

LEAST p TH POWER POLYNOMIALS ON A FINITE POINT SET⁽¹⁾

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The writers have recently made a study [6] of approximation to an arbitrary function by polynomials on a finite point set $E: (z_1, z_2, \dots, z_m)$, with especial reference to the study of polynomials $z^n + a_1 z^{n-1} + \dots$ of least norm on real E . The present paper is a continuation of that study of the anatomy of such polynomials, but where now we emphasize nonreal E . Notation and terminology are uniform with the preceding paper, to which the reader may refer for details.

In §1 we study the determination of T -polynomials of degree $m-1$ for the *norm*

$$(0.1) \quad \sum_{k=1}^m \mu_k |T_{m-1}(z_k)|^p, \quad \mu_k > 0, p > 0,$$

in the various cases $p=1$, $p>1$, $0<p<1$; a T -polynomial of degree n is merely a polynomial $T_n(z) \equiv z^n + \dots$ of least norm. In §§2 and 3 we study the geometry of zeros of extremal polynomials, including (§3) analogs and extensions of Fejér's Principle. In §4 we consider extremal polynomials on point sets which are modified by adjunction or deletion of various points, and in §5 study real polynomials, showing that for real E three sets of polynomials are identical: those whose zeros separate a subset of E , those minimizing norm (0.1) with $p=1$ and unprescribed weights, and the set of infrapolynomials on E . In §6 we establish the orthogonality conditions relating to (1), and in §7 consider for fixed degree the polynomials of best approximation on an interval as limits of polynomials of best approximation on a finite set E .

1. **Totality of T -polynomials**, $n=m-1$. For reference we state a result already established [6, Theorem 8]:

THEOREM 1.1. *Let $E: (z_1, z_2, \dots, z_m)$ consist of $m(>1)$ distinct points, and let the μ_k be positive. Then the totality of T -polynomials of degree $m-1$, namely $T_{m-1}(z) \equiv z^{m-1} + \dots$ of minimum norm*

$$(1.1) \quad \mu(T_{m-1}) = \sum_1^m \mu_k |T_{m-1}(z_k)|, \quad \mu_k > 0,$$

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is found as follows. With the notation $\omega(z) = \prod^m (z - z_k)$, let E' be the subset of E on which the numbers $\mu_k |\omega'(z_k)|$ take their least value on E . Choose the number λ_k ($k = 1, 2, \dots, m$) as zero if z_k belongs to $E - E'$ and as an arbitrary non-negative number if z_k belongs to E' , subject to the restriction $\sum_1^m \lambda_k = 1$. Then we have

$$(1.2) \quad T_{m-1}(z) \equiv \omega(z) \sum_1^m \frac{\lambda_k}{z - z_k}.$$

Equation (1.2) is essentially Lagrange's interpolation formula

$$T_{m-1}(z) \equiv \sum_1^m \frac{T_{m-1}(z_k)}{\omega'(z_k)} \frac{\omega(z)}{z - z_k},$$

where we have $T_{m-1}(z_k) = \lambda_k \omega'(z_k)$.

That (1.2) is essentially Lagrange's interpolation formula follows from inspection of the terms in that formula involving z^{m-1} :

$$1 = \sum_1^m \frac{T_{m-1}(z_k)}{\omega'(z_k)}.$$

It may be noticed that for each point z_k of E we have

$$(1.3) \quad \arg T_{m-1}(z_k) = \arg \omega'(z_k),$$

whenever the first member is defined. Reciprocally, if $T_{m-1}(z) \equiv z^{m-1} + \dots$ is a polynomial for which (1.3) holds for all k whenever the first member is defined, then (1.2) is valid with $\lambda_k \geq 0$, $\sum_1^m \lambda_k = 1$.

A necessary and sufficient condition that (1.2) with all λ_k subject merely to the requirements $\lambda_k \geq 0$, $\sum_1^m \lambda_k = 1$ should represent the totality of extremal polynomials is that $\mu_k |\omega'(z_k)|$ be independent of k .

In order to prove the analog of Theorem 1.1 for the case $p > 1$ we need the

LEMMA. *The minimum of the function $\mu(A_1, A_2, \dots, A_m) \equiv \sum_1^m \rho_k A_k^p$, $\rho_k > 0$, $p > 1$, subject to the conditions $A_k \geq 0$, $\sum_1^m A_k = 1$, is given by*

$$A_k = \rho_k^{-1/(p-1)} / \sum_1^m \rho_j^{-1/(p-1)}.$$

The existence of a minimum is immediate, for the function μ is continuous and the point set S to which the (A_1, A_2, \dots, A_n) are constrained is closed and bounded. Use of the Lagrange multiplier Λ and partial differentiation with respect to A_k of

$$\sum \rho_k A_k^p - \Lambda \sum A_k$$

yields a local minimum, unique by the convexity of the function μ and the convexity of S .

THEOREM 1.2. Under the conditions of Theorem 1.1 except that (1.1) is replaced by

$$(1.4) \quad \mu(T_{m-1}) \equiv \sum_1^m \mu_k |T_{m-1}(z_k)|^p, \quad p > 1,$$

the T -polynomial $T_{m-1}(z)$ of degree $m-1$ is unique and defined by

$$(1.5) \quad T_{m-1}(z_k) = [\mu_k |\omega'(z_k)|^p]^{-1/(p-1)} \omega'(z_k) / \sum_1^m [\mu_j |\omega'(z_j)|^p]^{-1/(p-1)}.$$

By definition the polynomial $T_{m-1}(z)$ is characterized as that polynomial $P(z)$ of degree $m-1$ with

$$1 = \sum_{k=1}^m \frac{P(z_k)}{\omega'(z_k)}$$

for which the norm

$$\sum_1^m \mu_k |\omega'(z_k)|^p \left| \frac{P(z_k)}{\omega'(z_k)} \right|^p$$

is a minimum. It follows from the lemma that the minimum of $\mu(A_1, A_2, \dots, A_n)$ subject to the condition $\sum A_k = A$, a constant, increases as A increases. Thus the minimum of $\mu(T_{m-1})$ is given by the choice

$$\sum_{k=1}^m \left| \frac{P(z_k)}{\omega'(z_k)} \right| = 1$$

and (1.5).

It is clear that in every case we have $T_{m-1}(z_k) \neq 0$, and equation (1.3) is valid. Consequently if the points of E are collinear they are strongly separated by the zeros of $T_{m-1}(z)$; for the numbers $\omega'(z_k)$ and likewise the numbers $T_{m-1}(z_k)$ alternate in sign on the line containing E .

Reciprocally, every polynomial $P(z) \equiv z^{m-1} + \dots$ for which $P(z_k) \neq 0$, $\arg P(z_k) = \arg \omega'(z_k)$, is a T -polynomial for suitable choice of the μ_i . These conditions on $P(z)$ imply the equation

$$1 = \sum_{k=1}^m \frac{P(z_k)}{\omega'(z_k)} = \sum_{k=1}^m \left| \frac{P(z_k)}{\omega'(z_k)} \right|,$$

so we need merely define the μ_k by the equations

$$T_{m-1}(z_k) = [\mu_k |\omega'(z_k)|^p]^{-1/(p-1)} \omega'(z_k).$$

In particular if the points of E are collinear and strongly separated by the zeros of $P(z) \equiv z^{m-1} + \dots$, then $P(z)$ is a T -polynomial $T_{m-1}(z)$ for this choice of the μ_k .

For the case $0 < p < 1$ we shall prove

THEOREM 1.3. *Under the conditions of Theorem 1.2 except that $p > 1$ is replaced by $0 < p < 1$, the T -polynomial $T_{m-1}(z)$ of degree $m - 1$ is a polynomial which vanishes in all the z_k except the one z_k (or a z_k) in which $\mu_k |\omega'(z_k)|^p$ takes its minimum value, and $T_{m-1}(z)$ equals $\omega'(z_k)$ in that one z_k .*

For $0 < p < 1$ and arbitrary positive ξ_1 and ξ_2 we have

$$(1.6) \quad \xi_1^p + \xi_2^p > (\xi_1 + \xi_2)^p.$$

The minimum of $\xi^p + (A - \xi)^p$ in the interval $0 \leq \xi \leq A$ occurs for $\xi = 0$ and $\xi = A$. The minimum of $\rho_1 \xi_1^p + \rho_2 \xi_2^p$ with $0 < \rho_1 < \rho_2$ and $\xi_1 + \xi_2 = A$ in the interval $0 \leq \xi \leq A$, which is the minimum of $\rho_1(\xi_1^p + \xi_2^p) + (\rho_2 - \rho_1)\xi_2^p$, occurs for $\xi_2 = 0$. Continued application of (1.6) under the hypothesis of the lemma except with $0 < p < 1$ instead of $p > 1$ shows that $\min \mu(A_1, A_2, \dots, A_m)$ is found by choosing $A_k = 0$ except for a single one of the values of k for which ρ_k is least. Theorem 1.3 follows by the method of proof of Theorem 1.2.

Each $T_{m-1}(z)$ is of form (1.2) with one λ_k unity and the others zero. Again it is true that (1.3) is valid whenever the first member is defined. If the points of E are collinear, they are weakly separated by the zeros of $T_{m-1}(z)$.

Conversely to Theorem 1.3, any polynomial $P(z) \equiv z^{m-1} + \dots$ which vanishes in all the z_k except one, say z_j , is a T -polynomial of degree $m - 1$ for some choice of the μ_k . From the equation

$$1 = \sum_{k=1}^m \frac{P(z_k)}{\omega'(z_k)}$$

it follows that we have $\arg P(z_j) = \arg \omega'(z_j)$, so to exhibit $P(z)$ as a $T_{m-1}(z)$ we need merely choose the μ_k so that $\mu_k |\omega'(z_k)|^p$ is least for $k = j$.

Theorem 1.3 is somewhat similar to [6, Theorem 6], but is more specific in defining $T_{m-1}(z)$. On the other hand, the previous result is more general in that the functions involved are not necessarily polynomials in z .

Theorem 1.3 enables us to illustrate the nonuniqueness of the T -polynomial of degree n for the case $0 < p < 1$; we need merely choose $E: (-1, +1)$, $\mu_k = 1, n = 1$.

2. Geometry of zeros of T -polynomials. We have discussed [6, §7] in some detail the separation properties of the points of *real* E by the zeros of T -polynomials. We now devote some attention to the corresponding study for arbitrary $E: (z_1, z_n, \dots, z_m), m > 1$.

Let $T_1(z) \equiv z - \alpha$ denote the polynomial of degree unity which minimizes

$$(2.1) \quad \mu [T_1(z)] = \sum_1^m \mu_i |z_i - \alpha|, \quad \mu_i > 0.$$

For every z, α_1, α_2 we have

$$\left| z - \frac{\alpha_1 + \alpha_2}{2} \right| = \left| \frac{z - \alpha_1}{2} + \frac{z - \alpha_2}{2} \right| \leq \frac{1}{2} |z - \alpha_1| + \frac{1}{2} |z - \alpha_2|,$$

and this is a strong inequality unless either $\arg(z - \alpha_1) = \arg(z - \alpha_2)$ or one of these arguments is not defined. Thus if we have two distinct T -polynomials $z - \alpha_1$ and $z - \alpha_2$ with the same norm (2.1), half their sum is a T -polynomial of degree unity for which the norm (2.1) is smaller unless $\arg(z_i - \alpha_1) = \arg(z_i - \alpha_2)$ for every i for which these arguments are defined; any z_i not coinciding with α_1 or α_2 must be collinear with α_1 and α_2 . Consequently *the polynomial $T_1(z)$ with norm (2.1) is unique unless all the z_i are collinear; moreover, if the z_i are collinear, then $T_1(z)$ is unique and α lies on E unless with the linear order (z_1, z_2, \dots, z_m) we have for some k*

$$(2.2) \quad \sum_1^k \mu_i = \sum_{k+1}^m \mu_i,$$

in which case $T_1(z) \equiv z - \alpha$ and α may be chosen arbitrarily on the interval (z_k, z_{k+1}) . Of course it is clear that if E is collinear and (2.2) holds for no value of k , the norm (2.1) can be decreased by moving α until it coincides with z_k , where

$$\sum_{i < k} \mu_i < \sum_{i \geq k} \mu_i, \quad \sum_{i \leq k} \mu_i > \sum_{i > k} \mu_i;$$

on the other hand if (2.2) is valid, the least norm is given by α at an arbitrary point of the closed interval (z_k, z_{k+1}) .

The conclusions just established do not extend for given m to arbitrary n , $1 < n < m - 1$, with norm

$$\mu(T_n) = \sum_1^m \mu_k |T_n(z_k)|, \quad \mu_k > 0.$$

For instance with the choice $n = m - 1$, it follows from Theorem 1.1 that the extremal polynomials T_n may fail to be unique even when the z_k are not collinear. Of course [6, Theorem 8] the extremal T_n may fail to be unique when the z_k are collinear.

We have already indicated that the T -polynomial of degree n , $0 < n \leq m - 1$ is unique, with the norm

$$(2.3) \quad \mu[T_n(z)] = \sum_1^m \mu_i |T_n(z_i)|^p, \quad p > 1.$$

There seem to be no simple conclusive results on uniqueness even in the case $n = 1$ ($m > 2$) if the norm is taken as (2.3) with $p > 1$ replaced by $0 < p < 1$, $T_1(z) \equiv z - \alpha$. For given z_i and μ_i the uniqueness of extrema α may depend on p , as we show by an example. Let E be the set $(-1, 0, 1)$ with respective weights $\mu_i = 1, \mu, 1$. For $0 < p < 1$, $n = 1$, every extremal polynomial $T_1(z)$

$\equiv z - \alpha$ must vanish in a point of E [6, Theorem 6]. The norm of z is 2, and the norm of $z - 1$ is $\mu + 2^p$. Thus the extremal polynomial $T_1(z)$ is unique and equal to z if we have $2 < \mu + 2^p$, $T_1(z)$ is $z \pm 1$ if we have $2 > \mu + 2^p$, $T_1(z)$ is $z \pm 1$ or z if we have $2 = \mu + 2^p$. This situation with respect to uniqueness or nonuniqueness of $T_1(z)$ is essentially the same for the nonreal set $E(-1, i\epsilon, 1)$, where $\epsilon (> 0)$ is sufficiently small. For arbitrary nonreal E and $\mu_i, m > 2$, the natural method of determining $T_1(z)$ would seem to be the use of standard methods of differentiation to determine a possible point α different from the z_i yielding an extremum, and then comparing the norms $\mu[z - \alpha], \mu[z - z_i]$.

In connection with the families of polynomials which appear in Theorem 1.1, we mention

THEOREM 2.1. *If the distinct points z_1, z_2, \dots, z_m are given, then the zeros of the function*

$$(2.4) \quad \sum_{i=1}^m \frac{\lambda_i}{z - z_i}, \quad \lambda_i \text{ real, } \lambda_i \neq 0,$$

are the foci of the curve of class $m - 1$ which touches each line segment $z_i z_j$ in a point dividing that segment in the ratio $\lambda_i : \lambda_j$. Moreover, if $\sum \lambda_i \neq 0$ the zeros of (2.4) are $m - 1$ points ζ_k such that for every i

$$(2.5) \quad \sum_{k=1}^{m-1} \arg(\zeta_k - z_i) = \sum_{k=1, k \neq i}^m \arg(z_k - z_i).$$

The first part of Theorem 2.1 is due to Siebeck, and has been later proved by numerous other writers; proof and references are given by Marden [4, p. 11]. For $m = 3$, equation (2.5) identifies the foci of the conics with isogonal conjugates and is due to Steiner; equation (2.5) will now be established for arbitrary m . With the choice $\sum \lambda_i = 1$ and $z_1 = 0$, from (2.4) we have

$$\zeta_1 \zeta_2 \cdots \zeta_{m-1} = \lambda_1 z_2 z_3 \cdots z_m,$$

from which (2.5) follows for $i = 1, z_1 = 0$, and hence follows in every case. If we have $\lambda_i > 0$ for every i , it is a consequence of a slightly generalized form of the theorem of Lucas concerning the zeros of the derivative of a polynomial, that all the ζ_i lie in the convex hull of the given set z_j .

In the case $m = 3$ an arbitrary point z_0 in this convex hull can be chosen as a point ζ_1 with $\lambda_i \geq 0$, for the vectors

$$(2.6) \quad \frac{1}{\bar{z}_0 - \bar{z}_1}, \frac{1}{\bar{z}_0 - \bar{z}_2}, \frac{1}{\bar{z}_0 - \bar{z}_3},$$

represent respective forces at z_0 due to particles at z_1, z_2, z_3 , each particle repelling according to the law of inverse distance; the arguments of these vec-

tors lie in no sector of angle less than π , so for suitable numbers $\lambda_i (\geq 0)$ we have

$$(2.7) \quad \sum_1^m \frac{\lambda_i}{z_0 - z_i} = 0, \quad \sum_1^m \frac{\lambda_i}{\bar{z}_0 - \bar{z}_i} = 0,$$

and z_0 is a point ζ_1 . Once ζ_1 is known, its isogonal conjugate ζ_2 with respect to z_1, z_2, z_3 is of course uniquely determined unless the z_i are collinear.

The reasoning just given extends to the case $m > 3$, to show that an arbitrary point z_0 in the interior of the convex hull of the z_i but distinct from the z_i can be chosen as a point ζ_1 with $\lambda_i \geq 0$; the point z_0 lies in the interior of the convex hull of three of the points z_i , say z_1, z_2, z_3 , which are not to be taken collinear unless all the z_i are collinear. If the non-negative numbers $\lambda_4, \lambda_5, \dots, \lambda_m$ are chosen arbitrarily, the non-negative numbers $\lambda_1, \lambda_2, \lambda_3$ can then be chosen so as to satisfy (2.7).

Whatever $m (> 2)$ may be, a point $z_0 \neq z_i$ on the boundary of the convex hull can be made a point ζ_1 if we allow $m - 2$ of the λ_i to be zero; but if the convex hull is not a line segment, such a z_0 cannot be a point ζ_1 if we continue to require $\lambda_i > 0$ for all i .

In connection with the geometry of zeros, we state the following theorem concerning approximation to an arbitrary function, a generalization of a previous result [5, §9] which the reader may establish by the methods used to prove that result:

THEOREM 2.2. *Let E be the point set z_1, z_2, \dots, z_m . Let $\delta[\delta_1, \delta_2, \dots, \delta_m]$ be a positive function of the non-negative variables δ_k for $\sum \delta_k > 0$, which decreases whenever all the δ_k not zero decrease and the δ_k which are zero remain unchanged. Let the function $f(z)$ be arbitrarily chosen on E , and let $f_1(z) \equiv B_0 z^{m-1} + \dots, B_0 \neq 0$ be the polynomial of degree $m - 1$ which coincides with $f(z)$ on E . Let $T_{m-1}(B_0, z) \equiv B_0 z^{m-1} + \dots$ be the (or a) polynomial of least norm $\delta[T_{m-1}(B_0, z_1), \dots, T_{m-1}(B_0, z_m)]$ on E , and let $t_{m-2}(z)$ be the (or a) polynomial of degree $m - 2$ of least deviation $\delta[|f(z_i) - t_{m-2}(z_i)|] = \delta[|f_1(z_i) - t_{m-2}(z_i)|]$.*

Let the zeros of $f_1(z)$ lie in the closed interior (respectively closed exterior) C_1 of the circle $|z - \alpha| = r_1$, and let the zeros of $T_{m-1}(B_0, z)$ lie in the closed interior C_2 of the circle $|z - \beta| = r_2$ (for which it is sufficient that E lie in C_2), where C_1 and C_2 are mutually disjoint. Then all zeros of $t_{m-2}(z)$ lie in the $m - 2$ closed regions

$$(2.8) \quad \left| z - \frac{\alpha - \epsilon\beta}{1 - \epsilon} \right| \leq \frac{r_1 + r_2}{|1 - \epsilon|},$$

$$(2.9) \quad \left| z - \frac{\alpha - \epsilon\beta}{1 - \epsilon} \right| \geq \frac{r_1 - r_2}{|1 - \epsilon|},$$

respectively, where ϵ takes all the values except unity of the $(m - 1)$ st roots of

unity. If the circles (2.8) are mutually exterior, they contain each one zero of $t_{m-2}(z)$.

The method of Fejér (compare also §3 below) shows that the zeros of $T_{m-1}(B_0, z)$ lie in C_2 if E lies there; we do not know and it is not essential to know with norm function δ that the extremal polynomial $T_{m-1}(B_0, z)$ is $B_0 T_{m-1}(1, z)$.

Of course if the polynomial $f_1(z)$ of degree $m - 1$ which coincides with $f(z)$ on E is of degree $m - 2$, then we have $B_0 = 0$, $t_{m-2}(z) \equiv f_1(z)$, so the zeros of $t_{m-2}(z)$ lie in C_1 .

All the circles (2.8) respectively (2.9) lie in the closed exterior of the hyperbola (ellipse) C whose foci are α and β , and whose transverse (major) axis is $r_1 + r_2$ (respectively $r_1 - r_2$). The centers of the circles (2.8) and (2.9) are equidistant from α and β , and the circles are doubly tangent (algebraically if not geometrically) to C .

There is also an intermediate case here, in which C_1 is a half-plane; the regions (2.8) and (2.9) are replaced by half-planes, and all their boundaries are tangent to a certain parabola which does not depend on m and whose focus is β .

3. An analog of Fejér's principle. The geometric results of §2 are in the main supplementary to Fejér's principle, which applies to the most general norm δ of the kind used in Theorem 2.2 and to an arbitrary $E: (z_1, z_2, \dots, z_m)$ $m > n \geq 1$. In a slightly generalized form this principle can be expressed: Let $T_n(z) \equiv z^n + \dots$ be an extremal polynomial for the set E , let $z = \alpha$ be a zero of $T_n(z)$, and let E' denote the (necessarily nonempty) subset of E on which $T_n(z)/(z - \alpha)$ is different from zero. Then α lies in the convex extension of E' . If α does not satisfy this last condition, α does not lie in E' and there exists some α' near α (for instance on the line segment joining α to the nearest point of the convex extension of E') such that at every point of E' we have

$$|z - \alpha'| < |z - \alpha|,$$

$$\left| \frac{T_n(z)(z - \alpha')}{z - \alpha} \right| < |T_n(z)|;$$

on $E - E'$ we have $T_n(z)(z - \alpha')/(z - \alpha) = T_n(z) = 0$, so the deviation of $T_n(z)(z - \alpha')/(z - \alpha)$ is less than that of $T_n(z)$, and (contrary to hypothesis) $T_n(z)$ is not extremal on E .

We prove an analogous result, again valid for the general norm considered by Fejér or for the general norm $\delta[P_n(z)] = \delta(|P_n(z_1)|, \dots, |P_n(z_m)|)$ of the kind used in Theorem 2.2:

THEOREM 3.1. Let α and β be zeros of the extremal polynomial $T_n(z) \equiv z^n + \dots$ for the finite set E consisting of more than $n (\geq 2)$ points, let H be an arbitrary equilateral hyperbola with the segment $\alpha\beta$ as diameter, and let E'

denote the subset of E on which $T_n(z)/(z-\alpha)(z-\beta)$ does not vanish. Then E' cannot lie wholly in the interior of H nor wholly in the exterior of H .

We choose $\alpha = -1, \beta = 1$, so that $z^2 - 1$ is a factor of $T_n(z)$. We compare $z^2 - 1$ as a factor of $T_n(z)$ with $(z^2 - 1 - \epsilon\eta)$ as a factor, $\epsilon > 0, |\eta| = 1$. If $w (\neq 0)$ is given, the inequality $|w - \epsilon\eta| < |w|$ is clear for ϵ sufficiently small, if and only if we have $|\arg w - \arg \eta| < \pi/2$. Thus the inequality

$$(3.1) \quad |z^2 - 1 - \epsilon\eta| < |z^2 - 1|$$

is valid in every point z (we assume $z^2 \neq 1$) in which

$$(3.2) \quad |\arg(z^2 - 1) - \arg \eta| < \pi/2,$$

but is valid in no other point; similarly the inequality

$$(3.3) \quad |z^2 - 1 + \epsilon\eta| < |z^2 - 1|$$

is valid in every point ($z^2 \neq 1$) in which

$$(3.4) \quad |\arg(z^2 - 1) - \arg \eta - \pi| < \pi/2,$$

but is valid in no other point.

The locus

$$(3.5) \quad \arg(z^2 - 1) - \arg \eta = \pm \pi/2$$

can be written

$$(3.6) \quad \tan^{-1} \frac{2xy}{x^2 - y^2 - 1} = \arg \eta \pm \frac{\pi}{2} = \tan^{-1} \frac{1}{\sigma},$$

$$(x^2 - y^2 - 1) - 2\sigma xy = 0,$$

which is an equilateral hyperbola with $\alpha\beta$ as diameter; moreover any proper equilateral hyperbola with $\alpha\beta$ as diameter can be expressed in this form. The inequalities (3.2) and (3.4) are respectively characteristic of the interior and exterior of the curve; of course $\arg(z^2 - 1)$ is not defined if $z = \pm 1$.

If E' lies in the interior of H , neither α nor β belongs to E' ; at every point of E' (which is necessarily nonempty) we have (3.2), (3.1), and

$$\left| \frac{T_n(z)[(z^2 - 1) - \epsilon\eta]}{z^2 - 1} \right| < |T_n(z)|.$$

On $E - E'$ we have $T_n(z)/(z^2 - 1) = T_n(z) = 0$, so the deviation on E of the polynomial $T_n(z)[(z^2 - 1) - \epsilon\eta]/(z^2 - 1)$ is less than that of $T_n(z)$, and $T_n(z)$ is not extremal, contrary to hypothesis. Similarly by use of (3.4) and (3.3) it follows that E' cannot lie in the exterior of H , so Theorem 3.1 is established.

The points $+1$ and -1 separate the branches of (3.6) each into two arcs on which we have respectively $\arg(z^2 - 1) = \arg \eta \pm \pi/2$. It is a consequence

of the reasoning already given, by means of (any essentially) new choice of η , that *under the conditions of Theorem 3.1 not all the points of E' can lie on either one of the two loci* (each consisting of two open arcs) $\arg [(z-\alpha)(z-\beta)] = \arg \eta \pm \pi/2$ *which together compose H with α and β deleted.*

For the special case in which $T_n(z)$ is the derivative of a polynomial whose zeros are the points of E (such a derivative is known to have extremal properties), the conclusion of Theorem 3.1 so far as concerns points of E where $T_n(z)$ does not vanish, is due to J. v. S. Nagy [7]; the special case of a set E and also H symmetric in a line had been previously considered by D. R. Curtiss [2].

The limiting form $xy=0$ of (3.6) represents a degenerate equilateral hyperbola having $\alpha\beta$ as diameter; such a degenerate curve is admitted under Theorem 3.1. There is a further limiting case under Theorem 3.1 in which $\alpha=\beta$ is a double zero of $T_n(x)$; here the conclusion is valid, and refers to *any* degenerate equilateral hyperbola (two mutually perpendicular lines) with center α ; pairs of opposite sectors (quadrants) are to be interpreted as the "interior" and "exterior" of the curve respectively.

Theorem 3.1 is not a corollary of the method of Fejér; for it is a consequence of Theorem 3.1 that $T_n(z)$ cannot have the two zeros $+1$ and -1 (supposed not in E') if E lies wholly interior to the first and third quadrants, a conclusion that does not follow by the method of Fejér.

The method of proof of Theorem 3.1 is of course equally applicable if we commence with *three or more* instead of two zeros of $T_n(z)$, but the algebraic curves involved may no longer be elementary. However, it follows by this method that *if $z=\alpha$ (not in E') is a zero of $T_n(z)$ of order at least ν , then each set of ν equally spaced lines through α either passes through all the points of E' or separates the plane into two sets each of ν alternate open sectors, of which neither set can contain all points of E' .* There is a certain "sharpness" in this result so far as concerns the number ν ; for instance α is a zero of $T_\nu(z) [\equiv (z-\alpha)^\nu]$ of order ν but not of higher order if E is the set of vertices of a regular $(\nu+1)$ -sided polygon with center α and where $\delta = \max \delta_k$; these vertices lie some in every set of alternate closed sectors bounded by ν equally spaced lines through α ; the vertices lie on a set S of $\nu+1$ equally spaced lines through α but none lies in a suitably chosen set of alternate open sectors bounded by $\nu+1$ equally spaced lines through α near S .

The result of Fejér can be phrased as the special case $\nu=1$ of the italicized statement above.

Our method here of modifying factors of $T_n(z)$ of degree greater than two is in strong contrast to the situation for real E , where [6, §8] modification of such factors gives no more information than does modification of factors of degree two.

An alternate form of the conclusion of Theorem 3.1 (including the later statement concerning arcs of H) is that *the origin must lie in the convex hull*

of the numbers $(z_i - \alpha)(z_i - \beta)$, for z_i in E' . In fact the negation of this condition is that (for $\alpha = 1, \beta = -1$) there exist some η for which (3.2) is valid for z on E' . Indeed, if $q(z)$ is any factor of $T_n(z)$ of positive degree the origin must lie in the convex hull of the numbers $\{q(z_i)\}$ for z_i in the subset of E on which $T_n(z)/q(z)$ does not vanish.

We have formulated Theorem 3.1 for the extremal polynomial $T_n(z)$; an equivalent formulation in the spirit of Fekete's characterization [3] is: *If the zeros of $q(z) \equiv (z - \alpha)(z - \beta)$ do not lie on a set E and if $q(z)$ has no underpolynomial for E , then neither the interior nor the exterior of any equilateral hyperbola H with diameter $\alpha\beta$ can contain all points of E .*

For the particular choice $\delta(\delta_1, \delta_2, \dots, \delta_m) \equiv \sum \mu_k \delta_k^p, \mu_k > 0, p > 0$, and for various other choices a stronger result than that of Theorem 3.1 can be established. We say that $\delta(\delta_1, \delta_2, \dots, \delta_m)$ satisfies condition A provided δ decreases whenever one or more of the δ_k not zero are decreased by infinitesimal amounts $\Delta\delta_k$, even if the other δ_j not zero are increased by amounts $\Delta\delta_j$ which are infinitesimal of order at least twice that of every $\Delta\delta_k$; during these changes we suppose that the δ_j which are initially zero remain unchanged. Although $\delta \equiv \sum \mu_k \delta_k^p, \mu_k > 0, p > 0$, satisfies condition A, $\delta \equiv \max [\mu_k \delta_k]$ does not.

THEOREM 3.2. *Let Theorem 3.1 be modified so that $\delta(\delta_1, \delta_2, \dots, \delta_m)$ satisfies condition A. Suppose neither α nor β belongs to E' . Then either E' lies on H or some points of E' lie interior to H and other points of E' lie exterior to H .*

If (3.2) or (3.4) holds, the difference between the two members of (3.1) or of (3.3) is an infinitesimal of the same order as ϵ , but if (3.5) holds, that difference is an infinitesimal of the same order as ϵ^2 . Suppose not every point of E' lies on H . If every point of E' satisfies either (3.2) or (3.5), replacement of $T_n(z)$ by $T_n(z)[(z^2 - 1) - \epsilon\eta]/(z^2 - 1)$ decreases the error $|T_n(z_k)|$ at every point z_k of E' interior to H (at least one such point exists) by an infinitesimal of the same order as ϵ and increases the error $|T_n(z_j)|$ at every point z_j of E' on H by an infinitesimal of the same order as ϵ^2 ; this replacement leaves unchanged the error (necessarily zero) at each point of $E - E'$. Consequently the norm of $T_n(z)$ is decreased and $T_n(z)$ is not extremal. Similarly it follows that if not every point of E' lies on H , not every point of E' can satisfy either (3.4) or (3.5). Theorem 3.2 is established.

The comments made concerning Theorem 3.1 apply also in stronger form under the conditions of Theorem 3.2: limiting cases that H is a pair of perpendicular lines and that $\alpha = \beta$ are admitted; conclusions corresponding to those of Theorem 3.2 are valid. Stronger conclusions than those previously formulated, relating to three or more (instead of two) zeros of $T_n(z)$, including zeros of multiplicity ν , likewise apply if δ satisfies condition A.

4. Variable point sets. In certain cases immediate results can be formulated concerning adjunction or deletion of points of the fundamental set E .

THEOREM 4.1. *If $T_n(z)$ is a T -polynomial of degree n for arbitrary sets E' and E'' with respective norms $\mu'[P_n(z)]$ and $\mu''[P_n(z)]$, then $T_n(z)$ is also a T -polynomial for the set $E = E' + E''$ with the norm $\mu[P_n(z)] = \mu'[P_n(z)] + \mu''[P_n(z)]$. Any other T -polynomial of degree n for E is then also a T -polynomial of degree n for E' and E'' .*

If $P_n(z) \equiv z^n + \dots$ is arbitrary, we have

$$(4.1) \quad \mu'[P_n(z)] \geq \mu'[T_n(z)], \quad \mu''[P_n(z)] \geq \mu''[T_n(z)],$$

whence $\mu[P_n(z)] \geq \mu[T_n(z)]$, so $T_n(z)$ is a T -polynomial for E . Moreover, if we have $\mu[P_n(z)] = \mu[T_n(z)]$, then the equality sign holds in both parts of (4.1), so $P_n(z)$ is a T -polynomial for both E' and E'' . Theorem 4.1 does not require that E' and E'' be disjoint, nor does it place any restrictions on E' , E'' , μ' , or μ'' . Theorem 4.1 obviously extends to the sum of any number (even under suitable conditions to an infinite number) of sets.

An immediate consequence of Theorem 4.1 is the

COROLLARY 4.1. *If $T_n(z)$ is a T -polynomial of degree n for the set $E_m: (z_1, z_2, \dots, z_m)$ with norm $\sum_1^m \mu_k |T_n(z_k)|^p$, $\mu_k > 0$, $p > 0$, and if z_0 is a zero of $T_n(z)$ not in E_m , then $T_n(z)$ is also a T -polynomial of degree n for the set $E_{m+1}: (z_0, z_1, \dots, z_m)$ with norm $\sum_0^m \mu_k |T_n(z_k)|^p$, $\mu_0 > 0$. Every T -polynomial of degree n for the set E_{m+1} vanishes in z_0 .*

Of course $T_n(z)$ is a T -polynomial for the set consisting merely of z_0 , with norm $\mu_0 |P_n(z_0)|^p$.

Even if every T -polynomial of degree n for the set E_{m+1} vanishes in z_0 , it is not necessarily true that such a T -polynomial is a T -polynomial also for E_m :

There exists a unique T -polynomial $T_1(z)$ of degree unity for any noncollinear set $E_3: (z_0, z_1, z_2)$ with suitable norm $\sum_0^2 \mu_k |T_1(z_k)|^p$, $\mu_k > 0$, $0 < p \leq 1$, and $T_1(z_0) = 0$, yet $T_1(z)$ is not a T -polynomial for $E_2: (z_1, z_2)$ with norm $\sum_1^2 \mu_k |T_1(z_k)|^p$.

We choose $\mu_0 > \mu_1 + \mu_2$ from which it follows by Theorem 4.2 below that the unique extremal polynomial is $T_1(z) \equiv z - z_0$; nevertheless $T_1(z)$ is not a T -polynomial of degree unity for the set E_2 .

Such a counter-example is not possible in the case $p > 1$:

COROLLARY 4.2. *If $T_n(z)$ is a T -polynomial of degree n for a set $E_{m+1}: (z_0, z_1, \dots, z_m)$, $m > n$, with norm $\sum_0^m \mu_k |T_n(z_k)|^p$, $\mu_k > 0$, $p > 1$, and if $T_n(z_0) = 0$, then $T_n(z)$ is a T -polynomial for the set $E_m: (z_1, z_2, \dots, z_m)$, with norm $\sum_1^m \mu_k |T_n(z_k)|^p$.*

We shall prove later (Theorem 6.1) that the extremal polynomial for an arbitrary E_m with $p > 1$ is uniquely characterized by the orthogonality condition

$$\sum_1^m \mu_k |T_n(z_k)|^{p-1} \text{sg}[T_n(z_k)] \bar{g}(z_k) = 0$$

for every polynomial $g(z)$ of degree $n-1$. This condition is obviously unchanged if we insert the term corresponding to $k=0$.

Corollary 4.2 can be considered a converse (for $p > 1$) of Corollary 4.1.

THEOREM 4.2. *Let $T_n(z)$ be a T -polynomial of degree $n (\geq 0)$ for the set $E_m: (z_1, z_2, \dots, z_m)$ with deviation $\sum_1^m \mu_i |T_n(z_i)|^p, 0 < p \leq 1$, and let z_0 not belong to E_m . Then $T_{n+1}(z) \equiv (z-z_0)T_n(z)$ is a T -polynomial of degree $n+1$ for the set $E_{m+1}: (z_0, \dots, z_m)$ with deviation $\sum_0^m \mu'_i |T_{n+1}(z_i)|^p, \mu'_i = \mu_i / |z_0 - z_i|^p, \mu'_0 \geq \sum_1^m \mu'_i$. If we have $\mu'_0 > \sum_1^m \mu'_i$, the totality of T -polynomials $T_{n+1}(z)$ for E_{m+1} is precisely the set $(z-z_0)T_n(z)$ where $T_n(z)$ ranges through the set of T -polynomials of degree n for E_m with norm $\sum_1^m \mu_i |T_n(z_i)|^p$; every $T_{n+1}(z)$ vanishes in z_0 .*

If $Q_n(z) \equiv z^n + \dots$ is arbitrary we have by hypothesis

$$\begin{aligned} & \sum_1^m \mu_i |Q_n(z_i)|^p \geq \sum_1^m \mu_i |T_n(z_i)|^p, \\ (4.2) \quad & \sum_1^m \mu'_i |z_i - z_0|^p |Q_n(z_0)|^p \geq \sum_1^m \mu'_i |z_i - z_0|^p |T_n(z_i)|^p. \end{aligned}$$

Let now $P_{n+1}(z) \equiv z^{n+1} + \dots$ be arbitrary; we write $P_{n+1}(z) \equiv (z-z_0)Q_n(z) + P_{n+1}(z_0)$, whence

$$\begin{aligned} & \sum_0^m \mu'_i |P_{n+1}(z_i)|^p \equiv \sum_0^m \mu'_i |(z_i - z_0)Q_n(z_i) + P_{n+1}(z_0)|^p \\ (4.3) \quad & \geq \sum_1^m \mu'_i |z_i - z_0|^p |Q_n(z_i)|^p + \left(\mu'_0 - \sum_1^m \mu'_i \right) |P_{n+1}(z_0)|^p \\ & \geq \sum_1^m \mu'_i |z_i - z_0|^p |Q_n(z_i)|^p \geq \sum_1^m \mu'_i |z_i - z_0|^p |T_n(z_i)|^p; \end{aligned}$$

the first inequality in (4.3) is a consequence of the well known inequality $a^p + b^p \geq (a+b)^p, a \geq 0, b \geq 0$; the second inequality in (4.3) holds by virtue of our hypothesis on μ'_0 ; the last inequality in (4.3) is (4.2), and the extremes of (4.3) complete the proof of the first part of Theorem 4.3. Here $T_{n+1}(z)$ is not a unique T -polynomial for E_{m+1} unless $T_n(z)$ is a unique T -polynomial for E_m .

We suppose now $\mu'_0 > \sum_1^m \mu'_i$. Then the second inequality in (4.3) is strong unless $P_{n+1}(z_0) = 0$. The norm of $(z-z_0)Q_n(z)$ on E_{m+1} is precisely the norm of $Q_n(z)$ on E_m , so the last part of Theorem 4.2 follows.

Theorem 4.2 does not extend to the case $p > 1$, as follows from Theorem

1.2; the T -polynomial $T_m(z)$ of degree m for E_{m+1} cannot have a zero on E_{m+1} .

It may be noted that *the inequalities of Theorem 4.2 cannot be replaced by $\mu'_0 \geq \theta \sum_1^m \mu'_i$ or $\mu'_0 > \theta \sum_1^m \mu'_i$ respectively with θ independent of the μ'_i , $0 < \theta < 1$* . We prove this statement by use of the specific choice $m=2$, $z_1=0$, $z_2=2$, $p=1$, $\mu_1=\mu_2=1$, whence $T_1(z) \equiv z-\alpha$, $0 \leq \alpha \leq 2$; $z_0=1$, $\mu'_1=\mu'_2=1$, $\omega(z) \equiv z(z-1)(z-2)$, $\mu'_0 |\omega'(z_0)| = \mu'_0$, $\mu'_1 |\omega'(z_1)| = 2$, $\mu'_2 |\omega'(z_2)| = 2$. Then by Theorem 1.1 we have $T_2(z) \equiv (z-1)(z-\alpha)$, $0 \leq \alpha \leq 2$, if $\mu'_0 > \sum_1^m \mu'_i = 2$; $T_2(z) \equiv \lambda_0 z(z-2) + \lambda_1(z-1)(z-2) + \lambda_2 z(z-1)$, $\sum_0^2 \lambda_i = 1$, if $\mu'_0 = 2$; $T_2(z) \equiv z(z-2)$ if $\mu'_0 < 2$. We remark especially that $T_2(z_0) \neq 0$ if $\mu'_0 < \sum_1^2 \mu'_i = 2$.

The latter part of Theorem 4.2 is obviously of interest independently of any relation to E_m ; it implies that for *an arbitrary E_{m+1} with norm*

$$\sum_0^m \mu'_i |T_{n+1}(z)|^p, \quad 0 < p \leq 1, \mu'_0 > \sum_1^m \mu'_i,$$

every $T_{n+1}(z)$ vanishes in z_0 .

In the special case $p=1$, $n=m-1$, Theorem 4.2 can be somewhat improved:

THEOREM 4.3. *Let $E_m: (z_1, z_2, \dots, z_m)$ and the numbers $\mu_i (>0)$ be given, defining the norm $\sum_1^m \mu_i |T(z_i)|$. Let z_0 be a point not in E_m , and let E_{m+1} denote the set (z_0, z_1, \dots, z_m) for which we have the norm $\sum_0^m \mu'_i |T(z_i)|$. With the choice $\mu'_i = \mu_i / |z_0 - z_i|$ for $i > 0$ and with $\mu'_0 |\omega_m(z_0)| > \min [\mu_i |\omega'_m(z_i)|, i > 0]$, $\omega_m(z) \equiv (z - z_1) \dots (z - z_m)$, the T -polynomials $T_m(z)$ of degree m for E_{m+1} are precisely $z - z_0$ multiplied by the T -polynomials $T_{m-1}(z)$ for E_m .*

We have $\omega_{m+1}(z) \equiv (z - z_0)\omega_m(z)$, $\omega'_{m+1}(z_i) = (z_i - z_0)\omega'_m(z_i)$ for $i > 0$, $\mu'_i |\omega'_{m+1}(z_i)| = \mu_i |\omega'_m(z_i)|$ for $i > 0$. The conclusion follows from Theorem 1.1. We note too that if we choose $\mu'_0 |\omega_m(z_0)| < \min [\mu_i |\omega'_m(z_i)|$ for $i > 0]$, the unique $T_m(z)$ for E_{m+1} is $\omega_m(z)$; if we choose $\mu'_0 |\omega_m(z_0)| = \min [\mu_i |\omega'_m(z_i)|$ for $i > 0]$, the totality of $T_m(z)$ for E_{m+1} is the set $\lambda \omega_m(z) + (1 - \lambda)(z - z_0)T_{m-1}(z)$, $0 \leq \lambda \leq 1$, where $T_{m-1}(z)$ ranges over the set of T -polynomials of degree $m - 1$ for E_m .

Theorem 4.3 is effectively stronger than Theorem 4.2 ($n = m - 1, p = 1$), as we now prove by showing that *the condition $\mu'_0 > \sum_1^m \mu_i / |z_i - z_0|$ implies $\mu'_0 > \min \mu_i |\omega'_m(z_i)| / |\omega_m(z_0)|$, but not conversely*. We shall prove

$$(4.4) \quad \sum_1^m \mu_i / |z_i - z_0| \geq \min \mu_i |\omega'_m(z_0)| / |\omega_m(z_0)|,$$

for which it is sufficient to prove

$$(4.5) \quad \sum_{i=1}^m \frac{|\omega_m(z_0)|}{|z_i - z_0| |\omega'_m(z_i)|} \geq 1.$$

Indeed, the first member of (4.4) when divided by the first member of (4.5) may be considered a weighted mean of the numbers whose minimum occurs in the second member of (4.4), with weights the individual terms of the first member of (4.5); the weighted mean is not less than the minimum. The function unity is a polynomial of degree $m - 1$ which is represented by Lagrange's interpolation formula as taking the value unity in the m points z_i , whence for $z = z_0$

$$\sum_{i=1}^m \frac{\omega_m(z_0)}{\omega_m'(z_i)(z_0 - z_i)} = 1,$$

which implies (4.5). This establishes the first part of the italicized statement; the latter part is now obvious.

Corollary 4.1 admits a second converse, now for all $p (>0)$, which is also a companion piece to Theorem 4.2:

THEOREM 4.4. *Let z_0 not be a zero of any T -polynomial $S_n(z)$ of degree $n (< m)$ for the set $E_m: (z_1, z_2, \dots, z_m)$ with norm $\sum_1^m \mu_i |S_n(z_i)|^p, p > 0$, and let z_0 not be a point of E_m . Then if $\mu_0 (> 0)$ is sufficiently small, z_0 is not a zero of any T -polynomial $T_n(z)$ of degree n for the set $E_{m+1}: (z_0, z_1, \dots, z_m)$ with norm $\sum_0^m \mu_i |T_n(z_i)|^p$.*

Let $T_{n-1}(z)$ be a T -polynomial of degree $n - 1$ for E_m with weights $\mu'_i = \mu_i |z - z_0|$, and $S_n(z)$ a T -polynomial of degree n for E_m with weights μ_i . We have

$$\sum_1^m \mu'_i |T_{n-1}(z_i)|^p = \sum_1^m \mu_i |z_0 - z_i| |T_{n-1}(z_i)|^p = \sum_1^m \mu_i |S_n(z_i)|^p + \Delta,$$

where Δ is positive. If $P_{n-1}(z) \equiv z^{n-1} + \dots$ is arbitrary we have

$$\begin{aligned} \sum_0^m \mu_i |S_n(z_i)|^p &= \mu_0 |S_n(z_0)|^p + \sum_1^m \mu_i |S_n(z_i)|^p \\ &= \mu_0 |S_n(z_0)|^p - \Delta + \sum_1^m \mu'_i |T_{n-1}(z_i)|^p \\ &\leq \mu_0 |S_n(z_0)|^p - \Delta + \sum_0^m \mu_i |(z_i - z_0)P_{n-1}(z_i)|^p. \end{aligned}$$

Consequently, whenever we have $\mu_0 < \Delta / |S_n(z_0)|^p$, the polynomial $S_n(z)$ which has no zero in z_0 has a smaller norm on E_{m+1} with weights μ_i than does an arbitrary polynomial $(z - z_0)P_{n-1}(z) \equiv z^n + \dots$ with a zero in z_0 , so the latter cannot be a T -polynomial. Theorem 4.4 is established. Of course $S_n(z)$ need not be unique ($p \leq 1$) and $|S_n(z_0)|$ need not be unique; the smallest $|S_n(z_0)|$ may be chosen. But Δ is unique. In the limiting case $\mu_0 = \Delta / |S_n(z_0)|^p$,

at least one T -polynomial (namely $S_n(z)$) for E_{m+1} with weights μ_i can be chosen to have no zero in z_0 .

The bound $\Delta/|S_n(z_0)|^p$ for μ_0 is sharp, in the sense that the conclusion may be false if the bound is replaced by a larger one, as we now indicate. Choose $m = 2, n = 1, z_1 = 0, z_2 = 1, \mu_1 = \mu_2 = 1, z_0 = -1, p = 1$. We have $T_{n-1}(z) \equiv 1, \sum_1^m \mu_i |T_{n-1}(z_i)| = 3, S_n(z) \equiv z - \alpha, 0 \leq \alpha \leq 1, \sum_1^m \mu_i |S_n(z_i)| = 1, \Delta = 2$. We choose $|S_n(z_0)|$ as small as possible, namely unity, so the inequality of the theorem is $\mu_0 < 2$. However, for the limiting value $\mu_0 = 2$ we may choose $T_1(z)$ for E_3 as $z + 1$ (whence $T_1(z_0) = 0$); such a choice is possible by the first part of Corollary 4.1.

The previous results of §4 have some intrinsic interest, and are to be applied in the sequel. As another application we prove

THEOREM 4.5. *Let integers m and n be given, $1 < n < m$. Then there exists a set $E_m: (z_1, z_2, \dots, z_m)$ whose points are not collinear and weights $\mu_i > 0$ for which the T -polynomial $T_n(z)$ of degree n with norm $\sum_1^m \mu_i |T_n(z_i)|$ is not unique.*

There exists (§2) a real set E_{m-n+1} of $m - n + 1$ points and suitable weights for which the T -polynomial of degree unity is not unique, so by Theorem 4.2 there exists a set E_{m-n+2} of $m - n + 2$ noncollinear points for which the T -polynomial of degree two is not unique. By continued application of Theorem 4.2 it follows that there exists a set of m noncollinear points for which the T -polynomial of degree n is not unique.

Theorem 4.5 is false (§2) without the restriction $n > 1$.

It is not merely possible to add points to a given set on which approximation is considered, but under suitable conditions (compare Theorem 4.2) to delete points:

THEOREM 4.6. *Let $T_n(z) \equiv z^n + \dots$ be extremal for the set $E: (z_1, \dots, z_m), m \geq n$, with norm $\sum_1^m \mu_i |T_n(z_i)|^p, \mu_i > 0, p > 0$, and suppose $T_n(z_1) = 0$. Then $T_n(z)/(z - z_1)$ is extremal for the set $E - z_1$ with weights $\mu_i |z_i - z_1|^p$.*

For an arbitrary polynomial $P_{n-1}(z) \equiv z^{n-1} + \dots$ of degree $n - 1$ we have by hypothesis

$$\begin{aligned} \sum_2^m \mu_i |z_i - z_1|^p |P_{n-1}(z_i)|^p &= \sum_1^m \mu_i |P_{n-1}(z_i)(z_i - z_1)|^p \geq \sum_1^m \mu_i |T_n(z_i)|^p \\ &= \sum_2^m \mu_i |z_i - z_1|^p \left| \frac{T_n(z_i)}{z_i - z_1} \right|^p; \end{aligned}$$

the extreme members express the conclusion. It may be noted too that if $T_n(z)$ as extremal polynomial is unique, which is necessarily the case if $p > 1$, then $T_n(z)/(z - z_1)$ is also a unique extremal polynomial.

5. **Totality of T -polynomials, real E .** The determination of the totality of the T -polynomial $T_n(z)$ of degree n with norm (0.1) and unprescribed weights for a set E of m points is trivial if $m = n$, for then the T -polynomial is unique and vanishes at every point of E ; the case $m < n$ is also trivial, for then $T_n(z)$ vanishes at every point of E but is otherwise arbitrary, hence not unique. Henceforth in §5 we treat only the case $n < m$.

THEOREM 5.1. *For a real point set $E: (z_1, z_2, \dots, z_m)$ and with given $p, 0 < p < 1$, the T -polynomials of degree $n (< m)$ for the norms (0.1) and unprescribed weights are precisely those polynomials $z^n + \dots$ which vanish in n points of E .*

It follows from [6, Theorem 6] that every T -polynomial of degree n vanishes in n points of E .

Conversely, let a polynomial $P_n(z) \equiv z^n + \dots$ be given that vanishes in the n points z_1, z_2, \dots, z_n ; we prove that $P_n(z)$ is a T -polynomial for E with suitable choices of the weights. For the point set $E_{m-n+1}: (z_n, z_{n+1}, \dots, z_m)$, we choose weights μ'_i with $\mu'_n > \sum_{n+1}^m \mu'_i$; it follows from Theorem 4.2 that the unique T -polynomial of degree unity is $T_1(z) \equiv z - z_n$. For the point set $E_{m-n+2}: (z_{n-1}, z_n, \dots, z_m)$ we choose weights $\mu''_i = \mu'_i / |z_{n-1} - z_i|^p$ for $i > n - 1$, $\mu''_{n-1} > \sum_n^m \mu''_i$; by Theorem 4.2 the unique T -polynomial of degree two is $T_2(z) \equiv (z - z_{n-1})(z - z_n)$. Continued application of Theorem 4.2 with suitable choice of weights shows that $P_n(z)$ is a T -polynomial for E , and completes the proof.

THEOREM 5.2. *For a real point set $E: (z_1, z_2, \dots, z_m)$ and with given $p (> 1)$, the T -polynomials of degree $n (< m)$ for the norms (0.1) and unprescribed weights are precisely those polynomials $T_n(z) \equiv z^n + \dots$ whose zeros are simple and lie interior to the smallest closed interval containing E , and whose zeros strongly separate a subset of E (containing $n + 1$ points) on which $T_n(z)$ is different from zero.*

These conditions are necessary. We have already shown [6, §9] that the zeros of an extremal polynomial $T_n(z)$ are simple and lie interior to the smallest closed interval containing E . If $t(z)$ is an arbitrary polynomial of degree $n - 1$, we have [6, §9]

$$(5.1) \quad \sum_1^m \mu_i |T_n(z_i)|^{p-1} \operatorname{sg} [T_n(z_i)] t(z_i) = 0.$$

In (5.1) we omit the points z_i in which $T_n(z_i) = 0$, denoting by \sum' the summation over the remaining set E' , and set $\mu'_i = \mu_i |T_n(z_i)|^{p-2}$ on E' :

$$(5.2) \quad \sum' \mu'_i |T_n(z_i)| \operatorname{sg} [T_n(z_i)] t(z_i) = 0$$

for every polynomial $t(z)$ of degree $n - 1$. Equations (5.2) are precisely the

conditions for classical orthogonality ($p=2$) of $T_n(z)$ to $t(z)$ on E' with weights μ_i' (>0), and (Theorem 6.1) determine $T_n(z)$ uniquely if E' contains at least $n+1$ points; if E' contains fewer than $n+1$ points, $T_n(z)$ must vanish at every point of E' , a situation excluded by our definition of E' . Thus $T_n(z)$ is a polynomial of degree n extremal ($p=2$) on E' with weights μ_i' , and is different from zero at all points of E' . Then [6, Theorem 7] the zeros of $T_n(z)$ strongly separate a subset of $n+1$ points of E' , and the necessity is established.

The conditions are sufficient. Let the zeros of $T_n(z)$ be simple and strongly separate the subset E'' of E containing precisely $n+1$ points. Then $T_n(z)$ is an extremal polynomial for E'' with given p (>1) and positive weights by the reciprocal of Theorem 1.2. Conceivably E'' is not uniquely determined, but every point of the subset E' of E on which $T_n(z)$ is not zero belongs to some E'' ; $T_n(z)$ is an extremal polynomial with positive weights for each of these sets E'' , hence by Theorem 4.1 is extremal for E' with suitably chosen positive weights. It then follows if necessary by repeated application of Corollary 4.1, that $T_n(z)$ is also extremal on E with positive weights, so Theorem 5.2 is established.

Under the conditions of Theorem 5.2 with $p>1$, any polynomial $T_n(z)$ extremal on E with positive weights μ_i is also extremal on the subset E' of E on which $T_n(z)$ is different from zero with the same positive weights on E' . This conclusion is contained in Corollary 4.2.

It is striking that in each of the two categories $0 < p < 1$ and $p > 1$, the class of polynomials of degree n extremal on real E with positive weights is independent of p . This situation for real E is to be contrasted with the corresponding facts for complex E , to which the present writers plan to return on a later occasion.

We say that the polynomial $z^n + \dots$ is an *infrapolynomial* for E if it has no underpolynomial on E . Fekete uses the term "extremal polynomial" here, a term which we consider too firmly established in its customary sense to be assigned a new meaning.

THEOREM 5.3. *If E is a real point set containing m ($\geq n+1$) points, then the three classes of polynomials of degree n are identical: the class C_1 of polynomials whose zeros separate a subset of E , the set C_2 of extremal polynomials $T_n(z)$ for the norm (0.1) with $p=1$ and positive unprescribed weights on E , and the set C_3 of infrapolynomials on E .*

It follows at once that every extremal polynomial is an infrapolynomial, so C_2 is contained in C_3 . The proof of [6, Theorems 2 and 7] is valid under the hypothesis that the given polynomial is an infrapolynomial, so C_3 is contained in C_1 . It remains to show that C_1 is contained in C_2 : *If the zeros of $T_n(z)$ weakly separate a subset of E , $m > n$, then $T_n(z)$ is a T -polynomial for E with $p=1$ and positive weights.*

Consider first the case of strong separation. If E' : $(z'_1, z'_2, \dots, z'_{n+1})$ is an appropriate subset of E containing $n+1$ points, the zeros of $T_n(z)$ strongly separate E' , and [6, Theorem 5] we have

$$T_n(z) \equiv \sum_1^{n+1} \frac{\lambda_i \omega(z)}{z - z'_i}, \quad \lambda_i > 0, \quad \sum_1^{n+1} \lambda_i = 1, \quad \omega(z) \equiv (z - z'_1) \cdots (z - z'_{n+1}).$$

Then [6, Theorem 8] the polynomial $T_n(z)$ is extremal for E' , $p=1$, with suitably chosen positive weights. A set E' can be chosen to contain any pre-assigned point of E , and a (finite) sum of such sets E' contains all points of E . Thus (as in the proof of Theorem 5.2) by Theorem 4.1 the polynomial $T_n(z)$ is extremal for E with suitably chosen positive weights.

Let next the zeros of $T_n(z)$ weakly separate a suitable subset of E ; the zeros of $T_n(z)$ can be moved continuously, each without leaving the original closed interval of the axis of reals bounded by successive points of E on which it lies, so that the zeros become disjoint from E and strongly separate this subset of E . Then (as we have already proved) the new polynomial $T_n(z)$ is extremal on E with positive weights. But [6, Theorem 3] zeros of such an extremal polynomial ($p=1$) can be moved continuously in their respective closed intervals of the axis of reals bounded by successive points of E , without affecting the extremal character of the polynomial. Thus the original polynomial $T_n(z)$ is extremal on E with positive weights, and the proof is complete.

Of course the original $T_n(z)$ may have a double zero, necessarily in a point of E ; the reasoning as given is valid.

6. Orthogonality conditions and uniqueness. The extremal polynomials satisfy certain orthogonality conditions not merely in the classical case $p=2$, but in every case $p>1$.

THEOREM 6.1. *With norm $\sum_1^m \mu_k |T_n(z_k)|^p$, $\mu_k > 0$, $p > 1$, $m \geq n$, E : (z_1, z_2, \dots, z_m) , every extremal polynomial $T_n(z) \equiv z^n + \dots$ satisfies the orthogonality conditions*

$$(6.1) \quad \sum_1^m \mu_i |T_n(z_i)|^{p-1} \operatorname{sg} [T_n(z_k)] \bar{g}(z_k) = 0,$$

for every polynomial $g(z)$ of degree less than n . The norm is a strongly convex function of the numbers $T_n(z_i)$, so the orthogonality conditions (6.1) determine $T_n(z)$ uniquely.

For an arbitrary polynomial $P(z) \equiv z^n + \dots$ and an arbitrary polynomial $g(z) \not\equiv 0$ of degree $n-1$, we compute at each point of E the deviation $|P + \epsilon g|^p$, $p > 1$, as a function of ϵ , where ϵ is real and positive.

Case 1: $g=0$. This is trivial:

$$|P + \epsilon g|^p = |P|^p.$$

Case 2: $g \neq 0, P = 0$. Here we have $|P + \epsilon g|^p \equiv \epsilon^p |g|^p$, and ϵ^p is convex in ϵ for $\epsilon \geq 0, p > 1$.

Case 3. $g \neq 0, P \neq 0$. We set $v = g/P$, whence for sufficiently small ϵ we have

$$\begin{aligned}
 |P + \epsilon g|^p &= [(1 + \epsilon v)^p (1 + \epsilon \bar{v})^p]^{1/2} |P|^p \\
 &= \left[\left(1 + p\epsilon v + \frac{p(p-1)}{2} \epsilon^2 v^2 + \dots \right) \right. \\
 &\quad \left. \cdot \left(1 + p\epsilon \bar{v} + \frac{p(p-1)}{2} \epsilon^2 \bar{v}^2 + \dots \right) \right]^{1/2} |P|^p \\
 (6.2) \quad &= \left[1 + p\epsilon(v + \bar{v}) + p^2 \epsilon^2 |v|^2 \right. \\
 &\quad \left. + \frac{p(p-1)}{2} \epsilon^2 (v^2 + \bar{v}^2) + \dots \right]^{1/2} |P|^p \\
 &= \left[1 + \frac{1}{2} p\epsilon(v + \bar{v}) + \frac{1}{2} p^2 \epsilon^2 |v|^2 + \frac{p(p-1)}{4} \epsilon^2 (v^2 + \bar{v}^2) \right. \\
 &\quad \left. - \frac{1}{8} p^2 \epsilon^2 (v + \bar{v})^2 + \dots \right] |P|^p.
 \end{aligned}$$

The coefficient of ϵ^2 here is one-eighth of

$$\begin{aligned}
 &p^2 [4|v|^2 - (v + \bar{v})^2] + 2p(p-1)(v^2 + \bar{v}^2) \\
 (6.3) \quad &= 2|v|^2 \left(p^2 + \frac{p(p-2)}{2} \left[\left(\frac{v}{|v|} \right)^2 + \left(\frac{\bar{v}}{|v|} \right)^2 \right] \right).
 \end{aligned}$$

The square bracket is not less than -2 nor greater than $+2$, so this entire expression is positive.

We have now shown that each term of the norm

$$\sum_1^m \mu_i |P(z_i)|^p, \quad p > 1,$$

is convex when considered as a function of the real and pure imaginary parts of the numbers $P(z_i)$. In the case $m \geq n$ not all the numbers $g(z_i)$ can vanish, so the deviation is strongly convex. In this case the orthogonality condition

$$(6.4) \quad \sum_1^m \mu_i [v(z_i) + \bar{v}(z_i)] |P(z_i)|^p = 0,$$

for every choice of $g(z)$, which is derived by equating to zero the coefficient of ϵ , is both necessary and sufficient that $P(z)$ be extremal. For an extremal polynomial is known to exist and to be unique, and a strongly convex func-

tion has at most one stationary point. The orthogonality condition can be written in various forms, such as

$$(6.5) \quad \sum_1^m \mu_i |P(z_i)|^{p-2} \operatorname{Re} [\bar{g}(z_i)P(z_i)] = 0.$$

A further equivalent form of the orthogonality condition is found by writing $v = g/P$ in (6.4), and then replacing g by ig . We may rewrite the resulting equations for the polynomial $P(z) \equiv T_n(z)$ as (6.1), a form that is especially useful if $T(z_i) \neq 0$. If $T(z_i) = 0$, of course $\operatorname{sg}[T(z_i)]$ is not defined, but nevertheless if $p > 1$ we may here interpret $|T(z_i)|^{p-1} \operatorname{sg}[T(z_i)]$ as zero.

We have previously [6, §9] used (6.1) when E is real and when therefore the unique $T_n(z)$ is real.

The relations (6.2) and (6.3) yield without further discussion some results even for $0 < p \leq 1$:

THEOREM 6.2. *Under the conditions of Theorem 6.1 except with $p > 1$ replaced by $0 < p \leq 1$, it remains true that every extremal polynomial $T_n(z)$ which is different from zero on E satisfies the orthogonality conditions (6.1).*

Theorem 6.1 has already been applied several times. We give two further results to show the power of Theorem 6.1.

THEOREM 6.3. *An extremal polynomial $T_n(z)$ of degree n on a set $E: (z_1, z_2, \dots, z_m)$, $m > n$, with norm $\sum_1^m \mu_k |T_n(z_k)|^p$, $\mu_k > 0$, $p > 1$, is also an extremal polynomial on E with norm $\sum_1^m \mu'_k |T_n(z_k)|^{p'}$ for arbitrary $p' (> 1)$ and suitably chosen $\mu'_k (> 0)$.*

The orthogonality condition characteristic of $T_n(z)$ with relation to p and μ_k is

$$\sum_1^m \mu_k |T_n(z_k)|^{p-1} \operatorname{sg}[T_n(z_k)] \bar{g}(z_k) = 0$$

for every polynomial $g(z)$ of degree $n - 1$, and even if T_n vanishes in some points of E this same condition can be written

$$\sum_1^m \mu'_k |T_n(z_k)|^{p'-1} \operatorname{sg}[T_n(z_k)] \bar{g}(z_k) = 0,$$

with $\mu'_k > 0$. The latter condition is likewise characteristic of $T_n(z)$ as extremal polynomial, now for exponent p' and weights μ'_k .

Theorem 6.3 gives a proof (for real E compare §5) of the fact that under the conditions of Theorem 6.3 with $p > 1$ but the μ_k unspecified, the class of extremal polynomials is independent of p .

Every extremal polynomial of degree n with norm (0.1), $1 < p < \infty$, on a finite set E containing $m (\geq n + 1)$ points is obviously an infrapolynomial. But

though the class C_1 of proper infrapolynomials (that is, infrapolynomials not vanishing in any point of E) is identical [Fekete, 3] with the class of proper extremal polynomials for $p = \infty$, namely for the norm

$$\max [\mu_k | T_k(z_k) |, z \text{ on } E], \quad \mu_k > 0,$$

and μ_k unspecified, this class need not coincide with the class C_2 of proper extremal polynomials with the norm (0.1), $1 < p < \infty$ (these classes coincide if E is real, by Theorems 5.2 and 5.3). For instance in the case $n = 1, m = 3$, let E consist of the vertices of a nondegenerate triangle; then C_2 consists of all linear polynomials $z + a$ which vanish interior to the triangle, whereas C_1 consists of C_2 plus those linear polynomials vanishing on the sides excluding the vertices.

THEOREM 6.4. *If the polynomial $T_n(z) \equiv (z - z_0)^n, n \geq 1$, is extremal for a set $E_m: (z_1, z_2, \dots, z_m)$ with norm $\sum_1^m \mu_k |T_n(z_k)|^p, \mu_k > 0, p > 1$, then $T_{n-1}(z) \equiv (z - z_0)^{n-1}$ is extremal for the set E_m with norm $\sum_1^m \mu_k |T_{n-1}(z_k)|^{np/(n-1)}$.*

We choose $z_0 = 0$, so the orthogonality conditions for $T_n(z)$ can be written

$$\sum \mu_k |z_k^n|^{p-1} \text{sg } z_k^n \cdot \bar{z}_k^j = 0, \quad j = 0, 1, \dots, n - 1.$$

However, $z_k = |z_k| \text{sg } z_k, \bar{z}_k = |z_k| / \text{sg } z_k$, so these equations for $j > 0$ become

$$\sum \mu_k |z_k^{n-1}|^{np/(n-1)-1} \text{sg } z_k^{n-1} \cdot \bar{z}_k^j = 0, \quad j = 0, 1, \dots, n - 2,$$

precisely the characteristic orthogonality conditions for $T_{n-1}(z)$ with the new norm.

Of course Theorem 6.4 can be iterated; the successive degrees of the polynomials are $n, n - 1, n - 2, \dots, 1$, and the successive exponents of the moduli of the polynomial in the norms are $p, np/(n - 1), np/(n - 2), \dots, np$.

Throughout our discussion we have consistently used non-negative weights, but under suitable conditions some of our results hold even if negative weights are admitted. We remark however that orthogonality conditions if negative weights are admitted are not of themselves sufficient to insure that a polynomial $P(x)$ has its zeros in the convex hull of E , even if $P(x)$ and E are real. As counter-example we exhibit the polynomial $P(x) \equiv x^2 + 1$, which is orthogonal ($p = 2$) on the set $E: (-1, 0, +1)$ with respective weights $(1, -4, 1)$ to both x and unity.

7. Approximation on an infinite set as the limit of approximation on a finite set. In [6] and the present paper we have investigated primarily approximation on a finite point set. It is ordinarily finite sets that are used in numerical computation, and such sets are important even in the study of approximation on an infinite set. According to a classical result due to de la Vallée Poussin, approximation in the sense of Tchebycheff on a closed bounded real set E to a function $f(z)$ continuous on E by a polynomial of

degree n is equivalent to approximation in the sense of Tchebycheff to $f(z)$ on a particular subset of E consisting of $n+2$ points by a polynomial of degree n . Other results can be proved, concerning approximation on a finite set as the set varies and approaches (in a certain sense) an infinite set.

THEOREM 7.1. *Let n be fixed, let E denote a closed bounded set containing infinitely many points, and let E_k ($k=1, 2, \dots$) denote a closed subset of E containing at least $n+1$ points such that given $\epsilon(>0)$, for every sufficiently large k each point of E is at a distance less than ϵ from some point of E_k .*

Let $\mu(z)$ be positive and continuous on E , let $f(z)$ be continuous on E , let $p(z)$ be the polynomial of degree n of best approximation to $f(z)$ on E in the sense that

$$(7.1) \quad \max [\mu(z) | f(z) - p(z) |, z \text{ on } E]$$

is least, and let $p_k(z)$ be the polynomial of degree n of best approximation to $f(z)$ on E_k in the analogous sense with $\mu(z)$ unchanged. Then for all z we have

$$\lim_{k \rightarrow \infty} p_k(z) = p(z),$$

uniformly on any closed bounded set of the plane.

The polynomials $p_k(z)$ are uniformly bounded on any closed bounded set of the plane. In fact, we may use Lagrange's interpolation formula

$$p_k(z) \equiv \sum_{i=1}^{n-1} \frac{p_k(z_i)\omega(z)}{\omega'(z_i)(z-z_i)}, \quad \omega(z) \equiv \prod_1^{n+1} (z-z_i).$$

Here we choose $n+1$ distinct points ζ_i of E , and require that the z_i although variable with k shall lie respectively in mutually disjoint closed neighborhoods of the ζ_i ; the numbers $\omega(z)/\omega'(z_i)(z-z_i)$ are uniformly bounded for all bounded z and for all z_i so chosen. For sufficiently large k , points z_i of E_k lie in the respective neighborhoods of the ζ_i . If we denote by M the least value of (7.1), we obviously have

$$(7.2) \quad M_k = \max [\mu(z) | f(z) - p_k(z) |, z \text{ on } E_k] \leq M,$$

for $p(z)$ is an admissible polynomial of degree n in considering approximation on E_k . Thus the numbers $p_k(z_i)$ are bounded uniformly with respect to i and k , and it follows from the boundedness of the $\omega(z)/\omega'(z_i)(z-z_i)$ that the $p_k(z)$ are uniformly bounded on any closed bounded set. The analyticity of the $p_k(z)$ implies that the $p_k(z)$ are equicontinuous on any closed bounded set.

Any sequence of the $p_k(z)$ admits a subsequence $p_{k_j}(z)$ which converges uniformly on any closed bounded set (in particular on E) to a function $p_0(z)$ which by Lagrange's interpolation formula is necessarily a polynomial of degree n . If z_0 lies on E , there exists z_{k_j} on E_{k_j} such that $z_{k_j} \rightarrow z_0$; we have $p_{k_j}(z_j) \rightarrow p_0(z_0)$, and also

$$\mu(z_{k_j}) | f(z_{k_j}) - p_{k_j}(z_{k_j}) | \rightarrow \mu(z_0) | f(z_0) - p_0(z_0) |.$$

The first members here are not greater than M , so the second member is not greater than M . Then $p_0(z)$ coincides with the (necessarily unique) polynomial $p(z)$, and the sequence $p_{k_j}(z)$ converges uniformly on E to $p(z)$. Every subsequence of the $p_k(z)$ admits a subsequence which converges uniformly on E to $p(z)$, so the sequence $p_k(z)$ converges uniformly on E (and on any bounded set of the plane) to $p(z)$.

Theorem 7.1 can be extended so that it applies even to sets E_k not subsets of E , provided we make the additional hypothesis that, given an ϵ -neighborhood of E , all E_k for k sufficiently large lie in that neighborhood. The functions $\mu(z)$ and $f(z)$ can be extended in definition so that they are defined and continuous over the entire plane; then $\mu(z)$ remains positive in the neighborhood of E , and M_k has a meaning for k sufficiently large. Inequality (7.2) is to be modified by adding $\epsilon_k (> 0)$ to the last member, where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$; this modified inequality follows by comparing $p_k(z)$ with $p(z)$ as an approximating polynomial on E_k , thanks to the continuity of $\mu(z)$, $f(z)$, and $p(z)$. The proof of the theorem can be carried out essentially as before. The set E may be finite, containing at least $n+1$ points. This extension of Theorem 7.1 is proved in the particular case $f(z) \equiv z^{n+1}$, $\mu(z) \equiv 1$ by Calugareanu [1].

We turn now to an analog of Theorem 7.1 for least p th powers. For the sake of simplicity, we choose E as a single interval of the axis of reals, but the theorem obviously extends to an arbitrary Riemann integral on a bounded set of the axis of reals, a line integral on a curve in the z -plane, or a double integral in the z -plane.

THEOREM 7.2. *Let E denote the interval $0 \leq x \leq 1$, and let E_k (namely the set $\{x_i\}$ depending on k) be a Riemann sequence for E associated with the intervals $\Delta_i x$. Let $\mu(x)$ be positive and continuous on E , $f(x)$ continuous on E , and let n and $p (> 0)$ be fixed. If $p_k(x)$ is a polynomial of degree n of best approximation to $f(x)$ on E_k in the sense that*

$$(7.3) \quad M_k = \sum_{E_k} \mu(x_i) |f(x_i) - p_k(x_i)|^p \Delta_i x$$

is least, then any sequence of the $p_k(x)$ admits a subsequence which converges uniformly on every closed bounded set of the complex plane, to a polynomial $p(x)$ of degree n of best approximation to $f(x)$ on E in the sense that

$$M = \int_0^1 \mu(x) |f(x) - p(x)|^p dx$$

is least. In the case $p > 1$, the polynomials $p_k(x)$ and $p(x)$ are unique, and the sequence $p_k(x)$ converges uniformly to $p(x)$ on every closed bounded set of the plane.

For each k divide E into a finite number of successive open or closed intervals, say of respective lengths $\Delta_1 x, \Delta_2 x, \dots, \Delta_m x$; the point x_i may be chosen

arbitrarily in the interval $\Delta_i x$ and the set E_k consists of these points x_i , assumed distinct. The entire sequence $\{E_k\}$ is to be chosen so that if $\delta (>0)$ is given, for k sufficiently large each interval $\Delta_i x$ is less than δ in length.

To prove that the polynomials $p_k(x)$ are uniformly bounded on any bounded set, we remark first that if $\Delta (>0)$ is arbitrary, then for k sufficiently large we have by comparison of $p_k(x)$ with $p(x)$ as approximating polynomials on E_k

$$(7.4) \quad M_k = \sum_{E_k} \mu(x_i) |f(x_i) - p(x_i)|^p \Delta_i x \leq M + \Delta;$$

this sum is a Riemann sum for M . As in the proof of Theorem 7.1, we next choose $n+1$ distinct points ζ_j of E , and mutually disjoint closed (one-dimensional) neighborhoods N_j of these points. For k sufficiently large, at least half of each N_j is covered by intervals $\Delta_i x$ having both initial and terminal points in that N_j . The sum of the lengths of these $\Delta_i x$ is at least half the total lengths of the N_j ; for these intervals the individual terms occur in the second member of (7.3), and the sum of such terms is by (7.4) not greater than $M+\Delta$. There exists at least one point (depending on k) in each N_j at which $p_k(x)$ is bounded uniformly for all k ; it followed by Lagrange's interpolation formula that the $p_k(x)$ are uniformly bounded on any closed bounded set and therefore equicontinuous on any closed bounded set.

If $F(x)$ is an arbitrary function continuous on E , the absolute value of the difference between the integral of $F(x)$ over E and a Riemann sum for $F(x)$ over E is not greater than $\omega(\delta)$, where $\omega(\delta)$ is a modulus of continuity for $F(x)$ on E , and δ is the greatest length of a subinterval in the mesh defining the Riemann sum.

If any subsequence of the $p_k(x)$ is given, there exists a new subsequence $p_{k_j}(x)$ converging uniformly on any closed bounded set of the plane to some polynomial $p_0(x)$ of degree n . The functions $\mu(x) |f(x) - p_k(x)|^p$ are equicontinuous on E , say with common modulus of continuity $\omega(\delta)$. Given $\epsilon (>0)$, for k_j sufficiently large we have

$$\left| M_{k_j} - \int_0^1 \mu(x) |f(x) - p_{k_j}(x)|^p dx \right| \leq \omega(\delta),$$

$$\left| \int_0^1 \mu(x) |f(x) - p_{k_j}(x)|^p dx - M_0 \right| \leq \epsilon,$$

$$M_0 = \int_0^1 \mu(x) |f(x) - p_0(x)|^p dx.$$

We can allow $\delta = \delta_{k_j}$ to approach zero as $k_j \rightarrow \infty$ whence $M_{k_j} \rightarrow M_0$. By the definition of M we have $M_0 \geq M$. But we also have by the definition of $p_{k_j}(x)$

$$M_{k_j} \leq \sum_{E_{k_j}} \mu(x_i) |f(x_i) - p(x_i)|^p \Delta_i x;$$

this last member approaches M , whence $M_0 \leq M$ and $M_0 = M$, so $p_0(x)$ is an extremal polynomial.

In the case $p > 1$, the extremal polynomial $p(x)$ is unique, every subsequence of the $p_k(x)$ admits a subsequence converging uniformly on every bounded set to $p(x)$, so the sequence $p_k(x)$ itself converges uniformly on every bounded set to $p(x)$.

In the case $0 < p < 1$, neither the extremal polynomials $p_k(x)$ nor $p(x)$ need be unique; but if $p = 1$ the $p_k(x)$ need not be unique although (as D. Jackson proved in 1921) $p(x)$ is unique, so every sequence $p_k(x)$ converges uniformly to $p(x)$ on every bounded set.

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