in \( K \). Then \( K \) is a splitting field for the \( u \)-algebra \( a \) of \( I \). Every irreducible \( a \)-module \( m \) is a direct sum of weight spaces \( m_\lambda \) belonging to the distinct weights of \( m \).

**Proof.** Let \( \Omega \) be the algebraic closure of \( K \); obviously \( \Omega \) is a splitting field. Moreover \( K \) is a finite field and hence perfect. By Lemma 5, \( K \) is a splitting field if we can prove that an arbitrary irreducible \( a^0 \)-module \( q \) is equal to \( m^a \), where \( m \) is an \( a-K \)-submodule of \( q \). Since \( \Sigma \) is a finite dimensional commutative subalgebra of \( I \), it is immediate that \( q \) has at least one weight \( \lambda \); let \( u \neq 0 \) belong to \( \lambda \). If \( x_\alpha \) is one of the root elements among the \( (x_i) \) then \( ux_\alpha \) is either zero, or belongs to the weight \( \lambda + \alpha \). Consider the \( K \) subspace \( m \) generated by the elements

\[
ux_{i_1} \cdots x_{i_r}, \quad r \geq 0,
\]

where either \( 1 \leq i_j \leq m \) or \( m+l+1 \leq i_j \leq n \); each of these elements is either a weight vector or zero. We observe that if \( w \) is any weight vector belonging to a weight \( \mu \), then because of (13), \( \mu(x_i)^0 = \mu(x_i) \), \( m + 1 \leq i \leq m + l \), and hence \( \mu(x_i) \subseteq \mathcal{R}_0 \), where \( \mathcal{R}_0 \) is the prime field contained in \( K \), for \( m + 1 \leq i \leq m + l \). It follows from this remark that \( ma \subseteq m \), and \( m^a a^0 \subseteq m^a \). By the irreducibility of \( q \), we have \( m^a = q \), and the first part of the theorem is proved.

Now let \( m \) be an irreducible \( a \)-module, \( \Omega \) the algebraic closure of \( K \); then \( m^a \) is irreducible, and hence \( m^a = m_1^a \), where \( m_1 \) is an \( a \)-submodule generated by weight vectors of the form (19). By the remarks preceding Lemma 4,
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m and m₁ are a-isomorphic. But since m₁ has a basis consisting of weight vectors, we have \( m₁ = \sum (m₁)_λ \), where the \( λ \) are the distinct weights of m₁. Since \( m ≅ m₁ \), the same statement applies to m, and the theorem is proved.

Remark. Let m be an arbitrary a-module such that m is spanned by weight vectors. If \( λ₁, \cdots, λ_ν \) are the distinct weights of m, then \( m = \sum_{λ=1}^{ν} mλ \) (direct sum). From this we see that if m has a basis consisting of weight vectors belonging to weights \( λ₁, \cdots, λ_ν \), then every weight of m appears among the \( λ_i \). The proofs of these facts are easy, and we omit them.

The (1-1) mapping \( α → \bar{α} \) of the roots of \( Φ \) onto the roots of I defines a linear order relation among the roots of I; thus we shall write \( α > 0 \) if \( α > 0 \), and \( \bar{α} < \bar{β} \) if \( α < β \).

Lemma 6. Let m be a a-module, and let m have a weight \( λ \) such that for some weight vector \( u ≠ 0 \) belonging to \( λ \), \( ux_i = 0 \), for \( 1 ≤ i ≤ m \). Then the subspace \( m_1 \) generated by all elements \( uz(0, 0, R) \) is an a-submodule.

Proof. Clearly \( m_1x_i ⊆ m_1 \) for \( m + 1 ≤ i ≤ n \). Let \( x_α \) be a root vector belonging to a positive root \( α \), and let \( uz(0, 0, R) = ux_1 \cdots xᵢ \) be a generator of \( m_1 \). Since \( ux_{α} = 0 \), we can assume that \( uz(0, 0, R')x_α = ux_1 \cdots xᵢx_α \subseteq m_1 \) whenever \( s < r \). If \( \bar{α} \) is the least positive root, then, if \( xᵢ = x_β \), we have

\[
ux_i \cdots xᵢx_α = ux_i \cdots xᵢ₋₁xᵢ₋₁ \cdots + ux_i \cdots xᵢ₋₁ \cdot [x_βx_α],
\]

which is in \( m_1 \) because of our induction hypothesis, and because \( [x_βx_α] \) is either zero, or in \( h \), or a multiple of \( x_β+α \) where \( β + α < 0 \), otherwise \( 0 < β + α < α \), and \( 0 < β + α < α \) since the mapping \( α → \bar{α} \) preserves sums of roots and the order relation. Thus we can also assume that \( m_1x_γ ⊆ m_1 \) for all \( γ < α \). Then by (20), \( ux_i \cdots x_i x_α = ux_i \cdots xᵢ₋₁ [x_βx_α] \) (mod \( m_1 \)), where \( [x_βx_α] \) is either zero, or in \( h \), or a multiple of \( x_β+α \), where \( β + α < α \). In all cases, \( ux_i \cdots x_i x_α \subseteq m_1 \) by our induction hypothesis, and the lemma is proved.

We shall call a weight \( λ \) of an a-module m a leading weight of m if there exists a nonzero vector \( u \) belonging to \( λ \) such that

\[
ux_i = 0, \quad 1 ≤ i ≤ m,
(21)
\]

\[
ux_i \cdots xᵢ = 0
(22)
\]

whenever \( r > 0 \) and the \( xᵢ \) belong to negative roots \( αᵢ \) such that \( \sum_{j=1}^{r-1} αᵢ = 0 \). Even though the system of weights of an a-module m may not admit a linear order relation, we shall call \( λ \) a highest weight of m if \( λ + α \) is not a weight for all roots \( α > 0 \).

We remark first that if \( λ \) is a highest weight of m, then \( λ \) is a leading weight of m. Obviously condition (21) is satisfied. If an expression of the form (22) is different from zero, then \( ux_i \cdots xᵢ₋₁ ≠ 0 \) belongs to the weight \( λ + α_i + \cdots + α_r = λ - α_r \), where \( -α_r > 0 \) since \( -α_r \) corresponds to \( -α_r \), and this contradicts our assumption that \( λ \) is a highest weight.
It is possible to verify that the three dimensional simple modular Lie algebra $I$ of characteristic $p$ obtained from the three dimensional unimodular Lie algebra of characteristic zero at a nonexceptional prime has exactly $p$ inequivalent irreducible restricted representations (counting the trivial one dimensional representation), of degrees $1, 2, \cdots, p$. The representations of degree $1, 2, \cdots, p-1$ all have a highest weight, while the representation of degree $p$ has a leading weight which is not a highest weight.

The significance of the concept of leading weight is clarified by the following result.

**Theorem 4.** Let $m$ be an irreducible $a$-module which possesses a leading weight $\lambda$. Then the dimension of $m_\lambda$ is equal to one. Let $m$ and $m'$ be irreducible $a$-modules which have leading weights $\lambda$ and $\lambda'$ respectively. Then $m$ and $m'$ are $a$-isomorphic if and only if $\lambda = \lambda'$.

**Proof.** Let $u \neq 0$ be a vector in $m$ belonging to $\lambda$ which satisfies conditions (21) and (22). Since $m$ is irreducible, $m$ has a basis consisting of elements of the form (19), each of which is a weight vector. The first assertion of the theorem will be proved if we can show that any vector of the form (19) which has weight $\lambda$ is a multiple of $u$. Actually we prove somewhat more, namely, that if $w = \alpha x_{i_1} \cdots x_{i_r}$ is an element of the form (19) of weight $\lambda$, then $w = \alpha u$, where $\alpha$ is an element of $K$ which depends only upon $\lambda$, and the sequence of root vectors $x_{i_1}, \cdots, x_{i_r}$, and not upon the action of $a$ upon $m$. Let $\beta_1, \cdots, \beta_r$ be the roots corresponding to $x_{i_1}, \cdots, x_{i_r}$. Then $\sum \beta_i = 0$. If all $\beta_i < 0$, then $w = 0$ by (22). We have thus reduced the problem to the situation considered by Cartan and Weyl (see [13, p. 282]), so that if $w \neq 0$, then at least one of the $\beta_i > 0$. The argument of Cartan and Weyl, together with the properties (21) and (22) of a leading weight, can be applied to prove the first assertion.

Now let $m$ and $m'$ be irreducible $a$-modules having the same leading weight. The result established in the first part of the proof, and Weyl's method [13, p. 283] lead at once to the statement that $m$ and $m'$ are $a$-isomorphic. It is not completely trivial, however, to prove that if $m$ and $m'$ are $a$-isomorphic irreducible modules which possess leading weights $\lambda$ and $\lambda'$, then $\lambda = \lambda'$. Suppose that $\lambda \neq \lambda'$, and suppose that there exists an $a$-isomorphism $S$ of $m'$ onto $m$. Let $u \neq 0$ belong to $\lambda$, and let $u' \neq 0$ belong to $\lambda'$, where both $u$ and $u'$ satisfy the defining properties (21) and (22) of a leading weight. Then $u'S$ belongs to $\lambda'$, and it follows that both $\lambda$ and $\lambda'$ are leading weights of $m$. Since $m$ is irreducible, by Lemma 6 and the remark after Theorem 3, there exist root vectors $x_{i_1}, \cdots, x_{i_r}$ belonging to negative roots such that $u_i = \alpha x_{i_1} \cdots x_{i_r}$ is a nonzero element of $m_{\lambda'}$. Moreover $r > 0$ since $\lambda \neq \lambda'$. Because $\lambda'$ is a leading weight, and $m_{\lambda'}$ is one dimensional, it follows that $u_i x_i = 0$, $1 \leq i \leq m$. We can apply Lemma 6 again, and obtain negative root vectors $x_{i_1}, \cdots, x_{i_s}$, $s > 0$, such that $u x_{i_1} \cdots x_{i_s} x_{i_1} \cdots x_{i_s}$ is a nonzero ele-
ment of \( m \). This statement contradicts our assumption that \( \lambda \) is a leading weight, and the theorem is proved.

We emphasize that all the statement in Theorem 4 are valid if the words "leading weight" are replaced by "highest weight."

8. The action of the Weyl group of \( L \) upon the system of weights of an \( a \)-module. The Weyl group of \( L \) with respect to \( F \) is the group of l.t. in the dual space \( F^* \) of \( F \) generated by the reflections \( S_\alpha \), where \( \Delta S_\alpha = \Lambda - \Lambda(H_\alpha)\alpha \), \( \Lambda \in F^* \), and \( \alpha \) is a root. It is known that the reflections determined by a fundamental system of roots generate the Weyl group, and that if \( \alpha \) and \( \beta \) are roots, then \( \alpha S_\beta \) is a root. Similarly we define the Weyl group of \( L \) with respect to \( F \) to be the group of l.t. in the dual \( F^* \) of \( F \) generated by the l.t. \( S_\alpha \), \( \alpha \) a root, where \( \lambda s_\alpha = \lambda - \lambda(h_\alpha)\alpha \), and \( h_\alpha = H_\alpha T \). We observe that \( s_\alpha^2 = 1 \) and that \( s_{-\alpha} = s_\alpha \). Moreover, if \( \alpha \) and \( \beta \) are roots, then \( \alpha s_\beta \) is a root. Since \( \alpha(h_\beta) = \phi(\alpha(H_\beta)) \) by (10), we have

\[
\phi((\alpha S_\beta)(H_i)) = \phi(\alpha(H_i)) - \phi(\alpha(H_\beta))\phi(\beta(H_i)) = (\alpha s_\beta)(H_i; T)
\]

for \( 1 \leq i \leq l \), proving that \( \alpha s_\beta \) is the root of \( L \) corresponding to the root \( \alpha S_\beta \) of \( F \).

The following result is based on a theorem of E. Cartan [3, p. 360] for representations of complex semi-simple algebras. As in the classical theory, a weight \( \lambda \) of an \( a \)-module \( m \) is called extreme if it is impossible to find a root \( \alpha \) such that both \( \lambda + \alpha \) and \( \lambda - \alpha \) are weights.

**Theorem 5.** Let \( m \) be an irreducible \( a \)-module. If \( m \) has an extreme weight, then \( m \) has a highest weight, and the Weyl group of \( L \) acts transitively upon the set of extreme weights of \( m \).

Before giving the proof, we establish some preliminary results.

**Lemma 7.** If \( \lambda \) is a weight of an \( a \)-module \( m \), then so is \( \lambda s_\alpha \) for every root \( \alpha \). If \( \lambda \) is extreme, then \( \lambda s_\alpha \) is extreme. If \( \lambda \) is extreme, and if \( \lambda + \alpha \) is a weight, then for any vector \( u \) belonging to \( \lambda \), the weights belonging to \( u x_\alpha^k \), \( i = 0, 1, \cdots \), where \( x \) is the root vector among the \( x_i \) belonging to \( \alpha \), include \( \lambda s_\alpha \).

**Proof.** Since \( H_\alpha \) is a linear combination of the \( H_i \) with coefficients in \( Q \cap \alpha \) for all roots \( \alpha \), it follows that \( \lambda(h_\alpha) = \lambda(H_\alpha T) \in \mathbb{K}_0 \). If \( \lambda + k\alpha \) is a weight for all \( k \in \mathbb{K}_0 \) then \( \lambda s_\alpha \) is a weight since \( \lambda(h_\alpha) \in \mathbb{K}_0 \). If not all \( \lambda + k\alpha \) are weights, then by Lemma 5.1 [12], the weights of this form lie in disjoint arithmetic progressions, each symmetric about \( \lambda - 2^{-1}\lambda(h_\alpha)\alpha \). Since \( \lambda \) and \( \lambda s_\alpha \) are symmetric, \( \lambda s_\alpha \) is a weight.

If \( \lambda s_\alpha \) is not extreme, then \( \lambda s_\alpha \pm \beta \) are weights for some root \( \beta \). Applying \( s_\alpha \), and using the first statement of the lemma, we infer that \( (\lambda s_\alpha \pm \beta)s_\alpha = \lambda \pm \beta s_\alpha \) are both weights. But we have shown that \( \beta s_\alpha \) is a root, hence \( \lambda \) is not extreme, and the second assertion is proved.

(*) This lemma is well known (see [13; 12]); we include the argument in order to emphasize those facts which we need for the proof of Theorem 5.
Finally let \( \lambda \) be extreme, and let \( \lambda + \alpha \) be a weight. If \( u \neq 0 \) belongs to \( \lambda \), then \( ux_{-\alpha} = 0 \) since \( \lambda \) is extreme. We form \( u_i = ux_{\alpha_i}^i, i = 0, 1, 2, \cdots \), and prove by induction that \( u_i x_{-\alpha} = c_i u_{i-1} \), where \( c_i = -i\lambda(h_\alpha) - i(i-1) \). For some \( r < p - 1 \) we have \( u_r \neq 0 \), \( u_{r+1} = 0 \), and it follows that \( c_{r+1} = -(r+1)\lambda(h_\alpha) - (r+1)r = 0 \). Since \( r+1 \neq 0 \), \( r = -\lambda(h_\alpha) \), and \( u_r \) belongs to \( \lambda s_\alpha \). This completes the proof.

**Lemma 8.** Let \( \alpha \) be the \( \mu \)-algebra of \( \mathfrak{l} \). Then the subalgebra \( \mathfrak{s} \) of \( \alpha \) generated by the root vectors \( x_i, 1 \leq i \leq m \), belonging to the positive roots, is a nilpotent algebra.

**Proof.** \( \mathfrak{s} \) is the enveloping algebra of the subspace \( \mathfrak{g} \) of \( \mathfrak{l} \) generated by the \( x_i, 1 \leq i \leq m \), and will be nilpotent by a result of Jacobson \([11]\) if we can prove that \( \mathfrak{g} \) is a (Lie) subalgebra of \( \mathfrak{l} \) all of whose elements are nilpotent in \( \alpha \). Now \( \mathfrak{g} = \Theta \mathfrak{T} \), where \( \Theta \) is the \( \sigma \)-submodule of \( \Sigma \) generated by the \( X_i, 1 \leq i \leq m \). Since \( [X_i X_j] \) is either zero or a multiple of a root vector belonging to a positive root, for \( 1 \leq i, j \leq m \), \( \Theta \) is closed under the bracket operation, and hence \( \mathfrak{g} \) is a subalgebra of \( \mathfrak{l} \).

We prove now that a \( p \)-power of every element of \( \Theta \) is in \( \mathfrak{x} \), the kernel of \( \mathfrak{T} \). By (13), \( X_i^p \in \mathfrak{x} \) for \( 1 \leq i \leq m \). We shall use the identity (see \([16, \text{p. 91}]\))

\[
(X + Y)^p = X^p + Y^p + s(X, Y) \quad (\text{mod } p \mathfrak{g})
\]

where \( X, Y \in \mathfrak{g} \), and \( s(X, Y) \) is a sum of commutators of \( p \)-factors which are either \( X \) or \( Y \). Now let \( Y = \sum_a x_i \) be an element of \( \Theta \); then from (23) we have

\[
Y^p = \sum a_i^p x_i^p + Y^{(1)} \quad (\text{mod } p \mathfrak{g})
\]

where \( Y^{(1)} \in \Theta \), and \( Y^{(1)} \) has the property that the minimum root \( \alpha_i \) whose coefficient in \( Y^{(1)} \) is not zero is greater than the minimum root whose coefficient in \( Y \) is not zero. If we iterate this process, and use the fact that the number of roots is finite, we obtain \( Y^{tp} \in \mathfrak{f} \) for some integer \( t \geq 0 \). This proves the lemma.

**Proof of Theorem 5.** Let \( \lambda \) be an extreme weight of \( \mathfrak{m} \). We prove that there exists an elements \( s \) of the Weyl group of \( \mathfrak{l} \) such that \( \lambda s \) is a highest weight of \( \mathfrak{m} \). Since \( \mathfrak{m} \) is irreducible, it follows from Theorem 4 that \( \mathfrak{m} \) has at most one highest weight, and we shall have proved that all the extreme weights of \( \mathfrak{m} \) lie in one system of transitivity relative to the Weyl group, namely the one that contains a highest weight.

If \( \lambda \) is a highest weight, there is nothing to prove. Suppose that \( \lambda + \alpha \) is a weight for some \( \alpha > 0 \), and let \( u \neq 0 \) be a vector belonging to \( \lambda \). By Lemma 7, there exists an integer \( r \geq 0 \) such that \( ux_{\alpha}^r \) belongs to \( \lambda s_\alpha \), which is again an extreme weight. We assert that \( r > 0 \), for if \( r = 0 \), \( \lambda s_\alpha = \lambda \), and the center of
symmetry for the arithmetic progressions of weights of the form \( \lambda + k\alpha \) is \( \lambda \). Therefore since \( \lambda - \alpha \) and \( \lambda + \alpha \) are symmetric about \( \lambda \), \( \lambda - \alpha \) is a weight, contrary to our assumption that \( \lambda \) is extreme.

We now repeat the argument with \( \lambda \), and the vector \( ux_\alpha^r \), \( r > 0 \), belonging to \( \lambda \). If a highest weight is not obtained by this process, then the construction yields products \( x_\alpha \cdots x_\alpha \neq 0 \), where the \( \alpha > 0 \), containing an arbitrarily large number of factors. But this is contrary to the fact that the subalgebra \( \mathfrak{g} \) generated by the \( x_i, 1 \leq i \leq m \), is nilpotent, by Lemma 8, and the theorem is proved.

9. Every irreducible \( \alpha \)-module with a leading weight is a constituent of a \( \phi \)-module. The main theorem can be stated as follows.

**Theorem 6.** Let \( \mathfrak{m} \) be an irreducible \( \alpha \)-module which has a leading weight \( \lambda \neq 0 \). Let \( \Lambda \) be the dominant integral function on \( H \) such that \( \Lambda(H) \) is the rational integer, \( 0 \leq \Lambda(H) \leq p - 1 \) such that \( \phi(\Lambda(H)) = \lambda(H,T) \), \( 1 \leq i \leq l \), and let \( \mathfrak{M} \) be an irreducible \( \mathfrak{M} \)-module whose highest weight is \( \Lambda \). Then \( \mathfrak{m} \) is \( \alpha \)-isomorphic to a composition factor of a \( \phi \)-module \( \mathfrak{M}_0/\mathfrak{M}_0' \) belonging to \( \mathfrak{M} \).

**Proof.** For simplicity we shall write \( h_i \) for \( H_i, T, \) and \( H_i \) for \( X_{m+i}, 1 \leq i \leq l \). Find \( h_{i_0}, 1 \leq i_0 \leq l \), such that \( uh_{i_0} \neq 0 \), where \( u \) is a fixed vector belonging to \( \mathfrak{m} \). Let \( h = \lambda(h_{i_0})^{-1}h_{i_0} \); then \( uh = u \). Let \( e^{(i)} \) be the idempotent element of \( \alpha \) such that \( ve^{(i)} = v \) for all \( v \) in \( \mathfrak{m} \), which was constructed in the discussion preceding Lemma 4. Let \( (0:u) = \{ f \mid f \in \alpha, uf = 0 \} \). Then there exists \( g \in (0:u) \) such that \( e^{(i)} = h + g \). Let \( g = \sum \alpha_i z(P_i, Q_i, R_i), \alpha_i \in \mathbb{R} \).

Let \( U \) be an element of weight \( \Lambda \) in \( \mathfrak{M} \) such that the elements \( UZ(0, 0, R) \) generate \( \mathfrak{M}_0 \). Let \( H = cH_{i_0} \), where \( \phi(c) = \lambda(h_{i_0})^{-1} \); then \( \phi(c\Lambda(H_{i_0})) = 1 \). Let \( G = \sum \alpha_i z(P_i, Q_i, R_i) \), where the \( \alpha_i \) are elements of \( \alpha \) such that \( \phi(\alpha_i) = \hat{\alpha}_i \), and let \( E = H + G \); then \( HT = h, GT = g, \) and \( ET = h + g = e^{(i)} \). By Lemma 4, it is sufficient to prove that \( UE \in \mathfrak{M}_0 \).

We write \( Y_j = a_j z(P_j, Q_j, R_j) \), and \( y_j = a_j z(P_j, Q_j, R_j) \); then \( G = \sum \alpha_i z(P_i, Q_i, R_i) \), \( g = \sum \alpha_i z(P_i, Q_i, R_i) \). We define two subsets \( J \) and \( J' \) of \( I \) as follows:

\[ J = \{ v \mid v \in I, Uy_v \neq 0, Uy_v \in \mathfrak{M}_0 \}, \]
\[ J' = \{ v \mid v \in I, uy_v \neq 0, uy_v \in \mathfrak{M}_0 \}. \]

We prove: (i) if \( v \in J \) then \( P_\star = R_\star = 0, \) and \( Y_\star = a_\star Z(0, Q_\star, 0) \); and (ii) \( J' \subseteq J \). First we observe that if \( P_\star = R_\star = 0 \), and if \( Q_\star = (j_1, \ldots, j_l) \), then

\[ UZ(0, Q_\star, 0) = \Lambda(H_1)h_1 \cdots \Lambda(H_l)h_l U \]
and

\[ uz(0, Q_\star, 0) = \lambda(h_1)h_1 \cdots \lambda(h_l)h_l U = \phi(\Lambda(H_1)h_1 \cdots \Lambda(H_l)h_l) u \]
since \( \phi(\Lambda(H_1)) = \lambda(h_1) \).

Now let \( v \in J' \), and consider \( uy_\star = \hat{\alpha}_i uz(P_\star, Q_\star, R_\star) \), \( uy_\star \in \mathfrak{M}_0 \), \( uy_\star \neq 0 \). If \( P_\star \neq 0 \), then \( uy_\star = 0 \). If \( P_\star = 0, R_\star \neq 0 \), then
\[ u y_r = \bar{a}_r \lambda(h_1)^i_1 \cdots \lambda(h_1)^i_n u z(0, 0, R, s) \in m_n, \]

and \( u y_r = 0 \) by (22), since \( \lambda \) is a leading weight of \( m \). Therefore \( P_s = R_s = 0 \). Similarly if \( v \in J \), then \( P_s = R_s = 0 \), proving (i). If \( v \in J' \), then

\[ u y_r = a_r \Lambda(h_1)^i_1 \cdots \Lambda(h_1)^i_n u z(0, 0, R, s) \in m_n, \]

and by (24),

\[ u z(0, Q, 0) = \phi(\Lambda(H_1)^i_1 \cdots \Lambda(H_1)^i_n) u \neq 0 \]

and \( a_r \neq 0 \), hence \( U Y_r = a_r \Lambda(H_1)^i_1 \cdots \Lambda(H_1)^i_n U \neq 0 \), and (ii) is proved.

If \( v \in J \), then \( P_v = R_v = 0 \) by (i), and hence \( u y_v \in m_n \).

Now let \( Y = \sum_{r \in J} Y_r \); we prove that \( U Y \in p \mathfrak{m}_0 \subset \mathfrak{m}_0 \mathfrak{x} \). By (i) we have \( U Y = d U \), where

\[ d = \sum_{r \in J} a_r \Lambda(H_1)^i_1 \cdots \Lambda(H_1)^i_n, \quad Q_r = (j_1, \ldots, j_i), \]

Let \( y = \sum_{r \in J} y_r \); then \( u y \in m_n \), and since \( J' \subseteq J \) by (ii), \( u y \) is the component of \( u g \) of weight \( \lambda \) in the expression of \( u g \) as a sum of vectors belonging to distinct weights. Since \( u g = 0 \), \( u y = 0 \) because vectors belonging to distinct weights are linearly independent. On the other hand, \( u y = e u \), where

\[ e = \sum_{r \in J} d \Lambda(h_1)^i_1 \cdots \Lambda(h_1)^i_n, \quad Q_r = (j_1, \ldots, j_i), \]

and \( e = \phi(d) = 0 \), and \( d \in p \).

Now write \( G = Y + Y' \), where \( Y' = \sum_{r \in J} Y_r \). Since \( E = H + G \) we have

\[ U E = U H + U Y + U Y' = (c \Lambda(H_1) + d) U + U Y', \]

and \( c \Lambda(H_1) + d \equiv 1 \) (mod \( p \)). From the definition of \( J \) it follows that if \( U Y' \) is expressed as an \( \alpha \)-linear combination of the generators \( U Z(0, 0, R, s) \) of \( \mathfrak{m}_n \), the coefficient of \( U \) is zero. If \( W \in \mathfrak{m}_n \), then although an expression \( W = \sum a_i U Z(0, 0, R, s), a_i \in \mathfrak{m}_0 \), is not uniquely determined, since \( U \) is the only generator of weight \( \Lambda \), the coefficient of \( U \) is uniquely determined, and will be called the \( \Lambda \)-component of \( W \). We have shown that the \( \Lambda \)-component of \( U E \) is \( \equiv 1 \) (mod \( p \)). The theorem will be proved if we can show that the \( \Lambda \)-component of any element of \( \mathfrak{m}_n \mathfrak{x} \) is in \( p \), and this we shall do by using the generators of \( \mathfrak{x} \) (see Lemma 2). An arbitrary element of \( \mathfrak{m}_n \mathfrak{x} \) is a sum of terms of the following types, and it is sufficient to verify in each case that the \( \Lambda \)-component is in \( p \).

(a) \( W \in p \mathfrak{m}_0 \) implies that the \( \Lambda \)-component is in \( p \).

(b) \( W = U F X_i^\prime G, F, G \in \mathfrak{B}, 1 \leq i \leq m \); then \( W \equiv U X_i^\prime F G \) (mod \( p \mathfrak{m}_0 \)) by (17), and hence \( W \equiv 0 \) (mod \( p \mathfrak{m}_0 \)).

(c) \( W = U (H_i^\prime - H_i) G, 1 \leq i \leq l \); then

\[ W \equiv U (H_i^\prime - H_i) F G \] (mod \( p \mathfrak{m}_0 \))

\[ \equiv (\Lambda(H_i^\prime)^p - \Lambda(H_i)) U F G \] (mod \( p \mathfrak{m}_0 \))

\[ \equiv 0 \] (mod \( p \mathfrak{m}_0 \))

by (17).

(d) \( W = U F X_i^\prime G, m + l \leq i \leq n \); then
\[ W = UFGX^p \pmod{\mathfrak{m}_0} \]
\[ = \sum a_k UZ(0, 0, R_k)X^p \pmod{\mathfrak{m}_0}, \quad a_k \in \mathfrak{o}, \]
and the \( \Lambda \)-component of \( \sum a_k UZ(0, 0, R_k)X^p \) is zero. This completes the proof of the theorem.

**Remarks.** In a subsequent paper we plan to show how a theorem analogous to Theorem 6, in combination with results of Seligman [12] and Harish-Chandra [7, Theorems 1 and 2], can be applied to prove that the Weyl matrix of a modular separable algebra determines the algebra up to (restricted) isomorphism, and that every separable algebra of characteristic \( p \) which satisfies certain conditions is isomorphic to a modular Lie algebra.

The following problem may be raised in connection with Theorem 6, but remains unsolved. Find necessary and sufficient conditions in order that a linear function \( \lambda \) on a Cartan subalgebra \( \mathfrak{h} \) of a modular separable Lie algebra \( \mathfrak{g} \) be a leading weight of an irreducible \( \mathfrak{g} \)-module, where \( \mathfrak{g} \) is the \( \mathfrak{u} \)-algebra of \( \mathfrak{g} \).

**References**